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SOME CONSTRUCTIONS OF MULTIPLICATIVE n -ARY HOM-NAMBU ALGEBRAS

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ABSTRACT. We show that given a Hom-Lie algebra one can construct the n -ary Hom-Lie bracket by means of an $(n-2)$ -cochain of given Hom-Lie algebra and find the conditions under which this n -ary bracket satisfies the Filippov-Jacobi identity, there by inducing the structure of n -Hom-Lie algebra. We introduce the notion of Hom-Lie n -uplet system which is the generalization of Hom-Lie triple system. We construct Hom-Lie n -uplet system using a Hom-Lie algebra.

INTRODUCTION

The first instances of n -ary algebras in Physics appeared with a generalization of the Hamiltonian mechanics proposed in 1973 by Nambu [26]. More recent motivation comes from string theory and M-branes involving naturally an algebra with ternary operation called Bagger-Lambert algebra which gives impulse to a significant development. It was used in [11] as one of the main ingredients in the construction of a new type of supersymmetric gauge theory that is consistent with all the symmetries expected of a multiple M2-brane theory: 16 supersymmetries, conformal invariance, and an $SO(8)$ R-symmetry that acts on the eight transverse scalars. On the other hand in the study of supergravity solutions describing M2-branes ending on M5-branes, the Lie algebra appearing in the original Nahm equations has to be replaced with a generalization involving ternary bracket in the lifted Nahm equations(see [10]).

In [4], generalizations of n -ary algebras of Lie type and associative type by twisting the identities using linear maps have been introduced. The notion of representations, derivations, cohomology and deformations are studies in [5, 17, 19, 24, 27]. These generalizations include n -ary Hom-algebra structures generalizing the n -ary algebras of Lie type including n -ary Nambu algebras, n -Lie algebra (called also n -ary Nambu-Lie algebras) and n -ary Lie algebras, and n -ary algebras of associative type including n -ary totally associative and n -ary partially associative algebras. In [7], a method was demonstrated of how to construct ternary multiplications from the binary multiplication of a Hom-Lie algebra, a linear twisting map, and a trace function satisfying certain compatibility conditions; and it was shown that this method can be used to construct ternary Hom-Nambu-Lie algebras from Hom-Lie algebras. This construction was generalized to n -Lie algebras and n -Hom-Nambu-Lie algebras in [8].

It is well known that the algebras of derivations and generalized derivations are very important in the study of Lie algebras and its generalizations. The notion of δ -derivations appeared in the paper of Filippov [16]. The results for δ -derivations and generalized derivations were studied by many authors. For example, R.Zhang and Y.Zhang [28] generalized the above results to the case of Lie superalgebras; Chen, Ma, Ni and Zhou considered the generalized derivations of color Lie algebras, Hom-Lie superalgebras and Lie triple systems [14, 15]. Derivations and generalized derivations of n -ary algebras were considered in [20, 21] and other. In [13], the authors generalize this results in Color n -ary Nambu Hom case.

This paper is organized as follows. In Section I we review basic concepts of Hom-Lie, n -ary Hom-Nambu algebras and n -Hom-Lie algebras. We also recall the definitions of derivations, α^k -derivations,

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α^k -quasiderivations and α^k -centroid. In Section II we provide a construction procedure of n-Hom-Lie algebras starting from a binary bracket of a Hom-Lie algebra and multilinear form satisfying certain conditions. To this end, we give the relation between α^k -derivations, (resp. α^k -quasiderivations and α^k -centroid) of Hom-Lie algebra and α^k -derivations, (resp. α^k -quasiderivations and α^k -centroid) of n-Hom-Lie algebras. In Section III we introduce the notion of Hom-Lie n -uplet system which is the generalisation of Lie n -uplet system which is introduced in [9]. We construct Hom-Lie n -uplet system using Hom-Lie algebra. Finally, we give a relation between α^k -quasiderivations of a Hom-Lie algebra and $(n + 1)$ -ary α^k -derivation of Hom-Lie n -uplet system associate.

1. HOM-LIE ALGEBRA AND N-ARY HOM-NAMBU ALGEBRAS

Throughout this paper, we will for simplicity of exposition assume that \mathbb{K} is an algebraically closed field of characteristic zero, even though for most of the general definitions and results in the paper this assumption is not essential.

1.1. Definitions. The notion of a Hom-Lie algebra was initially motivated by examples of deformed Lie algebras coming from twisted discretizations of vector fields(see [18, 22]). We will follow notation conventions in [25].

Definition 1.1. A Hom-Lie algebra is a triple $(\mathfrak{g}, [\cdot, \cdot], \alpha)$, where $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is a bilinear map and $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$ a linear map satisfying

$$[x, y] = -[y, x] \text{ (skew-symmetry)}$$

$$\bigcirc_{x,y,z} [\alpha(x), [y, z]] = 0 \text{ (Hom-Jacobi condition)}$$

for all x, y, z from \mathfrak{g} , where $\bigcirc_{x,y,z}$ denotes summation over the cyclic permutations of x, y, z .

Definition 1.2. A Hom-Lie algebras $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ is called multiplicative if $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ for all $x, y \in \mathfrak{g}$.

We define a linear map $ad : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ by $ad_x(y) = [x, y]$. Thus the Hom-Jacobi identity is equivalent to

$$(1.1) \quad ad_{[x,y]}(\alpha(z)) = ad_{\alpha(x)} \circ ad_y(z) - ad_{\alpha(y)} \circ ad_x(z), \text{ for all } x, y, z \in \mathfrak{g}.$$

Remark 1.3. An ordinary Lie algebras is a Hom-Lie algebras when $\alpha = id$.

Example 1.4. Let \mathcal{A} be the complex algebra where $\mathcal{A} = \mathbb{C}[t, t^{-1}]$ is the Laurent polynomials in one variable. The generators of \mathcal{A} are of the form t^n for $n \in \mathbb{Z}$.

Let $q \in \mathbb{C} \setminus \{0, 1\}$ and $n \in \mathbb{N}$, we set $\{n\} = \frac{1-q^n}{1-q}$, a q -number. The q -numbers have the following properties $\{n+1\} = 1 + q\{n\} = \{n\} + q^n$ and $\{n+m\} = \{n\} + q^n\{m\}$.

Let \mathfrak{A}_q be a space with basis $\{L_m, I_m | m \in \mathbb{Z}\}$ where $L_m = -t^m D$, $I_m = -t^m$ and D is a q -derivation on \mathcal{A} such that

$$D(t^m) = \{m\}t^m.$$

We define the bracket $[\cdot, \cdot]_q : \mathfrak{A}_q \times \mathfrak{A}_q \longrightarrow \mathfrak{A}_q$, with respect the super-skew-symmetry for $n, m \in \mathbb{Z}$ by

$$(1.2) \quad [L_m, L_n]_q = (\{m\} - \{n\})L_{m+n},$$

$$(1.3) \quad [L_m, I_n]_q = -\{n\}I_{m+n},$$

$$(1.4) \quad [I_m, I_n]_q = 0.$$

Let α be an even linear map on \mathfrak{A}_q defined on the generators by

$$\alpha_q(L_n) = (1 + q^n)L_n, \quad \alpha_q(I_n) = (1 + q^n)I_n,$$

The triple $(\mathfrak{A}_q, [\cdot, \cdot]_q, \alpha_q)$ is a Hom-Lie algebra, called the q -deformed Heisenberg-Virasoro algebra of Hom-type.

Example 1.5. We consider the matrix construction of the algebra $\mathfrak{sl}_2(\mathbb{R})$ generated by the following three vectors

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ; Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The defining relations are

$$[H, X] = 2X ; [H, Y] = -2Y ; [X, Y] = H$$

Let $\lambda \in \mathbb{R}^*$, consider the linear maps $\alpha_\lambda : \mathfrak{sl}_2(\mathbb{R}) \rightarrow \mathfrak{sl}_2(\mathbb{R})$ defined by

$$\alpha_\lambda(H) = H ; \alpha_\lambda(X) = \lambda^2 X ; \alpha_\lambda(Y) = \frac{1}{\lambda^2} Y$$

Note that α_λ is a Lie algebra automorphism.

In [3], the authors have shown that $(\mathfrak{sl}_2(\mathbb{R}))_\lambda = (\mathfrak{sl}_2(\mathbb{R}), [\cdot, \cdot]_{\alpha_\lambda}, \alpha_\lambda)$ is a family of multiplicative Hom-Lie algebras where the Hom-Lie bracket $[\cdot, \cdot]_{\alpha_\lambda}$ on the basis elements is given, for $\lambda \neq 0$ by

$$[H, X]_{\alpha_\lambda} = 2\lambda^2 X ; [H, Y]_{\alpha_\lambda} = -\frac{2}{\lambda^2} Y ; [X, Y]_{\alpha_\lambda} = H$$

Now, we recall the definitions of n -ary Hom-Nambu algebras and n -ary Hom-Nambu-Lie algebras, generalizing of n -ary Nambu algebras and n -ary Nambu-Lie algebras (called also Filippov algebras) respectively, which were introduced in [4] by Ataguema, Makhlouf and Silvestrov.

Definition 1.6. An n -ary Hom-Nambu algebra is a triple $(\mathcal{N}, [\cdot, \dots, \cdot], \tilde{\alpha})$ consisting of a vector space \mathcal{N} , an n -linear map $[\cdot, \dots, \cdot] : \mathcal{N}^n \rightarrow \mathcal{N}$ and a family $\tilde{\alpha} = (\alpha_i)_{1 \leq i \leq n-1}$ of linear maps $\alpha_i : \mathcal{N} \rightarrow \mathcal{N}$, satisfying

$$(1.5) \quad [\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1}), [y_1, \dots, y_n]] = \sum_{i=1}^n [\alpha_1(y_1), \dots, \alpha_{i-1}(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha_i(y_{i+1}), \dots, \alpha_{n-1}(y_n)],$$

for all $(x_1, \dots, x_{n-1}) \in \mathcal{N}^{n-1}$, $(y_1, \dots, y_n) \in \mathcal{N}^n$.

The identity (1.5) is called *Hom-Nambu identity*.

Let $X = (x_1, \dots, x_{n-1}) \in \mathcal{N}^{n-1}$, $\tilde{\alpha}(X) = (\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1})) \in \mathcal{N}^{n-1}$ and $y \in \mathcal{N}$. We define an adjoint map $\text{ad}(X)$ as a linear map on \mathcal{N} , such that

$$(1.6) \quad \text{ad}_X(y) = [x_1, \dots, x_{n-1}, y].$$

Then the Hom-Nambu identity (1.5) may be written in terms of adjoint map as

$$\text{ad}_{\tilde{\alpha}(X)}([x_n, \dots, x_{2n-1}]) = \sum_{i=n}^{2n-1} [\alpha_1(x_n), \dots, \alpha_{i-n}(x_{i-1}), \text{ad}_X(x_i), \alpha_{i-n+1}(x_{i+1}), \dots, \alpha_{n-1}(x_{2n-1})].$$

Definition 1.7. An n -ary Hom-Nambu algebra is a triple $(\mathcal{N}, [\cdot, \dots, \cdot], \tilde{\alpha})$ is called n -Hom-Lie algebra if the bracket $[\cdot, \dots, \cdot]$ is skewsymmetric i.e $[x_{\sigma(1)}, \dots, x_{\sigma(n)}] = (-1)^{\text{sign}(\sigma)} [x_1, \dots, x_n]$ for $\sigma \in S_n$.

Remark 1.8. When the maps $(\alpha_i)_{1 \leq i \leq n-1}$ are all identity maps, one recovers the classical n -ary Nambu algebras. The Hom-Nambu identity (1.5), for $n = 2$, corresponds to Hom-Jacobi identity (see [25]), which reduces to Jacobi identity when $\alpha_1 = \text{id}$.

Let $(\mathcal{N}, [\cdot, \dots, \cdot], \tilde{\alpha})$ and $(\mathcal{N}', [\cdot, \dots, \cdot]', \tilde{\alpha}')$ be two n -ary Hom-Nambu algebras where $\tilde{\alpha} = (\alpha_i)_{i=1, \dots, n-1}$ and $\tilde{\alpha}' = (\alpha'_i)_{i=1, \dots, n-1}$. A linear map $f : \mathcal{N} \rightarrow \mathcal{N}'$ is an n -ary Hom-Nambu algebras *morphism* if it satisfies

$$\begin{aligned} f([x_1, \dots, x_{2n-1}]) &= [f(x_1), \dots, f(x_{2n-1})]' \\ f \circ \alpha_i &= \alpha'_i \circ f \quad \forall i = 1, n-1. \end{aligned}$$

In the sequel we deal sometimes with a particular class of n -ary Hom-Nambu algebras which we call n -ary multiplicative Hom-Nambu algebras.

Definition 1.9. A *multiplicative n -ary Hom-Nambu algebra* (resp. *multiplicative n -Hom-Lie algebra*) is an n -ary Hom-Nambu algebra (resp. n -Hom-Lie algebra) $(\mathcal{N}, [\cdot, \dots, \cdot], \tilde{\alpha})$ with $\tilde{\alpha} = (\alpha_i)_{1 \leq i \leq n-1}$ where $\alpha_1 = \dots = \alpha_{n-1} = \alpha$ and satisfying

$$(1.7) \quad \alpha([x_1, \dots, x_n]) = [\alpha(x_1), \dots, \alpha(x_n)], \quad \forall x_1, \dots, x_n \in \mathcal{N}.$$

For simplicity, we will denote the n -ary multiplicative Hom-Nambu algebra as $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ where $\alpha : \mathcal{N} \rightarrow \mathcal{N}$ is a linear map. Also by misuse of language an element $x \in \mathcal{N}^n$ refers to $x = (x_1, \dots, x_n)$ and $\alpha(x)$ denotes $(\alpha(x_1), \dots, \alpha(x_n))$.

1.2. Derivations, Quasiderivations and centroids of multiplicative n -Hom-Lie algebras.

In this section we recall the definition of derivation, generalized derivation, quasiderivation and centroids of multiplicative n -Hom-Lie algebras.

Let $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ be an multiplicative n -Hom-Lie algebra. We denote by α^k the k -times composition of α (i.e. $\alpha^k = \alpha \circ \dots \circ \alpha$ k -times). In particular $\alpha^{-1} = 0$, $\alpha^0 = id$.

Definition 1.10. For any $k \geq 1$, we call $D \in \text{End}(\mathcal{N})$ an α^k -derivation of the multiplicative n -Hom-Lie algebra $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ if

$$(1.8) \quad [D, \alpha] = 0 \quad \text{i.e.} \quad D \circ \alpha = \alpha \circ D,$$

and

$$(1.9) \quad D[x_1, \dots, x_n] = \sum_{i=1}^n [\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)],$$

We denote by $\text{Der}_{\alpha^k}(\mathcal{N})$ the set of α^k -derivations of the multiplicative n -Hom-Lie algebra \mathcal{N} .

For $X = (x_1, \dots, x_{n-1}) \in \mathcal{N}^{n-1}$ satisfying $\alpha(X) = X$ and $k \geq 1$, we define the map $\text{ad}_X^k \in \text{End}(\mathcal{N})$ by

$$(1.10) \quad \text{ad}_X^k(y) = [x_1, \dots, x_{n-1}, \alpha^k(y)] \quad \forall y \in \mathcal{N}.$$

Then

Lemma 1.11. The map ad_X^k is an α^{k+1} -derivation, that we call inner α^{k+1} -derivation.

We denote by $\text{Inn}_{\alpha^k}(\mathcal{N})$ the space generate by all the inner α^{k+1} -derivations. For any $D \in \text{Der}_{\alpha^k}(\mathcal{N})$ and $D' \in \text{Der}_{\alpha^k}(\mathcal{N})$ we define their commutator $[D, D']$ as usual:

$$(1.11) \quad [D, D'] = D \circ D' - D' \circ D.$$

Set $\text{Der}(\mathcal{N}) = \bigoplus_{k \geq -1} \text{Der}_{\alpha^k}(\mathcal{N})$ and $\text{Inn}(\mathcal{N}) = \bigoplus_{k \geq -1} \text{Inn}_{\alpha^k}(\mathcal{N})$.

Definition 1.12. An endomorphism D of a multiplicative n -ary Hom-Nambu algebra $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ is called a generalized α^k -derivation if there exist linear mappings $D', D'', \dots, D^{(n-1)}, D^{(n)} \in \text{End}(\mathcal{N})$ such that

$$(1.12) \quad D^{(n)}([x_1, \dots, x_n]) = \sum_{i=1}^n [\alpha^k(x_1), \dots, D^{(i-1)}(x_i), \dots, \alpha^k(x_n)],$$

for all $x_1, \dots, x_n \in \mathcal{N}$. An $(n+1)$ -tuple $(D, D', D'', \dots, D^{(n-1)}, D^{(n)})$ is called an $(n+1)$ -ary α^k -derivation.

The set of generalized α^k -derivation is denoted by $\text{GDer}_{\alpha^k}(\mathcal{N})$. Set $\text{GDer}(\mathcal{N}) = \bigoplus_{k \geq -1} \text{GDer}_{\alpha^k}(\mathcal{N})$.

Definition 1.13. Let $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ be a multiplicative n -ary Hom-Nambu algebra and $\text{End}(\mathcal{N})$ be the endomorphism algebra of \mathcal{N} . An endomorphism $D \in \text{End}(\mathcal{N})$ is said to be an α^k -quasiderivation, if there exists an endomorphism $D' \in \text{End}(\mathcal{N})$ such that

$$\sum_{i=1}^n [\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_n)] = D'([x_1, \dots, x_n]),$$

for all $x_1, \dots, x_n \in \mathcal{N}$. We call D' the endomorphism associate to the α^k -quasiderivation D .

The set of α^k -quasiderivations will be denoted by $QDer_{\alpha^k}(\mathcal{N})$. Set $QDer(\mathcal{N}) = \bigoplus_{k \geq -1} QDer_{\alpha^k}(\mathcal{N})$.

Definition 1.14. Let $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ be a multiplicative n -ary Hom-Nambu algebra and $End(\mathcal{N})$ be the endomorphism algebra of \mathcal{N} . Then the following subalgebra of $End(\mathcal{N})$

$$Cent(\mathcal{N}) = \{\theta \in End(\mathcal{N}) : \theta([x_1, \dots, x_n]) = [\theta(x_1), \dots, \theta(x_n)], \forall x_i \in \mathcal{N}\}$$

is said to be the centroid of the n -ary Hom-Nambu algebra. The definition is the same for classical case of n -ary Nambu algebra. We may also consider the same definition for any n -ary Hom-Nambu algebra.

Now, let $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ be a multiplicative n -ary Hom-Nambu algebra.

Definition 1.15. An α^k -centroid of a multiplicative n -ary Hom-Nambu algebra $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ is a subalgebra of $End(\mathcal{N})$ denoted $Cent_{\alpha^k}(\mathcal{N})$, given by

$$Cent_{\alpha^k}(\mathcal{N}) = \{\theta \in End(\mathcal{N}) : \theta[x_1, \dots, x_n] = [\theta(x_1), \alpha^k(x_2), \dots, \alpha^k(x_n)], \forall x_i \in \mathcal{N}\}.$$

We recover the definition of the centroid when $k = 0$.

If \mathcal{N} is a multiplicative n -Hom-Lie algebra, then it is a simple fact that

$$\theta[x_1, \dots, x_n] = [\alpha^k(x_1), \dots, \theta(x_p), \dots, \alpha^k(x_n)], \forall p \in \{1, \dots, n\}.$$

2. n -HOM-LIE ALGEBRAS INDUCED BY HOM-LIE ALGEBRAS

In [5] and [6], the authors introduced a construction of a 3-Hom-Lie algebra from a Hom-Lie algebra, and more generally of an $(n+1)$ -Hom-Lie algebra from an n -Hom-Lie algebra. It is called $(n+1)$ -Hom-Lie algebra induced by n -Hom-Lie algebra. In this context, Abramov give a new approach of this construction (see [1]). Now we generalise this approach in the Hom case.

Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ a multiplicative Hom-Lie algebra and \mathfrak{g}^* be its dual space. Fix an element of the dual space $\varphi \in \mathfrak{g}^*$. Define the triple product as follows

$$(2.1) \quad [x, y, z] = \varphi(x)[y, z] + \varphi(y)[z, x] + \varphi(z)[x, y], \forall x, y, z \in \mathfrak{g}.$$

Obviously this triple product is skew-symmetric. Straightforward computation of the left hand side and the right hand side of the Filippov-Jacobi identity (1.5) if $\varphi \circ \alpha = \varphi$ and

$$(2.2) \quad \varphi(x)\varphi([y, z]) + \varphi(y)\varphi([z, x]) + \varphi(z)\varphi([x, y]) = 0.$$

Now we consider φ as a \mathbb{K} -valued cochain of degree one of the Chevalley-Eilenberg complex of a Lie algebra \mathfrak{g} . Making use of the coboundary operator $\delta : \wedge^k \mathfrak{g}^* \rightarrow \wedge^{k+1} \mathfrak{g}^*$ defined by

$$(2.3) \quad \delta f(u_1, \dots, u_{k+1}) = \sum_{i < j} (-1)^{i+j+1} f([u_i, u_j]_{\mathfrak{g}}, \alpha(u_1) \dots, \widehat{u_i}, \dots, \widehat{u_j}, \dots, \alpha(u_{k+1})),$$

for $f \in \wedge^k \mathfrak{g}^*$ and for all $u_1, \dots, u_{k+1} \in \mathfrak{g}$. We obtain that $\delta\varphi(x, y) = \varphi([x, y])$.

Finally we can define the wedge product of two cochains φ and $\delta\varphi$, which is the cochain of degree three by

$$\varphi \wedge \delta\varphi(x, y, z) = \varphi(x)\varphi([y, z]) + \varphi(y)\varphi([z, x]) + \varphi(z)\varphi([x, y]).$$

Hence (2.2) is equivalent to $\varphi \wedge \delta\varphi = 0$. Thus if an 1-cochain φ satisfies the equation (2.2) then the triple product (2.1) is the ternary Lie bracket and we will call this multiplicative 3-Hom-Lie bracket the quantum Nambu bracket induced by a 1-cochain.

Definition 2.1. Let $\phi \in \wedge^{n-2} \mathfrak{g}^*$, we define the n -ary product as follows

$$(2.4) \quad [x_1, \dots, x_n]_{\phi} = \sum_{i < j}^n (-1)^{i+j+1} \phi(x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_n) [x_i, x_j],$$

for all $x_1, \dots, x_n \in \mathfrak{g}$.

Then, we obtain.

Proposition 2.2. *The n -ary product $[\cdot, \dots, \cdot]_\phi$ is skew-symmetric.*

Proof. Let $x_1, \dots, x_n \in \mathfrak{g}$ and fixed two integer $i < j$, we have:

$$\begin{aligned}
[x_1, \dots, x_i, \dots, x_j, \dots, x_n]_\phi &= \sum_{k < l: k, l \neq i, j} (-1)^{k+l+1} \phi(x_1, \dots, x_i \cdots, \hat{x}_k, \dots, x_j, \dots, \hat{x}_l, \dots, x_n) [x_l, x_k] \\
&+ \sum_{i < l \neq j} (-1)^{i+l+1} \phi(x_1, \dots, \hat{x}_i \cdots, x_j, \dots, \hat{x}_l, \dots, x_n) [x_l, x_i] \\
&+ \sum_{l < i} (-1)^{i+l+1} \phi(x_1, \dots, \hat{x}_l, \dots, \hat{x}_i, \dots, x_j, \dots, x_n) [x_l, x_i] \\
&+ \sum_{j < l} (-1)^{j+l+1} \phi(x_1, \dots, x_i \cdots, \hat{x}_j, \dots, \hat{x}_l, \dots, x_n) [x_j, x_l] \\
&+ \sum_{l < j, i \neq l} (-1)^{j+l+1} \phi(x_1, \dots, x_i, \dots, \hat{x}_l, \dots, \hat{x}_j, \dots, x_n) [x_l, x_j] \\
&+ (-1)^{i+j+1} \phi(x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n) [x_i, x_j] \\
&= -[x_1, \dots, x_j, \dots, x_i, \dots, x_n]_\phi.
\end{aligned}$$

□

Given $X = (x_1, \dots, x_{n-3}) \in \wedge^{n-3} \mathfrak{g}$, $Y = (y_1, \dots, y_n) \in \wedge^n \mathfrak{g}$ and $z \in \mathfrak{g}$. We define the linear map ϕ_X by:

$$\phi_X(z) = \phi(X, z),$$

and

$$\begin{aligned}
\phi \wedge \delta \phi_X(Y) &= \sum_{i < j}^n (-1)^{i+j} \phi(y_1, \dots, \hat{y}_i \cdots \hat{y}_j \cdots, y_{n-1}) \delta \phi_X(y_i, y_j) \\
&= \sum_{i < j}^n (-1)^{i+j} \phi(y_1, \dots, \hat{y}_i \cdots \hat{y}_j \cdots, y_n) \phi_X([y_i, y_j]).
\end{aligned}$$

Theorem 2.3. *Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie algebra, \mathfrak{g}^* be its dual and ϕ be a cochain of degree $n-2$, i.e. $\phi \in \wedge^{n-2} \mathfrak{g}^*$. The vector space \mathfrak{g} equipped with the n -ary product (2.4) and the linear map α is a multiplicative n -Hom-Lie algebra if and only if*

$$(2.5) \quad \phi \wedge \delta \phi_X = 0, \quad \forall X \in \wedge^{n-3} \mathfrak{g},$$

$$(2.6) \quad \phi \circ (\alpha \otimes Id \otimes \dots \otimes Id) = \phi.$$

Proof. Firstly, let $(x_1, \dots, x_n) \in \wedge^n \mathfrak{g}$. We have:

$$\begin{aligned}
[\alpha(x_1), \dots, \alpha(x_n)]_\phi &= \sum_{i < j}^n (-1)^{i+j+1} \phi(\alpha(x_1), \dots, \alpha(\hat{x}_i), \dots, \alpha(\hat{x}_j), \dots, \alpha(x_n)) [\alpha(x_i), \alpha(x_j)] \\
&= \sum_{i < j}^n (-1)^{i+j+1} \phi(x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n) \alpha([x_i, x_j]) \\
&= \alpha([x_1, \dots, x_n]_\phi).
\end{aligned}$$

Secondly, for $(x_1, \dots, x_{n-1}) \in \wedge^{n-1} \mathfrak{g}$ and $(y_1, \dots, y_n) \in \wedge^n \mathfrak{g}$, we have

$$\begin{aligned} & [\alpha(x_1), \dots, \alpha(x_{n-1}), [y_1, \dots, y_n] \phi] \phi \\ &= \sum_{i < j} (-1)^{i+j+1} \phi(y_1, \dots, \widehat{y}_i, \dots, \widehat{y}_j, \dots, y_n) [\alpha(x_1), \dots, \alpha(x_{n-1}), [y_i, y_j]] \phi \\ &= \sum_{i < j} \sum_{k < l \leq n-1} (-1)^{i+j+k+l} \phi(\alpha(x_1), \dots, \widehat{\alpha(x_k)}, \dots, \widehat{\alpha(x_l)}, \dots, [y_i, y_j]) \phi(y_1, \dots, \widehat{y}_i, \dots, \widehat{y}_j, \dots, y_n) [\alpha(x_k), \alpha(x_l)] \\ &+ \sum_{i < j} \sum_{k < n} (-1)^{i+j+k} \phi(\alpha(x_1), \dots, \widehat{\alpha(x_k)}, \dots, \alpha(x_{n-1}), \dots, [y_i, y_j]) \phi(y_1, \dots, \widehat{y}_i, \dots, \widehat{y}_j, \dots, y_n) [\alpha(x_k), [y_i, y_j]]. \end{aligned}$$

The terms $[\alpha(x_k), [y_i, y_j]]$ are simplified by identity of Jacobi in the second half of the Filippov identity. Now, we group together the other terms according to their coefficient $[\alpha(x_i), \alpha(x_j)]$. For example, if we fixed (k, l) and, if we collect all the terms containing the commutator $[\alpha(x_k), \alpha(x_l)]$, then we get the expression

$$\left(\sum_{i < j} (-1)^{i+j+k+l} \phi(\alpha(x_1), \dots, \widehat{\alpha(x_k)}, \dots, \widehat{\alpha(x_l)}, \dots, [y_i, y_j]) \phi(y_1, \dots, \widehat{y}_i, \dots, \widehat{y}_j, \dots, y_n) \right) [\alpha(x_k), \alpha(x_l)].$$

Hence the n -ary product (2.4) will satisfy the n -ary Filippov-Jacobi identity if for any elements $X = (x_1, \dots, x_{n-3}) \in \wedge^{n-3} \mathfrak{g}$ and $Y = (y_1, \dots, y_n) \in \wedge^n \mathfrak{g}$ we require

$$\left(\sum_{i < j}^n (-1)^{i+j} \phi(\alpha(x_1), \dots, \alpha(x_{n-3}), [y_i, y_j]) \phi(y_1, \dots, \widehat{y}_i, \dots, \widehat{y}_j, \dots, y_n) \right) = 0.$$

□

Definition 2.4. Let $\phi : \mathfrak{g} \otimes \dots \otimes \mathfrak{g} \rightarrow \mathbb{K}$ a skew-symmetric multilinear form of the multiplicative Hom-Lie algebras $(\mathfrak{g}, [\cdot, \cdot], \alpha)$, then ϕ is called trace if:

$$\phi \circ (Id \otimes \dots \otimes Id \otimes [\cdot, \cdot]) = 0 \text{ and } \phi \circ (\alpha \otimes Id \otimes \dots \otimes Id) = \phi.$$

Corollary 2.5. Let $\phi : \mathfrak{g}^{\otimes n-2} \rightarrow \mathbb{K}$ be a trace of Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$, then $\mathfrak{g}_\phi = (\mathfrak{g}, [\cdot, \dots, \cdot]_\phi, \alpha)$ is a n -Hom-Lie algebra.

Proposition 2.6. Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra and $D \in Der(\mathfrak{g})$ be an α^k -derivation such that

$$\sum_{i=1}^{n-2} \phi(x_1, \dots, D(x_i), \dots, x_{n-2}) = 0.$$

Then D is an α^k -derivation of the n -Hom-Lie algebra $(\mathfrak{g}, [\cdot, \dots, \cdot]_\phi, \alpha)$.

Proof. Let $X = (x_1, \dots, x_n) \in \wedge^n \mathfrak{g}$, on the one hand we get

$$\begin{aligned} D([x_1, \dots, x_n]_\phi) &= D \left(\sum_{i < j} (-1)^{i+j+1} \phi(\alpha(x_1), \dots, \alpha(\widehat{x}_i), \dots, \alpha(\widehat{x}_j), \dots, \alpha(x_n)) [\alpha(x_i), \alpha(x_j)] \right) \\ &= \sum_{i < j} (-1)^{i+j+1} \phi(\alpha(x_1), \dots, \alpha(\widehat{x}_i), \dots, \alpha(\widehat{x}_j), \dots, \alpha(x_n)) D([\alpha(x_i), \alpha(x_j)]) \\ &= \sum_{i < j} (-1)^{i+j+1} \phi(x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_n) [\alpha(D(x_i)), \alpha^{k+1}(x_j)] \\ &+ \sum_{i < j} (-1)^{i+j+1} \phi(x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_n) [\alpha^{k+1}(x_i), \alpha(D(x_j))], \end{aligned}$$

and on the other hand, we have

$$\begin{aligned}
& \sum_{l=1}^n [\alpha^k(x_1), \dots, \alpha^k(x_{l-1}), D(x_l), \dots, \alpha^k(x_{l+1}), \dots, \alpha^k(x_n)]_\phi \\
&= \sum_{l=1}^n \sum_{i < j; i, j \neq l} (-1)^{i+j+1} \phi(\alpha^k(x_1), \dots, \widehat{\alpha^k(x_i)}, \dots, D(x_l), \dots, \widehat{\alpha^k(x_j)}, \dots, \alpha^k(x_n)) [\alpha^k(x_i), \alpha^k(x_j)] \\
&+ \sum_{l=1}^n \sum_{i < l} (-1)^{i+l+1} \phi(\alpha^k(x_1), \dots, \widehat{\alpha^k(x_i)}, \dots, \widehat{D(x_l)}, \dots, \alpha^k(x_n)) [\alpha^k(x_i), D(x_l)] \\
&+ \sum_{l=1}^n \sum_{l=i < j} (-1)^{j+l+1} \phi(\alpha^k(x_1), \dots, \widehat{D(x_l)}, \dots, \alpha^k(x_j), \dots, \alpha^k(x_n)) [D(x_l), \alpha^k(x_j)].
\end{aligned}$$

If D is an α^k -derivation then $D([x_1, \dots, x_n]_\phi) = \sum_{l=1}^n [\alpha^k(x_1), \dots, \alpha^k(x_{l-1}), D(x_l), \dots, \alpha^k(x_{l+1}), \dots, \alpha^k(x_n)]_\phi$, which gives

$$\sum_{i < j; i, j \neq l} (-1)^{i+j+1} \left(\sum_{l=1}^n \phi(\alpha^k(x_1), \dots, \widehat{\alpha^k(x_i)}, \dots, D(x_l), \dots, \widehat{\alpha^k(x_j)}, \dots, \alpha^k(x_n)) \right) [\alpha^k(x_i), \alpha^k(x_j)] = 0.$$

Finally if we fixed (i, j) we have

$$\sum_{l=1}^{n-2} \phi(\alpha^k(x_1), \dots, D(x_l), \dots, \alpha^k(x_{n-2})) = 0.$$

□

Proposition 2.7. Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra and $D \in QDer(\mathfrak{g})$ be an α^k -quasiderivation and $D' : \mathfrak{g} \rightarrow \mathfrak{g}$ the endomorphism associate to D such that

$$\sum_{i=1}^{n-2} \phi(x_1, \dots, D(x_i), \dots, x_{n-2}) = 0.$$

Then D is an α^k -quasiderivation of the n -Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \dots, \cdot]_\phi, \alpha)$ with the same endomorphism associate D' .

Proposition 2.8. Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra and $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ be an α^k -centroid such that

$$\phi(\theta(x_1), \dots, x_i, \dots, x_{n-2}) [\alpha^k(x), y] = \phi(x_1, \dots, x_i, \dots, x_{n-2}) [\theta(x), y].$$

Then D is an α^k -centroid on the n -Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \dots, \cdot]_\phi, \alpha)$.

Proof. Let $x_1, \dots, x_n \in \mathfrak{g}$, we have

$$\begin{aligned}
\theta([x_1, \dots, x_n]_\phi) &= \sum_{i < j}^n (-1)^{i+j+1} \phi(x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n) \theta([x_i, x_j]) \\
&= \sum_{i < j}^n (-1)^{i+j+1} \phi(x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n) [\theta(x_i), \alpha^k(x_j)].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
[\theta(x_1), \alpha^k(x_2), \dots, \alpha^k(x_n)]_\phi &= \sum_{i < j}^n (-1)^{i+j+1} \phi(\theta(x_1), \alpha^k(x_2), \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, \alpha^k(x_n)) [\alpha^k(x_i), \alpha^k(x_j)] \\
&= \sum_{i < j}^n (-1)^{i+j+1} \phi(x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n) [\theta(x_i), \alpha^k(x_j)] \\
&= \theta([x_1, \dots, x_n]_\phi).
\end{aligned}$$

□

3. HOM-LIE n -UPLET SYSTEM

3.1. Hom-Lie triple system.

In this section, we start by recalling the definitions of Lie triple systems and Hom-Lie triple systems.

Definition 3.1. ([23])

A vector space T together with a trilinear map $(x, y, z) \rightarrow [x, y, z]$ is called a Lie triple system (LTS) if

- (1) $[x, x, z] = 0$,
- (2) $[x, y, z] + [y, z, x] + [z, x, y] = 0$,
- (3) $[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]]$,

for all $x, y, z, u, v \in T$.

Definition 3.2. ([29]) A Hom-Lie triple system (Hom-LTS for short) is denoted by $(T, [\cdot, \cdot, \cdot], \alpha)$, which consists of an \mathbb{K} -vector space T , a trilinear product $[\cdot, \cdot, \cdot] : T \times T \times T \rightarrow T$, and a linear map $\alpha : T \rightarrow T$, called the twisted map, such that α preserves the product and for all $x, y, z, u, v \in T$,

- (1) $[x, x, z] = 0$,
- (2) $[x, y, z] + [y, z, x] + [z, x, y] = 0$,
- (3) $[\alpha(u), \alpha(v), [x, y, z]] = [[u, v, x], \alpha(y), \alpha(z)] + [\alpha(x), [u, v, y], \alpha(z)] + [\alpha(x), \alpha(y), [u, v, z]]$.

Remark 3.3. When the twisted map α is equal to the identity map, a Hom-LTS is a LTS. So LTS are special examples of Hom-LTS.

Definition 3.4. A Hom-Lie triple system $(T, [\cdot, \cdot, \cdot], \alpha)$ is called multiplicative if $\alpha([x, y, z]) = [\alpha(x), \alpha(y), \alpha(z)]$, for all $x, y, z \in T$.

Theorem 3.5. ([29])

Let $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie algebra. Then

$$\mathfrak{g}_T = (\mathfrak{g}, [\cdot, \cdot, \cdot] = [\cdot, \cdot] \circ ([\cdot, \cdot] \otimes \alpha), \alpha^2),$$

is a multiplicative Hom-Lie triple system.

3.2. Hom-Lie n -uplet system.

In this section we introduce the definitions of Lie n -uplet systems and multiplicative Hom-Lie n -uplet systems. We give the analogue of Theorem 3.5 in the Hom-Lie n -uplet systems case.

Definition 3.6. A vector space \mathcal{G} together with a n -linear map $(x_1, \dots, x_n) \rightarrow [x_1, \dots, x_n]$ is called a Lie n -uplet system if

- (1) $[x, x, y_1, \dots, y_{n-2}] = 0$, for all $x, y_1, \dots, y_{n-2} \in \mathcal{G}$.
- (2) $[x_1, \dots, x_{n-1}, [y_1, \dots, y_n]] = \sum_{i=1}^n [y_1, \dots, y_{i-1}, [x_1, \dots, x_{n-1}, y_i], y_{i+1}, \dots, y_n]$,

for all $x_1, \dots, x_{n-1}, y_1, \dots, y_n \in \mathcal{G}$.

Definition 3.7. A vector space \mathcal{G} together with a n -linear map $(x_1, \dots, x_n) \rightarrow [x_1, \dots, x_n]$ and a family $\tilde{\alpha} = (\alpha_i)_{1 \leq i \leq n-1}$ of linear maps $\alpha_i : \mathcal{G} \rightarrow \mathcal{G}$, is called a Hom-Lie n -uplet system if

- (1) $[x, x, y_1, \dots, y_{n-2}] = 0$, for all $x, y_1, \dots, y_{n-2} \in \mathcal{G}$.
- (2) $[\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1}), [y_1, \dots, y_n]] = \sum_{i=1}^n [\alpha_1(y_1), \dots, \alpha_{i-1}(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha_i(y_{i+1}), \dots, \alpha_{n-1}(y_n)]$,
for all $x_1, \dots, x_{n-1}, y_1, \dots, y_n \in \mathcal{G}$.

Definition 3.8. A Hom-Lie n -uplet system $(\mathcal{G}, [\cdot, \dots, \cdot], \tilde{\alpha})$ is called a multiplicative Hom-Lie n -uplet system if $\alpha_1 = \dots = \alpha_{n-1} = \alpha$ and $\alpha([x_1, \dots, x_n]) = [\alpha(x_1), \dots, \alpha(x_n)]$ for all $x_1, \dots, x_n \in \mathcal{G}$.

Remark 3.9. When the twisted maps α_i is equal to the identity map, a Hom-Lie n -uplet systems is a Lie n -uplet systems. So Lie n -uplet systems are special examples of Hom-Lie n -uplet systems.

Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra. We define the following n linear map:

$$(3.1) \quad \begin{aligned} [\cdot, \dots, \cdot]_n : \mathfrak{g}^{\otimes n} &\longrightarrow \mathfrak{g} \\ (x_1, \dots, x_n) &\longmapsto [x_1, \dots, x_n]_n = [[\dots [x_1, x_2], \alpha(x_3)], \alpha^2(x_4)] \cdots \alpha^{n-3}(x_{n-1}), \alpha^{n-2}(x_n) \end{aligned}$$

For $n=2$, $[x_1, x_2]_2 = [x_1, x_2]$ and for $n \geq 3$ we have $[x_1, \dots, x_n]_n = [[x_1, \dots, x_{n-1}]_{n-1}, x_n]$.

The following result gives a way to construct Hom-Lie n -uplet system starting from a classical Lie n -uplet system and algebra endomorphisms.

Proposition 3.10. *Let $(\mathcal{G}, [\cdot, \dots, \cdot])$ a Lie n -uplet system and $\alpha : \mathcal{G} \rightarrow \mathcal{G}$ a linear map such that $\alpha([x_1, \dots, x_n]) = [\alpha(x_1), \dots, \alpha(x_n)]$. Then $(\mathcal{G}, [\cdot, \dots, \cdot]_\alpha, \alpha)$ be a Hom-Lie n -uplet system, where $[x_1, \dots, x_n]_\alpha = [\alpha(x_1), \dots, \alpha(x_n)]$, for all $x_1, \dots, x_n \in \mathcal{G}$.*

Theorem 3.11. *Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie algebra. Then*

$$\mathfrak{g}_n = (\mathfrak{g}, [\cdot, \dots, \cdot]_n, \alpha^{n-1})$$

is a multiplicative Hom-Lie n -uplet system.

When $n = 3$ we obtain the multiplicative Hom-Lie triple system constructed in Theorem 3.5. To prove this theorem we need to the following lemma.

Lemma 3.12. *Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie algebra, and ad^2 the adjoint map defined by*

$$ad_x^2(y) = ad_x(y) = [x, y]$$

Then, we have

$$ad_{\alpha^{n-1}(x)}^2[y_1, \dots, y_n]_n = \sum_{k=1}^n [\alpha(y_1), \dots, \alpha(y_{k-1}), ad_x^2(y_k), \alpha(y_{k+1}), \dots, \alpha(y_n)]_n,$$

where $x \in \mathfrak{g}, y \in \mathfrak{g}$ and $(y_1, \dots, y_n) \in \mathfrak{g}^n$.

Proof. For $n = 2$, using the Hom-Jacobi identity we have

$$\begin{aligned} ad_{\alpha(x)}^2[y, z] &= [\alpha(x), [y, z]] = [[x, y], \alpha(z)] + [\alpha(y), [x, z]] \\ &= [ad_x^2(y), \alpha(z)] + [\alpha(y), ad_x^2(z)]. \end{aligned}$$

Assume that the property is true up to order n , that is

$$ad_{\alpha^{n-1}(x)}^2[y_1, \dots, y_n]_n = \sum_{k=1}^n [\alpha(y_1), \dots, \alpha(y_{k-1}), ad_X^2(y_k), \alpha(y_{k+1}), \dots, \alpha(y_n)]_n.$$

Let $x \in \mathfrak{g}$ and $(y_1, \dots, y_{n+1}) \in \mathfrak{g}^{n+1}$, we have

$$\begin{aligned} ad_{\alpha^n(x)}^2[y_1, \dots, y_{n+1}] &= ad_{\alpha^n(x)}^2[[y_1, \dots, y_n]_n, \alpha^{n-1}(y_{n+1})]_2 \\ &= [ad_{\alpha^{n-1}(x)}^2[y_1, \dots, y_n]_n, \alpha^n(y_{n+1})]_2 + [[\alpha(y_1), \dots, \alpha(y_n)]_n, ad_{\alpha^{n-1}(x)}^2(\alpha^{n-1}(y_{n+1}))]_2 \\ &= \sum_{k=1}^n [\alpha(y_1), \dots, \alpha(y_{k-1}), ad_x^2(y_k), \alpha(y_{k+1}), \dots, \alpha(y_n)]_n, \alpha^n(y_{n+1}) \\ &\quad + [[\alpha(y_1), \dots, \alpha(y_n)]_n, \alpha^{n-1}(ad_x^2(y_{n+1}))]_2 \\ &= \sum_{k=1}^n [\alpha(y_1), \dots, \alpha(y_{k-1}), ad_x^2(y_k), \alpha(y_{k+1}), \dots, \alpha(y_n), \alpha(y_{n+1})]_{n+1} \\ &\quad + [\alpha(y_1), \dots, \alpha(y_n), ad_x^2(y_{n+1})]_{n+1} \\ &= \sum_{k=1}^{n+1} [\alpha(y_1), \dots, \alpha(y_{k-1}), ad_x^2(y_k), \alpha(y_{k+1}), \dots, \alpha(y_{n+1})]_{n+1}. \end{aligned}$$

The lemma is proved. \square

Proof. (Proof of Theorem 3.11) Let $X = (x_1, \dots, x_{n-1}) \in \mathfrak{g}^{n-1}$ and $Y = (y_1, \dots, y_n) \in \mathfrak{g}^n$.

(i) Its easy to see that $[x_1, x_1, x_2, \dots, x_{n-1}]_n = [[\dots [[x_1, x_1]_2, \alpha(x_2)]_2, \alpha^2(x_3)]_2, \dots]_2, \alpha^{n-2}(x_{n-1})]_2 = 0$

(ii) Using lemma (3.12), we have

$$\begin{aligned} [\alpha^{n-1}(x_1), \dots, \alpha^{n-1}(x_{n-1}), [y_1, \dots, y_n]_n]_n &= [\alpha^{n-1}(x_1), \dots, \alpha^{n-1}(x_{n-1})]_{n-1}, [\alpha^{n-2}(y_1), \dots, \alpha^{n-2}(y_n)]_n]_2 \\ &= \text{ad}_{\alpha^{n-1}[x_1, \dots, x_{n-1}]}^2([\alpha^{n-2}(y_1), \dots, \alpha^{n-2}(y_n)]_n) \\ &= \sum_{k=1}^n [\alpha^{n-1}(y_1), \dots, \text{ad}_{[x_1, \dots, x_{n-1}]}^2(\alpha^{n-2}(y_k)), \dots, \alpha^{n-1}(y_n)]_n \\ &= \sum_{k=1}^n [\alpha^{n-1}(y_1), \dots, [[x_1, \dots, x_{n-1}], \alpha^{n-2}(y_k)]_2, \dots, \alpha^{n-1}(y_n)]_n \\ &= \sum_{k=1}^n [\alpha^{n-1}(y_1), \dots, [x_1, \dots, x_{n-1}, y_k]_n, \dots, \alpha^{n-1}(y_n)]_n. \end{aligned}$$

\square

Example 3.13. Using Exemple 1.5 and by the theorem (3.11), for $\lambda \in \mathbb{R}^*$ we have:

For $n = 3$, $(\mathfrak{sl}_2(\mathbb{R}), [\ , \], \alpha_\lambda^2)$ is a Hom-Lie triple system. The different brackets are as follows:

$$[H, X, Y]_3 = [[H, X]_{\alpha_\lambda}, \alpha_\lambda(Y)]_{\alpha_\lambda} = 2H \quad ; \quad [H, X, H]_3 = -4\lambda^4 X \quad ; \quad [H, Y, X]_3 = 4H.$$

$$[H, Y, H]_3 = -\frac{4}{\lambda^4} Y \quad ; \quad [X, Y, Y]_3 = -\frac{2}{\lambda^4} Y \quad ; \quad [X, Y, X]_3 = 2\lambda^4 X.$$

The other brackets are zeros.

For $n = 4$, $(\mathfrak{sl}_2(\mathbb{R}), [\ , \], \alpha_\lambda^3)$ is a 4-uplet Hom-Lie triple system. The different brackets are defined as follows:

$$[H, X, H, H]_4 = [[H, X, H]_3, \alpha^2(H)]_{\alpha_\lambda} = -4\lambda^4 [X, H]_{\alpha_\lambda} = 8\lambda^6 X \quad ; \quad [H, X, H, Y]_4 = -4H$$

$$[H, Y, H, H]_4 = -\frac{8}{\lambda^6} Y \quad ; \quad [H, Y, H, X]_4 = 4H \quad ; \quad [H, X, Y, X]_4 = 4\lambda^6 X \quad ; \quad [H, X, Y, Y]_4 = -\frac{2}{\lambda^6} Y$$

$$[H, Y, X, X]_4 = 8\lambda^6 X \quad ; \quad [H, Y, X, Y]_4 = -\frac{8}{\lambda^6} Y \quad ; \quad [X, Y, X, Y]_4 = 2H$$

$$[X, Y, X, H]_4 = -4\lambda^6 X \quad ; \quad [X, Y, Y, X]_4 = 2H \quad ; \quad [X, Y, Y, H]_4 = -\frac{4}{\lambda^6} Y.$$

The other brackets are zeros.

Proposition 3.14. Let $(\mathfrak{g}, [\ , \], \alpha)$ be a multiplicative Hom-Lie algebra and $D : \mathfrak{g} \rightarrow \mathfrak{g}$ an α^k -derivation of \mathfrak{g} for an integer k . Then D is an α^k -derivation of \mathfrak{g}_n .

Proof. By recurrence

For $n = 3$. Let $x, y, z \in \mathfrak{g}$, we have

$$\begin{aligned} D([x, y, z]) &= D([x, y], \alpha(z)) \\ &= [D([x, y]), \alpha^{k+1}(z)] + [[\alpha^k(x), \alpha^k(y)], D(\alpha(z))] \\ &= [[D(x), \alpha^k(y)], \alpha^{k+1}(z)] + [[\alpha^k(x), D(y)], \alpha^{k+1}(z)] + [[\alpha^k(x), \alpha^k(y)], \alpha(D(z))] \\ &= [D(x), \alpha^k(y), \alpha^k(z)] + [\alpha^k(x), D(y), \alpha^k(z)] + [\alpha^k(x), \alpha^k(y), D(z)] \end{aligned}$$

Now, suppose that the property is true to order $n - 1$, i.e:

$$D([x_1, \dots, x_{n-1}]_{n-1}) = \sum_{i=1}^n [\alpha^k(x_1), \dots, D(x_k), \dots, \alpha^k(x_{n-1})]_{n-1}.$$

Let $(x_1, \dots, x_n) \in \mathfrak{g}^n$, then

$$\begin{aligned} D([x_1, \dots, x_n]_n) &= D([x_1, \dots, x_{n-1}]_{n-1}, \alpha^{n-2}(x_n)) \\ &= [D([x_1, \dots, x_{n-1}]_{n-1}), \alpha^{n+k-2}(x_n)] + [[\alpha^k(x_1), \dots, \alpha^k(x_{n-1})]_{n-1}, D(\alpha^{n-2}(x_n))] \\ &= [D([x_1, \dots, x_{n-1}]_{n-1}), \alpha^{n-2}(\alpha^k(x_n))] + [[\alpha^k(x_1), \dots, \alpha^k(x_{n-1})]_{n-1}, \alpha^{n-2}(D(x_n))] \\ &= \sum_{i=1}^{n-1} [[\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_{n-1})]_{n-1}, \alpha^{n-2}(\alpha^k(x_n))] + [\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), D(x_n)]_n \\ &= \sum_{i=1}^{n-1} [\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_{n-1}), \alpha^k(x_n)]_n + [\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), D(x_n)]_n \\ &= \sum_{i=1}^n [\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_{n-1}), \alpha^k(x_n)]_n \end{aligned}$$

□

Proposition 3.15. *Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie algebra and $D, D', \dots, D^{(n-1)}$ an endomorphisms of \mathfrak{g} such that $D^{(i)}$ is an α^k -quasiderivation with associated endomorphism $D^{(i+1)}$ for $0 \leq i \leq n - 2$. Then the $(n + 1)$ -tuple $(D, D, D', D'', \dots, D^{(n-1)})$ is an $(n + 1)$ -ary α^k -derivation of \mathfrak{g}_n .*

Proof. Let $x_1, \dots, x_n \in \mathfrak{g}$, then

$$\begin{aligned} D^{(n-1)}([x_1, \dots, x_n]_n) &= D^{(n-1)}([x_1, \dots, x_{n-1}]_{n-1}, x_n) \\ &= [D^{(n-2)}([x_1, \dots, x_{n-1}]_{n-1}), \alpha^k(x_n)] + [[\alpha^k(x_1), \dots, \alpha^k(x_{n-1})]_{n-1}, D^{(n-2)}(x_n)] \\ &\vdots \\ &= [D(x_1), \alpha^k(x_2), \dots, \alpha^k(x_n)]_n + [\alpha^k(x_1), D(x_2), \dots, \alpha^k(x_n)]_n \\ &\quad + [\alpha^k(x_1), \alpha^k(x_2), D'(x_3), \dots, \alpha^k(x_n)]_n + \dots + [\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), D^{(n-2)}(x_n)]_n \end{aligned}$$

Therefore $(n + 1)$ -tuple $(D, D, D', D'', \dots, D^{(n-1)})$ is an $(n + 1)$ -ary α^k -derivation of \mathfrak{g}_n . □

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