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ON UNBOUNDED SOLUTIONS OF ERGODIC PROBLEMS FOR NON-LOCAL HAMILTON-JACOBI EQUATIONS

CRISTINA BRÄNDLE AND EMMANUEL CHASSEIGNE

ABSTRACT. We study an ergodic problem associated to a non-local Hamilton-Jacobi equation defined on the whole space $\lambda - \mathcal{L}[u](x) + |Du(x)|^m = f(x)$ and determine whether (unbounded) solutions exist or not. We prove that there is a threshold growth of the function f , that separates existence and non-existence of solutions, a phenomenon that does not appear in the local version of the problem. Moreover, we show that there exists a critical ergodic constant, λ_* , such that the ergodic problem has solutions for $\lambda \leq \lambda_*$ and such that the only solution bounded from below, which is unique up to an additive constant, is the one associated to λ_* .

1. INTRODUCTION

The starting point of this paper concerns some ergodic problems as they were studied in [2] and [11, 12]. There, the authors study the Hamilton-Jacobi equation

$$\lambda - \Delta u + H(x, Du) = 0,$$

where typically $H(x, Du) = |Du|^m - f(x)$, and $m > 1$ (or $m > 2$). Our initial aim was to consider a simple non-local version of this equation and try to see how similar results could be obtained: existence of solutions, critical ergodic constants, and some qualitative behaviour like growth estimates. The equation we dealt with is the following:

$$(EP) \quad \lambda - \mathcal{L}[u](x) + |Du(x)|^m = f(x) \quad \text{in } \mathbb{R}^N,$$

where the non-local operator is defined as a convolution with a regular kernel, $\mathcal{L}[u] := J * u - u$, and J is a continuous, compactly supported probability density. We explain below why, especially in those ergodic problems, this equation raises interesting questions and new phenomena, even compared to the more commonly studied fractional Laplacian, for which the kernel is singular, $J(x) = 1/|x|^{N+\alpha}$, $\alpha \in (0, 2)$.

Let us also mention that in the “standard” setting, studying ergodic problems is done either in bounded domains, or in the periodic case, [see for instance \[4, 3\]](#). Both situations allow for a better control of the solutions and using compactness arguments. The fact that in [2, 11, 12], the authors consider an unbounded domain (the whole space \mathbb{R}^N), with a possibly unbounded data f , leads to various difficulties in the process of constructing solutions, estimating their growth and getting comparison results. This is even more challenging and difficult in our non-local setting.

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Nevertheless, back to our initial intention, we managed to recover most of the results we had in mind for (EP), so that this work turned out to be much more interesting and demanding (and much more difficult) than a simple adaptation from the local case. We had to develop new methods and techniques to deal with the non-local term, and we found out that there is a natural limitation in the growth of solutions, a feature which is not present in local setting. We are also convinced that the ideas that we use here can be helpful in the local case as well, and improve some of the results found in [2, 11, 12].

By a solution of (EP) we understand a pair (λ, u) where $\lambda \in \mathbb{R}$ and u is a continuous viscosity solution of the equation. We also refer to $(EP)_\lambda$ when λ is given and the unknown is only u . Observe that (EP) is invariant by addition of constants to the solution, as is usually the case in ergodic problems. As we shall see, the solutions will be actually locally Lipschitz continuous so that the equation will hold almost everywhere and in the weak sense.

This kind of ergodic problems are known, [12], to be closely related to the asymptotic behaviour of solution of the associated evolution equations, which would read in our case

$$(1.1) \quad u_t - \mathcal{L}[u] + |Du|^m = f \quad \text{in } \mathbb{R}^N.$$

It is not the purpose of this paper to investigate this question, but let us just mention that in general, solutions of (1.1) are expected to behave like $u(x, t) = \bar{\lambda}t + v(x) + o(1)$ as $t \rightarrow \infty$ where $(\bar{\lambda}, v)$ is a solution of (EP). The specific value $\bar{\lambda}$ is usually obtained by taking the supremum of all the λ 's such that there is a solution v of $(EP)_\lambda$. And as for v , it is in general the unique (up to an additive constant) solution of (EP) which is bounded from below. In this paper we focus on the existence and properties of this pair $(\bar{\lambda}, v)$.

1.1. The framework. Throughout the paper, we make the following fundamental assumptions:

- (i) the kernel $J : \mathbb{R}^N \rightarrow \mathbb{R}$ is C^1 , symmetric, radially decreasing, compactly supported in $B_1(0)$, with $\int J(y) dy = 1$ and strictly positive in all $B_1(0)$;
- (ii) we restrict ourselves to the superquadratic case $m > 2$ (more on this below);
- (iii) the function f is assumed to be at least continuous and bounded from below.

We will give more precise assumptions on f later, but these basic three assumptions (i)–(iii) will always hold and we shall not recall them anymore.

ON THE NON-LOCAL TERM — We use here a convolution with a regular, compactly supported kernel. This choice has several consequences that need to be dealt with. First, no regularizing effect can be obtained from operator \mathcal{L} . Indeed, contrary to the Laplacian which is a second-order one, or a fractional Laplacian which would be of order $\alpha \in (0, 2)$, the operator \mathcal{L} can be understood as a zero-order term. However, this non-local operator still enjoys some strong maximum principle. We refer to [1, 7] and the references therein for general properties of this type of non-local operator. In particular, \mathcal{L} is known to be an approximation of the Laplacian as the support of J shrinks to the origin (when J is symmetric).

We also choose to consider here a compactly supported kernel.

When the non-local operator is defined through a fractional Laplacian, for instance, the tail of order $1/|x|^{N+\alpha}$ implies a power-type growth restriction for all

possible solutions, since $\mathcal{L}[u]$ has to be defined, at least. On the contrary, if J is compactly supported, $\mathcal{L}[u]$ is always defined, as long as u is locally bounded. But (see below), contrary to the local case, the presence of a non-local term in the equation implies some growth limitation, even in the case of a compactly supported kernel. A similar behaviour was also found in [5], where the growth of initial data is limited, and differs from the local heat equation.

ON THE HAMILTONIAN — In this paper we consider the case $H(x, p) = |p|^m - f(x)$, but most of the results are adaptable to more general cases, **for instance $H(x, p) = a(x)|p|^m - f(x)$ as is done in [11]**. Notice however that since our solutions are not necessarily bounded, the gradient is only locally bounded in principle, and as is well-known in Hamilton-Jacobi equations, mixing the x -dependence with the p -dependence leads to several difficult issues.

As we mentioned, we will restrict ourselves to the superquadratic case, $m > 2$. Actually, there is only one place where this specific condition seems to play a role, namely in the existence construction (proof of Proposition 3.1), when using results of [6] to deal with a *vanishing viscosity approximation*. It is not clear to us whether this restriction is purely technical and could be relaxed to the more general assumption $m > 1$. This is a clearly difference from the local equation, since in this case, the viscous term does not vanish.

1.2. Main results. We present now the main results of this paper, which can be summarized in three items.

A. GROWTH LIMITATIONS — As we said, since J is compactly supported, the non-local term $\mathcal{L}[u]$ is well-defined for any locally bounded function u . However, contrary to the local case, it turns out that problem (EP) is not solvable for arbitrary growths of function f . More precisely, if $f(x) = C \exp(a^{|x|})$, then (EP) is solvable only when $a \leq m$.

The formal explanation is that in order to solve the equation, the $|Du|^m$ -term has to be the leading term. But, at least for fast growing radial supersolutions ψ , this is not the case: the convolution looks like $-\mathcal{L}[\psi](r) \simeq -\psi(r+1)$, which grows faster than $|\psi|^m(r)$ and this implies that the supersolution inequality cannot be satisfied, see Lemma 2.14 and Corollary 2.15.

B. EXISTENCE OF SOLUTIONS AND OF A CRITICAL ERGODIC CONSTANT — We prove that for functions f in a suitable growth class, typically $f(x) \leq C \exp(m^{|x|})$ for some $C > 0$, problem (EP) is solvable. Moreover, there is a bound for the constructed solution, $u(x) \leq \Psi(x) := |x|f^{1/m}(x)$ for $|x|$ large, so that $u(x) \leq C|x| \exp(m^{|x|-1})$ for large $|x|$.

Getting this existence result requires to deconstruct all the methods that are used in [2, 11, 12] (and even [10], see Appendix). A big issue that we face is that we do not have an universal local gradient estimate, as it is the case in [2, 11, 12]. This is due to the fact that \mathcal{L} is just a zero-order operator. We manage to bypass this difficulty by using a supersolution (obtained by a modification of function Ψ above) in order to control the non-local term. But this implies several technical problems, since Ψ is only a supersolution of (EP) for $|x| \neq 0$, see the whole construction in Section 3.

Notice that there are bigger supersolutions, but this specific Ψ yields a kind of *minimal supersolution* in the sense that bounded from below solutions behave like Ψ (see below).

Once the existence result is proved, it is usual to consider the critical ergodic constant as the supremum of all λ 's such that $(\text{EP})_\lambda$ is solvable. However, we still face the difficulty of estimates here and we need to include the upper behaviour of solutions in the definition:

$$\bar{\lambda} := \sup \left\{ \lambda \in \mathbb{R} : \exists u, \text{ solution of } (\text{EP})_\lambda, \text{ such that } \limsup_{|x| \rightarrow \infty} \frac{u(x)}{\Psi(x)} < \infty \right\}.$$

We prove that $\bar{\lambda}$ is finite and that there exists a solution u associated to $\bar{\lambda}$. Again, it is natural to have a limitation for the growth of solutions. Indeed, if u grows too fast, then $\mathcal{L}[u]$ becomes the leading term of the equation, and we have seen already that this is not possible if we want to have a solution.

C. CHARACTERIZATION OF THE CRITICAL ERGODIC CONSTANT — As in the local case, we prove that the critical ergodic constant $\bar{\lambda}$ can be characterized by the fact that (λ, u) is a solution of (EP) such that u is bounded from below and $\limsup u(x)/\Psi(x) < \infty$ if and only if $\lambda = \bar{\lambda}$. And in this case, u is uniquely determined (up to an additive constant).

Notice that in [2], such results are obtained for solutions and functions f which grow like powers. In contrast, we are able to consider much faster growths, like $\exp(a^{|x|})$. A key step in this improvement is to prove a bound from below for solutions such that $\inf u > -\infty$. This is done in Lemma 7.1. This Lemma is a refinement of [2, Proposition 3.4], and allows to treat faster growths. Actually, this approach could also be applied to the local case in order to generalize various results in [2].

1.3. Organization of the paper. In Section 2 we state the main hypotheses on the function f , construct sub and supersolutions to the problem and prove a non-existence result which illustrates the fact that the growth of f is limited. Section 3 deals with auxiliary problems defined on a bounded domain. Then, in Section 4 we prove the existence of solutions of (EP) . The last four sections are devoted to the critical ergodic constant and bounded from below solutions. In particular, we establish the existence of a critical ergodic constant in Section 5 under some growth restriction. In Section 6 and 7 we prove that there are solutions that are bounded from below, that these are unique (up to an additive constant) and that they are associated to the critical ergodic constant. Finally, in Section 8 we extend the class of solutions associated with the critical ergodic constant and prove some continuous dependence of the critical ergodic constant with respect to f .

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2. PRELIMINARIES

BASIC NOTATIONS — In the following, B_R will stand for $B_R(0) = \{x \in \mathbb{R}^N : |x| < R\}$ and we use the notation $|x| \gg 1$ to say that a property is valid for $|x|$ sufficiently large. We will use also R_1 to denote any radius meaning “ $|x|$ big enough”.

We will denote $u(x) = o(v(x))$ to say that $u(x)/v(x) \rightarrow 0$ as $|x| \rightarrow \infty$. In particular, $o_\alpha(1)$ represents a quantity which goes to zero as the parameter α goes to zero (or $+\infty$, depending on the situation). If some uniformity with respect to some other parameter is required, this will be mentioned explicitly.

2.1. Definitions and hypotheses.

Definition 2.1. A locally bounded u.s.c function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is a viscosity subsolution of (EP) if for any C^1 -smooth function φ , and any point $x_0 \in \mathbb{R}^N$ where $u - \varphi$ reaches a maximum, there holds,

$$\lambda - \mathcal{L}[u](x_0) + |D\varphi(x_0)|^m - f(x_0) \leq 0.$$

A locally bounded l.s.c. function is a viscosity supersolution if the same holds with reversed inequalities and the maximum point replaced by a minimum. Finally a viscosity solution is a locally bounded function u such that its u.s.c. and l.s.c. envelopes are respectively sub- and super-solutions of (EP).

Notice that, in the above definitions we only need the test function φ to be C^1 in a neighborhood of x_0 (or even only at x_0), and we shall use this remark when we use test functions which are not C^1 in all \mathbb{R}^N .

Remark 2.2. If we consider $u : \Omega \rightarrow \mathbb{R}$ and the equation defined on a bounded domain Ω together with a boundary condition, say $u = \psi$ on $\partial\Omega$, then the definition of viscosity subsolution (respectively supersolution) has to parts, depending whether the maximum point x_0 is achieved inside Ω or on the boundary, $\partial\Omega$. In this latter case, the condition for u to be a subsolution reads

$$\max(\lambda - \mathcal{L}[u](x_0) + |D\varphi(x_0)|^m - f(x_0), u(x_0) - \psi(x_0)) \leq 0$$

(respectively min and \geq for supersolutions). However, we shall not use boundary conditions here: we have only a viscosity solution in a ball B_R and send $R \rightarrow \infty$, see Lemma 4.6.

We list now the complete set of assumptions we use throughout this paper. We want to stress at this point, that we could have simplified this list by making strong assumptions (for instance, assuming that f is radial and radially increasing). However, we opted to keep track of what was really necessary to assume to produce each result. We think that the methods we design, with these weak assumptions, can be helpful in other situations. We comment on each hypothesis and give typical examples.

The main hypothesis on the right hand-side f , that we use throughout the paper is the following:

(H0) $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is C^1 and $\inf\{f(x) : |x| = r\} \rightarrow \infty$ as $r \rightarrow \infty$.

In particular, we are assuming that f is *uniformly coercive* so that for x large enough we have $f > 0$ and we can set

$$\Phi(x) := |x|f(x)^{1/m}.$$

In addition we have to impose some extra hypothesis on f . The next set of assumptions are related to its growth. The first two hypotheses, **(H1)** and **(H2)**, are required to construct a supersolution in $\mathbb{R}^N \setminus \{0\}$. Hypothesis **(H1)** is fundamental here, it is were we see the limitation in the growth of f , see more below.

(H1) For $|x| \gg 1$, $\sup_{y \in B_1(x)} |D\Phi(y)| \leq |D\Phi(x)|^m$.

(H2) For $|x| \gg 1$, $x \cdot Df(x) \geq -f(x)$.

The following two hypotheses, **(H3)** and **(H4)**, control the behaviour of f from below. This is crucial in order to prove that solutions which are bounded from below, actually have a minimal behaviour at infinity, which is given by Φ .

(H3) As $|x| \rightarrow \infty$, $\Phi(x) = o(f(x))$.

(H4) There exists $\eta_0 \in (0, 1)$ such that for all $\eta \in (0, \eta_0)$, there exists $\underline{c}_\eta, \bar{c}_\eta > 0$ and $R_\eta > 0$ such that

$$\text{for } |x| \geq R_\eta, \text{ and } s \in B_\eta(0), \quad \underline{c}_\eta f((1 - \eta)x) \leq f(x + s|x|) \leq \bar{c}_\eta f((1 + \eta)x).$$

The last two hypotheses are needed in order to prove that for a close enough to 1 and x big enough (uniform with respect to a), if u is a supersolution then $a^N u(ax)$ is also a supersolution. This is a crucial step. If f verifies **(H7)**-fast in proving that the only solutions that are bounded from below correspond to λ^* . If **(H7)**-slow holds, we do a slightly different approach, proving first that $u^q(x)$ is a supersolution, if $q > 1$

(H5) There exists $a_0 > 1$: $\forall a \in (1, a_0)$, as $|x| \rightarrow \infty$, $\sup_{|z| \leq 1} \Phi(a(x + z)) \ll f(x)$.

(H6) There exists $R_0 > 0$ such that $\forall a \in (1, a_0)$ and $|x| > R_0$, $f(ax) \geq af(x)$.

(H7) One of the following holds:

Slow case – for all $a \in (1, a_0)$, $\limsup(f(ax)/f(x)) < \infty$ as $|x| \rightarrow \infty$;

Fast case – for all $a \in (1, a_0)$, $\liminf(f(ax)/f(x)) = +\infty$ as $|x| \rightarrow \infty$.

Examples and discussion on the hypotheses — As we said, **(H1)** highlights the limiting behaviour of f in order to get a solution. It can be checked that if $f(x) = \exp(p^{|x|})$, then **(H1)** is satisfied if and only if $p \leq m$. This hypothesis is essential in order to control the non-local term by the gradient term.

On the contrary, **(H3)** and **(H4)** both imply that f has a minimal growth. By the specific form of Φ , **(H3)** is equivalent to $f(x) \gg |x|^{m_*}$ where $m_* := m/(m - 1)$. Actually, this is not a real limitation since if f does not grow so fast, the methods in [2] readily adapt.

Hypothesis **(H5)**, though it is similar to **(H3)**, it is a bit more restrictive non-local version of **(H3)**. For power-type functions f , this condition also reduces to $f(x) \gg |x|^{m_*}$ and **(H5)** implies **(H3)**.

Finally, **(H2)**, **(H4)** and **(H6)** are automatically satisfied if f is radial and increasing. Hence, these hypotheses are needed to control how much the function f is allowed to deviate from this behaviour. Notice though, that they allow f to be quite “far” from radial and increasing.

In conclusion, the typical functions that satisfy all these hypotheses are the following:

$$\begin{aligned} f_1(x) &= c|x|^\alpha \text{ with } \alpha > m_*, \\ f_2(x) &= c \exp(\alpha|x|) \text{ with } \alpha > 0, \\ f_3(x) &= c \exp(p^{|x|}) \text{ with } p \leq m. \end{aligned}$$

Some non-radial as well as some non-monotone versions are allowed within the range of **(H2)**, **(H4)** and **(H6)**.

Hypothesis **(H7)** covers all cases f_1, f_2, f_3 above, but makes the distinction between *power-type* growths which satisfy **(H7)**-slow and exponentials for which **(H7)**-fast is fulfilled.

Remark 2.3. We assume that f is C^1 throughout the paper for simplicity: with this assumption we can compute and use the gradient of $\Phi(x) = |x|f^{1/m}(x)$, for $|x|$ large. In fact, regularity of f is not an issue here and we could consider only continuous functions to get exactly the same results by using smooth approximations of f .

Across Sections 4–6 we will assume that f verifies the three assumptions **(H0)**–**(H2)** without mentioning it anymore. In Section 7, where we prove uniqueness, we will use more assumptions on f and hence we will write for each result only what is really necessary.

2.2. Subsolutions and Supersolutions. It is straightforward to see that, for $\lambda \leq \min(f)$, typical subsolutions of $(EP)_\lambda$ are the constants. In this range of λ -values, there are also coercive subsolutions, which will help us to build solutions which tend to infinity at infinity (see Section 6).

Lemma 2.4. *Let f verify **(H0)**. Then for any $\lambda \leq \min(f)$ there exists a Lipschitz subsolution Θ_λ of $(EP)_\lambda$ such that $\Theta_\lambda(x) \rightarrow \infty$ as $|x| \rightarrow \infty$.*

Proof. Since $\lambda \leq \min(f)$ and f is uniformly coercive, there exists $R_* > 0$ which depends on λ such that $f(x) \geq \lambda + \kappa^m$, if $|x| \geq R_*$, for some $\kappa > 0$. We define

$$\Theta_\lambda(x) := \kappa(|x| - R_*)_+$$

which is (globally) Lipschitz. Using Lemma A.4 we see that for any $x \in \mathbb{R}^N$, $-\mathcal{L}[\Theta_\lambda](x) \leq 0$. Moreover, since $|D\Theta_\lambda| = \kappa$ or $|D\Theta_\lambda| = 0$ almost everywhere, we get in any case

$$(2.1) \quad \lambda - \mathcal{L}[\Theta_\lambda] + |D\Theta_\lambda|^m \leq \lambda + \kappa^m \leq f.$$

Notice that the exact proof has to be done in the sense of viscosity, but at the points where $|x| = R_*$, no testing is done for the subsolution condition, while at other points the function is smooth. So, Θ_λ is indeed a coercive subsolution in \mathbb{R}^N . \square

Remark 2.5. The parameter κ is somewhat free if we allow R to be big. For $|x|$ big enough $f(x)$ is big and we can choose κ big. Thus we can build subsolutions with arbitrary big linear growth.

If we try now to construct a supersolution, the first thing that we face is that it is not possible to do it, if $\lambda \leq \min(f)$.

Lemma 2.6. *Let f verify **(H0)** and $\lambda \leq \min(f)$. There are no smooth, coercive supersolutions of (EP) in all \mathbb{R}^N .*

Proof. Assume by contradiction that there is such a supersolution, say ψ , and let x_0 be a minimum point of ψ . Then, since $\psi(y) \geq \psi(x_0)$ for all $y \in B_1(x_0)$

$$0 \leq f - \lambda \leq -\mathcal{L}[\psi](x_0) - |D\psi(x_0)|^m = - \int_{B_1(x_0)} J(x_0 - y)(\psi(y) - \psi(x_0)) \, dy \leq 0.$$

This implies that $\psi(y) = \psi(x_0)$ for all $y \in B_1(x_0)$.

We can repeat now the argument using as center any $y \in B_1(x_0)$. We finally get that $\psi(y) = \psi(x_0)$ for $y \in \mathbb{R}^N$. This yields a contradiction with f being coercive. \square

However we are able to build supersolutions in $\mathbb{R}^N \setminus \{0\}$ (without any restriction on λ). In order to do it, we look first at $(\text{EP})_0$,

$$(\text{EP})_0 \quad -\mathcal{L}[u](x) + |Du(x)|^m = f(x) \quad \text{in } \mathbb{R}^N.$$

In this construction we assume that **(H0)**–**(H2)** hold, so there exists $R^* > 0$ such that for any $|x| \geq R^*$, $f(x) \geq 1$ and **(H1)**, **(H2)** hold for such x .

Remark 2.7. If we take $\kappa = 1$ and $\lambda = 0$ in the construction of the subsolution Θ_λ , then $R_* = R^*$. We will use this fact in Section 6 in order to construct bounded from below solutions.

Up to fixing the constants, we use the following construction: we set $\Psi_{\text{int}} := b|x|$ for $|x| < R^* + 1$; $\Psi_{\text{ext}} = c\Phi$ for $|x| > R^*$ and then combine Ψ_{int} and Ψ_{ext} in the intermediate region $R^* \leq |x| \leq R^* + 1$, in order to get a supersolution of $(\text{EP})_0$ for all $|x| \neq 0$. We finally define $\Psi_\lambda = c_\lambda \Psi$ which yields a supersolution to (EP) for $|x| \neq 0$, provided c_λ is well-chosen.

Lemma 2.8. *There exists $c_0 > 0$ such that for any $c \geq c_0$, $\Psi_{\text{int}}(x) := c|x|$ is a supersolution of $(\text{EP})_0$ for $0 < |x| \leq R^* + 1$.*

Proof. The proof is straightforward: we first have

$$D\Psi_{\text{int}}(x) = c \frac{x}{|x|} \quad \text{and} \quad |\mathcal{L}[\Psi_{\text{int}}](x)| \leq \sup_{y \in B_1(x)} |D\Psi_{\text{int}}(x+y)| = c.$$

Then, since $m > 1$, for any c big enough we have

$$c^m - c \geq \max_{B_{R^*+1}} f,$$

and we get $-\mathcal{L}[\Psi_{\text{int}}](x) + |D\Psi_{\text{int}}(x)|^m \geq -c + c^m \geq f$. \square

Lemma 2.9. *Let f verify **(H0)**–**(H2)**. There exists $c_1 > 0$ such that for any $c \geq c_1$, $\Psi_{\text{ext}} = c\Phi$ is a supersolution of $(\text{EP})_0$ for $|x| \geq R^*$.*

Proof. We estimate each term in $(\text{EP})_0$ separately. On one hand we have

$$D\Psi_{\text{ext}} = cD\Phi = c \left(\frac{x}{|x|} f^{1/m} + \frac{|x|}{m} f^{1/m-1} \cdot Df \right).$$

Using **(H2)**,

$$(2.2) \quad \frac{x}{|x|} \cdot D\Phi = \frac{x}{|x|} \cdot \frac{x}{|x|} \left(f^{1/m} + \frac{1}{m} f^{1/m-1} x \cdot Df \right) \geq f^{1/m} \left(1 - \frac{1}{m} \right),$$

from where we get that $|D\Phi| \geq \left| \frac{x}{|x|} \cdot D\Phi \right| \geq f^{1/m} \left(1 - \frac{1}{m} \right)$.

On the other hand, in order to estimate the non-local term we use **(H1)** and get

$$|\mathcal{L}[\Phi](x)| \leq \sup_{y \in B_1(x)} |D\Phi(x+y)| \leq |D\Phi(x)|^m.$$

Therefore, if c is such that

$$c^m - c \geq \frac{m}{m-1},$$

we get

$$-\mathcal{L}[\Psi_{\text{ext}}](x) + |D\Psi_{\text{ext}}(x)|^m \geq -c|D\Phi(x)|^m + c^m|D\Phi(x)|^m \geq \frac{m-1}{m}(c^m - c)f \geq f,$$

and conclude that Ψ_{ext} is a supersolution of $(\text{EP})_0$ for $|x| \geq R^*$. \square

Finally, the construction ends by interpolating between Ψ_{int} and Ψ_{ext} in the region $R^* \leq |x| \leq R^* + 1$. To this aim, let $\chi : [0, \infty) \mapsto [0, \infty)$ be a regular, radial and non-decreasing function that verifies $\chi(r) = 0$ for $r \leq R^*$ and $\chi(r) = 1$ for $r \geq R^* + 1$. We set

$$(2.3) \quad \Psi := (1 - \chi)\Psi_{\text{int}} + \chi\Psi_{\text{ext}} \quad \text{in } \mathbb{R}^N.$$

Lemma 2.10. *Let f verify **(H0)**–**(H2)**. There exists $c_2 > 0$ such that for any $c \geq c_2$, Ψ is a supersolution of $(\text{EP})_0$ for $R^* \leq |x| \leq R^* + 1$.*

Proof. We first give a rough estimate of the non-local term. Notice that since $|x| \geq R^*$, $f(x) \geq 1$ so that $\Psi_{\text{ext}}(x) \geq c|x| \geq 0$. Since also $\Psi_{\text{int}}(x) = c|x| \geq 0$, it follows that for any $R^* \leq |x| \leq R^* + 1$, $\Psi(x) \geq 0$. Hence, for such x ,

$$-\mathcal{L}[\Psi](x) \geq -(J * \Psi)(x) \geq -c(R^* + 1) - c \sup_{B_{R^*+2} \setminus B_{R^*}} \Phi = -Kc$$

for some constant K depending only on f (through R^* and Φ).

We now turn to the gradient term. As we noticed, $\Psi_{\text{ext}}(x) \geq c|x| = \Psi_{\text{int}}(x)$ for $R^* \leq |x| \leq R^* + 1$. And since χ is radially nondecreasing, we get, using for the last inequality (2.2) and the fact that $f(x) \geq 1$ for $|x| \geq R^*$,

$$\begin{aligned} \frac{x}{|x|} D\Psi(x) &= (1 - \chi) \frac{x}{|x|} D\Psi_{\text{int}} + \chi \frac{x}{|x|} D\Psi_{\text{ext}} + \chi'(\Psi_{\text{ext}} - \Psi_{\text{int}}) \\ &\geq (1 - \chi) \frac{x}{|x|} D\Psi_{\text{int}} + \chi \frac{x}{|x|} D\Psi_{\text{ext}} \\ &\geq (1 - \chi)c + \chi c(1 - 1/m) = c(1 - \chi/m) \geq c(1 - 1/m). \end{aligned}$$

This gives a lower estimate for the gradient of Ψ : for any $R^* \leq |x| \leq R^* + 1$,

$$|D\Psi(x)|^m \geq \left| \frac{x}{|x|} D\Psi(x) \right|^m \geq c^m(1 - 1/m)^m$$

To conclude, we get that for any $R^* \leq |x| \leq R^* + 1$,

$$-\mathcal{L}[\Psi] + |D\Psi|^m - f \geq -Kc + c^m(1 - 1/m)^m - \sup_{B_{R^*+1} \setminus B_{R^*}} f.$$

Hence, if c is big enough, since $m > 1$, we obtain that the right-hand side is non-negative which yields the result. \square

We are now ready to construct a supersolution for $(\text{EP})_\lambda$ for $|x| \neq 0$. We first fix $c_* = \max(c_0, c_1, c_2)$ where c_0, c_1 and c_2 are defined in the lemmas above. Then the corresponding function Ψ is C^1 -smooth and it is a supersolution of $(\text{EP})_0$ in $\mathbb{R}^N \setminus \{0\}$. In order to deal with a nonzero ergodic constant λ , it is enough to multiply Ψ by some constant (depending on λ). We set

$$(2.4) \quad \Psi_\lambda := c_\lambda \Psi, \quad \text{where } c_\lambda = (2 + \lambda^-) \text{ and } \lambda^- = \max(0, -\lambda) \geq 0.$$

Proposition 2.11. *Let f verify **(H0)**–**(H2)**, then Ψ_λ is a strict supersolution for $(\text{EP})_\lambda$ for all $|x| \neq 0$.*

Proof. Recall that for $|x| > R^*$ $f(x) \geq 1$ and since Ψ is a supersolution of $(EP)_0$, for such x we have

$$\begin{aligned} \lambda - \mathcal{L}[\Psi_\lambda] + |D\Psi_\lambda|^m &= \lambda - (2 + \lambda^-)\mathcal{L}[\Psi] + (2 + \lambda^-)^m |D\Psi|^m \\ &\geq \lambda + (2 + \lambda^-)(-\mathcal{L}[\Psi] + |D\Psi|^m) \\ &\geq \lambda + (2 + \lambda^-)f \\ &\geq \lambda + \lambda^- + 1 + f \geq f + 1. \end{aligned}$$

On the other hand, if $|x| \leq R^*$, $\Psi_\lambda = (2 + \lambda_-)\Psi_{\text{int}}$. Then, as in Lemma 2.8 we obtain

$$\begin{aligned} -\mathcal{L}[\Psi_\lambda] + |D\Psi_\lambda|^m &\geq \lambda + (2 + \lambda^-)^m c^m - (2 + \lambda_-)c \geq \lambda + (2 + \lambda^-)(c^m - c) \\ &\geq \lambda + (2 + \lambda^-) \max_{B_{R^*+1}} f \geq f + 1. \end{aligned}$$

Notice that in the last inequality, we use that the maximum of f on B_{R^*+1} is greater than or equal to one since at least $f(x) \geq 1$ for $|x| \geq R^*$. \square

Remark 2.12. For $|x| \gg 1$, $\Psi_\lambda = (2 + \lambda_-)\Phi$. Hence, the supersolution Ψ_λ inherits all the properties that we assume on Φ .

Remark 2.13. Notice that for any $c \geq c_\lambda$, $c\Psi$ is also a strict supersolution of $(EP)_\lambda$.

2.3. Non-Existence results. We end this section by showing, at least in a radial case, that solutions to (EP) only exist if f grows less than a specific function (independently of the value of λ).

Lemma 2.14. *Let $\alpha, \beta, r_0, c, a > 0$, $m > 1$ and consider the inequation*

$$(2.5) \quad \alpha\psi(r) - \beta\psi(r+1) + (\psi'(r))^m \geq f(r) := ce^{ar} \quad \text{for } r > r_0.$$

Then if $a > m$, there is no function $\psi \in C^1([r_0, \infty))$ with $\psi' \geq 0$ satisfying (2.5).

Proof. We proceed by contradiction, assuming that such a function ψ exists.

We first claim that $\psi(r) \rightarrow +\infty$ as $r \rightarrow \infty$. Indeed, since ψ is nondecreasing, it is bounded from below on $[r_0, +\infty)$ and if in addition we assume that it is bounded from above, we get that for some constant $C > 0$, $(\psi'(r))^m \geq f(r) - C$. Hence $\psi'(r) \geq 1$ for r big enough, which is a contradiction with the boundedness of ψ . Hence ψ is not bounded and since it is nondecreasing, the claim holds.

Thus we can assume that for some $r_1 > r_0$, $\psi(r) \geq 1/\alpha$ on $[r_1, \infty)$, which implies that for $r \geq r_1$,

$$(2.6) \quad (\alpha\psi(r) + \psi'(r))^m \geq f(r) + \beta\psi(r+1).$$

From this inequality we prove, by an iteration process, that ψ has to blow-up for any $r > r_1 + 1$, which leads to a contradiction.

The first iteration neglects the term $\beta\psi(r+1)$ in (2.6). Hence, integrating the expression

$$\left[(e^{\alpha r} \psi)' e^{-\alpha r} \right]^m \geq ce^{ar}$$

we obtain for $r > r_1$

$$\psi(r+1) \geq ce^{-\alpha(r+1)} \int_{r_1}^{r+1} e^{(a^s)/m} e^{\alpha s} ds.$$

We fix $\delta \in (0, 1)$ such that $a^\delta/m := \gamma > 1$, which is possible since $a > m$. Thus,

$$(2.7) \quad \psi(r+1) \geq c e^{-\alpha(r+1)} \int_{r+\delta}^{r+1} e^{(a^s)/m} e^{\alpha s} ds \geq c(1-\delta) e^{\alpha(\delta-1)} e^{a^r \cdot a^\delta/m} \geq C e^{\gamma a^r},$$

where the constant $C > 0$ depends on c, δ, a, m, α .

For the next iterations, we repeat the integration and bounding process. We use (2.6), neglecting now the term $f(r)$, and (2.7) and get

$$\psi(r+1) \geq \beta C(c, \delta, a, m, \alpha)^2 e^{\gamma^2 a^r}.$$

Then by direct induction, we get that for any $n \geq 2$ and $r > r_1$, $\psi(r+1) \geq \beta^{n-1} C(c, \delta, a, m, \alpha)^n e^{\gamma^n a^r}$. Since $\gamma > 1$, we conclude by sending n to infinity, which yields blow up for ψ for any r (even if possibly, $\beta, C < 1$). \square

Corollary 2.15. *If $f(x) = C \exp(a^{|x|})$ for $|x| > r_0$ with $a > m$, then there is no C^1 solution u of*

$$\lambda - \mathcal{L}[\psi](r) + (\psi'(r))^m = f$$

such that for $|x| \gg 1$, u is radial and radially increasing.

Proof. For $|x| \gg 1$, $\bar{f} := f - \lambda \geq (C/2) \exp(a^{|x|})$. Then we use Lemma A.5 with $\alpha = 1$, $\beta = c_\epsilon$ and \bar{f} instead of f in (2.5). \square

This result is restrictive in the sense that we cannot avoid the possibility of having non radial solutions u , or that they are not radially increasing. But this is improbable with $f(r) = C \exp(a^r)$, and this result is at least a good hint that a more general non-existence statement should hold.

3. APPROXIMATE PROBLEMS IN BOUNDED DOMAINS

In this section we settle and solve some approximate problems that are defined in B_R . Solving such problems is quite standard for local equations (see [2, 11]), but we have to adapt here several steps to deal with the non-local term. In particular, we will use the Perron's method, following the standard construction; however, we do not want to skip it, since, because of the non-local operator which is involved, we have to check and adapt every step carefully. Those approximate problems will be the key in order to construct solutions of (EP) in the whole space, Section 4.

First of all we adapt the definition of the non-local term to the bounded domain B_R . When solving the Dirichlet problem we need to consider the usual boundary condition (function $g \in C^{0,\gamma}(\partial B_R)$ for any $\gamma \in (0, 1)$ see (3.2) below), but also an *outer condition* ψ , which enters in the non-local operator, see [7, 8]. Thus for $R > 1$ we define

$$(3.1) \quad \mathcal{L}_R^\psi[v](x) := \int_{B_R} J(x-y)v(y) dy + \int_{B_R^c} J(x-y)\psi(y) dy - v(x).$$

It is clear that, if v is defined in the whole space, then $\mathcal{L}_R^v[v] = \mathcal{L}[v]$. Moreover, since J is compactly supported on B_1 , for $x \in B_R$, the outer term in (3.1) becomes

$$\int_{B_R^c} J(x-y)\psi(y) dy = \int_{B_{R+1} \setminus B_R} J(x-y)\psi(y) dy,$$

so that the function ψ needs only to be, say, continuous on $B_{R+1} \setminus B_R$ for \mathcal{L}_R^ψ to be defined.

Now, for $\varepsilon > 0$ and $R > 1$ fixed, we consider the approximate problem

$$(3.2) \quad \begin{cases} \lambda - \varepsilon \Delta v - \mathcal{L}_R^\psi[v] + |Dv|^m = f, & x \in B_R, \\ v = g, & x \in \partial B_R. \end{cases}$$

The fundamental existence result is the following:

Proposition 3.1. *Let $\varepsilon > 0$, $R > 1$, $f \in C^1(B_R)$, $\psi \in C^0(B_{R+1} \setminus B_R)$ and $g \in C^{0,\gamma}(\partial B_R)$. If there exists a subsolution $\underline{v} \in C^2(B_R) \cap C^0(\overline{B_R})$ of (3.2), then there exists a solution, $v \in C^2(B_R) \cap C^0(\overline{B_R})$ of (3.2).*

Remark 3.2. We opted to state this result assuming $f \in C^1$, since this is the general assumption we made in order to have supersolutions (see Section 2). The same proof holds with a regularizing argument if f is only continuous or even $f \in W^{1,\infty}(\mathbb{R}^N)$.

In order to use Perron's method, we have to provide, as a first step, a supersolution to problem (3.2). To this aim, let us consider the linearized problem

$$(3.3) \quad \begin{cases} -\varepsilon \Delta \phi - \mathcal{L}_R^\psi[\phi] = M, & x \in B_R, \\ \phi = g, & x \in \partial B_R. \end{cases}$$

Existence and uniqueness of a solution $\phi \in C^{2,\gamma}(B_R) \cap C^0(\overline{B_R})$ for any $\gamma \in (0, 1)$ for problem (3.3) is obtained through a variant of [10, Theorem 6.8] that includes the non-local operator. We detail in the Appendix the construction and adaptations, see Lemma C.1.

Lemma 3.3. *Let $M > \|f\|_{L^\infty(B_R)} + |\lambda|$ and let $\bar{v} \in C^{2,\gamma}(B_R) \cap C^0(\overline{B_R})$ be the solution of (3.3) and \underline{v} a subsolution of (3.2). Then $\underline{v} \leq \bar{v}$ in B_R .*

Proof. Due to the choice of M , it is straightforward that \bar{v} is a strict supersolution of (3.2) and that $\bar{v} = \underline{v}$ on ∂B_R .

To get the comparison result, we define $w := \underline{v} - \bar{v} \in C^2(B_R) \cap C^0(\overline{B_R})$, which verifies

$$-\varepsilon \Delta w - \mathcal{L}_R^0[w] < 0 \quad \text{in } B_R \quad \text{and} \quad w = 0 \quad \text{on } \partial B_R.$$

Notice that the exterior term involving ψ cancels after subtracting the equations, hence we obtain a zero-Dirichlet problem for w , i.e. with $g = \psi = 0$.

Let x_0 be a maximum point of w in $\overline{B_R}$. If $w(x_0) \leq 0$ the result follows immediately, so let us assume that $w(x_0) > 0$. In this case, notice that $x_0 \in \partial B_R$ is impossible since $w = 0$ on the boundary.

Since $x_0 \in B_R$ and w is a C^2 -smooth function, we have $\Delta w(x_0) \leq 0$ and the equation yields

$$-\mathcal{L}_R^0[w](x_0) < 0.$$

But since x_0 is a point where w reaches its positive maximum, we get

$$0 < w(x_0) \left(\int_{B_R} J(x_0 - y) \, dy - 1 \right),$$

which is a contradiction since $w(x_0) > 0$ and $\int_{B_R} J(x_0 - y) \, dy \leq 1$. \square

In the following, we introduce the critical exponent $\alpha_* := (m-2)/(m-1) \in (0, 1)$ as found in [6], which is where $m > 2$ is needed.

Proof of Proposition 3.1. Let \bar{v} be a supersolution of (3.2) and consider the set

$$\mathcal{S} := \{v_S \in C^{0,\alpha^*}(\overline{B_R}) : v_S \text{ subsolution of (3.2) with } \underline{v} \leq v_S \leq \bar{v} \text{ in } \overline{B_R}\}.$$

The set is non-empty since $\underline{v} \in \mathcal{S}$. For any $x \in B_R$, we set

$$v(x) := \sup_{v_S \in \mathcal{S}} v_S(x)$$

which is well-defined, since all the functions v_S in \mathcal{S} are bounded above by \bar{v} . We first notice that $\underline{v} \leq v \leq \bar{v}$ in B_R and that necessarily $v = \underline{v} = \bar{v}$ on ∂B_R .

Concerning the regularity of v , we use the estimates of [6, Theorem 1.1]. Writing (3.2) as

$$v_S - \varepsilon \Delta v_S + |Dv_S|^m \leq F(x),$$

where

$$F(x) := \int_{B_R} v_S(y) J(x-y) dy + \int_{B_{R+1} \setminus B_R} \psi(y) J(x-y) dy + f(x) - \lambda,$$

we see that there exists a constant K depending only on $\|(v_S)^-\|_{L^\infty(B_R)}$ and $\|F\|_{L^\infty(B_R)}$ such that

$$|v_S(x) - v_S(y)| \leq K|x-y|^{\alpha^*} \quad \text{in } \overline{B_R}.$$

Now, for any $v_S \in \mathcal{S}$ we have $(v_S)^- \leq (\underline{v})^-$ and $v_S \leq \bar{v}$. So, both functions $F(x)$ and $(v_S)^-(x)$ are uniformly bounded in B_R with respect to $v_S \in \mathcal{S}$. We deduce that the subsolutions v_S in \mathcal{S} are uniformly Hölder continuous up to the boundary which implies that $v \in C^{0,\alpha^*}(\overline{B_R})$.

CLAIM – v is a viscosity solution of (3.2). It is standard that, being defined as a supremum of subsolutions, v is also a subsolution of (3.2). Hence $v \in \mathcal{S}$ and it remains to prove that v is a supersolution of (3.2). Since by construction $v = g$ on ∂B_R , we only need to check the supersolution condition inside B_R , which is done as usual through the construction of a bump function.

We proceed by contradiction: let us assume that v is not a supersolution of (3.2). Then, there exists a fixed $\bar{x} \in B_R$ and $\varphi \in C^2(B_R)$ such that $v - \varphi$ has a minimum at \bar{x} and

$$(3.4) \quad -\varepsilon \Delta \varphi(\bar{x}) - \mathcal{L}_R^\psi[v](\bar{x}) + |D\varphi(\bar{x})|^m - f(\bar{x}) + \lambda < 0.$$

We can assume with no restriction that $v(\bar{x}) = \varphi(\bar{x})$, hence $v \geq \varphi$ in B_R .

Moreover, we claim that $v(\bar{x}) < \bar{v}(\bar{x})$. Indeed, assuming otherwise that $v(\bar{x}) = \bar{v}(\bar{x})$, since $v \leq \bar{v}$ we would have $v(\bar{x}) = \varphi(\bar{x}) = \bar{v}(\bar{x})$ and $\varphi \leq v \leq \bar{v}$, which together imply that $\varphi - \bar{v}$ has a minimum at \bar{x} . Consequently $D\varphi(\bar{x}) = D\bar{v}(\bar{x})$ and $\Delta\varphi(\bar{x}) - \Delta\bar{v}(\bar{x}) \geq 0$. Replacing φ by \bar{v} in (3.4) at the point \bar{x} we would get

$$-\varepsilon \Delta \bar{v}(\bar{x}) - \mathcal{L}_R^\psi[\bar{v}](\bar{x}) + |D\bar{v}(\bar{x})|^m - f(\bar{x}) + \lambda < 0.$$

But since $v(\bar{x}) = \bar{v}(\bar{x})$ and $v \leq \bar{v}$ in B_R , we have $\mathcal{L}_R^\psi[v](\bar{x}) \leq \mathcal{L}_R^\psi[\bar{v}](\bar{x})$ and we see that

$$-\varepsilon \Delta \bar{v}(\bar{x}) - \mathcal{L}_R^\psi[\bar{v}](\bar{x}) + |D\bar{v}(\bar{x})|^m - f(\bar{x}) + \lambda < 0.$$

This contradicts the fact that \bar{v} is a supersolution of (3.2). Hence $v(\bar{x}) < \bar{v}(\bar{x})$ and we define, for any $y \in B_R$, the bump function

$$v_\delta(y) := \max\{v(y); \varphi(y) + \delta - |\bar{x} - y|^2\}.$$

Notice that by construction,

$$(3.5) \quad \begin{cases} v_\delta(y) = v(y) & \text{if } y \notin B_{\delta^{1/2}}(\bar{x}), \\ v(y) \leq v_\delta(y) \leq v(y) + \delta & \text{if } y \in B_{\delta^{1/2}}(\bar{x}). \end{cases}$$

The strategy is to prove that for $\delta > 0$ small enough, $v_\delta \in \mathcal{S}$, which contradicts the definition of v as a sup, since $v_\delta(\bar{x}) = \varphi(\bar{x}) + \delta > v(\bar{x})$. We divide this into four steps.

(i) REGULARITY – The function v_δ , defined as a maximum of two functions which belong to $C^{0,\alpha^*}(\overline{B_R})$, belongs itself to $C^{0,\alpha^*}(\overline{B_R})$.

(ii) BOUNDS – It is clear by construction that $v_\delta \geq v$ in B_R .

On the other hand, if $|y - \bar{x}|^2 \geq \delta$ then $v_\delta(y) = v(y)$, see (3.5). Hence, outside $B_{\delta^{1/2}}(\bar{x})$, we have $v_\delta \leq \bar{v}$. Moreover, since $v(\bar{x}) < \bar{v}(\bar{x})$, for δ small, we have $v_\delta \leq \bar{v}$ for $y \in B_{\delta^{1/2}}(\bar{x})$. Therefore $v_\delta \leq \bar{v}$ in B_R for δ small enough.

(iii) SUBSOLUTION CONDITION – As before, outside $B_{\delta^{1/2}}(\bar{x})$, we have that $v_\delta = v$. Hence v_δ is a subsolution in $B_{\delta^{1/2}}^C(\bar{x})$.

Let $y \in B_{\delta^{1/2}}(\bar{x})$ and $\varrho \in C^2(B_R)$ be a test function such that $v_\delta - \varrho$ has a (strict) maximum zero at y . We have to prove that in this situation v_δ verifies

$$-\varepsilon \Delta \varrho(y) - \mathcal{L}_R^\psi[v_\delta](y) + |D\varrho(y)|^m - f(y) + \lambda \leq 0.$$

Since $y \in B_{\delta^{1/2}}(\bar{x})$ we have $\varrho(y) = \varrho(\bar{x}) + o_\delta(1)$ uniformly with respect to $y \in B_{\delta^{1/2}}(\bar{x})$. Similarly, $|D\varrho(y)|^m = |D\varrho(\bar{x})|^m + o_\delta(1)$ and $\varepsilon \Delta \varrho(y) = \varepsilon \Delta \varrho(\bar{x}) + o_\delta(1)$.

Moreover, $-\mathcal{L}_R^\psi[v_\delta](y) = -\mathcal{L}_R^\psi[v_\delta](\bar{x}) + o_\delta(1)$, and the facts that $v_\delta \geq v$ and $v_\delta(\bar{x}) = v(\bar{x}) + \delta$ imply that $-\mathcal{L}_R^\psi[v_\delta](\bar{x}) \leq -\mathcal{L}_R^\psi[v](\bar{x}) + \delta$. Finally, since f is continuous, $f(y) = f(\bar{x}) + o_\delta(1)$. Gathering these estimates we obtain that for any $y \in B_{\delta^{1/2}}(\bar{x})$,

$$\begin{aligned} & -\varepsilon \Delta \varrho(y) - \mathcal{L}_R^\psi[v_\delta](y) + |D\varrho(y)|^m - f(y) + \lambda \\ & \leq -\varepsilon \Delta \varrho(\bar{x}) - \mathcal{L}_R^\psi[v](\bar{x}) + |D\varrho(\bar{x})|^m - f(\bar{x}) + \lambda + o_\delta(1). \end{aligned}$$

Finally, since $\varrho(\bar{x}) = v_\delta(\bar{x}) + o_\delta(1) = \varphi(\bar{x}) + o_\delta(1)$, we deduce that for $\delta > 0$ small enough, according to (3.4), the bump function v_δ is a subsolution of (3.2) in $B_{\delta^{1/2}}(\bar{x})$. Therefore, $v_\delta = \max(v, \varphi_\delta)$ is a subsolution in B_R .

(iv) CONTRADICTION – The above points (i) – (ii) – (iii) imply that $v_\delta \in \mathcal{S}$, which is a contradiction with the definition of v , since $v_\delta(\bar{x}) > v(\bar{x})$.

We conclude that $v \in C^{0,\alpha^*}(\overline{B_R})$ is a supersolution of (3.2) in B_R and since it is also a subsolution, it is a (viscosity) solution of (3.2).

Finally, to get that $v \in C^2(B_R)$ we use some standard bootstrap regularity estimates. Notice first that, since v and ψ are continuous, both integrals terms in $\mathcal{L}_R^\psi[v]$ are at least Lipschitz continuous. Since, by assumption, f is also Lipschitz, we apply [6, Theorem 3.1], which implies that v is at least Lipschitz continuous, locally in B_R . Thus, the Lipschitz function v satisfies an equation of the form $-\varepsilon \Delta v = \tilde{F} \in L_{\text{loc}}^\infty(B_R)$ in the viscosity sense.

This implies that v is actually also a *weak* solution of this equation. This is a quite straightforward and standard statement in viscosity solutions' theory which comes from the fact that we can regularize v as v_n and pass to the limit in the weak sense in the equation.

By standard regularity results, it follows that $v \in W_{\text{loc}}^{2,p}(B_R)$ for any $p > 1$ so that $v \in C^{1,\alpha}$ for any $\alpha \in (0,1)$. Hence \tilde{F} is in fact in $C^{0,\alpha}(B_R)$ and from this we deduce that $v \in C^{2,\alpha}(B_R)$ for any $\alpha \in (0,1)$. \square

We end this section by introducing a uniform (in ε) estimate of the solution v :

Lemma 3.4. *For any $R > 1$ there exists a constant $C = C(R, \psi, g, f) > 0$ such that for $\varepsilon > 0$ small enough, $\|v\|_{L^\infty(B_R)} \leq C(R, \psi, g, f)$.*

Proof. Consider the following equation

$$(3.6) \quad -\mathcal{L}_R^\psi[\chi] = M + 1 \quad \text{in } B_R,$$

where M is defined in Lemma 3.3. We refer to [9, Appendix] for existence of a solution $\chi \in L^1(B_R)$ of (3.6). Notice that, since both integrals in the non-local term are at least continuous, we have $\chi \in C^0(\overline{B_R})$. Then we consider a resolution of the identity $(\rho_k)_{k \in \mathbb{N}}$ and set

$$\bar{\chi}_k := \rho_k * (\chi + c), \quad c := \|g\|_{L^\infty(B_R)} + \|\chi\|_{L^\infty(B_R)},$$

so that for any $k \in \mathbb{N}$, $\bar{\chi}_k \in C^2(B_R) \cap C^0(\overline{B_R})$ and $\bar{\chi}_k \geq g$ on ∂B_R . Since $\bar{\chi}_k \rightarrow \chi + c$ uniformly in B_R and $c > 0$, it follows that

$$-\mathcal{L}_R^\psi[\bar{\chi}_k] = -\mathcal{L}_R^\psi[\chi] - \mathcal{L}_R^0[c] + o_k(1) \geq M + 1 + o_k(1),$$

where $o_k(1)$ vanishes as $k \rightarrow \infty$, uniformly with respect to $x \in B_R$. We first choose $k = k_0$ big enough (but fixed) so that the right-hand side above is greater than $M + 1/2$. Then, it follows that for $0 < \varepsilon < \varepsilon_0(k_0)$ we have $\varepsilon \|\Delta \bar{\chi}_{k_0}\|_{L^\infty(B_R)} < 1/2$. Hence, setting $\omega(x) := \bar{\chi}_{k_0}(x)$ we find that for ε small enough,

$$-\varepsilon \Delta \omega - \mathcal{L}_R^\psi[\omega] > M$$

which means that ω is a supersolution of (3.3) such that $\omega \geq g$ on ∂B_R . By the comparison principle (which can be proved exactly as in the proof of Lemma 3.3 and Theorem B.1) we get that $\bar{v} \leq \omega \leq \|\omega\|_{L^\infty(B_R)} := C(R, \psi, g, f)$ in B_R . And the result follows since by construction $v \leq \bar{v}$. \square

Remark 3.5. We will use later (see Lemmas 4.3 and 4.5) that the above estimate is uniform with respect to the data ψ, g, f provided they remain bounded. Especially, this holds true if we take approximations ψ_n, g_n, f_n that converge uniformly in B_R .

4. EXISTENCE RESULTS FOR (EP)

The aim of this section is to prove that there exists at least one value of λ for which problem (EP) is solvable. Moreover, the solution turns out to be bounded by the supersolution Ψ_λ constructed in Section 2, see (2.4).

Theorem 4.1. *If there exists a strict viscosity subsolution, $\underline{u} \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^N)$, of (EP) $_\lambda$ such that $\underline{u} \leq \Psi_\lambda$ in \mathbb{R}^N and $\underline{u}(0) = 0$, then there exists a viscosity solution, $u_\lambda \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^N)$, of (EP) $_\lambda$, such that $u_\lambda \leq \Psi_\lambda$ in \mathbb{R}^N and $u_\lambda(0) = 0$.*

We reduce the proof of this result to solve the approximate problem defined on B_R , see (3.2), and then pass to the limit first as ε tends to zero, and second as R tends to $+\infty$. To perform this we need two main ingredients: (i) smooth the data (the right-hand side f and the subsolution \underline{u}); (ii) get some local uniform bounds independent of ε and R to pass to the limit.

Let ρ_n be a resolution of the identity and set $f_n := \rho_n * f$, $\psi_n := \rho_n * \underline{u}$. Then both f_n and ψ_n are smooth and converge uniformly in B_R to f and \underline{u} respectively. In the following, $o_n(1)$ stands for any quantity which vanishes as $n \rightarrow \infty$, uniformly with respect to $x \in B_R$. Thus, $\psi_n = \underline{u} + o_n(1)$ and $f_n = f + o_n(1)$.

Observe that, as we mentioned in Remark 2.3, our general assumption on f is C^1 . This will be not enough here, since at some steps in this section we will have to compute $\Delta\Psi_\lambda$. This is why we have to do an approximation argument and define f_n .

Consider also $(3.2)_n$; i.e. problem (3.2) with data $f = f_n$, outer condition $\psi = \psi_n$, and boundary data $g = \psi_n$ on ∂B_R .

The first result we prove states that we can use ψ_n as a subsolution to $(3.2)_n$ in B_R . Notice that during the proof we will choose ε depending on n . This will not be a problem afterwards, since we will first send ε to zero (Lemma 4.6) and then n to infinity.

Lemma 4.2. *There exists $\eta > 0$ such that for n big enough and ε small enough, the smooth function ψ_n satisfies*

$$(4.1) \quad \lambda - \varepsilon\Delta\psi_n - \mathcal{L}_R^{\psi_n}[\psi_n] + |D\psi_n|^m \leq f_n - \eta/2 \quad \text{in } B_R.$$

Moreover, $\psi_n(0) = o_n(1)$ and $\psi_n \leq \Psi_\lambda + o_n(1)$ uniformly in B_R .

Proof. Since ψ_n converges uniformly in B_R to \underline{u} , it is a direct consequence of the assumptions on \underline{u} that $\psi_n(0) = o_n(1)$ and $\psi_n \leq \Psi_\lambda + o_n(1)$.

In order to prove the first part of the lemma, notice that, since \underline{u} is a locally Lipschitz strict subsolution of (EP) in \mathbb{R}^N , there exists $\eta > 0$ such that for almost any $x \in B_R$,

$$\lambda - \mathcal{L}[\underline{u}] + |D\underline{u}|^m \leq f - \eta.$$

We can then estimate the terms in (4.1) as follows: for the non-local term we notice that $\mathcal{L}_R^{\psi_n}[\psi_n] = \mathcal{L}[\psi_n] = \mathcal{L}[\underline{u}] + o_n(1)$. Moreover, $|D\psi_n|^m = |D\underline{u}|^m + o_n(1)$ and $-\varepsilon\Delta\psi_n$ is as small as we want if we choose ε small enough, once n is fixed. Hence, we deduce that

$$\lambda - \varepsilon\Delta\psi_n - \mathcal{L}_R^{\psi_n}[\psi_n] + |D\psi_n|^m \leq f_n - \eta + o_n(1) + \varepsilon\|\Delta\psi_n\|_{L^\infty(B_R)} \quad \text{in } B_R.$$

Choosing first n big enough, then $\varepsilon = \varepsilon(n)$ small enough yields (4.1). \square

A supersolution to $(3.2)_n$ is obtained as in Section 3, but now using $(3.3)_n$, which is (3.3) with ψ replaced by ψ_n and $g = \psi_n$. Hence, using Proposition 3.1, we find a solution of $(3.2)_n$, that we denote $v_{R,n,\varepsilon}$. By construction we have $v_{R,n,\varepsilon} \geq \psi_n = \underline{u} + o_n(1)$. We define also

$$w_{R,n,\varepsilon}(x) := v_{R,n,\varepsilon}(x) - v_{R,n,\varepsilon}(0),$$

so that $w_{R,n,\varepsilon}(0) = 0$ and $w_{R,n,\varepsilon}(x) = \psi_n(x) - v_{R,n,\varepsilon}(0)$ on ∂B_R . Moreover, $w_{R,n,\varepsilon}$ satisfies

$$(4.2) \quad \lambda - \varepsilon\Delta w_{R,n,\varepsilon} - \mathcal{L}_R^{\psi_n}[w_{R,n,\varepsilon}] + |Dw_{R,n,\varepsilon}|^m = f_n + \mu_{R,n,\varepsilon}, \quad x \in B_R,$$

where $\mu_{R,n,\varepsilon}(x) := \mathcal{L}_R^0[v_{R,n,\varepsilon}(0)](x)$. Concerning this term (which does not exist neither in the local case nor in the problem defined in the whole space), we have

a first estimate which follows directly from the construction of $v_{R,n,\varepsilon}$ and the fact that $\underline{u}(0) = 0$:

$$(4.3) \quad \begin{aligned} \mu_{R,n,\varepsilon}(x) &= v_{R,n,\varepsilon}(0) \left(\int_{B_R} J(x-y) dy - 1 \right) \\ &\geq (\underline{u}(0) + o_n(1)) \left(\int_{B_R} J(x-y) dy - 1 \right) = o_n(1). \end{aligned}$$

Moreover, introducing the indicator function $\mathbf{1}_A$ of A , we have:

Lemma 4.3. *For any $R > 1$, there exists a constant $\nu(R) > 0$ such that, for any $\varepsilon > 0$ small enough and n big enough,*

$$|\mu_{R,n,\varepsilon}(x)| \leq \nu(R) \cdot \mathbf{1}_{B_R \setminus B_{R-1}}(x).$$

Proof. Notice that, for any $|x| \leq R-1$, since J is compactly supported in B_1 ,

$$\mu_{R,n,\varepsilon}(x) = v_{R,n,\varepsilon}(0) \left(\int_{B_R} J(x-y) dy - 1 \right) = 0.$$

Hence, we first deduce that $\mu_{R,n,\varepsilon}$ is compactly supported in $B_R \setminus B_{R-1}$. Second, Lemma 3.4 gives an estimate of $v_{R,n,\varepsilon}(0)$ by a constant which is independent of ε (small enough). Actually, this estimate can be found uniform in n , since ψ_n and f_n converge uniformly in B_R to \underline{u} and f respectively. \square

We now prove a local bound for $w_{R,n,\varepsilon}$, independent of ε, n and R , in terms of the supersolution Ψ_λ .

Lemma 4.4. *For any $R > 1$ fixed, $w_{R,n,\varepsilon} \leq \Psi_\lambda + o_n(1)$ in \overline{B}_R .*

Proof. By Lemma 4.2 we have that $\psi_n \leq \Psi_n + o_n(1)$ uniformly in B_R . Hence

$$\mathcal{L}_R^{\psi_n}[\Psi_\lambda] \leq \mathcal{L}_R^{\Psi_\lambda + o_n(1)}[\Psi_\lambda] \leq \mathcal{L}[\Psi_\lambda] + o_n(1).$$

Since Ψ_λ is a strict supersolution of $(EP)_\lambda$ in $\mathbb{R}^N \setminus \{0\}$, we deduce that for $\varepsilon \ll 1$ and n big enough

$$(4.4) \quad \lambda - \varepsilon \Delta \Psi_\lambda - \mathcal{L}_R^{\psi_n}[\Psi_\lambda] + |D\Psi_\lambda|^m > f_n + o_n(1), \quad \text{for } |x| \neq 0.$$

Let $x_0 \in \overline{B}_R$ be such that $(w_{R,n,\varepsilon} - \Psi_\lambda)(x_0) \geq (w_{R,n,\varepsilon} - \Psi_\lambda)(x)$ for all $x \in \overline{B}_R$.

If $x_0 \in B_R \setminus \{0\}$ and $(w_{R,n,\varepsilon} - \Psi_\lambda)(x_0) \leq 0$, the result follows. On the other hand, if $x_0 \in B_R \setminus \{0\}$ is a point where $w_{R,n,\varepsilon} - \Psi_\lambda$ achieves a positive maximum, then at this point $Dw_{R,n,\varepsilon} = D\Psi_\lambda$, $\Delta w_{R,n,\varepsilon} \leq \Delta \Psi_\lambda$ and $\mathcal{L}_R^{\psi_n}[w_{R,n,\varepsilon}] \leq \mathcal{L}_R^{\psi_n}[\Psi_\lambda]$. Using that $w_{R,n,\varepsilon}$ satisfies (4.2) together with (4.3) we get a contradiction with (4.4).

Hence, either $x_0 = 0$ or $x_0 \in \partial B_R$. In both cases we get $w_{R,n,\varepsilon} \leq \Psi_\lambda + o_n(1)$ in \overline{B}_R as follows:

(i) If $x_0 \in \partial B_R$ then, since by construction $v_{R,n,\varepsilon}(x_0) = \psi_n(x_0) \leq \Psi_\lambda(x_0) + o_n(1)$ and $v_{R,n,\varepsilon}(0) \geq \psi_n(0) = o_n(1)$, we get $w_{R,n,\varepsilon}(x_0) \leq \Psi_\lambda(x_0) + o_n(1)$.

(ii) If $x_0 = 0$ then $w_{R,n,\varepsilon}(0) = 0 \leq \Psi_\lambda(0) + o_n(1)$. \square

Our next aim is letting $\varepsilon \rightarrow 0$. To do so we need estimates that are independent of ε both from above and below.

Lemma 4.5. *For any $R > 1$ fixed, there exists $C_1(R), C_2(R) > 0$, independent of ε small enough, such that $-C_1(R) + o_n(1) \leq w_{R,n,\varepsilon} \leq C_2(R) + o_n(1)$ in B_R .*

Proof. The upper bound is a direct consequence of Lemma 4.4.

For the lower bound we use that, by construction, $v_{R,n,\varepsilon} \leq \bar{v}$. This implies, see the proof of Lemma 3.4, that $v_{R,n,\varepsilon} \leq C(R)$ in B_R . Here, as in Lemma 4.3, we notice that the estimate is uniform with respect to n , since ψ_n and f_n converge uniformly in B_R . Now, using this bound, $w_{R,n,\varepsilon}(x) = v_{R,n,\varepsilon}(x) - v_{R,n,\varepsilon}(0) \geq \psi_n(x) - C(R) = \underline{u} - C(R) + o_n(1) \geq -C_1(R) + o_n(1)$ in B_R . \square

Lemma 4.6. *For any $R > 1$ fixed, the sequence of solutions (up to a subsequence) $\{w_{R,n,\varepsilon}\}_\varepsilon$ of (4.2) converges locally uniformly in B_R as $\varepsilon \rightarrow 0$ to a continuous viscosity solution $w_{R,n}$ of*

$$(4.5) \quad \lambda - \mathcal{L}_R^{\psi_n}[w_{R,n}] + |Dw_{R,n}|^m = f_n + \mu_{R,n}, \quad x \in B_R,$$

where $\mu_{R,n}$ is compactly supported in $B_R \setminus B_{R-1}$. Moreover, $w_{R,n}(0) = 0$ and $w_{R,n} \leq \Psi_\lambda$ in B_R .

Proof. By adding and subtracting the term $w_{R,n,\varepsilon}$, we rewrite equation (4.2) under the form

$$w_{R,n,\varepsilon} - \varepsilon \Delta w_{R,n,\varepsilon} + |Dw_{R,n,\varepsilon}|^m = F_{R,n,\varepsilon},$$

where $F_{R,n,\varepsilon} := f_n + \mu_{R,n,\varepsilon} + \mathcal{L}_R^{\psi_n}[w_{R,n,\varepsilon}] + w_{R,n,\varepsilon} - \lambda$. Using the estimates of [6, Theorem 3.1] we have that for any $R' < R$ there exists a constant K depending only on R' , $\|F_{R,n,\varepsilon}\|_{L^\infty(B_R)}$ and $\|(w_{R,n,\varepsilon})^-\|_{L^\infty(B_R)}$ such that

$$|w_{R,n,\varepsilon}(x) - w_{R,n,\varepsilon}(y)| \leq K|x - y| \quad \text{in } B_{R'}.$$

By Lemma 4.5, we know that $w_{R,n,\varepsilon}$ is bounded uniformly by some $C(R)$ in B_R , with respect to $\varepsilon > 0$ small enough (and n). Moreover, Lemma 4.3 provides a uniform estimate for $\mu_{R,n,\varepsilon}$. Finally, we can estimate $\mathcal{L}_R^{\psi_n}[w_{R,n,\varepsilon}]$ using the bounds given in Lemma 4.5, and we get for some constant $C(R) > 0$,

$$\|F_{R,n,\varepsilon}\|_{L^\infty(B_R)} \leq \|f_n\|_{L^\infty(B_R)} + |\lambda| + C(R).$$

From this we deduce that K can be chosen independent of $\varepsilon > 0$ small and we obtain a local uniform bound in $C^{0,\alpha}(B_R)$ as $\varepsilon \rightarrow 0$. Passing to the limit is done by Ascoli's Theorem and the stability property of viscosity solutions: up to an extraction, $w_{R,n,\varepsilon} \rightarrow w_{R,n}$ in B_R which is a viscosity solution of (4.5). \square

The last steps consist in sending $n, R \rightarrow +\infty$, for which we have to find other local estimates, now independent of R and n . This time we use gradient estimates, which are provided by a sort of *implicit control* of the equation.

Lemma 4.7. *Fix $R_0 > 0$. Then for any n big enough and any $R > R_0 + 1$, there exists a constant $C = C(R_0)$ such that $\|w_{R,n}\|_{W^{1,\infty}(B_{R_0})} \leq C$.*

Proof. The continuous viscosity solution $w_{R,n}$ of (4.5) is Lipschitz continuous in B_R . Hence it is differentiable almost everywhere and equation (4.5) holds almost everywhere. Moreover, since all the terms in (4.5) are continuous, the equation holds everywhere in B_R .

As a consequence, since $\mu_{R,n} = 0$ on $B_{R_0} \subset B_{R-1}$, we can estimate the gradient term as follows,

$$(4.6) \quad \sup_{B_{R_0}} |Dw_{R,n}|^m \leq \sup_{B_{R_0}} |f_n| + |\lambda| + \sup_{B_{R_0}} |\mathcal{L}_R^{\psi_n}[w_{R,n}]|.$$

On $B_{R_0+1} \setminus B_{R_0}$ we use the bound $\psi_n \leq \Psi_\lambda + o_n(1)$ for the non-local term, so that for n big enough, we have the *implicit estimate*

$$\begin{aligned} \sup_{B_{R_0}} |Dw_{R,n}|^m &\leq C_0(R_0) + 2 \sup_{B_{R_0}} |w_{R,n}| + \sup_{B_{R_0+1} \setminus B_{R_0}} |\Psi_\lambda| \\ &\leq C_1(R_0) + C_2(R_0) \sup_{B_{R_0}} |Dw_{R,n}|, \end{aligned}$$

where we have used the Mean Value Theorem and the fact that $w_{R,n}(0) = 0$ to get the last inequality.

Setting now $X := \sup_{B_{R_0}} |Dw_{R,n}|$ we have $X^m \leq C_2 X + C_1$ for some constants $C_1(R_0), C_2(R_0)$ independent of R . So, since $m > 1$, there exists a positive constant $C_3 = C_3(R_0)$, depending only on R_0 , such that $\sup_{B_{R_0}} |Dw_{R,n}| \leq C_3(R_0)$.

Using again that $w_{R,n}(0) = 0$, we deduce also that $\|w_{R,n}\|_{L^\infty(B_{R_0})} \leq C_4(R_0)$ for some $C_4(R_0) > 0$. Gathering these estimates, we get $\|w_{R,n}\|_{W^{1,\infty}(B_{R_0})} \leq C(R_0)$ for some $C(R_0) > 0$, which is the desired result. \square

We can finally complete the existence result:

Proof of Theorem 4.1. Since the sequence $\{w_{R,n}\}$ is locally bounded in $W^{1,\infty}(B_R)$, independently of n , by using Ascoli's Theorem we can pass to the limit as $n \rightarrow \infty$ in B_R . We skip the details of this passage to the limit, which is straightforward, and yields a solution (passing to the limit in (4.5)) w_R of

$$(4.7) \quad \lambda - \mathcal{L}_R^u[w_R] + |Dw_R|^m = f + \mu_R, \quad x \in B_R,$$

where μ_R is still supported on $B_R \setminus B_{R-1}$. Then, we send $R \rightarrow \infty$ and get that the functions w_R converge locally uniformly to a function $u_\lambda \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^N)$, such that $u_\lambda \leq \Psi_\lambda$ in \mathbb{R}^N and $u_\lambda(0) = 0$.

Moreover u_λ verifies $(EP)_\lambda$. Indeed, as a consequence of the Dominated Convergence Theorem we have $\mathcal{L}_R^u[w_R] \rightarrow \mathcal{L}[u_\lambda]$ locally uniformly and the correction term μ_R vanishes locally uniformly as $R \rightarrow \infty$. So, we can pass to the limit in (4.7) in the viscosity sense to get the result. \square

We conclude this section by the following result which yields solutions of (EP) for certain values of λ .

Corollary 4.8. *For any $\lambda \leq \min(f)$ the problem $(EP)_\lambda$ is solvable and the constructed solution u_λ satisfies $u_\lambda(0) = 0$, $u_\lambda \leq \Psi_\lambda$.*

Proof. If $\lambda < \min(f)$, we can take $\underline{u} = 0$ as a strict subsolution of $(EP)_\lambda$ in Theorem 4.1.

If $\lambda = \min(f)$, $\underline{u} = 0$ is not a strict subsolution of $(EP)_\lambda$, but it is a regular subsolution. So in this case we do not need to use the strict subsolution property to regularize \underline{u} (see Lemma 4.2) into a smooth subsolution. Hence the above construction also works. \square

5. EXISTENCE OF A CRITICAL CONSTANT

In this Section we investigate the existence of a critical ergodic constant λ_* . Following [2, 11], it would seem natural to consider the supremum of all λ 's such that there exist a solution (or a subsolution) u of (EP):

$$\lambda_{\sharp} := \sup \{ \lambda \in \mathbb{R} : (\text{EP})_{\lambda} \text{ is solvable} \}.$$

However, due to the non-local character of the equation, it seems impossible to prove that λ_{\sharp} is finite, because such a result would require a uniform control of the growth of possible solutions. Nevertheless, we have seen in Subsection 2.3 that the growth of (super)solutions is restricted somehow. Following this remark, let us define for $\mu > 0$ the class

$$\mathcal{E}(\mu) := \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} : \limsup_{|x| \rightarrow \infty} \frac{u(x)}{\Psi(x)} \leq \mu \right\}$$

where Ψ has been defined in (2.3). Hence, instead of λ_{\sharp} , we will deal with

$$\Lambda(\mu) := \{ \lambda \in \mathbb{R} : \text{there exists } u \in \mathcal{E}(\mu) \text{ solution of } (\text{EP})_{\lambda} \}$$

and define a critical ergodic constant under restrictive growth $\lambda_*(\mu) := \sup \Lambda(\mu)$. We will then relax the growth condition in Section 8, after we have collected more information on bounded from below solutions and uniqueness.

Let us set $\mu_0 := 2 + (\min f)^-$, so that $\mu_0 = c_{\lambda}$ for $\lambda = \min(f)$, see (2.4). As we noticed, see Remark 2.13, for all $\mu \geq \mu_0$, $\mu\Psi$ is a strict supersolution of (EP) for $|x| \neq 0$.

Lemma 5.1. *Assume that $u \in \mathcal{E}(\mu)$ is a solution of (EP) for some $\mu \geq \mu_0$. Then $u(x) \leq \mu\Psi + u(0)$ for all $x \in \mathbb{R}^N$.*

Proof. We follow the same ideas as in the proof of Lemma 4.4. First, we define $\tilde{u} := u - u(0)$, which is still a solution of (EP) such that $\tilde{u} \in \mathcal{E}(\mu)$.

By the limsup property, for any $\eta > 0$ there exists R_{η} such that for $|x| \geq R_{\eta}$, $\tilde{u}(x) \leq (1 + \eta)\mu\Psi(x)$. Now, by continuity in $\overline{B}_{R_{\eta}}$, the maximum of $\tilde{u} - (1 + \eta)\mu\Psi$ is attained at some point $x_0 \in \overline{B}_{R_{\eta}}$. If the maximum is attained at the boundary, then $\tilde{u} \leq (1 + \eta)\mu\Psi$ in \mathbb{R}^N . Similarly, if the maximum is attained at $x_0 = 0$ we have $\tilde{u}(0) = (1 + \eta)\mu\Psi(0) = 0$.

Finally, if the maximum is attained at a point x_0 such that $0 < |x_0| < R_{\eta}$, we use the comparison principle, Theorem B.1: \tilde{u} is a (sub)solution of (EP) while $(1 + \eta)\mu\Psi$ is a C^1 -smooth, strict supersolution of (EP) and $\tilde{u} \leq (1 + \eta)\mu\Psi$ outside the ball $B_{R_{\eta}}$. We reach a contradiction by using $(1 + \eta)\mu\Psi$ as a test function for \tilde{u} at x_0 .

The conclusion is that the maximum of $\tilde{u} - (1 + \eta)\mu\Psi$ in \mathbb{R}^N is nonpositive, and the result follows after letting η tend to zero: $\tilde{u} \leq \mu\Psi$ which implies the estimate on u . \square

Lemma 5.2. *Let $\lambda_1 \in \Lambda(\mu)$ for some $\mu > 0$. Then $\lambda_2 \in \Lambda(\mu)$ for all $\lambda_2 < \lambda_1$.*

Proof. Since $\lambda_1 \in \Lambda(\mu)$, there exists a solution $u_1 \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^N) \cap \mathcal{E}(\mu)$ of $(\text{EP})_{\lambda_1}$. But since $\lambda_2 < \lambda_1$, it follows that $u_1 - u_1(0)$ is a strict subsolution of $(\text{EP})_{\lambda_2}$. Then, Theorem 4.1 with $\underline{u} = u_1 - u_1(0)$ and $\lambda = \lambda_2$ yields a solution $u \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^N)$ of $(\text{EP})_{\lambda_2}$, such that $u(0) = 0$ and $u \leq u_1$. Hence, $u \in \mathcal{E}(\mu)$ which implies that $\lambda_2 \in \Lambda(\mu)$. \square

Lemma 5.3. *For any $\mu \geq \mu_0$, we have $\min(f) \leq \lambda^*(\mu) < \infty$.*

Proof. We begin with the bound from below: from Corollary 4.8 we have that if $\lambda = \min(f)$, then u_λ is a solution of (EP) such that $u_\lambda(0) = 0$ and $u_\lambda \leq c_\lambda \Psi$. But for this specific λ , $c_\lambda = \mu_0$. Hence, u_λ belongs to $\mathcal{E}(\mu_0) \subset \mathcal{E}(\mu)$ for any $\mu \geq \mu_0$, which proves that for any $\mu \geq \mu_0$, $\lambda_*(\mu) \geq \min(f)$.

Assume now that $\lambda_*(\mu) = \infty$. Then, there exists a sequence of solutions $\{(\lambda_n, v_n)\}$ such that $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$, and thus we can assume that $\lambda_n \geq \min(f)$ for all n sufficiently large.

Following [2], we set $\psi_n := \lambda_n^{-1/m}(v_n - v_n(0))$ so that

$$-\lambda_n^{1/m} \mathcal{L}[\psi_n] + \lambda_n |D\psi_n|^m = f - \lambda_n$$

and after dividing by λ_n we get

$$(5.1) \quad |D\psi_n|^m = \lambda_n^{-1} f + \lambda_n^{1/m-1} \mathcal{L}[\psi_n] - 1.$$

Now we fix $R_0 > 0$ and use the *implicit estimates* as in the proof of Lemma 4.7, but here we take into account the uniform estimate given by Lemma 5.1 in order to control the convolution on $B_{R_0+1} \setminus B_{R_0}$: since $\psi_n(0) = 0$ and $\lambda_n \geq 1$ for n big enough, we have $\psi_n \leq \lambda_n^{-1} \mu \Psi \leq \mu \Psi$. Hence

$$\begin{aligned} \sup_{B_{R_0}} |D\psi_n|^m &\leq \lambda_n^{-1} \sup_{B_{R_0}} |f| + \lambda_n^{1/m-1} \sup_{B_{R_0}} |\mathcal{L}[\psi_n]| \\ &\leq C_1(R_0) + \mu \sup_{B_{R_0+1} \setminus B_{R_0}} |\Psi| + 2 \sup_{B_{R_0}} |\psi_n|. \end{aligned}$$

Recall that $\mu > 0$ is fixed so that, setting $X := \sup_{B_{R_0}} |D\psi_n|$, there exist some constants $a(R_0)$, $b(R_0)$ such that for n big enough

$$X^m \leq a(R_0) + b(R_0)X.$$

This yields a uniform bound (*i.e.* independent of n) for the gradient of ψ_n in any fixed ball B_{R_0} .

Using the the fact that $\psi_n(0) = 0$, up to extraction of a subsequence, we can assume that $\psi_n \rightarrow \psi$ locally uniformly for some $\psi \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^N)$. Then, sending $n \rightarrow \infty$ in (5.1), we obtain a contradiction: $|D\psi|^m \leq -1$. The conclusion is that necessarily $\lambda_*(\mu) < \infty$. \square

Lemma 5.4. *For any $\mu \geq \mu_0$, there exists a solution $v \in \mathcal{E}(\mu)$ of (EP) $_\lambda$ for the critical ergodic constant $\lambda = \lambda^*(\mu)$.*

Proof. Consider a sequence of solutions $\{\lambda_n, v_n\}$ such that $\lambda_n \rightarrow \lambda_*(\mu)$. Since for any n , $\tilde{v}_n := v_n - v_n(0) \in \mathcal{E}(\mu)$ and $\tilde{v}_n(0) = 0$, this allows to use again the same *implicit estimates* as in Lemma 5.3. This implies that the sequence (\tilde{v}_n) is locally uniformly bounded in $W_{\text{loc}}^{1,\infty}(\mathbb{R}^N)$. Hence, up to extraction of a subsequence, we get local uniform convergence of \tilde{v}_n to some $\tilde{v} \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^N)$ and passing to the limit in the viscosity sense, \tilde{v} is a solution of (EP) $_\lambda$ for $\lambda = \lambda_*(\mu)$. Finally, since for any $n \in \mathbb{N}$, $\tilde{v}_n \leq \mu \Psi$ (see Lemma 5.1), we have $\tilde{v} \in \mathcal{E}(\mu)$ so that $\lambda_*(\mu) \in \Lambda(\mu)$. \square

6. BOUNDED FROM BELOW SOLUTIONS

Along this section we follow [11, 12] but, as mentioned before, we have to adapt the arguments to take into account the non-local character of the problem. Here we use the specific sub and supersolutions that we constructed in Section 2. We denote by $\Theta := \Theta_0$ the subsolution constructed for $\lambda = 0$ and $\kappa = 1$, see Lemma 2.4.

Let us consider for $\sigma \in (0, 1)$ the following equation, defined in \mathbb{R}^N :

$$(6.1) \quad -\mathcal{L}[v] + |Dv|^m + \sigma v = f + \sigma\Theta.$$

Lemma 6.1. *There exists $c_1 > 0$ such that for any $\sigma > 0$, $\theta_0 := \Theta - c_1\sigma^{-1}$ is a strict subsolution of (6.1).*

Proof. From Lemma 2.4 we see that Θ is not necessarily a subsolution of $(EP)_0$, since f could be negative. However there exists $c \geq 0$ such that, in the viscosity sense,

$$(6.2) \quad -\mathcal{L}[\Theta] + |D\Theta|^m - f \leq c \quad \text{in } \mathbb{R}^N.$$

Indeed, if $|x| > R_*$, using (2.1) with $\kappa = 1$, then $-\mathcal{L}[\Theta] + |D\Theta|^m - f \leq 1 - f \leq 0$, while for $|x| < R_*$, we have $-\mathcal{L}[\Theta] + |D\Theta|^m - f \leq -\min(f)$. Finally, recall that (see Lemma 2.4) if $|x| = R_*$, no smooth test function can touch from above so that we do not need to check the subsolution condition in the viscosity sense. Hence, (6.2) holds true with $c = (\min(f))^- \geq 0$.

Now, choosing $c_1 > c$, we have for θ_0

$$\begin{aligned} -\mathcal{L}[\theta_0] + |D\theta_0| + \sigma\theta_0 - f &= -\mathcal{L}[\Theta] + |D\Theta|^m + \sigma(\Theta - c_1\sigma^{-1}) - f \\ &< c - c_1 + \sigma\Theta = \sigma\Theta, \end{aligned}$$

which proves the result. \square

Now, in order to construct a supersolution to the viscous version of (6.1) we use a C^2 -regularization of Ψ as follows: let $\bar{\Psi} \in C^2(\mathbb{R}^N)$, such that $\bar{\Psi} = 2\Psi$ if $|x| \geq R_*$ and $\bar{\Psi} \geq 0$ if $|x| \leq R_*$. Notice that such a $\bar{\Psi}$ exists, since $\Psi \geq 0$ in \mathbb{R}^N and it is C^2 -regular for $|x| \geq R_*$. Notice also that the constant 2 corresponds to the choice $\lambda = 0$ in (2.4).

Lemma 6.2. *There exists $c_2 > 0$ such that for any $R > R_*$, $\sigma > 0$ and $0 < \varepsilon < \varepsilon_0(c_2, R)$, $\psi_0 := \bar{\Psi} + c_2\sigma^{-1}$ is a strict supersolution of*

$$-\varepsilon\Delta v - \mathcal{L}[v] + |Dv|^m + \sigma v = f + \sigma\Theta \quad \text{in } B_R.$$

Proof. Notice first that since for $|x| > R_*$, $\Theta(x) = (|x| - R_*)$ while $\Psi(x) = |x|f^{1/m}(x) \geq |x|$ and $\Psi \geq 0$, then $\Theta \leq \bar{\Psi}$. Now, in a similar way as in the proof of Lemma 6.1, using that 2Ψ is a strict supersolution of (EP) (with $\lambda = 0$, see Proposition 2.11) for $|x| \geq R_* > 0$, and that for $|x| \leq R_*$, both f and $\bar{\Psi}$ are regular, we obtain $-\mathcal{L}[\bar{\Psi}] + |D\bar{\Psi}|^m - f \geq -c$ for some $c > 0$ independent of ε, σ, R . Hence,

$$\begin{aligned} -\varepsilon\Delta\psi_0 - \mathcal{L}[\psi_0] + |D\psi_0| + \sigma\psi_0 - f &= -\varepsilon\Delta\bar{\Psi} - \mathcal{L}[\bar{\Psi}] + |D\bar{\Psi}|^m + \sigma(\bar{\Psi} + c_2\sigma^{-1}) - f \\ &\geq -\varepsilon\|\Delta\bar{\Psi}\|_{L^\infty(B_R)} - c + c_2 + \sigma\bar{\Psi} > \sigma\Theta, \end{aligned}$$

provided we choose $c_2 > c$ and $\varepsilon < \varepsilon_0(c_2, R)$ small enough. \square

Lemma 6.3. *For any $\sigma > 0$, there exists a viscosity solution $v_\sigma \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^N)$ of (6.1). Moreover, $\sigma v_\sigma(0)$ is bounded independently of σ .*

Proof. Using θ_0 and ψ_0 as sub/supersolutions, we follow exactly the proofs of Proposition 3.1 and Theorem 4.1 to construct a solution with the desired properties. We sketch only the main modifications that we make here:

(i) We use θ_0 as a strict subsolution in B_R , regularize it as $(\theta_0)_n$ to play the role of ψ_n in Proposition 3.1. This yields that for any $\varepsilon \in (0, 1)$ and $R > 1$ fixed, we have a solution $v_{\sigma, R, n, \varepsilon}$ of the approximate problem

$$(6.3) \quad \begin{cases} -\varepsilon \Delta v - \mathcal{L}_R^{\theta_0}[v] + |Dv|^m + \sigma v = f_n + \sigma \Theta = \tilde{f}_n, & x \in B_R, \\ v = \theta_0, & x \in \partial B_R. \end{cases}$$

Notice that Θ is Lipschitz so that we do not need to regularize it in the right-hand side.

(ii) As we already noticed, $\theta_0 \leq \Theta \leq \bar{\Psi}$ in \mathbb{R}^N . Thus it follows that $-\mathcal{L}_R^{\theta_0}[\bar{\Psi}] \geq -\mathcal{L}[\bar{\Psi}]$. Hence, using Lemma 6.2 and the fact $\psi_0 = \bar{\Psi} + c_2 \sigma^{-1} \geq \bar{\Psi}$, we obtain that for n big enough and ε small enough (depending on c_2 and R), ψ_0 is a supersolution of (6.3).

(iii) Using the classical comparison result in B_R , see Theorem B.3, for $R > R_*$, we deduce that

$$\Theta - \frac{c_1}{\sigma} = \theta_0 \leq v_{\sigma, R, n, \varepsilon} \leq \psi_0 = \bar{\Psi} + \frac{c_2}{\sigma},$$

which yields directly local uniform bounds for the solution, independent of $R > R_*$, n and ε (provided R is fixed). Actually, this step is easier than in Theorem 4.1.

(iv) Passing first to the limit as $\varepsilon \rightarrow 0$ (with $R > R_*$ fixed), then as $n, R \rightarrow \infty$, we conclude that there exists a function $v_\sigma \in W_{\text{loc}}^{1, \infty}(\mathbb{R}^N)$, viscosity solution of (6.1), which verifies

$$(6.4) \quad \Theta - \frac{c_1}{\sigma} \leq v_\sigma \leq \bar{\Psi} + \frac{c_2}{\sigma}.$$

This implies that $\sigma v_\sigma(0)$ is bounded between $-c_1$ and c_2 , which are constants independent of σ . \square

The last step consists in sending $\sigma \rightarrow 0$ and get a bounded from below solution of $(\text{EP})_\lambda$ for a certain λ . We define $w_\sigma := v_\sigma - v_\sigma(0)$, which verifies $w_\sigma(0) = 0$ and is a viscosity solution of

$$(6.5) \quad \lambda_\sigma - \mathcal{L}[w_\sigma] + |Dw_\sigma|^m + \sigma w_\sigma = f + \sigma \Theta \quad \text{in } \mathbb{R}^N,$$

where $\lambda_\sigma = \sigma v_\sigma(0)$. Again, we need a uniform bound from above in order to control the non-local term as $\sigma \rightarrow 0$:

Lemma 6.4. *There exists $\mu > 0$ such that for any $\sigma \in (0, 1)$, $w_\sigma \leq \mu \Psi$.*

Proof. The argument is similar to that of Lemma 4.4, except that we are in the whole space \mathbb{R}^N .

Let us first notice that since λ_σ is bounded from below, we can find $\mu > 0$ such that for any $\sigma > 0$, $c_{\lambda_\sigma} = 2 + (\lambda_\sigma)^- < \mu$. This implies in particular that $\mu \Psi$ is a (strict) supersolution of $(\text{EP})_{\lambda_\sigma}$.

Now, we keep $\sigma > 0$ fixed. Since $w_\sigma \leq 2\Psi + c_2 \sigma^{-1}$ for $|x|$ big and $\mu > 2$, it follows that $w_\sigma - \mu \Psi$ reaches a maximum at some point $x_0 \in \mathbb{R}^N$.

(i) If $x_0 = 0$, the result follows by using that $w_\sigma(0) = 0$ and $\mu \Psi \geq 0$.

(ii) Let $x_0 \neq 0$. Up to a constant, we can assume that the maximum is such that $w_\sigma(x_0) > \mu \Psi(x_0)$. Otherwise we are done. Since w_σ is a viscosity solution of (6.5), we can use the subsolution condition at x_0 with $\mu \Psi$ as test function (recall that by construction $\mu \Psi$ is C^1 -smooth). We get

$$(6.6) \quad \lambda_\sigma - \mathcal{L}[w_\sigma] + |D(\mu \Psi)|^m + \sigma w_\sigma - f \leq \sigma \Theta.$$

Since $w_\sigma - \mu\Psi$ reaches a maximum at x_0 , we have $-\mathcal{L}[w_\sigma](x_0) \geq -\mathcal{L}[\mu\Psi](x_0)$. Hence we get

$$\lambda_\sigma - \mathcal{L}[\mu\Psi] + |D(\mu\Psi)|^m + \sigma\mu\Psi - f \leq \sigma\Theta.$$

But since $\mu\Psi$ is a supersolution of $(EP)_{\lambda_\sigma}$ and $\mu\Psi > \Theta$, we reach a contradiction.

The conclusion is that for any $\sigma > 0$, we have $w_\sigma \leq \mu\Psi$ in \mathbb{R}^N for some $\mu > 0$ fixed. \square

In order to pass to the limit, we need local uniform estimates.

Lemma 6.5. *Let $R_0 > 0$. For any $R > R_0 + 1$, there exists a constant $C = C(R_0)$ such that $\|w_\sigma\|_{W^{1,\infty}(B_{R_0})} \leq C$.*

Proof. We use the same *implicit estimate* technique as in the proof of Lemma 4.7 with only two minor modifications. The first one comes from the extra term $\sigma\Theta$ in equation (6.6), which does not pose any problem in B_{R_0+1} . The second comes from the non-local operator, which is defined now on the whole space *i.e.*, \mathcal{L} instead of \mathcal{L}_R^ψ . In order to deal with this latter issue, we use the uniform bound $w_\sigma \leq \mu\Psi$ on $B_{R_0+1} \setminus B_{R_0}$, see Lemma 6.4. Hence, the equivalent to (4.6) reads now

$$\begin{aligned} \sup_{B_{R_0}} |Dw_\sigma|^m &\leq \sup_{B_{R_0}} |f| + \sup_{B_{R_0}} |\sigma\Theta_0| + |\lambda_\sigma| + \sup_{B_{R_0}} |\mathcal{L}[w_\sigma]| \\ &\leq C_0(R_0) + 2 \sup_{B_{R_0}} |w_\sigma| + \sup_{B_{R_0+1} \setminus B_{R_0}} |\mu\Psi| \\ &\leq C_1(R_0) + C_2(R_0) \sup_{B_{R_0}} |Dw_\sigma|. \end{aligned}$$

We conclude the proof as in Lemma 4.7, using that here also $w_\sigma(0) = 0$ and get $\|w_\sigma\|_{W^{1,\infty}(B_{R_0})} \leq C(R_0)$. \square

Finally, we also need to control w_σ uniformly from below:

Lemma 6.6. *There exists $M > 0$ such that for any $\sigma > 0$,*

$$w_\sigma \geq \Theta - M \quad \text{in } \mathbb{R}^N.$$

Proof. First of all, observe that, thanks to the estimate in $W_{\text{loc}}^{1,\infty}$, for fixed $R > R_*$, there exists $M = M(R) > 0$ such that

$$\sup_{0 < \sigma < 1} \sup_{B_R} (|\Theta| + |w_\sigma|) \leq M.$$

In order to prove (6.6), we fix $\delta \in (1/2, 1)$ and show that $w_\sigma \geq \delta\Theta - M$ in \mathbb{R}^N . We distinguish three cases:

(i) If $|x| \leq R$, it is straightforward that $\delta\Theta - w_\sigma \leq \sup_{B_R} (|\Theta| + |w_\sigma|) \leq M$, and the result follows.

(ii) Using (6.4) we have that $\inf_{\mathbb{R}^N} (w_\sigma - \Theta) > -\infty$. Then, since $M < \infty$ and $\Theta \rightarrow \infty$ as $|x| \rightarrow \infty$, we get that

$$w_\sigma - \delta\Theta + M = (w_\sigma - \Theta) + (1 - \delta)\Theta + M \rightarrow \infty$$

as $|x| \rightarrow \infty$. Hence, there exists $R_1 = R_1(\sigma, \delta) > R$, such that $w_\sigma \geq \delta\Theta - M$ for $|x| > R_1$.

(iii) Finally, let $\mathcal{A} := \{x \in \mathbb{R}^N : R < |x| < R_1\}$. The idea here is to apply a comparison argument to the functions w_σ and $\delta\Theta - M$, which will imply the result.

We observe that neither w_σ nor $(\delta\Theta - M)$ are a super or subsolution of (6.1). But consider

$$(6.7) \quad -\mathcal{L}[\nu] + |D\nu|^m + \sigma\nu - f = \sigma\Theta - \sigma M \quad \text{in } \mathcal{A}.$$

Since $\lambda_\sigma = \sigma w_\sigma(0)$ is bounded by σM , from (6.5) we get that w_σ is a supersolution of (6.7). On the other hand, since $R > R_*$, we have $\Theta = (|x| - R_*)$ and $f > 1$. Hence, using that $-\mathcal{L}[\Theta] \leq 0$,

$$\begin{aligned} -\mathcal{L}[\delta\Theta - M] + |D(\delta\Theta - M)|^m + \sigma(\delta\Theta - M) - f &= -\delta\mathcal{L}[\Theta] + \delta^m + \sigma(\delta\Theta - M) - f \\ &\leq \sigma\Theta + (\delta^m - \sigma M - 1) \leq \sigma\Theta - \sigma M. \end{aligned}$$

We conclude that $\delta\Theta - M$ is a subsolution of (6.7).

Thanks to (i) and (ii) above, we have that $w_\sigma \geq \delta\Theta - M$ on $\partial\mathcal{A}$ so we can apply a comparison argument, see Theorem B.1, to get that $w_\sigma \geq \delta\Theta - M$ in \mathcal{A} .

From (i)–(iii) we have that $w_\sigma \geq \delta\Theta - M$ in \mathbb{R}^N and we conclude by letting $\delta \rightarrow 1$. \square

We can finally prove the existence of a solution of (EP) that is bounded from below.

Theorem 6.7. *There exists a solution (λ, u) of (EP) such that $\inf_{\mathbb{R}^N}(u - \Theta) > -\infty$ and $u \in \mathcal{E}(\mu)$ for some $\mu > 2$.*

Proof. In order to pass to the limit in (6.5), we use that $\sup_\sigma |\lambda_\sigma|$ is bounded independently of σ and the bounds of Lemma 6.5. This yields a sequence σ_n , with $\sigma_n \rightarrow 0$ as $n \rightarrow \infty$, a constant λ and $u \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^N)$, such that $\lambda_\sigma \rightarrow \lambda$ and $w_\sigma \rightarrow u$ as $n \rightarrow \infty$. By passing to the limit in (6.5) we also get that (λ, u) is a solution of (EP).

Moreover, since $w_\sigma \geq \Theta - M$ for all $\sigma > 0$ we get that $\inf_{\mathbb{R}^N}(u - \Theta) > -\infty$. Finally, we pass to the limit in $w_\sigma(0) = 0$ and in the estimate $w_\sigma \leq \mu\Psi$ of Lemma 6.4 to conclude that $u \in \mathcal{E}(\mu)$ for some $\mu > 2$. \square

7. UNIQUENESS

We are finally concerned with the uniqueness (up to addition of constants) of solutions to (EP), for λ fixed. To this aim, we have to develop first some tools related to the behaviour and comparison results of bounded from below solutions. Let us recall that we have constructed the supersolution $\mu\Psi$ for large values of x , under the conditions (H0)–(H2) and moreover, for that values of x , $\mu\Psi$ is just $\mu|x|f^{1/m}$, see Section 2.

We begin with a lower-estimate. In [2, Proposition 3.4] the authors consider functions with power-type growth. Here we use a similar technique, but we have to refine it, due to the fact that we are considering functions whose growth is far bigger than power-type.

Lemma 7.1. *Let f verify (H0)–(H4) and let u be a supersolution of (EP) for $|x| \gg 1$ such that $u \in \mathcal{E}(\mu)$ for some $\mu > 0$ and $\inf_{\mathbb{R}^N} u > -\infty$. Then, for any $\eta \in (0, \eta_0)$, there exists a constant $C_\eta > 0$, such that*

$$u(x) \geq C_\eta \Psi((1 - \eta)x) - \frac{1}{C_\eta}, \quad \text{for } |x| \gg 1.$$

Proof. Since u is bounded from below, we can assume without loss of generality (just by adding a constant), that $u \geq 0$. Then we argue by contradiction; *i.e.* we assume that there exists a sequence $|x_\varepsilon| \rightarrow \infty$ such that

$$\frac{u(x_\varepsilon)}{\Psi((1-\eta)x_\varepsilon)} \rightarrow 0.$$

Let $\alpha = 1 - \eta$ and define

$$v_\varepsilon(s) := \frac{u(x_\varepsilon + s|x_\varepsilon|)}{\Psi(\alpha x_\varepsilon)}, \quad s \in B_\eta.$$

Then $v_\varepsilon(0) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and

$$|Dv_\varepsilon|^m(s) = \left(\frac{|x_\varepsilon|}{\Psi(\alpha x_\varepsilon)} \right)^m |Du(x_\varepsilon + s|x_\varepsilon|)|^m.$$

We use now that u is a nonnegative supersolution of (EP) to estimate the gradient from below: $J * u \geq 0$ so that $|Du(y)|^m \geq f(y) - u(y) - \lambda$. On the other hand, using Lemma 5.1 we have $u(y) \leq \mu\Psi(y) + u(0)$.

We combine these inequalities at $y = x_\varepsilon + s|x_\varepsilon|$. Notice that $\alpha|x_\varepsilon| \leq |y| \leq (1+\eta)|x_\varepsilon|$, so that $|y|$ is large provided $\varepsilon > 0$ is small enough and in this case,

$$\frac{|x_\varepsilon|}{\Psi(\alpha x_\varepsilon)} = \frac{1}{\alpha f^{1/m}(\alpha x_\varepsilon)}.$$

Hence it follows that

$$\begin{aligned} |Dv_\varepsilon|^m(s) &\geq \frac{1}{\alpha^m f(\alpha x_\varepsilon)} \left(f(x_\varepsilon + s|x_\varepsilon|) - u(x_\varepsilon + s|x_\varepsilon|) - \lambda \right) \\ &\geq \frac{f(x_\varepsilon + s|x_\varepsilon|)}{\alpha^m f(\alpha x_\varepsilon)} - \frac{\mu\Psi(x_\varepsilon + s|x_\varepsilon|) + u(0)}{\alpha^m f(\alpha x_\varepsilon)} - \frac{\lambda}{\alpha^m f(\alpha x_\varepsilon)} \\ &\geq \frac{f(x_\varepsilon + s|x_\varepsilon|)}{\alpha^m f(\alpha x_\varepsilon)} \left(1 - o_\varepsilon(1) \frac{\mu}{\alpha^m} \frac{f(x_\varepsilon + s|x_\varepsilon|)}{f(\alpha x_\varepsilon)} + o_\varepsilon(1) \right) \\ &\geq \frac{C_\eta}{\alpha^m} (1 - o_\varepsilon(1)) \geq \frac{C_\eta}{2\alpha^m}, \quad \text{for } \varepsilon \ll 1, \end{aligned}$$

where the last two lines follow from (H3) and (H4). Therefore, we conclude that for ε small

$$|Dv_\varepsilon|(s) \geq \frac{1}{\alpha} \left(\frac{C_\eta}{2} \right)^{1/m} > 0.$$

Let w be a solution to $|Dw| = \frac{1}{\alpha} \left(\frac{C_\eta}{2} \right)^{1/m}$ in B_η with boundary data $w = 0$. By a standard comparison (in the viscosity sense) for the equation $|Du| = \text{constant}$, we deduce that $v_\varepsilon \geq w$ for any $\varepsilon \ll 1$. But this leads to a contradiction, since $w(0) > 0$ while $v_\varepsilon(0) \rightarrow 0$. \square

Lemma 7.2. *Let f verify (H0)–(H3) and (H5)–(H6). Let u be a supersolution of (EP) such that $u \in \mathcal{E}(\mu)$ for some $\mu > 0$ and $\inf_{\mathbb{R}^N} u > -\infty$. Then, there exist $a_0 > 1$ and $R_1 > 0$ such that for all $a \in (1, a_0)$, $\bar{u}(x) := a^N u(ax)$ is a supersolution of (EP) for $|x| \geq R_1$.*

Proof. For simplicity, we reduce the computations to the case $u(0) = 0$ and use the estimate $u \leq \mu\Psi$ given by Lemma 5.1. Define $\mathcal{A}[\bar{u}](x) := \lambda - \mathcal{L}[\bar{u}](x) + |D\bar{u}|^m(x) - f(x)$. Using that u is a supersolution of (EP) and $a > 1$, we have that

$$\begin{aligned} \mathcal{A}[\bar{u}](x) &= \lambda - \mathcal{L}[u](ax) + a^{(N+1)m} |Du|^m(ax) - f(ax) \\ (7.1) \quad &\quad + \mathcal{L}[u](ax) - \mathcal{L}[\bar{u}](x) + f(ax) - f(x) \\ &\geq f(ax) - f(x) + \mathcal{L}[u](ax) - \mathcal{L}[\bar{u}](x). \end{aligned}$$

Changing variables and using that J is compactly supported in B_1 , we can estimate the difference $\mathcal{L}[u](ax) - \mathcal{L}[\bar{u}](x)$ as follows

$$\begin{aligned} \mathcal{L}[u](ax) - \mathcal{L}[\bar{u}](x) &\geq \int_{\mathbb{R}^N} J(ax - y)u(y) \, dy - a^N \int_{\mathbb{R}^N} J(x - y)u(ay) \, dy \\ &= a^N \int_{\mathbb{R}^N} (J(az) - J(z))u(a(x - z)) \, dz \\ &= a^N \int_{|z| < 1} (J(az) - J(z))u(a(x - z)) \, dz. \end{aligned}$$

Observe now that, since J is radially decreasing, $J(az) < J(z)$ for all $a > 1$ and moreover $|J(az) - J(z)| \leq \|DJ\|_\infty(a - 1)$, for $|z| < 1$. Thus,

$$\begin{aligned} \mathcal{L}[u](ax) - \mathcal{L}[\bar{u}](x) &\geq -a^N \|DJ\|_\infty(a - 1) \int_{|z| < 1} u(a(x - z)) \, dz \\ &\geq -a^N \|DJ\|_\infty(a - 1) \mu \sup_{|z| < 1} \Psi(a(x + z)). \end{aligned}$$

Plugin this estimate into (7.1) and using **(H5)** and **(H6)** we get

$$\begin{aligned} (7.2) \quad \mathcal{A}[\bar{u}](x) &\geq f(ax) - f(x) - a^N(a - 1) \|DJ\|_\infty \mu \sup_{|z| < 1} \Psi(a(x + z)) \\ &\geq af(x) - f(x) - a^N(a - 1) \|DJ\|_\infty \mu f(x) o_x(1) \end{aligned}$$

where $o_x(1)$ tends to 0 as x tends to infinity (this o_x is uniform with respect to a).

To conclude the proof take $|x| > R_1$ such that $\mu \|DJ\|_\infty a_0^N o_x(1) \leq 1/2$. Then (7.2) becomes

$$\mathcal{A}[\bar{u}](x) \geq f(x)(a - 1) \left(1 - \mu \|DJ\|_\infty a_0^N o_x(1)\right) \geq f(x) \frac{a - 1}{2} \geq 0$$

and \bar{u} is a supersolution of (EP). \square

Observe that Lemma 7.2 remains true independently of the hypothesis **(H7)**. However, this Lemma will be not enough to prove the comparison result when f has a ‘‘slow’’ growth, see the proof of Lemma 7.5 below. Therefore, we have to do a different approach in this latter case. Indeed, when **(H7)**-slow holds, we use a similar argument as in the subquadratic case of [2], proving that u^q is a supersolution of (EP). However, under our general hypotheses here, we have to be more precise in the control of the constants, so that the computations are more tedious. Notice also that we still assume **(H3)**, which implies that f cannot grow too slow: typically faster than $f(x) = |x|^{m_*}$, where $m_* = m/(m - 1) \in (1, 2)$.

The following estimate allows to control the non-local term in the case of **(H7)**-slow. We get that $\mathcal{L}[\Psi]$ can be controlled using the following estimate:

Lemma 7.3. *Let f verify **(H0)**–**(H4)** and **(H7)**-slow. Then there exists $C > 0$ such that for all $|x| \gg 1$,*

$$\sup_{B_1(x)} \Psi(y) \leq C \Psi(x).$$

Proof. Fix $\eta \in (0, \eta_0)$. Then, for $|x| > 1/\eta$, the ball $B_1(x)$ is contained in $\{x + s|x| : s \in B_\eta(0)\}$. Thus, using **(H4)** we have

$$\sup_{B_1(x)} f(y) \leq \sup_{s \in B_\eta(0)} f(x + s|x|) \leq \bar{c}_\eta f((1 + \eta)x).$$

Now, by **(H7)**-slow, there exists $C > 0$ such that for $|x|$ big enough, $f((1 + \eta)x) \leq C f(x)$, which implies that

$$\sup_{B_1(x)} f(y) \leq \bar{c}_\eta C f(x).$$

Finally, using that $\Psi(x) = |x|f^{1/m}(x)$ for $|x|$ large, we get the same result for Ψ (with another constant). \square

Lemma 7.4. *Let f verify **(H0)**–**(H6)** and **(H7)**-slow. Let u be a supersolution of (EP) such that $u \in \mathcal{E}(\mu)$ for some $\mu > 0$ and $\inf_{\mathbb{R}^N} u > -\infty$. Then, there exist $q_0 > 1$ and $R_1 > 0$ such that for all $q \in (1, q_0)$, the function u^q is a supersolution of (EP) for $|x| \geq R_1$.*

Proof. We first notice that under our assumptions, we can assume with no restriction that $u \geq 0$ so that $|Du^q|^m = (qu^{q-1})^m |Du|^m$. Now take $\eta \in (0, \eta_0)$. By Lemma 7.1 there exists R_1 such that for $|x| > R_1$, $u \geq (C_\eta/2)\Psi = C_0\Psi$. Hence, for such x ,

$$|Du^q|^m \geq (qC_0^{q-1})^m \Psi^{m(q-1)} |Du|^m \geq C_0^{m(q-1)} \Psi^{m(q-1)} |Du|^m.$$

For the non-local term we use Lemma 7.3 as above and the fact that $u \in \mathcal{E}(\mu)$:

$$\begin{aligned} -\mathcal{L}[u^q] &= - \int J(x-y)u^q(y) dy + u^q(x) \\ &\geq -\mu^{q-1} \int J(x-y)\Psi^{q-1}(y)u(y) dy + C_0^{q-1}\Psi^{q-1}(x)u(x) \\ &\geq -(\mu C)^{q-1}\Psi^{q-1}(x) \int J(x-y)u(y) dy + C_0^{q-1}\Psi^{q-1}(x)u(x) \\ &\geq -C_1^{q-1}\Psi^{q-1}(x)(J * u)(x) + C_0^{q-1}\Psi^{q-1}(x)u(x) \end{aligned}$$

where C_0, C_1 are uniform with respect to $q \in (1, q_0)$.

Using that u is a supersolution of (EP) to replace $(J * u)$ below we get

$$\begin{aligned} (7.3) \quad &\lambda - \mathcal{L}[u^q] + |Du^q|^m \\ &\geq \lambda - C_1^{q-1}\Psi^{q-1}(J * u) + C_0^{q-1}\Psi^{q-1}u + C_0^{m(q-1)}\Psi^{m(q-1)}|Du|^m \\ &\geq \lambda + C_1^{q-1}\Psi^{q-1}(f - \lambda - |Du|^m) + C_0^{m(q-1)}\Psi^{m(q-1)}|Du|^m \\ &\quad + (C_0^{q-1} - C_1^{q-1})\Psi^{q-1}u \\ &\geq \lambda + |Du|^m(C_0^{m(q-1)}\Psi^{m(q-1)} - C_1^{q-1}\Psi^{q-1}) \\ &\quad + C_1^{q-1}\Psi^{q-1}(f - \lambda) + (C_0^{q-1} - C_1^{q-1})\Psi^{q-1}u. \end{aligned}$$

If we choose $|x|$ big enough such that $C_0^m \Psi^m > 2C_1 \Psi$, which is possible since Ψ is coercive, then, $(C_0^{m(q-1)}\Psi^{m(q-1)} - C_1^{q-1}\Psi^{q-1}) > \frac{1}{2}C_0^{m(q-1)}\Psi^{m(q-1)}$.

Moreover, since u is a supersolution, $|Du|^m \geq f(x) - \lambda + J * u - u$. Using again Lemma 7.3 together with the fact that $u \leq \mu\Psi + u(0)$ we get $J * u \leq \mu C\Psi + u(0)$, which implies $|Du|^m \geq f - \lambda - C'\Psi(x) - u(0)$ for some $C' > 0$. Moreover, by **(H3)**, $\Psi(x) \ll f(x)$ so that for $|x|$ big enough, $|Du|^m(x) \geq f(x)/2$. We plug this into the last line in (7.3) and get

$$(7.4) \quad \begin{aligned} \lambda - \mathcal{L}[u^q] + |Du^q|^m &\geq \lambda + \frac{1}{2}C_0^{m(q-1)}\Psi^{m(q-1)}f + C_1^{q-1}\Psi^{q-1}(f - \lambda) \\ &\quad + (C_0^{q-1} - C_1^{q-1})\Psi^{q-1}u. \end{aligned}$$

We first take $|x|$ big enough so that $f(x) > \lambda$ and $C_1\Psi(x) \geq 1$. Then, by **(H3)**, $u \leq \mu\Psi \leq f/2$ for $|x|$ big enough. And using again the fact that Ψ is coercive, for $|x|$ big enough we also have

$$C_0^m \Psi^m \geq 2|C_0^{q-1} - C_1^{q-1}|^{1/(q-1)} \Psi.$$

Thus, replacing in (7.4) yields

$$\lambda - \mathcal{L}[u^q] + |Du^q|^m \geq \lambda + (f - \lambda) = f$$

and the result holds. \square

We are now ready to perform comparison results.

Lemma 7.5. *Let f verify **(H0)**–**(H7)** (slow or fast) and let $u_1, u_2 \in \mathcal{E}(\mu)$ for some $\mu > 0$ be respectively a subsolution and a supersolution of (EP) with $\inf_{\mathbb{R}^N} u_2 > -\infty$. There exists $R_1 > 0$ such that if $u_1 < u_2$ on ∂B_{R_1} , then $u_1 \leq u_2$ in $B_{R_1}^C$.*

Proof. Let us begin by assuming **(H7)**-slow. By Lemma 7.4, for any $q > 1$ (close enough to 1), u_1^q is a supersolution in $\{|x| > 1\}$. Since $u_1, u_2 \in \mathcal{E}(\mu)$, by Lemma 7.1 we see that $u_2^q \gg u_1$ as $|x| \rightarrow \infty$. So, the maximum of $u_1 - u_2^q$ in $B_{R_1}^C$ is attained at some point x_0 . If $x_0 \in \partial B_{R_1}$, then since $u_1(x_0) < u_2(x_0)$ we deduce the result. On the other hand, if $|x_0| > 0$ we can use the equations and the Strong Maximum Principle, see Theorem B.3, to reach a contradiction. The conclusion is that for any $q \in (1, q_0)$, $u_1 \leq u_2^q$ and the comparison follows by sending $q \rightarrow 1$.

Let us now turn to the case of **(H7)**-fast and define, for $a > 1$, $\bar{u}_2 := a^N u_2(ax)$. From Lemma 7.2, we know that there exists $R_1 > 0$ and $a_0 > 0$, such that \bar{u}_2 is a supersolution of (EP) in $B_{R_1}^C$, for any $a \in (1, a_0)$. Moreover if a is chosen close enough to 1, by continuity of both functions we have $\bar{u}_2 \geq u_1$ in ∂B_{R_1} .

Observe first that, by Lemma 7.1, for any $a > 1$ fixed and $\eta \in (0, 1)$, there exists a constant $c_{a,\eta} > 0$ such that for $|x|$ large enough,

$$\bar{u}_2(x) \geq c_{a,\eta} \Psi(a(1-\eta)x) - 1/c_{a,\eta},$$

while on the other hand, since $u_1 \in \mathcal{E}(\mu)$ we have

$$u_1(x) \leq \mu \Psi(x) + u_1(0) \text{ in } \mathbb{R}^N.$$

Therefore, for $|x| \gg 1$,

$$\begin{aligned} (u_1 - \bar{u}_2)(x) &\leq \mu \Psi(x) - c_{a,\eta} \Psi(a(1-\eta)x) + 1/c_{a,\eta} + u_1(0) \\ &\leq \mu |x| \left(f^{1/m}(x) - \frac{c_{a,\eta}}{\mu} a(1-\eta) f^{1/m}(a(1-\eta)x) \right) + c(a, \eta, u_1). \end{aligned}$$

Now, for $a \in (1, a_0)$ we fix $\eta > 0$ small enough such that $a(1-\eta) > 1$. Hypothesis **(H7)**-fast implies that $\liminf [f^{1/m}(a(1-\eta)x)/f^{1/m}(x)] = +\infty$. Hence, for any $c > 0$, there exists $c_0 > 0$ such that provided $|x|$ is big enough we have

$$f^{1/m}(a(1-\eta)x) \geq c f^{1/m}(x) + c_0.$$

From this, choosing conveniently c , it follows that as $|x| \rightarrow \infty$, $\limsup(u_1 - \bar{u}_2) = -\infty$. This implies that the supremum of $u_1 - \bar{u}_2$ is attained at a point $x_0 \in B_{R_1}^C$ or on the boundary ∂B_{R_1} .

In the first case, *i.e.* if $x_0 \in B_{R_1}^C$ we get a contradiction by using the Maximum Principle (see Theorem B.3). In the second case, by assumption, $(u_1 - \bar{u}_2)(x) \leq (u_1 - \bar{u}_2)(x_0) \leq 0$ for $x \in B_{R_1}^C$ and hence $u_1 \leq \bar{u}_2$.

The proof concludes by sending $a \searrow 1$, which implies $u_1 \leq u_2$ in $B_{R_1}^C$. \square

The next two theorems show, not only the uniqueness of bounded from below solutions, but also that this unique solution corresponds to the solution associated with the critical ergodic constant $\lambda_*(\mu)$.

Theorem 7.6. *Let f verify (H0)–(H7) (slow or fast). Let (λ_1, u_1) and (λ_2, u_2) be two solutions of $(\text{EP})_{\lambda_1}$ and $(\text{EP})_{\lambda_2}$, such that $u_1, u_2 \in \mathcal{E}(\mu)$ for some $\mu > 0$ and $\inf_{\mathbb{R}^N} u_1 > -\infty$, $\inf_{\mathbb{R}^N} u_2 > -\infty$. Then $\lambda_1 = \lambda_2$ and $u_1 = u_2 + c$ for some constant $c \in \mathbb{R}$.*

Proof. Assume that $\lambda_2 \leq \lambda_1$. Then (λ_1, u_1) can be seen as a subsolution of $(\text{EP})_{\lambda_2}$. Moreover, by adding a constant, if necessary, we can ensure that $w = u_1 - (u_2 + C)$ verifies $\sup_{\partial B_{R_1}} w = 0$. Therefore, for any $\varepsilon > 0$, $u_1 - (u_2 + C + \varepsilon) < 0$ on ∂B_{R_1} and we can apply Lemma 7.5 which gives that $u_1 - (u_2 + C + \varepsilon) \leq 0$ in $B_{R_1}^C$. After sending ε to zero, we get that $w \leq 0$ in $B_{R_1}^C$.

Now, if we consider the function $g(s) = |s|^m$, then, by convexity, $g(p+q) \geq g(p) + Dg(p) \cdot q$. Using this inequality with $p = Du_1$ and $q = Du_2 - Du_1$ yields that w is a subsolution of $(\lambda_1 - \lambda_2) - \mathcal{L}[w] + c(x)|Dw| = 0$, with $c(x) = m|Du_1|^m$. And since $\lambda_1 \geq \lambda_2$, w is a subsolution of $-\mathcal{L}[w] + c(x)|Dw| = 0$. Moreover, since $w \leq 0$ outside B_{R_1} , we can use the comparison property in B_{R_1} (see Theorem B.1) and deduce that $w \leq 0$ in B_{R_1} .

Hence, w reaches a maximum at some point in ∂B_{R_1} . But w satisfies $-\mathcal{L}[w] + c(x)|Dw| \leq 0$ where $c(x) \geq 0$.

Thus, applying the Strong Maximum Principle, see Theorem B.3, we infer that $w = \max w$ in \mathbb{R}^N . This implies that $u_1 = u_2 + C$ and consequently, that $\lambda_1 = \lambda_2$. \square

Theorem 7.7. *Let f verify (H0)–(H7) (slow or fast) and let (λ, u) be a solution of (EP) such that $u \in \mathcal{E}(\mu)$ for some $\mu > 0$ and $\inf_{\mathbb{R}^N} u > -\infty$. Then $\lambda = \lambda_*(\mu)$.*

Proof. The proof is done exactly as in [2] and uses arguments which are similar to that of Theorem 7.6. Assume that (λ, u) is a solution of (EP) such that $u \in \mathcal{E}(\mu)$ for some $\mu > 0$ and $\inf u > -\infty$. Let also v be a solution associated with the critical ergodic constant $\lambda_*(\mu)$. We already know that $\lambda \leq \lambda_*(\mu)$ so we only need to prove the converse inequality.

Take R_1 as in Lemma 7.5. We can choose $C \in \mathbb{R}$ such that $\max_{\partial B_{R_1}} (v - (u + C)) \leq 0$. Considering the function $w := \max(u + C + \varepsilon, v)$, it turns out that w is bounded from below because of u . It is also a subsolution of $(\text{EP})_{\lambda_*(\mu)}$ because $u + C + \varepsilon$ is a subsolution of this equation since $\lambda \leq \lambda_*(\mu)$. And moreover, $w \in \mathcal{E}(\mu)$.

Using Lemma 7.5 we deduce that for any $\varepsilon > 0$, $w \leq u + C + \varepsilon$ in $B_{R_1}^C$, so that finally $w \leq u + C$ in $B_{R_1}^C$. The comparison in B_{R_1} implies also that $w \leq u + C$ in B_{R_1} . Thus, $w - (u + C)$ reaches its maximum on ∂B_{R_1} which implies that $w = u + C$ in \mathbb{R}^N . The conclusion is that $v = u + C$, and thus $\lambda = \lambda_*(\mu)$. \square

8. CRITICALITY REVISITED

In this section, we first extend the results of Section 5 on critical ergodic constants to the more general class

$$\bar{\mathcal{E}} := \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} : \limsup_{|x| \rightarrow \infty} \frac{u(x)}{\Psi(x)} < \infty \right\}.$$

Notice that $\bar{\mathcal{E}} \supset \mathcal{E}(\mu)$ for any $\mu > 0$, and that in $\bar{\mathcal{E}}$, contrary to $\mathcal{E}(\mu)$, we do not have a uniform control of the behaviour of solutions (indeed for any $c > 0$, $c\Psi \in \bar{\mathcal{E}}$).

Let us define the critical ergodic constant in $\bar{\mathcal{E}}$ as usual:

$$\bar{\lambda} := \sup \left\{ \lambda \in \mathbb{R} : \text{there exists } u \in \bar{\mathcal{E}}, \text{ solution of } (\text{EP})_\lambda \right\}.$$

We will prove here in particular that $\bar{\lambda}$ is finite.

Lemma 8.1. *For any $\mu > \mu_0$, $\lambda_*(\mu) = \lambda_*(\mu_0)$.*

Proof. This is a consequence of the uniqueness and characterization of bounded from below solutions in $\mathcal{E}(\mu)$. We know that (up to a constant) there exists a unique $u \in \mathcal{E}(\mu_0)$ such that $\inf u > -\infty$ and u is a solution of $(\text{EP})_{\lambda_*(\mu_0)}$. Similarly, there is a unique $v \in \mathcal{E}(\mu)$ such that $\inf v > -\infty$ and v is a solution of $(\text{EP})_{\lambda_*(\mu)}$. Now, since $\mathcal{E}(\mu_0) \subset \mathcal{E}(\mu)$, we can apply Theorem 7.6 to conclude that $u = v$ (up to a constant) and $\lambda_*(\mu) = \lambda_*(\mu_0)$. \square

Corollary 8.2. *Assume that f satisfies **(H0)**–**(H7)**. Then $\min(f) \leq \bar{\lambda} < \infty$.*

Proof. Take any pair (λ, u) solution of (EP) such that $u \in \bar{\mathcal{E}}$. Since $u \in \mathcal{E}(\mu)$ for some μ , we deduce that, by definition of $\lambda_*(\mu)$, $\lambda \leq \lambda_*(\mu)$. But using Lemma 8.1, we get that necessarily $\lambda \leq \lambda_*(\mu_0)$. Hence, taking the supremum, $\bar{\lambda} \leq \lambda_*(\mu) < \infty$. \square

Now, following [2], let us give some Lipschitz estimate of the critical ergodic constant $\bar{\lambda}$. We denote by $\bar{\lambda}(f)$ the constant $\bar{\lambda}$ that corresponds to the equation with right-hand side f . The following result extends [2, Proposition 4.4] to more general cases that we cover here.

Lemma 8.3. *Let f_1, f_2 verify **(H0)**–**(H2)**. Assume that there exists a constant $c > 0$ and a function $g > 0$ such that $f_1(x), f_2(x) \geq cg(x)$ and*

$$m := \sup_{x \in \mathbb{R}^N} \frac{|f_1(x) - f_2(x)|}{g(x)} < \infty.$$

then

$$(8.1) \quad |\bar{\lambda}(f_2) - \bar{\lambda}(f_1)| \leq \frac{m}{c+m} \max\{\bar{\lambda}(f_1), \bar{\lambda}(f_2)\}$$

Proof. The proof is exactly the same as in [2] and we omit it. We only want to point out that, once we know that there is a solution to $(\text{EP})_{\bar{\lambda}}$, due to **(H0)**–**(H2)**, the key points of the proof rely on the boundness of m and the lower bound for f given by g . \square

A typical application is to power-type functions f as in [2], but we also have a similar result for faster growths, for instance in the limiting case:

Corollary 8.4. *Assume that $f_i(x) \leq c_i \exp(m|x|)$, $i = 1, 2$, and define the function g as $g(x) = c_0 \exp(m|x|)$ where $c_0 := \min(c_1, c_2)$. Then (8.1) holds.*

We end this section by a remark, more than a result, concerning the scaling properties of $\bar{\lambda}$. Let $f(x) = |x|^\alpha$, with $\alpha > m_*$. Then, for any $c > 1$, it seems reasonable to think that

$$\bar{\lambda}(cf) = c^{m_*N/(\alpha - m_*N)} \bar{\lambda}(f).$$

The idea of the proof follows again [2]. Our main difficulty to complete the proof comes from the non-local term. Indeed, let u_1 be a solution to $(EP)_{\bar{\lambda}}$. We would like to construct a solution (or subsolution) to (EP) with right-hand side $\tilde{f} = cf$. To this aim consider $u_2(x) = a^{-\beta}u_1(ax)$, with $a = c^{1/(\alpha - m_*N)} < 1$ and $\beta = (N + m)/(m - 1)$. The fact is that we are not able to prove that $-\mathcal{L}[u_2](x) \leq -\mathcal{L}[u_1](ax)$ for all $x \in \mathbb{R}^N$, and we only have, following the proof of Lemma 7.2, that

$$-\mathcal{L}[a^{N+\beta}u_2](x) \leq -\mathcal{L}[u_1](ax) + o(|x|^\alpha).$$

Hence

$$\begin{aligned} -\mathcal{L}[u_2](x) + |Du_2(ax)|^m &\leq a^{-m_*N}(-\mathcal{L}[u_1](ax) + |Du_1(ax)|^m) + o(|x|^\alpha) \\ &= a^{-m_*N}(f(ax) - \bar{\lambda}(f(ax))) + o(|x|^\alpha), \end{aligned}$$

which implies that, u_2 is a subsolution of

$$a^{-m_*N}\bar{\lambda}(a|x|^\alpha) - \mathcal{L}[u_2](x) + |Du_2(ax)|^m = a^{-m_*N+\alpha}|x|^\alpha$$

only for x big enough. If we could prove that u_2 is a subsolution for all x we would conclude, due to definition of $\bar{\lambda}$ as a supremum, that

$$a^{-m_*N}\bar{\lambda}(a|x|^\alpha) \leq \bar{\lambda}(a^{-m_*N+\alpha}|x|^\alpha).$$

In a similar way as in [2] we could get the reverse inequality. So, while it is not clear whether there is really a scaling property for $\bar{\lambda}$ with power functions f , at least it seems that an approximating scaling property should hold, with a different exponent than in the local case.

APPENDIX

A. PROPERTIES OF THE NON-LOCAL OPERATOR \mathcal{L}

Across the paper we use several times basic properties and technical estimates of \mathcal{L} or \mathcal{L}_R^ψ . We summarize them here, for the reader's convenience.

Let $R > 0$ and $\psi \in C^0(\mathbb{R}^N)$. Recall that we have defined non-local operator \mathcal{L} and the Dirichlet non-local operator, respectively, as

$$\begin{aligned} \mathcal{L}[v](x) &:= \int_{\mathbb{R}^N} J(x-y)v(y) dy - v(x), \\ \mathcal{L}_R^\psi[v](x) &:= \int_{B_R} J(x-y)v(y) dy + \int_{B_{R+1} \setminus B_R} J(x-y)\psi(y) dy - v(x), \end{aligned}$$

where J is a symmetric and compactly supported in B_1 kernel.

Lemma A.1. *Let c be a positive constant and u, v two positive functions. Then*

- (i) $\mathcal{L}[c] = 0$ and $\mathcal{L}_R^0[c] \leq 0$ if $c \geq 0$.
- (ii) $\mathcal{L}_R^\psi[u + c] = \mathcal{L}_R^\psi[u] + \mathcal{L}_R^0[c]$.
- (iii) $\mathcal{L}_R^\psi[\psi] = \mathcal{L}[\psi]$.
- (iv) $\mathcal{L}_R^\psi[u] \leq \mathcal{L}[u]$ if $\psi \leq u$.
- (v) $\mathcal{L}[u](x_0) \leq \mathcal{L}[v](x_0)$ and $\mathcal{L}_R^\psi[u](x_0) \leq \mathcal{L}_R^\psi[v](x_0)$ if $u(x_0) = v(x_0)$ and $u \geq v$.
- (vi) $\mathcal{L}[u](x) \leq \mathcal{L}[u](y) + o_\delta(1)$ and $\mathcal{L}_R^\psi[u](x) \leq \mathcal{L}_R^\psi[u](y) + o_\delta(1)$ if $|x - y|^2 \leq \delta$.

We omit the proof, since it follows straightforward from the definition of the non-local operators.

Lemma A.2. *Let x_0 be a point where u attains a positive maximum, respectively minimum. Then $\mathcal{L}[u](x_0) \leq 0$ and $\mathcal{L}_R^0[u](x_0) \leq 0$, respectively ≥ 0 .*

Proof. At the point x_0 where u attains a positive maximum we have

$$\mathcal{L}[u](x_0) = \int_{\mathbb{R}^N} J(x-y)u(y) dy - u(x) \leq u(x_0) \left(\int_{\mathbb{R}^N} J(x-y) - 1 \right) = 0.$$

We do a similar computation for $\mathcal{L}_R^0[u]$. \square

Lemma A.3. *If $g \in C^1(\mathbb{R}^N)$ then $|\mathcal{L}[g](x)| \leq \sup_{z \in B_1(x)} |Dg(z)|$.*

Proof. We use the fact that, for all $y \in B_1(x)$, $|g(x) - g(y)| \leq \sup_{z \in B_1(x)} |Dg(z)|$.

Then, by direct computation, we obtain

$$\begin{aligned} |\mathcal{L}[g](x)| &\leq \int_{B_1(x)} J(x-y)|g(y) - g(x)| dy \leq \int_{B_1(x)} J(x-y) \sup_{z \in B_1(x)} |Dg(z)| dy \\ &= \sup_{z \in B_1(x)} |Dg(z)|, \end{aligned}$$

since J is compactly supported on B_1 . \square

Lemma A.4. *Let ψ be convex. Then $-\mathcal{L}[\psi] \leq 0$.*

Proof. The result follows from Jensen's inequality,

$$-\mathcal{L}[\psi](x) = \int_{\mathbb{R}^N} \psi(x+z) d\nu(z) - \psi(x) \leq \psi \left(\int_{\mathbb{R}^N} (x+z) d\nu(z) \right) - \psi(x) \leq 0.$$

where ν denotes the probability measure associated to J , $d\nu(z) = J(z) dz$. \square

Lemma A.5. *Let ψ be nondecreasing and for $\epsilon \in (0, 1)$, let $c_\epsilon = \mu(B_1 \setminus B_{1-\epsilon})$. Then*

$$-\psi(|x|+1) + \psi(|x|) \leq -\mathcal{L}[\psi](x) \leq -c_\epsilon \psi(|x|+1-\epsilon) + \psi(|x|).$$

Proof. Since ψ is nondecreasing we have

$$\begin{aligned} -\mathcal{L}[\psi](x) &= - \int_{B_1} J(y)(\psi(|x-y|) - \psi(|x|)) dy \geq - \int_{B_1} J(y)(\psi(|x|+1) - \psi(|x|)) dy \\ &= -\psi(|x|+1) + \psi(|x|). \end{aligned}$$

The other inequality yields as follows

$$\begin{aligned} -\mathcal{L}[\psi](x) &= - \int_{B_1} J(y)(\psi(|x-y|) - \psi(|x|)) dy \leq - \int_{B_1 \setminus B_{1-\epsilon}} J(y)\psi(|x-y|) dy + \psi(|x|) \\ &\leq -c_\epsilon \psi(|x|+1-\epsilon) + \psi(|x|). \end{aligned}$$

\square

B. COMPARISON RESULTS

We prove here two comparison results that we use in several places across the paper. To this aim let us consider the general equation

$$(B.1) \quad -\mathcal{L}[w] + c(x)|Dw| + \alpha w = 0, \quad \alpha \geq 0.$$

Observe that this equation appears in different contexts. For instance, it turns out to be satisfied (with $\alpha = 0$) by $w = v_1 - v_2$ if v_1, v_2 are a subsolution and a supersolution, respectively, of (EP).

Theorem B.1. *Let $v \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^N)$ be a subsolution of (B.1), such that for $R > 1$, $v \leq 0$ in $B_{R+1} \setminus B_R$. Then $v \leq 0$ in B_R .*

Proof. Let $x_0 \in B_R$ be a point where v reaches a positive maximum. Hence, the constant function $\varphi(x) := v(x_0)$ is an admissible test function for v at x_0 ; i.e $v(x) - \varphi(x)$ reaches a maximum at x_0 and $v(x_0) = \varphi(x_0) > 0$. Hence, since $|D\varphi| = 0$ and $v \leq 0$ in $B_{R+1} \setminus B_R$

$$\begin{aligned} 0 &\geq -\mathcal{L}[v](x_0) - c(x_0)|D\varphi(x_0)| + \alpha v(x_0) \\ &= -\int_{B_R} J(x_0 - y)v(y) \, dy - \int_{B_{R+1} \setminus B_R} J(x_0 - y)v(y) \, dy + v(x_0) + \alpha v(x_0) \\ &\geq v(x_0) \left(1 - \int_{B_R} J(x_0 - y) \, dy + \alpha \right) > 0, \end{aligned}$$

which is a contradiction. Hence $v(x_0) \leq 0$ and the result follows. \square

Remark B.2. The result holds true even if we replace the gradient in (B.1) by $c(x)|Dw|^{m-1}$, with $m > 1$. Moreover, it is also true for the approximate problems that have a $-\varepsilon\Delta$ -term. Indeed, it is straightforward, since at a maximum point $-\varepsilon\Delta v(x_0) \geq 0$.

Theorem B.3 (Strong Maximum Principle). *Let $v \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^N)$ be a subsolution of (B.1), which reaches a maximum at $x_0 \in \mathbb{R}^N$. Then $v \equiv v(x_0)$ in \mathbb{R}^N .*

Proof. Let $x_0 \in B_1$ be a point where v reaches a maximum. As in the previous proof, the constant function $\varphi(x) := v(x_0)$ is an admissible test function for v at x_0 . Hence, since $v(y) \leq v(x_0)$ for all $y \in B_1(x_0)$, we have

$$0 \geq -\mathcal{L}[v](x_0) - c(x_0)|D\varphi(x_0)| + \alpha v(x_0) \geq -\int_{B_1(x_0)} J(x_0 - y)(v(y) - v(x_0)) \, dy \geq 0.$$

This implies that $v(y) = v(x_0)$ for all $y \in B_1(x_0)$.

We can repeat now the argument using as center any $y \in B_1(x_0)$ and get that $v(y) = v(x_0)$ for $y \in \mathbb{R}^N$. \square

C. EXISTENCE RESULT FOR AN AUXILIARY PROBLEM

We devote this last section of the Appendix to prove the existence of solutions of equation (3.3). To this aim we fix $\gamma \in (0, 1)$, $\varepsilon > 0$, $R > 1$ and we consider the following problem

$$(C.1) \quad \begin{cases} -\varepsilon\Delta\phi - \mathcal{L}_R^\psi[\phi] = f, & x \in B_R, \\ \phi = g & x \in \partial B_R, \end{cases}$$

where $\psi \in C^0(B_{R+1} \setminus B_R)$, $g \in C^{0,\gamma}(\partial B_R)$ and $f \in C^{0,\gamma}(B_R)$.

Lemma C.1. *There exists a unique solution $\phi \in C^{2,\gamma}(B_R) \cap C^0(\overline{B_R})$ of (C.1).*

Uniqueness comes from the comparison principle, see Theorem B.1. Actually we do a similar argument in the proof of Lemma 3.3, so that we skip the details here.

In order to prove the existence part of the result, we consider the unique function $\varphi \in C^{2,\gamma}(B_R) \cap C^0(\overline{B_R})$ such that of $-\Delta\varphi = 0$ in B_R with boundary data $\varphi = g$ on ∂B_R (see for instance [10, Theorem 6.13]). Then we set $\rho = \phi - \varphi$, which is a solution of

$$(C.2) \quad \begin{cases} -\varepsilon\Delta\rho - \mathcal{L}_R^0[\rho] = F, & x \in B_R, \\ \rho = 0 & x \in \partial B_R, \end{cases}$$

where $F := f + \varepsilon\Delta\varphi + \mathcal{L}_R^\psi[\varphi] \in C^{0,\gamma}(B_R)$. It is clear that, if $\rho \in C^{2,\gamma}(\overline{B_R})$ is a solution of (C.2), then $\phi \in C^{2,\gamma}(B_R) \cap C^0(\overline{B_R})$ and it verifies problem (C.1). Notice that the boundary data for ρ is zero, so that it belongs to $C^{2,\gamma}(\partial B_R)$.

Hence we reduce the proof of Lemma C.1 to proving existence for (C.2). This is based on the Continuity Method for elliptic operators (see [10, Theorem 5.2]) and two a priori bounds that we show next.

Lemma C.2. *Let $\rho \in C^2(\overline{B_R})$ be a solution to (C.2) in B_R . Then, there exists a constant $C = C(\varepsilon, R)$ such that*

$$(C.3) \quad \sup_{B_R} |\rho| \leq \sup_{\partial B_R} |\rho| + C \sup_{B_R} |F|$$

Proof. Though the proof is essentially the same as [10, Lemma 3.7], it has to be adapted carefully in some places in order to take into account the non-local term. To this aim, let $\alpha > 0$, $\mathcal{L}_0 = -\varepsilon\Delta - \mathcal{L}_R^0$ and define for $x_1 \in [-R, R]$

$$\tilde{\rho} := \sup_{\partial B_R} \rho^+ + (e^{\alpha R} - e^{\alpha x_1}) \sup_{B_R} (|F^+|/\varepsilon).$$

We first observe that, if $\alpha = \alpha(\varepsilon, R)$ is chosen big enough, we get

$$\mathcal{L}_0[e^{\alpha x_1}] = -\alpha^2 \varepsilon e^{\alpha x_1} - \mathcal{L}_R^0[e^{\alpha x_1}] \leq -\alpha^2 \varepsilon e^{\alpha x_1} + e^{\alpha x_1} \leq -\varepsilon.$$

Moreover, since for any constant $k \geq 0$, $\mathcal{L}_R^0[k] \leq 0$, we have,

$$\mathcal{L}_0[\tilde{\rho}] = \mathcal{L}_0[\sup_{\partial B_R} \rho^+ + e^{\alpha R} \sup_{B_R} (|F^+|/\varepsilon)] - \mathcal{L}_0[e^{\alpha x_1} \sup_{B_R} (|F^+|/\varepsilon)] \geq \sup_{B_R} |F^+|.$$

Now, since $\mathcal{L}_0[\tilde{\rho} - \rho] \geq \sup_{B_R} |F^+| - F \geq 0$ in B_R and $\tilde{\rho} - \rho \geq 0$ on ∂B_R , by the Maximum Principle, see [9, Theorem 6], we get $\tilde{\rho} - \rho \geq 0$ in B_R , which yields

$$\sup_{B_R} \rho \leq \sup_{\partial B_R} \rho^+ + C \sup_{B_R} F^+$$

Replacing ρ by $-\rho$ we get (C.3). \square

Lemma C.3. *Let $\rho \in C^{2,\alpha}(\overline{B_R})$ be a solution to (C.2). Then, there exists a constant $C = C(N, \alpha, \varepsilon, R) > 0$ such that*

$$\|\rho\|_{C^{2,\alpha}(\overline{B_R})} \leq C \|F\|_{C^{0,\alpha}(\overline{B_R})}$$

Proof. Writing the equation as $\rho - \varepsilon\Delta\rho = \int_{B_R} J(x-y)\rho(y) dy + F$ we get, using [10, Theorem 6.6], that there exists a constant $C_1 > 0$ such that

$$\|\rho\|_{C^{2,\alpha}(\overline{B_R})} \leq C_1 \left(\|\rho\|_{C^0(\overline{B_R})} + \|F\|_{C^{0,\alpha}(\overline{B_R})} + \left| \int_{B_R} J(x-y)\rho(y) dy \right|_{C^{0,\gamma}(\overline{B_R})} \right).$$

On the other hand $\|\int_{B_R} J(x-y)\rho(y) dy\|_{C^{0,\gamma}(\overline{B_R})} \leq C_2\|\rho\|_{C^0(\overline{B_R})}$, for some positive constant C_2 . Combing these bounds with the previous one shown in Lemma C.2, we finally get

$$\|\rho\|_{C^{2,\alpha}(\overline{B_R})} \leq C\|F\|_{C^{0,\alpha}(\overline{B_R})}$$

□

Proof of Lemma C.1. It is a direct adaptation of [10, Theorem 6.8].

□

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