



**HAL**  
open science

# Sharp decay estimates for critical Dirac equations

William Borrelli

► **To cite this version:**

| William Borrelli. Sharp decay estimates for critical Dirac equations. 2018. hal-01865427v1

**HAL Id: hal-01865427**

**<https://hal.science/hal-01865427v1>**

Preprint submitted on 31 Aug 2018 (v1), last revised 11 Dec 2018 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# SHARP DECAY ESTIMATES FOR CRITICAL DIRAC EQUATIONS

WILLIAM BORRELLI

ABSTRACT. We prove sharp decay estimates for critical Dirac equations on  $\mathbb{R}^n$ , with  $n \geq 2$ .

They appear, e.g., in the study of critical Dirac equations on compact spin manifolds, describing blow-up profiles (the so-called *bubbles*) in the associated variational problem. We establish regularity and integrability properties of  $L^{2^\sharp}$ -solutions (where  $2^\sharp$  is the Sobolev critical exponent of the embedding of  $H^{\frac{1}{2}}(\mathbb{R}^n, \mathbb{C}^N)$  into Lebesgue spaces) and prove decay estimates, which are shown to be optimal proving the existence of a family of solutions having the prescribed asymptotic behavior.

## CONTENTS

1. Introduction and preliminaries	1
The Dirac operator	4
Lorentz spaces	5
The Green function of $\mathcal{D}$	6
2. A Liouville-type lemma	6
3. Decay estimates and regularity	8
4. Existence of a family of optimizers	12
References	17

## 1. INTRODUCTION AND PRELIMINARIES

This paper is devoted to the study of decay properties of solutions to nonlinear Dirac equations of the form

$$\mathcal{D}\psi = |\psi|^{2^\sharp-2}\psi, \quad \text{on } \mathbb{R}^n, \quad n \geq 2, \quad (1)$$

$2^\sharp := \frac{2n}{n-1}$  being the critical exponent for the embedding of

$$H^{\frac{1}{2}}(\mathbb{R}^n, \mathbb{C}^N) \hookrightarrow L^p(\mathbb{R}^n, \mathbb{C}^N).$$

We prove that  $L^{2^\sharp}$ -solutions to (1) have polynomial decay at infinity, also showing that the exact prescribed rate is attained by a family a solutions. This is in contrast with the case of *massive* nonlinear Dirac equation, for which it is showed in [11] that solutions have exponential decay, generalizing the method of [7] to deal with nonlinear bound states in any dimensions. Estimates for second order elliptic equations can be found e.g. in [32, 41, 40, 39] and in references therein. We also mention that sharp localization properties for eigenfunctions of perturbed Dirac operators have been investigated in [14]. Moreover,

recently critical Dirac equations in two dimensions have been studied as effective models for honeycomb structures. The reader can refer to [10, 9, 8, 19, 21] and references therein.

Equations of the form (1) appear, for instance, in the blow-up analysis of the variational problem associated with the equation

$$\mathcal{D}\psi = \mu\psi + |\psi|^{2^\sharp-2}\psi, \quad \text{on } M, \quad \mu \in \mathbb{R}, \quad (2)$$

where  $(M, g, \Sigma)$  is a compact spin manifold, that is, a compact riemannian manifold  $(M, g)$  carrying a spin structure  $\Sigma$  [33, 22]. In that case the  $L^2$ -spectrum of  $\mathcal{D}$  is discrete and composed of eigenvalues of finite multiplicity accumulating at  $\pm\infty$  (see e.g. [22, 33, 28]). Solutions to (2) can be found as critical point of the following  $C^1$ -functional

$$\mathcal{L}(\psi) = \frac{1}{2} \int_M \langle \mathcal{D}\psi, \psi \rangle d\text{vol}_g - \frac{\mu}{2} \int_M |\psi|^2 d\text{vol}_g - \frac{1}{2^\sharp} \int_{\mathbb{R}^n} |\psi|^{2^\sharp} d\text{vol}_g, \quad (3)$$

defined on  $H^{\frac{1}{2}}(\Sigma M)$  the space of  $H^{\frac{1}{2}}$ -sections of the spinor bundle  $\Sigma M$  of the manifold. Here  $d\text{vol}_g$  stands for the volume measure of  $(M, g)$ .

We remark that for  $\mu = 0$  equation (2) is referred to as the *spinorial Yamabe equation* and has been studied in [1, 4, 2]. Spinorial Yamabe-type equations have been studied on manifolds of bounded geometry in [25]. Such equations also arise in the study of the conformal geometry of manifolds, see e.g. [3, 2, 24, 35], and have been investigated by different techniques in the case of spheres  $\mathbb{S}^n$  [29, 30]. General existence and multiplicity results for subcritical equations on compact spin manifolds are contained in [27]. We also mention that for  $n = 2$  the spinorial Yamabe equation is related to a spinorial analogue of the supersymmetric extension of harmonic maps, the *Dirac-harmonic maps* [17, 16]. In [28] Isobe proved existence and  $C^{1,\alpha}$ -regularity of a (non-trivial) solution to (2), when  $\dim M \geq 4$  and  $\lambda \notin \sigma(\mathcal{D})$ . In the same paper [28] he analyzed the behavior of a generic Palais-Smale sequence  $(\psi_n)_{n \in \mathbb{N}} \subseteq H^{\frac{1}{2}}(\Sigma M)$  for the functional  $\mathcal{L}$  and proved that it decomposes as

$$\psi_n = \psi_\infty + \sum_{j=1}^M \omega_n^j + o(1), \quad \text{in } H^{\frac{1}{2}}(\Sigma M), \quad (4)$$

where  $\psi_\infty$  is the weak limit of  $(\psi_n)_n$  and the  $\omega_n^j$  are suitably rescaled spinors obtained mapping solutions to (1) to spinors on the manifold  $M$ . Moreover, the spinors  $\omega_n^j$  are centered around points  $a_n^j \rightarrow a^j \in M$ , as  $n \rightarrow \infty$ , and

$$\omega_n^j \rightharpoonup \delta_{a^j}, \quad n \rightarrow \infty,$$

weakly in the sense of measure,  $\delta_{a^j}$  being the delta measure concentrated in  $a^j$ . The above decomposition is related to the fact that the equation (2) is *critical* as we are dealing with the limiting Sobolev embedding  $H^{\frac{1}{2}}(\Sigma M) \hookrightarrow L^{2^\sharp}(\Sigma M)$ , where the (local) loss of compactness is due to the invariance with respect to scaling. Suitably identifying spinors on manifolds, as described in [26], one can map, after scaling, solutions to (1) to bubbles  $\omega_k^j$  on the manifold (see (4)). In this case invariance by scaling manifests itself in the symmetry of (1) with respect to the transformation

$$\psi(\cdot) \mapsto \lambda^{\frac{n-1}{2}} \psi(\lambda \cdot), \quad \lambda > 0. \quad (5)$$

More generally, as showed in [28], (1) is conformally invariant.

Remark that (4) is the spinorial analogue of the result of Struwe [36] for the Brezis-Nirenberg problem [13].

Solutions to the limiting problem (1) belonging to  $\dot{H}^{\frac{1}{2}}(\mathbb{R}^n, \mathbb{C}^N)$  correspond to critical points of the action functional

$$\mathcal{L}_0(\psi) = \frac{1}{2} \int_{\mathbb{R}^n} \langle \mathcal{D} \psi, \psi \rangle dx - \frac{1}{2^\sharp} \int_{\mathbb{R}^n} |\psi|^{2^\sharp} dx, \quad (6)$$

defined for spinors  $\psi \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^n, \mathbb{C}^N)$  (see Section 1). Invariance by scaling (5) makes the variational argument to find critical points of (6) more delicate, as can be seen from [10] where we treated the two-dimensional case, and whose proof can be easily adapted to the case  $n \geq 2$  (see also Remark 1.3). On the other hand, such symmetry guarantees the existence of non-trivial solutions. Indeed, as shown in [31], subcritical Dirac equations

$$\mathcal{D} \psi = |\psi|^{p-2} \psi, \quad \text{on } \mathbb{R}^n, \quad 2 < p < \frac{2n}{n-1}, \quad (7)$$

admit no non-trivial weak solutions  $\psi \in L^p(\mathbb{R}^n, \mathbb{C}^N)$ , essentially as a consequence of the fact that the two terms in (7) scale differently.

Another result proved in [28] consists in the following energy gap estimate for solutions to (1)

$$\mathcal{L}_0(\psi) \geq \frac{1}{2n} \left(\frac{n}{2}\right)^n \omega_n, \quad (8)$$

where  $\omega_n$  is the volume of the unit ball in  $\mathbb{R}^n$ .

However, to our knowledge besides this estimate general qualitative properties of solutions to (1) seem still to be investigated. We believe that understanding decay properties might be useful in further investigations of critical Dirac equations on spin manifolds.

**Definition 1.1.** *We say that  $\psi \in L^{2^\sharp}(\mathbb{R}^n, \mathbb{C}^N)$  is a weak solution to (1) if there holds*

$$\int_{\mathbb{R}^n} \langle \mathcal{D} \varphi, \psi \rangle dx = \int_{\mathbb{R}^n} |\psi|^{2^\sharp-2} \langle \varphi, \psi \rangle dx, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{C}^N). \quad (9)$$

The main result of the paper is the following

**Theorem 1.2.** *Any weak solution  $\psi \in L^{2^\sharp}(\mathbb{R}^n, \mathbb{C}^N)$  of (1) is such that*

$$\psi \in C^{1,\alpha}(\mathbb{R}^n, \mathbb{C}^N) \cap L^{\frac{n}{n-1},\infty}(\mathbb{R}^n, \mathbb{C}^N)$$

for some  $0 < \alpha < 1$  and satisfies the following decay estimate

$$|\psi(x)| \leq \frac{C}{(1+|x|^{n-1})}, \quad \forall x \in \mathbb{R}^n, \quad (10)$$

for some  $C > 0$ . As a consequence,

$$\psi \in L^p(\mathbb{R}^n, \mathbb{C}^N), \quad \text{for } \frac{n}{n-1} < p \leq \infty. \quad (11)$$

Moreover estimate (10) is optimal, as there exists a family of smooth solutions  $\varphi \in L^{2^\sharp}(\mathbb{R}^n, \mathbb{C}^N)$  to (1) such that

$$|\varphi(x)| \sim \frac{1}{|x|^{n-1}}, \quad \text{as } |x| \rightarrow +\infty. \quad (12)$$

This also proves that in general

$$\psi \notin L^{\frac{n}{n-1}}(\mathbb{R}^n, \mathbb{C}^N). \quad (13)$$

The proof of the theorem is achieved in two steps. First, regularity and decay estimates are proved in Section 3. Then Section 4 is devoted to the existence of solutions having the asymptotic behavior (12).

**Remark 1.3.** *The family of optimizers mentioned in the above theorem can be characterized as critical point of the action functional (6). More precisely, those solutions are ground states of  $\mathcal{L}_0$  in the sense that they have least action among all possible critical points of the action. However, they do not minimize the functional (6) which is strongly indefinite, even if one fixes the  $L^{2^\sharp}$ -norm, as a consequence of the unbounded negative spectrum of the Dirac operator (see e.g. [37]). The proof follows as in [10], with minor changes, combining duality and Nehari manifold arguments.*

**Remark 1.4.** *In [10] we studied a class of critical (cubic) Dirac equations in 2D with a nonlinearity of the form*

$$G_{\beta_1, \beta_2}(\psi)\psi = ((2\beta_2|\psi_1|^2 + \beta_1|\psi_2|^2)\psi_1, (\beta_1|\psi_1|^2 + 2\beta_2|\psi_2|^2)\psi_2)^T, \quad 0 < \beta_2 \leq \beta_1, \quad (14)$$

*which appears as effective model, e.g., in nonlinear optics and condensed matter physics (see [5, 20] and references therein). We proved the existence of a family of solutions satisfying (10), as already recalled in the previous Remark. It is easy to see that Theorem 1.2 applies to the case of the nonlinearity (14). More generally, we believe that Theorem 1.2 can be proved for more general nonlinearities of critical growth, under suitable assumptions, adapting the strategy developed in this paper. However, this is beyond the scopes of the present work.*

**The Dirac operator.** The Dirac operator on  $\mathbb{R}^n$  is defined as

$$\mathcal{D} := -i\boldsymbol{\alpha} \cdot \nabla = -i \sum_{j=1}^n \alpha_j \partial_{x_j}, \quad (15)$$

with  $\alpha_j$  being  $N \times N$  hermitian matrices satisfying the anticommutation relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{j,k} \mathbb{I}_N, \quad 1 \leq j, k \leq n, \quad (16)$$

$\delta_{j,k}$  and  $\mathbb{I}_N$  being the Kronecker symbol and the  $N \times N$  identity matrix, respectively. Here  $N = 2^{\lfloor \frac{n+1}{2} \rfloor}$ , where  $\lfloor \cdot \rfloor$  denotes the integer part of a real number. Matrices  $\alpha_j$  form a representation of the *Clifford algebra* of the euclidean space (see e.g. [22]). Different choices of the matrices satisfying (16) correspond to unitarily equivalent representations. In the sequel we will suppose the matrices  $(\alpha_j)_{j=1}^n$  to be fixed, without any reference to their particular form as our results are not affected by that choice. This only amounts to a unitary transformation on the space of Dirac spinors  $\mathbb{C}^N$ . For later purposes (see Section 4) we only assume matrices  $\alpha_j$  to have the following block-antidiagonal structure

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad 1 \leq j \leq n, \quad (17)$$

where the  $\sigma_j$  are  $(\frac{N}{2} \times \frac{N}{2})$  hermitian matrices satisfying analogous anticommutation relations as in (16):

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{j,k} \mathbb{I}_{N/2}, \quad 1 \leq j, k \leq n.$$

By (16) one can easily check that

$$\mathcal{D}^2 = (-\Delta) \mathbb{I}_N. \quad (18)$$

A more detailed presentation of Dirac operators and Clifford algebras can be found e.g. in [22, 33].

**Lorentz spaces.** In this section we collect some definitions and results about Lorentz spaces needed in the paper. They extend to Banach space-valued functions, replacing the absolute value with the norm of the space. We state them in the  $\mathbb{C}$ -valued case for simplicity.

Let  $\Omega \subseteq \mathbb{R}^n$  be a Lebesgue-measurable set and  $\sigma \in (0, \infty), \tau \in (0, \infty]$ . The *Lorentz space*  $L^{\sigma, \tau}(\Omega)$  is defined as the set of measurable function  $f : \Omega \rightarrow \mathbb{C}$  such that  $\|f\|_{L^{\sigma, \tau}(\Omega)} < +\infty$ , where

$$\|f\|_{L^{\sigma, \tau}(\Omega)} := \begin{cases} \sigma^{1/\tau} \left( \int_0^\infty h^{\tau-1} \mu(\{|f| > h\})^{\tau/\sigma} dh \right)^{1/\tau}, & \text{if } \tau < \infty \\ \sup_{h>0} \left( h^\sigma \mu(\{|f| > h\})^{1/\sigma} \right), & \text{if } \tau = \infty. \end{cases} \quad (19)$$

Here  $\mu$  denotes the  $n$ -dimensional Lebesgue measure. Recall that there holds

$$L^{\sigma, \sigma}(\Omega) = L^\sigma(\Omega), \quad \forall \sigma \in (0, \infty), \quad (20)$$

and

$$L^{\sigma, \tau_2}(\Omega) \subseteq L^{\sigma, \tau_1}(\Omega), \quad \forall \sigma \in (0, \infty], \quad \forall \tau_1, \tau_2 \in (0, \infty], \quad \tau_1 < \tau_2. \quad (21)$$

The reader can refer to the book of Grafakos [23] for a detailed presentation.

The following results contained in [34] extend Hölder and Young inequalities to Lorentz spaces.

**Lemma 1.5** (Hölder inequality). *Let  $\Omega \subseteq \mathbb{R}^n$  be Lebesgue-measurable and  $\sigma_1, \sigma_2, \sigma \in (0, \infty), \tau_1, \tau_2, \tau \in (0, \infty]$  such that*

$$\frac{1}{\sigma_1} + \frac{1}{\sigma_2} = \frac{1}{\sigma}, \quad \frac{1}{\tau_1} + \frac{1}{\tau_2} \geq \tau, \quad (22)$$

*adopting the convention that  $1/\infty = 0$ . Then there exists  $C > 0$  such that for any  $f_1 \in L^{\sigma_1, \tau_1}(\Omega)$  and  $f_2 \in L^{\sigma_2, \tau_2}(\Omega)$  there holds  $f_1 f_2 \in L^{\sigma, \tau}(\Omega)$ , with*

$$\|f_1 f_2\|_{L^{\sigma, \tau}(\Omega)} \leq C \|f_1\|_{L^{\sigma_1, \tau_1}(\Omega)} \|f_2\|_{L^{\sigma_2, \tau_2}(\Omega)}. \quad (23)$$

**Lemma 1.6** (Young inequality). *Let  $\sigma_1, \sigma_2, \sigma \in (0, \infty), \tau_1, \tau_2, \tau \in (0, \infty]$  such that*

$$\frac{1}{\sigma_1} + \frac{1}{\sigma_2} = \frac{1}{\sigma} + 1, \quad \frac{1}{\tau_1} + \frac{1}{\tau_2} \geq \tau, \quad (24)$$

*adopting the convention that  $1/\infty = 0$ . Then there exists  $C > 0$  such that for any  $f_1 \in L^{\sigma_1, \tau_1}(\mathbb{R}^n)$  and  $f_2 \in L^{\sigma_2, \tau_2}(\mathbb{R}^n)$  there holds  $f_1 * f_2 \in L^{\sigma, \tau}(\mathbb{R}^n)$ , with*

$$\|f_1 * f_2\|_{L^{\sigma, \tau}(\mathbb{R}^n)} \leq C \|f_1\|_{L^{\sigma_1, \tau_1}(\mathbb{R}^n)} \|f_2\|_{L^{\sigma_2, \tau_2}(\mathbb{R}^n)}, \quad (25)$$

where  $*$  denotes the convolution of two functions.

**Lemma 1.7** (A limit case of the Young inequality). *Let  $\sigma \in (0, \infty)$ . There exists  $C > 0$  such that for any  $f_1 \in L^1(\mathbb{R}^n)$  and  $f_2 \in L^{\sigma, \infty}(\mathbb{R}^n)$  there holds  $f_1 * f_2 \in L^{\sigma, \infty}(\mathbb{R}^n)$ , with*

$$\|f_1 * f_2\|_{L^{\sigma, \infty}(\mathbb{R}^n)} \leq C \|f_1\|_{L^1(\mathbb{R}^n)} \|f_2\|_{L^{\sigma, \infty}(\mathbb{R}^n)}. \quad (26)$$

For simplicity, in the following we will frequently omit the domain in the notation of functional spaces when this is clear from the context.

**The Green function of  $\mathcal{D}$ .** The *Green function*  $\Gamma$  of the Dirac operator  $\mathcal{D}$  is the matrix valued kernel

$$\Gamma : \mathbb{R}_x^n \times \mathbb{R}_y^n \setminus \{x = y\} \longrightarrow \text{End}(\mathbb{C}^N)$$

given by the distributional solution of the equation

$$\mathcal{D}_x \Gamma(x, y) = \delta(x - y) \mathbb{I}_N, \quad x \in \mathbb{R}^n \quad (27)$$

where  $\delta(x)$  is the delta distribution at  $x$ . The Green kernel is explicitly given, with an abuse of notation, by

$$\Gamma(x, y) = \Gamma(x - y) := -\frac{i}{n\omega_n} \boldsymbol{\alpha} \cdot \frac{x - y}{|x - y|^n} \quad (28)$$

where  $\omega_n$  is the volume of the unit ball in  $\mathbb{R}^n$  and  $\boldsymbol{\alpha} = (\alpha_j)_{j=1}^n$ . Formula (28) can be immediately obtained recalling that

$$\mathcal{D}^2 = (-\Delta) \mathbb{I}_N,$$

and the well-known expression for the Green function of the laplacian. Moreover, by (28) one can easily check that

$$\Gamma \in L^{\frac{n}{n-1}, \infty}(\mathbb{R}^n, \mathbb{C}^N). \quad (29)$$

## 2. A LIOUVILLE-TYPE LEMMA

**Lemma 2.1** (A Liouville-type lemma for harmonic spinors). *Fix  $p \geq 1$ . Let  $\psi \in L^p(\mathbb{R}^n, \mathbb{C}^N)$  be a weakly harmonic spinor, that is, a weak solution to*

$$\mathcal{D} \psi = 0. \quad (30)$$

Then  $\psi \equiv 0$ .

*Proof.* Let  $\psi \in L^p(\mathbb{R}^n, \mathbb{C}^N)$  be a weak solution to (30). Then

$$\int_{\mathbb{R}^n} \langle \psi, \mathcal{D} \chi \rangle dx = 0, \quad \forall \chi \in C_c^\infty(\mathbb{R}^n, \mathbb{C}^N). \quad (31)$$

Consider a family  $(\rho_\varepsilon)_{\varepsilon > 0}$  of smooth mollifiers  $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^n, \mathbb{R}_+)$  (see [18, Appendix C.5]) and define

$$\psi_\varepsilon(x) := (\rho_\varepsilon * \psi)(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(y) \psi(x - y) dy.$$

Since  $\psi \in L^1_{\text{loc}}(\mathbb{R}^n, \mathbb{C}^N)$ , by standard arguments there holds  $\psi_\varepsilon \in C^\infty(\mathbb{R}^n, \mathbb{C}^N)$ .

Given  $\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{C}^N)$ , we have by Fubini theorem

$$\begin{aligned} \int_{\mathbb{R}^n} \langle \psi_\varepsilon(x), \mathcal{D}_x \varphi(x) \rangle dx &= \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \rho_\varepsilon(y) \langle \psi(x-y), \mathcal{D}_x \varphi(x) \rangle \\ &= \int_{\mathbb{R}^n} dy \rho_\varepsilon(y) \int_{\mathbb{R}^n} dx \langle \psi(x-y), \mathcal{D}_x \varphi(x) \rangle \stackrel{z=x-y}{=} \int_{\mathbb{R}^n} dy \rho_\varepsilon(y) \int_{\mathbb{R}^n} dz \langle \psi(z), \mathcal{D}_z \varphi(y+z) \rangle. \end{aligned} \quad (32)$$

Remark that for all  $y \in \mathbb{R}^n$ ,  $\varphi(y + \cdot) \in C_c^\infty$ , and then by (31)

$$\int_{\mathbb{R}^n} dz \langle \psi(z), \mathcal{D}_z \varphi(y+z) \rangle = 0,$$

and thus

$$\int_{\mathbb{R}^n} \langle \psi_\varepsilon, \mathcal{D} \varphi \rangle dx = 0.$$

Being  $\varphi \in C_c^\infty$  arbitrary and since  $\psi_\varepsilon \in C^\infty$ , we conclude that

$$\mathcal{D} \psi_\varepsilon = 0,$$

in classical sense, and by (18)

$$-\Delta \psi_\varepsilon = 0.$$

Let  $\psi_\varepsilon = (\psi_\varepsilon^1, \dots, \psi_\varepsilon^N)^T$ , then all components  $\psi_\varepsilon^j : \mathbb{R}^n \rightarrow \mathbb{C}$  of the spinor  $\psi_\varepsilon$  are harmonic functions. Fix  $j \in \{1, \dots, N\}$ . Take  $x_0 \in \mathbb{R}^n$  and  $x \in B_r(x_0)$ , for a fixed  $r > 0$ . Well-known estimates on derivatives of harmonic functions (see [18, Theorem 4, Chapter 2.2]) give

$$|D^\beta \psi_\varepsilon^j(x)| \leq \frac{C_k}{r^{n+k}} \|\psi_\varepsilon^j\|_{L^1(B_r(x))} \leq \frac{C_k}{r^{n+k}} \|\psi_\varepsilon^j\|_{L^1(B_{2r}(x_0))}, \quad \forall x \in B_r(x_0), \quad (33)$$

for each multi-index  $\beta$  of order  $|\beta| = k \in \mathbb{N}$ .

Since  $\psi_\varepsilon^j \rightarrow \psi^j$  in  $L_{\text{loc}}^1$  as  $\varepsilon \rightarrow 0^+$  ([18, Appendix C.5]), in particular

$$\|\psi_\varepsilon^j\|_{L^1(B_{2r}(x_0))} \leq C,$$

uniformly in  $\varepsilon$ . Then by (33) one concludes that

$$\|D^\beta \psi_\varepsilon^j(x)\|_{L^\infty(B_r(x_0))} \leq C_k, \quad \forall x \in B_r(x_0),$$

for each multi-index  $\beta$  of order  $|\beta| = k \in \mathbb{N}$ .

Ascoli-Arzelà's theorem imply that we can extract a subsequence  $(\psi_k^j)_{k \in \mathbb{N}}$  such that

$$\psi_k^j \rightarrow \psi^j, \quad \text{in } C^2(B_r(x_0)), \quad \text{as } k \rightarrow +\infty.$$

In particular, we infer that  $\psi^j \in C^2$ , and  $-\Delta \psi^j = 0$ , for all  $j = 1, \dots, N$ . Moreover, recall that  $\psi^j \in L^p$ . Fix  $x \in \mathbb{R}^n$ . The mean-value property of harmonic functions and the Hölder inequality give

$$|\psi^j(x)| \leq \frac{1}{\omega_n r^n} \int_{B_r(x)} |\psi^j(y)| dy \lesssim \frac{1}{r^{n/p}} \|\psi^j\|_{L^p}, \quad \forall r > 0. \quad (34)$$

Letting  $r \rightarrow +\infty$  one gets  $\psi^j \equiv 0$ , thus concluding the proof.  $\square$



## 3. DECAY ESTIMATES AND REGULARITY

In this section we prove the  $C^1$ -regularity and the decay estimates stated in Theorem 1.2, dividing the proof in several intermediate steps. To this aim we borrow some ideas from [32, 41].

Let  $\psi \in L^{2^\sharp}(\mathbb{R}^n, \mathbb{C}^N)$  be a weak solution to (1).

**Lemma 3.1.** *There holds*

$$\psi = \Gamma * (|\psi|^{2^\sharp-2}\psi)$$

*Proof.* Recall that  $\Gamma \in L^{\frac{n}{n-1}, \infty}$ . Since  $\psi \in L^{2^\sharp}$  and  $2^\sharp = \frac{2n}{n-1}$ , we have  $|\psi|^{2^\sharp-2}\psi \in L^{\frac{2n}{n+1}}$ . Define

$$\varphi := \Gamma * (|\psi|^{2^\sharp-2}\psi).$$

Then Young inequality (24) gives

$$\varphi \in L^{2^\sharp}(\mathbb{R}^n, \mathbb{C}^N),$$

and there holds

$$\mathcal{D}\varphi = |\psi|^{2^\sharp-2}\psi,$$

in distributional sense. This implies that

$$\mathcal{D}(\psi - \varphi) = 0,$$

that is, the spinor  $(\psi - \varphi) \in L^{2^\sharp}$  is weakly harmonic and then the claim follows by Lemma 2.1.  $\square$

**Lemma 3.2.** *We have*

$$\psi \in L^\sigma(\mathbb{R}^n, \mathbb{C}^N), \quad \text{for all } \sigma \in \left(\frac{n}{n-1}, \frac{2n}{n-2}\right).$$

*Proof.* We argue by contradiction. Let  $\sigma \in \left(\frac{n}{n-1}, \frac{2n}{n-2}\right)$ , and assume that  $\psi \notin L^\sigma(\mathbb{R}^n, \mathbb{C}^N)$ . Then there exists a sequence  $(\varphi_k)_{k \in \mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^n, \mathbb{C}^N)$  such that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \langle \psi, \varphi_k \rangle dx = +\infty, \quad \text{and} \quad \|\varphi_k\|_{L^{\sigma'}} \leq 1, \quad \forall k \in \mathbb{N}, \quad \sigma' = \frac{\sigma}{\sigma-1}. \quad (35)$$

Moreover, since  $\psi \in L^{2^\sharp}$ , thanks to (35) we can choose  $\varphi_k$  so that

$$2 \int_{\mathbb{R}^n} \langle \psi, \varphi_k \rangle dx \geq S_k := \sup_{\varphi \in A_k} \int_{\mathbb{R}^n} \langle \psi, \varphi \rangle dx, \quad (36)$$

where

$$A_k := \left\{ \varphi \in L^{\sigma'} \cap L^{2^\sharp} : \|\varphi\|_{L^{\sigma'}} \leq \|\varphi_k\|_{L^{\sigma'}}, \|\varphi\|_{L^{2^\sharp}} \leq \|\varphi_k\|_{L^{2^\sharp}} \right\}$$

Remark that  $|\psi|^{2^\sharp-2} \in L^n(\mathbb{R}^n, \mathbb{R})$ . Then there exist

$$(f_h)_{h \in \mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^n, \mathbb{R}), \quad (g_h)_{h \in \mathbb{N}} \subseteq L^n(\mathbb{R}^n, \mathbb{R}),$$

such that

$$\lim_{h \rightarrow \infty} \|g_h\|_{L^n} = 0 \quad \text{and} \quad |\psi|^{2^\sharp-2} = f_h + g_h, \quad \forall h \in \mathbb{N}. \quad (37)$$

Using Lemma 3.1 we can write

$$\begin{aligned} \int_{\mathbb{R}^n} \langle \psi, \varphi_k \rangle dx &= \int_{\mathbb{R}^n} \langle \Gamma * (|\psi|^{2^\sharp-2} \psi), \varphi_k \rangle dx \\ &= \int_{\mathbb{R}^n} \langle \Gamma * (f_h \psi), \varphi_k \rangle dx + \int_{\mathbb{R}^n} \langle \Gamma * (g_h \psi), \varphi_k \rangle dx =: I_1 + I_2 \end{aligned} \quad (38)$$

Let us estimate  $I_1$ , using Hölder and then Young inequalities

$$\begin{aligned} |I_1| &\leq \left| \int_{\mathbb{R}^n} \langle \Gamma * (f_h \psi), \varphi_k \rangle dx \right| \leq \| \Gamma * (f_h \psi) \|_{L^\sigma} \| \varphi_k \|_{L^{\sigma'}} \\ &\leq \| \Gamma \|_{L^{\frac{n}{n-1}, \infty}} \| f_h \psi \|_{L^{\frac{n\sigma}{n+\sigma}}} \| \varphi_k \|_{L^{\sigma'}} \leq \| \Gamma \|_{L^{\frac{n}{n-1}, \infty}} \| f_h \psi \|_{L^{\frac{n\sigma}{n+\sigma}}} \end{aligned} \quad (39)$$

**Remark 3.3.** *In the above formula we need  $\sigma > \frac{n}{n-1}$ , in order to apply the Young inequality.*

Now we turn to the term  $I_2$ . Using Fubini theorem one finds, recalling that for all  $x, y \in \mathbb{R}^n$  with  $x \neq y$ ,  $\Gamma(x-y)$  is a hermitian matrix

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^n} \langle \Gamma * (g_h \psi), \varphi_k \rangle dx = \int_{\mathbb{R}^n} \left\langle \int_{\mathbb{R}^n} \Gamma(x-y) (g_h(y) \psi(y)) dy, \varphi_k(x) \right\rangle dx \\ &= \int_{\mathbb{R}^n} dy \int_{\mathbb{R}^n} dx \langle \Gamma(x-y) (g_h(y) \psi(y)), \varphi_k(x) \rangle = \int_{\mathbb{R}^n} dy \int_{\mathbb{R}^n} dx \langle g_h(y) \psi(y), \Gamma(x-y) \varphi_k(x) \rangle \\ &= \int_{\mathbb{R}^n} dy \langle g_h(y) \psi(y), \int_{\mathbb{R}^n} \Gamma(x-y) \varphi_k(x) dx \rangle = - \int_{\mathbb{R}^n} \langle g_h(y) \psi(y), (\Gamma * \varphi_k)(y) \rangle dy \end{aligned} \quad (40)$$

Using again Lemma 3.1 and arguing as for (40) we can rewrite the last integral in (40) as

$$\begin{aligned} \int_{\mathbb{R}^n} \langle g_h \psi, \Gamma * \varphi_k \rangle dy &= \int_{\mathbb{R}^n} g_h \langle \Gamma * (|\psi|^{2^\sharp-2} \psi), \Gamma * \varphi_k \rangle dy \\ &= - \int_{\mathbb{R}^n} \langle |\psi|^{2^\sharp-2} \psi, \Gamma * (g_h(\Gamma * \varphi_k)) \rangle dy \end{aligned} \quad (41)$$

Then one finds

$$I_2 = \int_{\mathbb{R}^n} \langle \psi, |\psi|^{2^\sharp-2} \Gamma * (g_h(\Gamma * \varphi_k)) \rangle dy. \quad (42)$$

Our aim now is to prove that

$$\chi_h := |\psi|^{2^\sharp-2} \Gamma * (g_h(\Gamma * \varphi_k)) \in A_k, \quad (43)$$

for  $h$  suitably large, but independent of  $k$ . This will be achieved by a repeated use of Hölder and Young inequalities, as follows.

We can estimate

$$\| \chi_h \|_{L^{2^+}} \leq \| |\psi|^{2^\sharp-2} \|_{L^n} \| \Gamma * (g_h(\Gamma * \varphi_k)) \|_{L^{\frac{2n}{n-1}, \frac{2n}{n+1}}}, \quad (44)$$

and

$$\| \Gamma * (g_h(\Gamma * \varphi_k)) \|_{L^{\frac{2n}{n-1}, \frac{2n}{n+1}}} \leq \| \Gamma \|_{L^{\frac{n}{n-1}, \infty}} \| g_h(\Gamma * \varphi_k) \|_{L^{\frac{2n}{n+1}}}. \quad (45)$$

Moreover, there holds

$$\| g_h(\Gamma * \varphi_k) \|_{L^{\frac{2n}{n+1}}} \leq \| g_h \|_{L^n} \| \Gamma * \varphi_k \|_{L^{\frac{2n}{n-1}, \frac{2n}{n+1}}} \leq \| g_h \|_{L^n} \| \Gamma \|_{L^{\frac{n}{n-1}, \infty}} \| \varphi_k \|_{L^{\frac{2n}{n+1}}}. \quad (46)$$

Combining (44,45,46) we get

$$\|\chi_h\|_{L^{2^+}} = o(\|\varphi_k\|_{L^{2^+}}), \quad (47)$$

for  $h \in \mathbb{N}$  large, thanks to (37). Arguing similarly, one finds

$$\|\chi_h\|_{L^{\sigma'}} = o(\|\varphi_k\|_{L^{\sigma'}}). \quad (48)$$

By (38,39,47,48) we deduce that

$$\int_{\mathbb{R}^n} \langle \psi, \varphi_k \rangle dx = \mathcal{O}(\|f_h \psi\|_{L^{\frac{n\sigma}{n+\sigma}, \sigma}}), \quad \text{for } h \text{ large but fixed, independently of } k \quad (49)$$

Recall that  $f_h \in C_c^\infty$ , and let  $\Omega = \text{supp}(f)$ . Then using (20,21) and the Hölder inequality

$$\|f_h \psi\|_{L^{\frac{n\sigma}{n+\sigma}, \sigma}(\Omega)} \leq \|f_h \psi\|_{L^{\frac{n\sigma}{n+\sigma}}(\Omega)} \lesssim \|f_h \psi\|_{L^{2^\sharp}(\mathbb{R}^n)} \lesssim \|\psi\|_{L^{2^\sharp}(\mathbb{R}^n)}, \quad (50)$$

thus contradicting (35)  $\square$

**Remark 3.4.** In (50) we used the assumption  $\sigma < \frac{2n}{n-2}$ .

**Lemma 3.5.** *There holds*

$$\psi \in L^{\frac{n}{n-1}, \infty}(\mathbb{R}^n, \mathbb{C}^N).$$

*Proof.* By Lemma 3.1 and (37) we have

$$\psi = \Gamma * (|\psi|^{2^\sharp-2} \psi) = \Gamma * (f_h \psi) + \Gamma * (g_h \psi) = \Gamma * (f_h \psi) + \Gamma * (g_h (\Gamma * (|\psi|^{2^\sharp-2} \psi))) =: F_h + G_h. \quad (51)$$

Let  $\sigma \in \left(\frac{n}{n-1}, \frac{2n}{n-2}\right)$ . A repeated use of Young and Hölder inequalities gives

$$\begin{aligned} \|G_h\|_{L^{\sigma, \infty}} &\leq \|\Gamma\|_{L^{\frac{n}{n-1}, \infty}} \|g_h (\Gamma * (|\psi|^{2^\sharp-2} \psi))\|_{L^{\frac{n\sigma}{n+\sigma}, \infty}} \\ &\leq \|\Gamma\|_{L^{\frac{n}{n-1}, \infty}} \|g_h\|_{L^n, \infty} \|\Gamma * (|\psi|^{2^\sharp-2} \psi)\|_{L^{\sigma, \infty}} \\ &\leq \|\Gamma\|_{L^{\frac{n}{n-1}, \infty}}^2 \|g_h\|_{L^n, \infty} \|\psi\|_{L^{\frac{n\sigma}{n+\sigma}}}, \end{aligned} \quad (52)$$

By Lemma 3.2, our choice of  $\sigma$  ensures that  $\|\psi\|_{L^{\frac{n\sigma}{n+\sigma}}} < \infty$  and (37) gives

$$\|G_h\|_{L^{\sigma, \infty}} = o(1), \quad \text{as } h \rightarrow +\infty. \quad (53)$$

Then (51) and (53) imply that

$$\|\psi\|_{L^{\sigma, \infty}} \leq 2\|\Gamma * (f_h \psi)\|_{L^{\sigma, \infty}},$$

and taking the limit as  $\sigma \rightarrow \left(\frac{n}{n-1}\right)^+$  we get

$$\|\psi\|_{L^{\frac{n}{n-1}, \infty}} \leq 2\|\Gamma * (f_h \psi)\|_{L^{\frac{n}{n-1}, \infty}}. \quad (54)$$

Now, the limit case of the Young inequality (26) gives

$$\|\Gamma * (f_h \psi)\|_{L^{\frac{n}{n-1}, \infty}} \leq \|\Gamma\|_{L^{\frac{n}{n-1}, \infty}} \|f_h \psi\|_{L^1} \lesssim \|\Gamma\|_{L^{\frac{n}{n-1}, \infty}} \|\psi\|_{L^{2^\sharp}}, \quad (55)$$

exploiting the fact that  $f_h \in C_c^\infty$ , as done for (50). By (54) and (55) we deduce the claim.  $\square$

**Lemma 3.6.** *The function  $\psi$  is of class  $C^{1, \alpha}(\mathbb{R}^n, \mathbb{C}^N)$ , for some  $0 < \alpha < 1$ .*

*Proof.* Lemma 3.2 gives that  $\psi \in L^\sigma$ , for  $\frac{n}{n-1} < \sigma < \frac{2n}{n-2}$ .

Observe that if  $n = 2$ , we have  $\psi \in L^p$ , for  $2 < p < \infty$ . In the case  $n \geq 3$  and we will obtain higher integrability by a bootstrap argument.

The assumptions on  $\sigma$  ensure that

$$|\psi|^{2^\sharp-2}\psi \in L^{2+\varepsilon_n}, \quad \text{for all } 0 < \varepsilon_n < \frac{4}{(n+1)(n-2)}.$$

Remark that by (16) there holds

$$|\mathcal{D}\psi| = |\nabla\psi|,$$

and since

$$\mathcal{D}\psi = |\psi|^{2^\sharp-2}\psi,$$

we conclude, using the Sobolev embedding, that

$$\psi \in \dot{W}^{1,2+\varepsilon_n}(\mathbb{R}^n, \mathbb{C}^N) \hookrightarrow L^{p_n}(\mathbb{R}^n, \mathbb{C}^N), \quad \text{with } p_n > \frac{2n}{n-2}.$$

Iterating this argument, one deduces that there exists  $r_n > n$  such that

$$\psi \in \dot{W}^{1,r_n}(\mathbb{R}^n, \mathbb{C}^N) \hookrightarrow C^{0,\alpha_n}(\mathbb{R}^n, \mathbb{C}^N), \quad \text{for some } 0 < \alpha_n < 1,$$

by Morrey embedding theorem. Then the claim follows using Schauder estimates for the Dirac operator [1, Chapter 3].  $\square$

**Remark 3.7.** *One may also prove  $C^{1,\alpha}$ -regularity of solutions to (1) adapting the result proved in [28, Appendix].*

**Lemma 3.8.** *The closed set  $Z := \{x \in \mathbb{R}^n : \psi(x) = 0\}$  has zero Lebesgue measure, and*

$$\psi \in C^2(\mathbb{R}^n \setminus Z, \mathbb{C}^N).$$

*Proof.* The main result of [6] ensures that  $Z$  has Hausdorff dimension at most  $(n-2)$ , and then its  $n$ -dimensional Lebesgue measure is zero. Then we have

$$|\psi|^{2^\sharp-2}\psi \in C^1(\mathbb{R}^n \setminus Z, \mathbb{C}^N),$$

and local Schauder estimates [1, Chapter 3] for (1) give the claim.  $\square$

**Lemma 3.9.** *Estimate (10) holds.*

*Proof.* Take  $R \geq 1$  and define

$$\psi_R(x) = R^{n-1}\psi(Rx), \quad \forall x \in \mathbb{R}^n. \quad (56)$$

By Lemma 3.8,  $\psi_R$  is of class  $C^2$  outside its nodal set  $Z$ , which has zero Lebesgue measure. Then a direct computation using (1,18,56) gives

$$\begin{aligned} -\Delta |\psi_R(x)| &= \frac{1}{|\psi_R(x)|} \langle \psi_R(x), -\Delta \psi_R(x) \rangle = \frac{1}{|\psi_R(x)|} \langle \psi_R(x), \mathcal{D}^2 \psi_R(x) \rangle \\ &= \frac{R^{n+1}}{|\psi(Rx)|} \left[ \underbrace{\langle \psi(Rx), -i(\boldsymbol{\alpha} \cdot \nabla (|\psi(Rx)|^{2^\sharp-2})\psi(Rx)) \rangle}_{=0} + \langle \psi(Rx), |\psi(Rx)|^{2(2^\sharp-2)}\psi(Rx) \rangle \right] \\ &= R^{n+1} \langle \psi(Rx), |\psi(Rx)|^{2 \times 2^\sharp - 3} \psi(Rx) \rangle = R^{-2} |\psi_R(x)|^{\frac{n+3}{n-1}}, \quad \text{for almost every } x \in \mathbb{R}^n. \end{aligned} \quad (57)$$

This implies

$$-\Delta |\psi_R| \leq |\psi_R|^{\frac{n+3}{n-1}}, \quad \text{for almost every } x \in \mathbb{R}^n, \quad (58)$$

and then  $\psi_R$  is a weak subsolution to equation (58).

Weak Harnack-type inequalities proved in [38] imply that for  $\sigma > 1$ ,

$$\|\psi_R\|_{L^\infty(B_4 \setminus B_2)} \leq C_\sigma \|\psi_R\|_{L^\sigma(B_5 \setminus B_1)}, \quad (59)$$

where  $B_r := \{x \in \mathbb{R}^n : |x| \leq r\}$ ,  $r > 0$ . Then if we choose  $1 < \sigma < \frac{n}{n-1}$ , Hölder inequality gives

$$\|\psi_R\|_{L^\sigma(B_5 \setminus B_1)} \leq \tilde{C}_{\sigma'} \|\psi_R\|_{L^{\frac{n}{n-1}\infty}(B_5 \setminus B_1)}. \quad (60)$$

Remark that there holds

$$\|\psi_R\|_{L^{\frac{n}{n-1}\infty}(B_5 \setminus B_1)} = \|\psi\|_{L^{\frac{n}{n-1}\infty}(B_{5R} \setminus B_R)} \leq \|\psi\|_{L^{\frac{n}{n-1}\infty}(\mathbb{R}^n)}, \quad \forall R \geq 1. \quad (61)$$

Combining (59,60,61) and Lemma 3.5 we get

$$\|\psi_R\|_{L^\infty(B_4 \setminus B_2)} \leq C, \quad \forall R \geq 1. \quad (62)$$

Then (56) and the continuity of  $\psi$  give

$$|\psi(x)| \leq \frac{C}{R^{n-1}}, \quad \text{if } 2R \leq |x| \leq 4R, \quad (63)$$

for all  $R \geq 1$ , and thus

$$|\psi(x)| \leq \frac{C}{|x|^{n-1}}, \quad \text{if } |x| \geq 2. \quad (64)$$

The claim (10) follows thanks to the continuity of  $\psi$ .  $\square$

This last step concludes the first part of the proof of Theorem 1.2.

#### 4. EXISTENCE OF A FAMILY OF OPTIMIZERS

In order to complete the proof of Theorem 1.2, we now prove the existence of a family of smooth solutions to (1) satisfying (12). We adapt the method of [10] which allows, exploiting a suitable ansatz, to convert (1) into a dynamical system and to get the desired result. As already mentioned, those solutions admit a variational characterization as *ground states* of the functional (6), adapting the proof given in [10] for the two-dimensionale case.

Let us consider the following *Soler/Wakano-type* ansatz as, e.g., in [12]. Take  $\mathbf{n} \in \mathbb{C}^{N/2}$  with  $|\mathbf{n}| = 1$ , and define

$$\varphi(x) = \begin{pmatrix} v(r)\mathbf{n} \\ iu(r)\left(\frac{x}{r} \cdot \boldsymbol{\sigma}\right)\mathbf{n} \end{pmatrix}, \quad r = |x|, \quad u, v : (0, \infty) \longrightarrow \mathbb{R}, \quad \boldsymbol{\sigma} = (\sigma_j)_{j=1}^n, \quad (65)$$

with  $\sigma_j$  as in (17). Plugging (65) into (1) one obtains the following system for  $(u, v)$

$$\begin{cases} u' + \frac{n-1}{r}u = v(u^2+v^2)^{1/(n-1)} \\ v' = -u(u^2+v^2)^{1/(n-1)} \end{cases} \quad (66)$$

where  $u' := \frac{du}{dr}$ , and similarly for  $v$ .

Thus we are lead to study the flow of the above system. In particular, since we are looking for  $L^{2\sharp}$ -solutions, we are interested in solutions to (66) such that

$$(u(r), v(r)) \longrightarrow (0, 0) \quad \text{as} \quad r \rightarrow +\infty$$

In order to avoid singularities and to get non-trivial solutions, we choose as initial conditions

$$u(0) = 0 \quad , \quad v(0) = \lambda \neq 0 \tag{67}$$

The symmetry of the system allows us to consider only the case  $\lambda > 0$ . An analogous result holds for  $\lambda < 0$ . We are going to prove the following

**Proposition 4.1.** *For any  $\lambda > 0$  there exists a unique solution*

$$(u_\lambda, v_\lambda) \in C^\infty([0, +\infty), \mathbb{R}^2)$$

*of the Cauchy problem (66,67).*

*Moreover, there holds*

$$u_\lambda(r), v_\lambda(r) > 0, \quad \forall r > 0, \tag{68}$$

*and*

$$u_\lambda(r) \sim \frac{1}{r^{n-1}}, \quad v_\lambda(r) \sim \frac{1}{r^n}, \quad \text{as} \quad r \rightarrow +\infty. \tag{69}$$

The proof of Prop. 4.1 is divided in several intermediate steps.

Multiplying the first equation in (66) by  $r^{n-1}$  allows to rewrite (66) in integral form as

$$\begin{cases} u(r) = \frac{1}{r^{n-1}} \int_0^r s^{n-1} v(s) (u^2(s) + v^2(s))^{1/(n-1)} ds \\ v(r) = \lambda - \int_0^r u(s) (u^2(s) + v^2(s))^{1/(n-1)} ds \end{cases} \tag{70}$$

where the integrands in the r.h.s. are locally Lipschitz continuous functions of  $(u, v)$ . Then a contraction mapping argument as in [15] gives the following

**Lemma 4.2.** *For any  $\lambda > 0$  there exist  $0 < R_\lambda \leq +\infty$  and  $(u, v) \in C^1([0, R_\lambda), \mathbb{R}^2)$  unique maximal solution to (66), which depends continuously on  $\lambda$  and uniformly on  $[0, R]$  for any  $0 < R < R_\lambda$ .*

Let  $(u_\lambda, v_\lambda)$  be the (maximal) solution corresponding to a fixed  $\lambda > 0$ . Dropping the singular term in (66) we obtain a hamiltonian system

$$\begin{cases} \dot{u} = v(u^2 + v^2)^{1/n-1} \\ \dot{v} = -u(u^2 + v^2)^{1/(n-1)} \end{cases} \tag{71}$$

with hamiltonian

$$H(u, v) = \frac{n-1}{2n} (u^2 + v^2)^{n/(n-1)}. \tag{72}$$

Consider

$$H_\lambda(r) := H(u_\lambda(r), v_\lambda(r)) \tag{73}$$

then a simple computation gives

$$H'_\lambda(r) = -\frac{n-1}{r} u_\lambda^2(r) (u_\lambda^2(r) + v_\lambda^2(r))^{1/n-1} \leq 0 \tag{74}$$

so that the energy  $H$  is non-increasing along the solutions of (66). Then  $\forall r \in [0, R_x)$ ,  $(u_\lambda(r), v_\lambda(r)) \in \{H(u, v) \leq H(0, \lambda)\}$ , the latter being a compact set. Thus there holds

**Lemma 4.3.** *Every solution to (66) is global.*

**Remark 4.4.** *Additionally, basic ODE theory implies smoothness of solutions to (66), that is*

$$(u_\lambda, v_\lambda) \in C^\infty([0, \infty), \mathbb{R}^2).$$

In view of the above remarks, one expects the solutions to (66) to be close to the hamiltonian flow (71). This is indeed true, as stated in the following

**Lemma 4.5.** *Let  $(f, g)$  be the solution of (71) with initial data  $(f_0, g_0)$ . Let  $(u_k^0, v_k^0)$  and  $\rho_k$  be such that*

$$\rho_k \xrightarrow{k \rightarrow +\infty} +\infty \quad \text{and} \quad (u_k, v_k) \xrightarrow{k \rightarrow +\infty} (f_0, g_0)$$

*Consider the solution of*

$$\begin{cases} u'_k + \frac{u_k}{r + \rho_k} = v_k(u_k^2 + v_k^2)^{1/(n-1)} \\ v'_k = -u_k(u_k^2 + v_k^2)^{1/(n-1)} \end{cases}$$

*such that  $u_k(0) = u_k^0$  and  $v_k(0) = v_k^0$ . Then  $(u_k, v_k)$  converges to  $(f, g)$  uniformly on bounded intervals.*

The proof is the same as in [15]. The above results allows us to obtain some informations on the asymptotic behavior of  $(u_\lambda, v_\lambda)$ .

**Proposition 4.6.** *For any  $\lambda > 0$ , we have*

$$u_\lambda(r), v_\lambda(r) > 0, \quad \forall r > 0. \quad (75)$$

*and*

$$\lim_{r \rightarrow +\infty} (u_\lambda(r), v_\lambda(r)) = (0, 0). \quad (76)$$

*Proof.* A direct computation using (66) gives

$$\frac{d}{dr}(r^{n-1}u_\lambda(r)v_\lambda(r)) = r^{n-1}(v_\lambda^2 - u_\lambda^2)(u_\lambda^2 + v_\lambda^2)^{1/(n-1)}, \quad (77)$$

and

$$\frac{d}{dr}(r^n H_\lambda(r)) = \left(\frac{n-1}{2}\right) r^{n-1}(v_\lambda^2 - u_\lambda^2)(u_\lambda^2 + v_\lambda^2)^{1/(n-1)}. \quad (78)$$

Combining (77) and (78) and integrating gives

$$\left(\frac{n-1}{2}\right) u_\lambda(r)v_\lambda(r) = rH_\lambda(r) \quad (79)$$

and (75) follows,  $H_\lambda$  being positive definite.

Combining (75) and the second equation in (66) one sees that  $v'_\lambda(r) \leq 0$  for all  $r > 0$ , and then

$$\exists \lim_{r \rightarrow +\infty} v_\lambda(r) =: \mu \geq 0. \quad (80)$$

Moreover, since  $u$  is bounded, there exists a sequence  $r_n \uparrow +\infty$  such that

$$\exists \lim_{k \rightarrow +\infty} u_\lambda(r_k) = \delta \geq 0. \quad (81)$$

We claim that

$$\lim_{r \rightarrow +\infty} u_\lambda(r) = \delta. \quad (82)$$

By contradiction, suppose that (82) does not hold. Then there exist  $\varepsilon > 0$  and another sequence  $s_k \uparrow +\infty$  such that

$$|u_\lambda(s_k) - \delta| \geq \varepsilon \geq 0, \quad \forall k \in \mathbb{N}. \quad (83)$$

Up to subsequences, we can suppose that

$$\lim_{k \rightarrow +\infty} u_\lambda(s_k) = \gamma \neq \delta, \quad (84)$$

for some  $\gamma \geq 0$ . Recall that  $H$  decreases along the flow of (66), as shown in (74), and then

$$\exists \lim_{r \rightarrow +\infty} H_\lambda(r) = h \geq 0. \quad (85)$$

Then it follows that

$$(\delta, \mu), (\gamma, \mu) \in \{H(u, v) = h\}. \quad (86)$$

It is easy to see that the algebraic equation for  $u$

$$H(u, \mu) = h, \quad (87)$$

has only one non-negative solution and thus  $\delta = \gamma$ , reaching a contradiction. This proves the claim (82), and then there holds

$$\lim_{r \rightarrow +\infty} (u_\lambda(r), v_\lambda(r)) = (\delta, \mu). \quad (88)$$

Let  $(\rho_k)_k \subseteq \mathbb{R}$  be a sequence such that

$$\lim_{k \rightarrow +\infty} \rho_k = +\infty, \quad \lim_{k \rightarrow +\infty} (u_\lambda(\rho_k), v_\lambda(\rho_k)) = (\delta, \mu) \quad (89)$$

and consider the solution  $(U, V)$  to (71) such that

$$(U(0), V(0)) = (\delta, \mu).$$

By (Lemma 4.5), it follows that  $(u_\lambda(\rho_k + \cdot), v_\lambda(\rho_k + \cdot))$  converges uniformly to  $(U, V)$  on bounded intervals. But since

$$\lim_{k \rightarrow +\infty} (u_\lambda(\rho_k + r), v_\lambda(\rho_k + r)) = (\delta, \mu), \quad \forall r > 0, \quad (90)$$

this implies that

$$(U(r), V(r)) = (\delta, \mu), \quad \forall r > 0 \quad (91)$$

and thus  $(\delta, \mu) = (0, 0)$  as the latter is the only equilibrium of the hamiltonian system (71). This proves (76).  $\square$

**Proposition 4.7.** *For large  $r > 0$ , we have*

$$\frac{1}{r^{2(n-1)}} \lesssim u_\lambda^2(r) + v_\lambda^2(r) \lesssim \frac{1}{r^{n-1}}. \quad (92)$$



*Proof.* By (74) we easily get

$$H'_\lambda(r) \geq -\frac{2n}{r}H_\lambda(r),$$

and thus by the comparison principle for ODE

$$H_\lambda(r) \gtrsim \frac{1}{r^{2n}}, \quad \text{for } r > 0 \text{ large.}$$

Then (72) gives the lower bound in (92). Moreover, using (79) and the definition of  $H$  we have

$$\frac{(n-1)}{2n}r(u_\lambda^2(r) + v_\lambda^2(r))^{n/(n-1)} = \frac{(n-1)}{2}u_\lambda(r)v_\lambda(r) \leq \frac{(n-1)}{2} \frac{(u_\lambda^2(r) + v_\lambda^2(r))}{2}, \quad \forall r > 0,$$

and the second inequality in (92) easily follows.  $\square$

We are now in a position to prove (12). Multiplying the first equation in (66) by  $r^{n-1}$  one can rewrite it as

$$\frac{d}{dr}(r^{n-1}u_\lambda(r)) = r^{n-1}v_\lambda(r)(u_\lambda^2(r) + v_\lambda^2(r))^{1/(n-1)}. \quad (93)$$

Then (93) implies that the function  $f(r) := r^{n-1}u_\lambda(r)$  is strictly increasing, as  $v_\lambda > 0$ , and thus

$$\exists \lim_{r \rightarrow +\infty} f(r) =: l \in (0, \infty]. \quad (94)$$

Suppose that

$$l = +\infty, \quad (95)$$

and then

$$u_\lambda(r) \geq \frac{1}{r^{n-1}}, \quad \text{for } r > 0 \text{ large.} \quad (96)$$

Combining (96) and the lower bound in (4.7), the second equation in (66) gives

$$v'(r) \lesssim -\frac{1}{r^{n+1}}, \quad (97)$$

and then

$$v_\lambda(r) \lesssim \frac{1}{r^n}, \quad \text{for } r > 0 \text{ large.} \quad (98)$$

Integrating (93) we obtain

$$f(r) = \int_0^r v_\lambda(s)(u_\lambda^2(s) + v_\lambda^2(s))^{1/(n-1)}s^{n-1}ds \lesssim \int_1^{+\infty} \frac{ds}{s^2} < +\infty, \quad \forall r > 0, \quad (99)$$

where we have used the upper bound in (92) and (98). This contradicts (95) and thus  $0 < l < +\infty$  and

$$u_\lambda(r) \sim \frac{1}{r^{n-1}}, \quad \text{for } r > 0 \text{ large.} \quad (100)$$

By (98) and the second equation in (66), one gets

$$v'_\lambda(r) \sim -\frac{1}{r^{n+1}},$$

and then

$$v_\lambda(r) \sim \frac{1}{r^n}, \quad \text{for } r > 0 \text{ large.} \quad (101)$$

Then (12) is proved, thus concluding the proof of Theorem 1.2.

## REFERENCES

- [1] B. AMMANN, *A variational problem in conformal spin geometry*, Habilitationsschrift, Universität Hamburg, 2003.
- [2] ———, *The smallest Dirac eigenvalue in a spin-conformal class and cmc immersions*, *Comm. Anal. Geom.*, 17 (2009), pp. 429–479.
- [3] B. AMMANN, J.-F. GROSJEAN, E. HUMBERT, AND B. MOREL, *A spinorial analogue of Aubin’s inequality*, *Math. Z.*, 260 (2008), pp. 127–151.
- [4] B. AMMANN, E. HUMBERT, AND B. MOREL, *Mass endomorphism and spinorial Yamabe type problems on conformally flat manifolds*, *Comm. Anal. Geom.*, 14 (2006), pp. 163–182.
- [5] J. ARBUNICH AND C. SPARBER, *Rigorous derivation of nonlinear Dirac equations for wave propagation in honeycomb structures*, *J. Math. Phys.*, 59 (2018), pp. 011509, 18.
- [6] C. BÄR, *Zero sets of solutions to semilinear elliptic systems of first order*, *Invent. Math.*, 138 (1999), pp. 183–202.
- [7] A. BERTHIER AND V. GEORGESCU, *On the point spectrum of Dirac operators*, *J. Funct. Anal.*, 71 (1987), pp. 309–338.
- [8] W. BORRELLI, *Stationary solutions for the 2D critical Dirac equation with Kerr nonlinearity*, *J. Differential Equations*, 263 (2017), pp. 7941–7964.
- [9] ———, *Multiple solutions for a self-consistent Dirac equation in two dimensions*, *J. Math. Phys.*, 59 (2018), pp. 041503, 13.
- [10] ———, *Weakly Localized States for Nonlinear Dirac Equations*, submitted - ArXiv e-prints, (2018).
- [11] N. BOUSSAÏD AND A. COMECH, *On spectral stability of the nonlinear Dirac equation*, *J. Funct. Anal.*, 271 (2016), pp. 1462–1524.
- [12] ———, *Nonrelativistic asymptotics of solitary waves in the Dirac equation with Soler-type nonlinearity*, *SIAM J. Math. Anal.*, 49 (2017), pp. 2527–2572.
- [13] H. BRÉZIS AND L. NIRENBERG, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, *Comm. Pure Appl. Math.*, 36 (1983), pp. 437–477.
- [14] B. CASSANO, *Sharp exponential localization for eigenfunctions of the Dirac Operator*, ArXiv e-prints, (2018).
- [15] T. CAZENAVE AND L. VÁZQUEZ, *Existence of localized solutions for a classical nonlinear Dirac field*, *Comm. Math. Phys.*, 105 (1986), pp. 35–47.
- [16] Q. CHEN, J. JOST, J. LI, AND G. WANG, *Regularity theorems and energy identities for Dirac-harmonic maps*, *Math. Z.*, 251 (2005), pp. 61–84.
- [17] ———, *Dirac-harmonic maps*, *Math. Z.*, 254 (2006), pp. 409–432.
- [18] L. C. EVANS, *Partial differential equations*, vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, second ed., 2010.
- [19] C. L. FEFFERMAN AND M. I. WEINSTEIN, *Honeycomb lattice potentials and dirac points*, *J. Amer. Math. Soc.*, 25 (2012), pp. 1169–1220.
- [20] ———, *Waves in honeycomb structures*, *Journées équations aux dérivées partielles*, (2012).
- [21] ———, *Wave packets in honeycomb structures and two-dimensional Dirac equations*, *Comm. Math. Phys.*, 326 (2014), pp. 251–286.
- [22] T. FRIEDRICH, *Dirac operators in Riemannian geometry*, vol. 25 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2000. Translated from the 1997 German original by Andreas Nestke.
- [23] L. GRAFAKOS, *Classical Fourier analysis*, vol. 249 of Graduate Texts in Mathematics, Springer, New York, third ed., 2014.
- [24] N. GROSSE, *On a conformal invariant of the Dirac operator on noncompact manifolds*, *Ann. Global Anal. Geom.*, 30 (2006), pp. 407–416.

- [25] ———, *Solutions of the equation of a spinorial Yamabe-type problem on manifolds of bounded geometry*, Comm. Partial Differential Equations, 37 (2012), pp. 58–76.
- [26] N. HITCHIN, *Harmonic spinors*, Advances in Math., 14 (1974), pp. 1–55.
- [27] T. ISOBE, *Existence results for solutions to nonlinear Dirac equations on compact spin manifolds*, Manuscripta Math., 135 (2011), pp. 329–360.
- [28] ———, *Nonlinear Dirac equations with critical nonlinearities on compact Spin manifolds*, J. Funct. Anal., 260 (2011), pp. 253–307.
- [29] ———, *A perturbation method for spinorial Yamabe type equations on  $S^m$  and its application*, Math. Ann., 355 (2013), pp. 1255–1299.
- [30] ———, *Spinorial Yamabe type equations on  $S^3$  via Conley index*, Adv. Nonlinear Stud., 15 (2015), pp. 39–60.
- [31] ———, *Morse-Floer theory for superquadratic Dirac equations, I: relative Morse indices and compactness*, J. Fixed Point Theory Appl., 19 (2017), pp. 1315–1363.
- [32] E. JANNELLI AND S. SOLIMINI, *Concentration estimates for critical problems*, Ricerche Mat., 48 (1999), pp. 233–257. Papers in memory of Ennio De Giorgi (Italian).
- [33] J. JOST, *Riemannian geometry and geometric analysis*, Universitext, Springer, Heidelberg, sixth ed., 2011.
- [34] R. O’NEIL, *Convolution operators and  $L(p, q)$  spaces*, Duke Math. J., 30 (1963), pp. 129–142.
- [35] S. RAULOT, *A Sobolev-like inequality for the Dirac operator*, J. Funct. Anal., 256 (2009), pp. 1588–1617.
- [36] M. STRUWE, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z., 187 (1984), pp. 511–517.
- [37] B. THALLER, *The Dirac equation*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1992.
- [38] N. S. TRUDINGER, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math., 20 (1967), pp. 721–747.
- [39] J. VÉTOIS, *Decay estimates and a vanishing phenomenon for the solutions of critical anisotropic equations*, Adv. Math., 284 (2015), pp. 122–158.
- [40] ———, *A priori estimates and application to the symmetry of solutions for critical  $p$ -Laplace equations*, J. Differential Equations, 260 (2016), pp. 149–161.
- [41] ———, *Decay estimates and symmetry of finite energy solutions to elliptic systems in  $R^n$* , Indiana University Mathematics Journal. To appear, (2018).

UNIVERSITÉ PARIS-DAUPHINE, PSL RESEARCH UNIVERSITY, CNRS, UMR 7534, CEREMADE, F-75016 PARIS, FRANCE.

*E-mail address:* borrelli@ceremade.dauphine.fr