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# Meta-mathematical knowledge about proof

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*We deal in this paper with a particular difficulty with proof and proving at the undergraduate level, which concerns knowledge about proof at a meta-level. Some undergraduate students' difficulties or mistakes observed in their proof texts have been related to lack of that meta-knowledge. In order to test this hypothesis, interviews with a sample of students have been performed. Relationships with the didactic contract have been discussed.*

*Keywords: Proof, meta-knowledge, theorem, undergraduate students.*

## Introduction

Most mathematics teaching at all school levels is concentrated on teaching content; at the university level, students learn about functions, differential equations, matrices and integrals, by manipulating definitions and theorems. In order to assimilate the content, students are asked to solve problems and prove statements. The difficulties of students with proof have been largely investigated in research (Moore, 1994; Epp, 2003; Selden & Selden 2007; Harel & Sowder, 1998); some of these difficulties are related with the fact that students do not know mathematics at the meta-level, particularly as it concerns proof (Morselli, 2007; Hemmi, 2008). Several students do not see clearly the difference between a definition and a theorem, the difference between an example and a counter example. Knowledge about proof at the meta-level is neither presented in textbooks nor in courses of specific mathematical disciplines, but it makes one of the most important differences between mathematicians and students. In this paper, we will consider in particular the meta-knowledge about proof (MKP), such as the knowledge of the notion of proof and the rules related to how a proof must be organized. Many researchers acknowledge the fact that high school and university students do not understand what is meant by “proof” and “proving” (e.g. Schoenfeld 1989, Harel & Sowder 1998). “To most undergraduates, convincing their teacher (and thereby earning satisfactory grades) is typically the most important reason for constructing a proof” (Weber, 2004, p. 429) and “unfortunately, many students believe that they either know how to solve a problem (prove a theorem) or they don't, and thus, if they don't make progress within a few minutes, they give up” (Selden & Selden 2007, p. 96); students often believe that non-deductive arguments constitute a proof, or “an argument is a proof if it is presented by or approved by an established authority, such as a teacher or a famous mathematician” (Weber, 2003, p. 3); other different interpretations and conceptions of students regarding proof are described in Harel & Sowder (1998) and in Recio & Godino (2001). Meta-knowledge about proof is used implicitly by mathematicians when they construct proofs, “what may be assumed contextually and what needs to be explicitly proved, using logical deduction and previously established results, is highly non-trivial and, I would suggest, is implicit rather than explicit in the minds of most mathematicians” (Tall, 2002, p. 3). Our focus in this paper will be on the lack of MKP and what it might cause as difficulties to students when constructing proofs. The present study is developed using a past empirical study with undergraduates that consisted of investigating students' difficulties by analyzing their proof texts

responding to different tests (Azrou, 2015). We would like to examine if the following hypothesis is supported by an interview analysis: Is lack of MKP one of the reasons behind the messy proof texts? Moreover, we would like to answer the following question: Why students do not develop MKP?

## **Theoretical framework**

We choose the definition of proof stated by Durand-Guerrier et al. in (Durand-Guerrier, Boero, Douek, Epp, Tanguay, 2012), inspired from the Vergnaud's conceptual fields (Vergnaud, 1990). According to Vergnaud, a concept (in our case proof) is learnt by acquiring three components: the set of different representations (oral, written, formal, etc), the situations of reference (proof in geometry, in algebra, in calculus, etc.) and the operatory invariants (related to the logical structure of proof: legitimated inference rules, status of hypotheses, thesis, axioms, etc). Mastering the MKP is mastering the concept of proof according to Vergnaud, as it was stated by Durand-Guerrier et al. (2012).

We are interested in comparing MKP that students acquire with how they have been presented proofs and how they have been taught MKP by their teachers. As we consider the relationship with the teaching regarding proof and proving, we will be referring to the didactic contract (Brousseau, 1988) that is defined as a set of rules framing the mathematical practices of teachers and students under the constraints of the teacher-students institutional relationships. Most of these rules regarding how, why and what teachers do mathematically (and students should learn to do) are implicit and thereby not declared by teachers, who often suppose that students would assimilate them over time and practice. Often, teachers use some particular intentions and rules with the proofs exposed to students, without being aware of and without feeling the importance of explaining them; consequently, sometimes students are misguided to make correct proofs. Let us take the example of proof writing: 'the processes used by mathematicians are often rough and informal, but students typically see proofs in their final forms, and rarely witness the process of creating a "rough draft", as a result, students often do not know where to begin when writing their own proofs' (Moore, R. C., 1994). We will examine what kind of MKP students learn from their teachers and how they manifest it.

## **Methods**

We are more interested in examining in students' proof texts their MKP considering the three components of the definition of Durand-Guerrier et al. (2012); students' behaviors will be checked regarding definitions and other used mathematical statements (mathematical argument), regarding their modes of reasoning and argumentation and how they expressed them and presented them. Especially, how lack of MKP is manifested through students' proof texts and their interviews. A test composed of open questions (to which the proof cannot be procedural but rather syntactic or semantic (Weber, 2004)) have been submitted to 98 undergraduates during their third academic year, for a complex analysis course in a high level school of engineers in Algeria. The written language is French, but often, Arabic dialect is used, along with French, in the oral form. The analysis of students' proof texts indicates that one of the difficulties behind writing messy and disorganized proof texts to open questions was lack of MKP (Azrou, 2015). To receive further evidence for our findings, interviews were performed with a sample of students. We have chosen

fourteen students to interview whose proof texts contained well-organized, less organized and very disorganized proof texts. Our aim was to investigate, by analyzing students' words, whether they master the concept of proof at a conscious level, in other words if they have a mastery of its operator invariants (Vergnaud, 1990).

### **A-priori analysis of the test**

The test contained three questions, but in scope of this paper we can only include the first question.

*1- Is it possible to find a holomorphic function that admits 0 as a simple pole such that Residue of f at 0 is 0 ( $\text{Res}(f, 0) = 0$ )?*

By designing such questions, we aimed at ascertaining if students were able to construct the proof, based on known definitions and theorems, in a clear argumentation form by providing their own way of expressing the answering to the question. The question is about the possibility of having  $(P \Rightarrow Q)$  and its negation  $(P \text{ and } \overline{Q})$  at the same time, which results in a contradiction and thus it is impossible. Logically speaking: the fact that the residue at a point is not 0 is a direct consequence of that point being a simple pole for the function. We have chosen to refer to the point 0 to simplify the formula. There was no doubt that students knew all these concepts because they had used them many times before, but always when performing direct calculations and procedures. However no request of identifying and exploring the links between concepts had been made, especially in a written form.

### **A preliminary analysis of students' proof texts**

We have observed in students' proof texts, among others, the following behaviors related to the concept of proof:

- Lack of justifications: students do not know when the justification is necessary and when it is not; they might give a justification for an obvious fact and miss to justify a non-obvious statement.
- Students turn around confusing the hypothesis and the thesis (forwards and backwards between the premise and the result).
- An example is given instead of a justification to prove that the statement is true.
- Incomplete mathematical statements and/or formulas.
- Missing details that make holes in the proof.
- Lack of organization of proof steps.
- Disconnection between statements.
- Writing the proof text like a draft or a sketch.

### **Examples of students' proof texts**

The following excerpts show some of the difficulties cited before; the language used is French.

**Proof text 1**

Exercice 1: Est-il possible de trouver une fonction qui admet (0) comme pôle simple tel que  $\text{Res}(f, 0) = 0$ ? Justifier.

il est impossible de trouver cette fonction  
 $\text{Res}(f, a) = \lim_{z \rightarrow a} (z-a) f(z)$   
~~on ne peut pas le trouver~~  
~~non on ne peut pas le trouver~~  
 exp.  $f(z) = \frac{1}{z(z-1)}$   $\text{Res } f(z) = \lim_{z \rightarrow 0} z \cdot \frac{1}{z(z-1)} = \lim_{z \rightarrow 0} \frac{1}{z-1} = -1$

In this proof, the student responds by saying that such function does not exist and gives an example of a function that does not verify the conditions given by the statement. Clearly, the existence of a function that does not verify the conditions does not tell why these conditions cannot go together. This student considers that giving such example is the proof of the inexistence of functions verifying the two conditions.

**Proof text 2**

Exercice 1: Est-il possible de trouver une fonction qui admet (0) comme pôle simple tel que  $\text{Res}(f, 0) = 0$ ? Justifier.

Soit  $z_0$  est pôle simple  $\Rightarrow \lim_{z \rightarrow z_0} f(z) \neq 0 \Rightarrow \text{Res}(f, z_0) = \frac{1}{z_0} \lim_{z \rightarrow z_0} z f(z)$   
 $\Rightarrow \lim_{z \rightarrow z_0} z f(z) = 0 \Rightarrow \lim_{z \rightarrow 0} z f(z) = \lim_{z \rightarrow 0} z f(z) = 0$

The student does not provide an answer to the question, the proof is a series of statements; each one derived from the previous one by an implication, but without any justification; moreover the last three ones are similar but incorrect, they present the simple pole definition, but the limit should be not 0.

**Interviews**

Based on the analysis of students' written productions, in the wider investigation this study belongs to, we have conducted interviews to address our previous questions, but also to receive more information about students' points of view. We will present only the questions of the interviews that deal with MKP about proof. The main interview contained three questions, each with three or four sub-questions.

Q1- If your answer would have been addressed to another teacher, would you have written it the same way?

a- What is important, to a teacher, to see in a student response to questions like this one?

b- Do you think another teacher, not familiar with the course, would have understood the answer?

c- How can the teacher know if the answer is right or not?

d- If the question has been proposed in homework, would you have presented it in a different way?

Q3- If a rigorous mathematician would have answered to this question, how would he presented his answer?

a- What is the difference from your answer text and those we find in mathematics books?

b- After this time, looking again to your answer, is there something you would add or change in yours answer or would you keep it as it is?

### Results of interviews

**Q1:** Four students said ‘yes’, while the rest (ten) said ‘no’. They intended their responses to be given especially to their own teacher, so they made their responses intentionally focusing on what is important to the teacher which is, according to them, their reasoning and showing that they got the idea of the process and understood enough the concept at stake in order to get the credit or a part of it. *‘I know that my teacher will understand it even if it’s not complete’.*

**Q1.a:** All students responded that the teacher would check in a proof whether a student got the whole idea of the solution or not: *‘the teacher would see always the method’; ‘the reasoning’; ‘the process of the proof’.*

**Q1.b:** Half of the students said ‘yes’ and the others said ‘no’: *‘no, because we are used to respond to get the credit, so we address the response to our teacher’.*

**Q1.c:** All of them responded mentioning the reasoning of the student (method, the logic in his response, whether it is convincing, if there is no contradiction): *‘the teacher would follow the reasoning of the student to find out if his understanding is clear or not about the concept’.*

**Q1.d:** Twelve students answered ‘no’ and only two students said ‘yes’. They would keep the idea or the method the same but make better the organization or the presentation: *‘I would have changed the way I wrote, ... the organization’; ‘I would have given more details’.*

**Q3:** Two students (good ones) said ‘the same’; one didn’t answer clearly, six said with more details and/or better organization; two said with better reasoning and three said with more symbols: *‘a mathematician would have another goal, mine is to give the response and get the credit’; ‘he would use only symbols till getting the final result, you see, I wrote a lot of comments’.*

**Q3.a:** Twelve students said that they would contain more symbols and less comments; with an academic rigorous style: *‘it’s different’; ‘my answer is addressed to the teacher while mathematics books are addressed to all’; ‘with more symbols and less comments’.*

**Q3.b:** Four students among fourteen answered by keeping their text as they are. Five said they would improve the organization, three said they would add more details and two said they would make the explanation better: *‘I might keep the idea, but I will give more details’; ‘I would write it better’.*

## Data analysis

### Analysis of the written texts

Different students' weaknesses emerge from the analysis of students' written texts (difficulty of communication, lack of justification, using incomplete mathematical statements (or formulas) and lack of organization of the proof); the last three are of particular interest for MKP. Failing to give justification may be caused by the didactic contract supposing that the teacher would not mind it, by a lack of concept mastery or by a lack of meta-knowledge about proof. Mathematical statements are given incomplete because students might suppose that they are clear for the teacher, or because they are not well mastered by them or not important to be given complete in a proof text, which is related to MKP. The lack of organization of the statements displayed by students might be originated in didactic contract, in lack of concepts mastery, but also in lack of MKP.

### Results of interviews analysis

The answers to Q1, Q1.b and Q1.c. confirm that students, when writing their proof texts, intend to address it particularly to their own teacher. The answers to Q1.d show that students are aware of their unclear text and possible missing details. According to them, they have to focus on two important points that have the same objective: how to get the most part of the credit and show to their teacher that they understood the concept at stake by presenting the main idea or the method of the proof; because they believe that the teacher will focus on that. This shapes their meta-knowledge about proof writing. Most answers to the third question and to Q3.a support more details would be given by a rigorous mathematician and mathematics textbooks, students mention that the organization would be better in both cases than theirs – but they reveal how their conception about proofs in mathematics concerns superficial aspects when they say that proofs in textbooks contain more symbols and less comments and words in comparison with their proofs and do not mention the structure of the proof. Answers to question Q3.b confirm that students are aware that their proof texts need improvement – but it must be related to previous consideration about superficial aspects.

## Conclusion

Students' texts and interviews offered strong evidence for students' lack of MKP and its influence on proof writing. Students have many situations of reference for proofs at their disposal but do not master the operatorial invariants of the proof concept and the form of the proof texts as conscious objects. Findings suggest particularly that the influence of the didactic contract is strong. Teachers generally write proofs in a direct linear way, making unfolding the steps till the conclusion. Students learn to do the same: when they first set some ideas about how to solve a problem, they write their first exploratory draft as a final text because they were never shown how to go further to the written proof text. Here, the didactic contract works against to the development of MKP because the contractual knowledge substitutes the knowledge about the concept of proof. An important element emerged in the interviews, which is the intention of the students to write the proof text only for their teachers, which supports our hypothesis of lack of MKP. Students acknowledge that their texts miss details, but do not see that these missing details would make the organization of the different parts of the proof clearer. This shows that the MKP and the didactical contract are strongly related. When students compare their texts with mathematicians' or textbooks' proofs, they only point out to symbols and comments, they do not see that in these perfect proofs, the statements are

linked through a deductive process from the hypothesis to the proof end, the proof text is organized, not only in its form, but also in the structure; avoiding holes, disconnections and missing justifications. This is evidence of students' superficial perception of proof texts, which indicates lack of mastery of proof structure and representation as a concept, which is related to lack at the operatory invariants level. As MKP is also built up through language, we hypothesize that students' weak mastery of French language, especially in the oral form, which should be translated to the written form, might have contributed to their unclear written texts.

Let us examine now why students are not able to develop their MKP; it seems that they are stuck in a constant perception of proof that does not help them to overcome their difficulties, if not causing some of them, and as long as there are not alternative ways of presenting proofs, they will hold on it. "Students need to understand that proofs are not generally conceived of in the order they are written" (Selden & Selden, 2007, p. 114) and that "successful reasoning can be carried out both by relying on the logic and formal structures of syntactic reasoning, and by relying on the informal representations of mathematical objects of semantic reasoning" (CadwalladerOlsker, 2011, p. 48). Changing or adjusting the didactic contract may favor students' autonomy to understand and make proofs; university teachers often mistakenly think that undergraduates understand what a proof is and how to make proofs by following the standard presented proofs. In fact, "while a traditional definition-theorem-proof style of lecture presentation may convey the content in the most efficient way, there are other ways of presenting proofs that may enable students to gain more insight" (Selden & Selden, 2010, p. 411). Teachers should provide samples of proof construction instead of final products, to be clear about what do they expect from students when they are asked to prove and to provide an opportunity to learn how to make proofs. "In general, professors should avoid "dumbing down" their assessments by asking routine questions that can be answered by mimicking. One needs to modify the "didactic contract" in order to achieve this; otherwise, questions requiring genuine problem solving and proving will be considered "unfair" " (Selden & Selden, 2010, p.414). We support that "university teachers should consider including a good deal of student-student and teacher-student interaction regarding students' proof attempts, as opposed to just presenting their own or textbook's proofs" (cf. Selden & Selden, 2007, p. 114). Finally, in order to gain control, students need to master meta-knowledge about proof; "the difficulty of students to reach a structural axiomatic proof scheme suggests that a capstone course including some attention to meta-mathematics as a topic might be of value to mathematics majors" (Harel and Sowder, 1998, p. 280).

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