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Stability of Planar Piecewise Linear Systems: A Geometric Condition

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Any planar piecewise linear (two-state, multi-modal) system is shown to be globally asymptotically stable just in case each linear mode satisfies certain conditions that solely depend on how its eigenvectors stand relative to the cone on which it is defined. The conditions are in terms of the eigenvalues, eigenvectors, and the cone.

Keywords: conewise linear; piecewise linear; planar; switched linear; stability; basins

1. Introduction

A thorough study of piecewise linear systems appeared to be the logical next step after the proven success of linear systems. The concern with such systems has also been application oriented because there are so many naturally hybrid, multi-modal plants all around. Examples are not limited to physical systems and sophisticated switched systems come up in the domain of social sciences, (Özgüler, 2013; Sezer & Özgüler, 2006). Since piecewise linearity is the simplest form of nonlinearity that can be imagined, such a study also promised to be very fruitful. The introduction of “conewise systems” as a worthwhile object of study has helped focus attention on this particular class of nonlinear systems, (Çamlıbel, Pang, & Shen, 2006). One can claim that, the most complete result so far on, for instance, stability of piecewise linear systems has turned out to be on the very special case of two-state systems (Araposthasis & Broucke, 2007; Iwatani & Hara, 2006) (also see (Çamlıbel, Heemels, & Schumacher, 2003)). Although Lyapunov approach has provided many sufficient conditions for stability of more general cases (Liberzon, 2003), the method becomes quickly stagnant by the requirement to concoct Lyapunov functions for a set of systems, (Johansson, 2003; Liberzon & Morse, 1999). Nevertheless, the piecewise linear interest is alive, and by now there are a number of good books and survey papers devoted to the subject (Abdullahi, 2015; Johansson, 2003; Liberzon, 2003; Lin & Antsaklis, 2009; van der Schaft & Schumacher, 2000; Sun, 2010; Sun & Ge, 2011).

The result on stability of Iwatani and Hara (Iwatani & Hara, 2006) on planar, multi-modal systems is complete from a test of stability point of view but it is still worth a second look for many reasons. The condition offered for stability is in terms of the zeros of a subsidiary system, the relation of which to the original plant is indirect. This prevents an insightful interpretation of the condition. They devise a very useful notion of ‘transitive modes’ but also use a, not so natural, notion of ‘weakly-transitive’ modes which are also transitive so that a clear distinction between the two notions is not possible. What are the ‘non-transitive modes’? The main result of (Araposthasis & Broucke, 2007), obtained independently of (Iwatani & Hara, 2006), is based on a clean characterization of trajectories escaping convex cones (i.e., transitive modes) and brings

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the eigenvectors of each mode into the picture via the notions of “visible eigenvectors” and “stable eigenspaces”. However, the result in (Araposthasis & Broucke, 2007), similar to that of (Iwatani & Hara, 2006), does not provide any intuition concerning “non-transitive cones.” In (Polderman & Langerak, 2012), a hybrid automaton approach is followed to study the stability of planar systems and a decision algorithm that is based on “contractive cycles” that are, in essence, stable transitive trajectories are given. Also in (Nishiyama & Hayakawa, 2008) and in (Liu, Yao, Yang, Balakrishnan & Guo, 2006), integral expressions are derived to characterize the “expansion factors” when trajectories go through transitive modes.

Here, we take a different approach to the same problem and obtain a new set of necessary and sufficient conditions. Any planar piecewise linear system is shown to be globally asymptotically stable just in case each linear mode satisfies certain conditions that only depend on how its eigenvectors stand relative to the cone on which it is defined. The conditions are in terms of the eigenvalues, eigenvectors, and the cone. The improvements on both (Iwatani & Hara, 2006) and (Araposthasis & Broucke, 2007) are the following: *i*) The condition is directly in terms of the “givens” of the problem. *ii*) Non-transitive modes are identified. *iii*) Initial states and their trajectories are classified (basins of attraction and repulsion are indicated). *iv*) The known condition for bimodal systems is obtained as an easy corollary of the main result.

We denote the real numbers, n -dimensional real vector space, and the set of real $n \times m$ matrices by \mathbf{R} , \mathbf{R}^n , and $\mathbf{R}^{n \times m}$, respectively. The norm of a vector $\mathbf{v} \in \mathbf{R}^n$ will be denoted by $|\mathbf{v}|$. The natural basis vectors in \mathbf{R}^n will be denoted by $\mathbf{e}_i, i = 1, \dots, n$. In particular, when $n = 3$, we will use $\mathbf{k} := \mathbf{e}_3$. If $\mathbf{v}, \mathbf{w} \in \mathbf{R}^3$, then $\mathbf{v} \times \mathbf{w}$ will denote the cross product of the vectors and $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$, their dot product, where ‘T’ denotes ‘transpose.’ If $\mathbf{v}, \mathbf{w} \in \mathbf{R}^2$, then by $\mathbf{v} \times \mathbf{w}$, we mean $\det[\mathbf{v} \ \mathbf{w}] \mathbf{k}$, where ‘det’ means ‘determinant,’ i.e., cross product of vectors in the plane will be computed by imbedding them in the space. The set of complex n -vectors will be \mathbf{C}^n and $j \in \mathbf{C}$ will be the imaginary number. For convenience, we will use the cross product of $\mathbf{v}, \mathbf{w} \in \mathbf{C}^2$ as well and define $\mathbf{v} \times \mathbf{w} := \det[\mathbf{v} \ \mathbf{w}] \mathbf{k}$. By $\log z, z \in \mathbf{C}$, we denote the complex principal logarithm $\log z = \ln |z| + j \angle z$ with $-\pi < \angle z \leq \pi$.

2. Planar Piecewise Linear Systems

The class of systems considered are

$$\dot{\mathbf{x}} = \begin{cases} A_1 \mathbf{x} & \text{if } \mathbf{x} \in \mathbf{S}_1, \\ A_2 \mathbf{x} & \text{if } \mathbf{x} \in \mathbf{S}_2, \\ \vdots & \vdots \\ A_m \mathbf{x} & \text{if } \mathbf{x} \in \mathbf{S}_m, \end{cases} \quad (1)$$

where $A_i \in \mathbf{R}^{2 \times 2}$ and, with $C_i \in \mathbf{R}^{2 \times 2}$,

$$\mathbf{S}_i := \{\mathbf{x} \in \mathbf{R}^2 : C_i \mathbf{x} \geq 0\},$$

for $i = 1, 2, \dots, m$. We assume that each C_i is nonsingular and is such that $\det C_i > 0$. Note that the latter causes no loss of generality and only requires a permutation of rows of C_i if necessary. The nonsingularity assumption implies that (1) is truly multi-modal ($m \geq 2$) and that the interior of each \mathbf{S}_i , $\text{int } \mathbf{S}_i$, is nonempty. We further assume that the interior of each pairwise intersection $\text{int } \mathbf{S}_i \cap \mathbf{S}_k, i \neq k$ is empty and that $\mathbf{S}_1 \cup \dots \cup \mathbf{S}_m = \mathbf{R}^2$. These assumptions ensure, in the terminology

of (Iwatani & Hara, 2006), that (1) is *memoryless*. Further, let

$$S_i = [\mathbf{s}_{i1} \quad \mathbf{s}_{i2}] := C_i^{-1} = \left[\begin{array}{c} \mathbf{c}_{i1}^T \\ \mathbf{c}_{i2}^T \end{array} \right]^{-1}$$

so that $\det S_i > 0$. It is easy to see that, if each \mathbf{S}_i , $i = 1, \dots, m$ is strictly contained in a half-plane, then \mathbf{S}_i is a convex cone

$$\mathbf{S}_i = \{ \alpha \mathbf{s}_{i1} + \beta \mathbf{s}_{i2} : \alpha, \beta \geq 0 \}$$

and the boundary of \mathbf{S}_i is the union two rays

$$\mathbf{B}_{ik} = \{ \alpha \mathbf{s}_{ik} : \alpha \geq 0 \}, \quad k = 1, 2.$$

Note that because $\det S_i > 0$, the cross product $\mathbf{s}_{i1} \times \mathbf{s}_{i2}$ points upward using the right-hand rule, i.e., positively oriented. This allows us to label \mathbf{B}_{i1} and \mathbf{B}_{i2} as the *right* and *left border*, respectively.

If, in (1), there is a mode defined on a half-plane or a sector larger than a half-plane, then it can be split into two modes having the same dynamics (the same A-matrix) so that each is still defined on a cone. The splitting must be done with care, as we will clarify later.

Given a mode i , its eigenvalues will be denoted by $\lambda_{i1}, \lambda_{i2} \in \mathbf{C}$ and, in case of real and distinct eigenvalues, they will be indexed so that $\lambda_{i1} > \lambda_{i2}$.

2.1 Single Mode

We now focus on a single mode i (and temporarily discard the index i) to consider

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x} \in \mathbf{S} \subset \mathbf{R}^2, \quad \mathbf{S} = \{ \alpha \mathbf{s}_1 + \beta \mathbf{s}_2 : \alpha, \beta \geq 0 \}, \quad (2)$$

where $\det S > 0$ for $S = [\mathbf{s}_1 \quad \mathbf{s}_2]$. Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{R}^2$ be such that

$$AV = V\Lambda, \quad V = [\mathbf{v}_1 \quad \mathbf{v}_2],$$

where Λ is equal to

$$\left[\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array} \right], \quad \left[\begin{array}{cc} \lambda & 0 \\ 1 & \lambda \end{array} \right], \quad \text{and} \quad \left[\begin{array}{cc} \sigma & -\omega \\ \omega & \sigma \end{array} \right].$$

respectively, when eigenvalues are such that $\lambda_1 > \lambda_2$ (real and distinct), $\lambda := \lambda_1 = \lambda_2$ (real and repeated), and $\lambda_1 = \lambda_2 = \sigma + j\omega$ (non-real) with $\omega > 0$. It follows that if the eigenvalues are distinct, then $\mathbf{v}_1, \mathbf{v}_2$ are the eigenvectors associated with the larger and smaller eigenvalues, respectively. If they are repeated, then \mathbf{v}_2 is an eigenvector and \mathbf{v}_1 is a generalized eigenvector. If the eigenvalues are non-real, then $\mathbf{v}_1 + j\mathbf{v}_2$ is the eigenvector associated with $\sigma - j\omega$. We define

$$W = \left[\begin{array}{c} \mathbf{w}_1^T \\ \mathbf{w}_2^T \end{array} \right] := V^{-1}.$$

Note that, $\det V > 0$ if and only if $\mathbf{v}_1 \times \mathbf{v}_2$ is positively oriented.

The trajectory at $t \geq 0$ of (2) starting at $\mathbf{x}(0) = \mathbf{b} \in \mathbf{S}$ at time 0 can be written as

$$\mathbf{x}(t, \mathbf{b}) = \begin{cases} e^{\lambda_1 t} \mathbf{w}_1^T \mathbf{b} \mathbf{v}_1 + e^{\lambda_2 t} \mathbf{w}_2^T \mathbf{b} \mathbf{v}_2, \\ e^{\lambda t} [\mathbf{w}_1^T \mathbf{b} \mathbf{v}_1 + (t \mathbf{w}_1^T \mathbf{b} + \mathbf{w}_2^T \mathbf{b}) \mathbf{v}_2], \\ e^{\sigma t} \{ [\mathbf{w}_1^T \mathbf{b} \cos(\omega t) - \mathbf{w}_2^T \mathbf{b} \sin(\omega t)] \mathbf{v}_1 + [\mathbf{w}_1^T \mathbf{b} \sin(\omega t) + \mathbf{w}_2^T \mathbf{b} \cos(\omega t)] \mathbf{v}_2 \}. \end{cases} \quad (3)$$

for the three cases. Examining the sign of the derivative of the angle of $\mathbf{x}(t, \mathbf{b})$, we can determine the direction the trajectory moves at time t .

Fact 1: *Trajectory $\mathbf{x}(t, \mathbf{b})$ moves in a positive direction at time $t \geq 0$ if for every real eigenvector $\mathbf{v}_k \times \mathbf{b} \neq 0$ for $k = 1, 2$ and*

$$\begin{cases} \det V(\mathbf{v}_1 \times \mathbf{b} \cdot \mathbf{v}_2 \times \mathbf{b}) > 0 & \text{if eigenvalues are distinct,} \\ \det V > 0 & \text{otherwise.} \end{cases} \quad (4)$$

Proof. Since $\mathbf{w}_l^T \mathbf{v}_k = 0$ for $l \neq k$, a trajectory moves radially along an eigen-direction if and only if $\mathbf{v}_k \times \mathbf{b} = 0$, by (3). For any other \mathbf{b} , the angle of the vector $\mathbf{v}_k \times \mathbf{b}$, and hence, the direction of a trajectory is well-defined. Suppose the eigenvalues are real and distinct. Let \mathbf{e}_i denote the i -th natural vector for $i = 1, 2$ and let $\mathbf{x}(t, \mathbf{b}) = \rho(t) \angle \psi(t)$ be in polar representation. Then,

$$\dot{\psi}(t) = \frac{(\mathbf{e}_2^T \dot{\mathbf{x}})(\mathbf{e}_1^T \mathbf{x}) - (\mathbf{e}_1^T \dot{\mathbf{x}})(\mathbf{e}_2^T \mathbf{x})}{\rho^2} = -\frac{\det V(\lambda_1 - \lambda_2)(\mathbf{w}_1^T \mathbf{b})(\mathbf{w}_2^T \mathbf{b})}{\rho^2 e^{-(\lambda_1 + \lambda_2)t}}$$

so that $\dot{\psi}(t) > 0$ if and only if $\det V(\mathbf{w}_1^T \mathbf{b})(\mathbf{w}_2^T \mathbf{b}) < 0$. Now, we note that $\mathbf{v}_1 \times \mathbf{b} = \det V(\mathbf{w}_2^T \mathbf{b}) \mathbf{k}$, $\mathbf{v}_2 \times \mathbf{b} = -\det V(\mathbf{w}_1^T \mathbf{b}) \mathbf{k}$, where \mathbf{k} denotes the (positively oriented) cross product of the unit vectors in x_1 and x_2 directions. It follows that $\dot{\psi}(t) > 0$ if and only if the first condition in (4) holds. If the eigenvalues are repeated or non-real, then the expressions in (3) give

$$\dot{\psi}(t) = \frac{\det V(\mathbf{w}_1^T \mathbf{b})^2}{\rho^2 e^{-2\lambda t}}, \quad \dot{\psi}(t) = \frac{\det V \omega [(\mathbf{w}_1^T \mathbf{b})^2 + (\mathbf{w}_2^T \mathbf{b})^2]}{\rho^2 e^{-2\sigma t}},$$

respectively. It follows that, in the last two cases, $\dot{\psi}(t) > 0$ if and only if the second condition in (4) holds. \square

That is, if the eigenvalues are non-real or repeated, then the direction is independent of the initial state \mathbf{b} and is determined by the sign of $\det V$ only. In case of real and distinct eigenvalues, how the initial state is situated with respect to the two eigenvectors also matters. For instance, if $\det V > 0$ then the trajectory moves in negative direction if and only if \mathbf{b} is in between \mathbf{v}_1 and \mathbf{v}_2 .

Let us now classify the cases of trajectories hitting the boundary, one of the two borders of \mathbf{S} .

Fact 2: *(i) There exists (a finite) $t_1 > 0$ such that $\mathbf{x}(t_1, \mathbf{b})$ intersects \mathbf{B}_1 if and only if*

$$\begin{cases} \det V(\mathbf{v}_1 \times \mathbf{b} \cdot \mathbf{v}_2 \times \mathbf{b}) < 0 \ \& \ \mathbf{v}_1 \times \mathbf{b} \cdot \mathbf{v}_1 \times \mathbf{s}_1 > 0, \\ \det V < 0 \ \& \ \mathbf{v}_2 \times \mathbf{b} \cdot \mathbf{v}_2 \times \mathbf{s}_1 > 0, \\ \det V < 0 \end{cases} \quad (5)$$

respectively, when eigenvalues are such that $\lambda_1 > \lambda_2$ (real and distinct), $\lambda := \lambda_1 = \lambda_2$ (real and repeated), and $\lambda_1 = \bar{\lambda}_2 = \sigma + j\omega$ (non-real).

(ii) There exists (a finite) $t_2 > 0$ such that $\mathbf{x}(t_2, \mathbf{b})$ intersects \mathbf{B}_2 if and only if

$$\begin{cases} \det V(\mathbf{v}_1 \times \mathbf{b} \cdot \mathbf{v}_2 \times \mathbf{b}) > 0 \ \& \ \mathbf{v}_1 \times \mathbf{b} \cdot \mathbf{v}_1 \times \mathbf{s}_2 > 0, \\ \det V > 0 \ \& \ \mathbf{v}_2 \times \mathbf{b} \cdot \mathbf{v}_2 \times \mathbf{s}_2 > 0, \\ \det V > 0. \end{cases} \quad (6)$$

respectively, when eigenvalues are such that $\lambda_1 > \lambda_2$ (real and distinct), $\lambda := \lambda_1 = \lambda_2$ (real and repeated), and $\lambda_1 = \bar{\lambda}_2 = \sigma + j\omega$ (non-real).

(iii) Let μ denote λ_1 or σ and let \mathbf{v} denote \mathbf{v}_2 or $\mathbf{v}_1 + j\mathbf{v}_2$ in the cases of real and non-real eigenvalues, respectively. Then, in the situations of items (i) and (ii), we have

$$\mathbf{x}(t_1, \mathbf{b}) = \mathbf{s}_1 \frac{(\mathbf{v} \times \mathbf{b}) \cdot \mathbf{k}}{(\mathbf{v} \times \mathbf{s}_1) \cdot \mathbf{k}} e^{\mu t_1}, \quad \mathbf{x}(t_2, \mathbf{b}) = \mathbf{s}_2 \frac{(\mathbf{v} \times \mathbf{b}) \cdot \mathbf{k}}{(\mathbf{v} \times \mathbf{s}_2) \cdot \mathbf{k}} e^{\mu t_2}. \quad (7)$$

Proof. (i) Suppose the eigenvalues are real and distinct. Such a $t_1 > 0$ exists just in case

$$\mathbf{c}_2^T \mathbf{x}(t_1, \mathbf{b}) = e^{\lambda_1 t_1} (\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{w}_1^T \mathbf{b}) + e^{\lambda_2 t_1} (\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_2^T \mathbf{b}) = 0,$$

which gives

$$e^{(\lambda_1 - \lambda_2)t_1} = -\frac{(\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_2^T \mathbf{b})}{(\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{w}_1^T \mathbf{b})} > 1, \quad (8)$$

where the inequality is by $\lambda_1 > \lambda_2$. Note that $\mathbf{c}_2^T \mathbf{v}_1 \neq 0$ and $\mathbf{w}_1^T \mathbf{b} \neq 0$ since otherwise either $\mathbf{v}_1 \times \mathbf{s}_1 = 0$ or $\mathbf{v}_2 \times \mathbf{b} = 0$, i.e., either there is a sliding mode or the initial condition is along an eigenvector. Now, the condition (8) is equivalent to

$$1 + \frac{(\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_2^T \mathbf{b})}{(\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{w}_1^T \mathbf{b})} = \frac{\det V \det S \mathbf{c}_2^T \mathbf{b}}{\mathbf{v}_2 \times \mathbf{b} \cdot \mathbf{v}_1 \times \mathbf{s}_1} < 0 \quad (9)$$

by the identities $(\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{w}_1^T \mathbf{b}) + (\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_2^T \mathbf{b}) = \mathbf{c}_2^T \mathbf{b}$, $\det S(\mathbf{c}_2^T \mathbf{v}_1)\mathbf{k} = -\mathbf{v}_1 \times \mathbf{s}_1$, $\det V(\mathbf{w}_1^T \mathbf{b})\mathbf{k} = -\mathbf{v}_2 \times \mathbf{b}$.

In (9), $\mathbf{c}_2^T \mathbf{b} > 0$ because, as $\mathbf{s}_1 \times \mathbf{s}_2$ is positively oriented and \mathbf{b} is in the interior of \mathbf{S} , $\mathbf{s}_1 \times \mathbf{b} = \det S(\mathbf{c}_2^T \mathbf{b})\mathbf{k}$ is also positively oriented. Using the condition from (4) that $\det V(\mathbf{v}_1 \times \mathbf{b} \cdot \mathbf{v}_2 \times \mathbf{b}) < 0$ is necessary for an intersection with \mathbf{B}_1 , we obtain the first condition in (5). Suppose, next, that the eigenvalues are repeated. Then, $t_1 > 0$ exists if and only if $\mathbf{c}_2^T \mathbf{x}(t_1, \mathbf{b}) = e^{\lambda t_1} [(\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{w}_1^T \mathbf{b}) + (\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_2^T \mathbf{b}) + t_1(\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_1^T \mathbf{b})] = 0$, which gives

$$t_1 = -\frac{(\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{w}_1^T \mathbf{b}) + (\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_2^T \mathbf{b})}{(\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_1^T \mathbf{b})} = -\frac{\det V \det S \mathbf{c}_2^T \mathbf{b}}{\mathbf{v}_2 \times \mathbf{b} \cdot \mathbf{v}_2 \times \mathbf{s}_1} > 0. \quad (10)$$

Using the condition from (4) that $\det V < 0$ is necessary for trajectory to intersect \mathbf{B}_1 and the fact that $\mathbf{c}_2^T \mathbf{b} > 0$ as in the previous case, we obtain the second condition in (5). In the final case that eigenvalues are non-real, there exists such t_1 if and only if $\det V < 0$, by (4) and by the fact that the trajectories are always foci or centers.

(ii) The proof is analogous to the proof in (i).

(iii) Let us first consider the case of non-real eigenvalues. In the situation depicted by (i), at $t = t_1 > 0$, we have

$$\mathbf{c}_2^T \mathbf{x}(t, \mathbf{b}) = e^{\sigma t} \{ [\mathbf{w}_1^T \mathbf{b} \cos(\omega t) - \mathbf{w}_2^T \mathbf{b} \sin(\omega t)] \mathbf{c}_2^T \mathbf{v}_1 + [\mathbf{w}_1^T \mathbf{b} \sin(\omega t) + \mathbf{w}_2^T \mathbf{b} \cos(\omega t)] \mathbf{c}_2^T \mathbf{v}_2 \} = 0, \quad (11)$$

which gives

$$\tan(\omega t_1) = \frac{(\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{w}_1^T \mathbf{b}) + (\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_2^T \mathbf{b})}{(\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{w}_2^T \mathbf{b}) - (\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_1^T \mathbf{b})} = \frac{\mathbf{c}_2^T \mathbf{b}}{(\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{w}_2^T \mathbf{b}) - (\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_1^T \mathbf{b})} \quad (12)$$

and

$$\mathbf{x}(t_1, \mathbf{b}) = \frac{\det V}{\det S} \frac{e^{\sigma t_1}}{\mathbf{c}_2^T \mathbf{v}_2} [\mathbf{w}_1^T \mathbf{b} \cos(\omega t_1) - \mathbf{w}_2^T \mathbf{b} \sin(\omega t_1)] \mathbf{s}_1. \quad (13)$$

In obtaining (12) from (3), we have used the identity

$$\mathbf{v}_1(\mathbf{c}_2^T \mathbf{v}_2) - \mathbf{v}_2(\mathbf{c}_2^T \mathbf{v}_1) = \mathbf{c}_2^T \mathbf{v}_2 (\mathbf{v}_1 + \mathbf{v}_2 \frac{\mathbf{w}_2^T \mathbf{s}_1}{\mathbf{w}_1^T \mathbf{s}_1}) = \frac{\mathbf{c}_2^T \mathbf{v}_2}{\mathbf{w}_1^T \mathbf{s}_1} \mathbf{s}_1.$$

Noting, with $\Delta := \sqrt{[(\mathbf{w}_1^T \mathbf{b})^2 + (\mathbf{w}_2^T \mathbf{b})^2][(\mathbf{c}_2^T \mathbf{v}_1)^2 + (\mathbf{c}_2^T \mathbf{v}_2)^2]}$, that

$$\cos(\omega t_1) = \frac{(\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{w}_2^T \mathbf{b}) - (\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_1^T \mathbf{b})}{\Delta}, \quad \sin(\omega t_1) = \frac{\mathbf{c}_2^T \mathbf{b}}{\Delta},$$

and substituting in (13), we get

$$\begin{aligned} \mathbf{x}(t_1, \mathbf{b}) &= -\frac{\det V}{\det S} e^{\sigma t_1} \sqrt{\frac{(\mathbf{w}_1^T \mathbf{b})^2 + (\mathbf{w}_2^T \mathbf{b})^2}{(\mathbf{c}_2^T \mathbf{v}_1)^2 + (\mathbf{c}_2^T \mathbf{v}_2)^2}} \mathbf{s}_1 \\ &= -\frac{\det V}{\det S} \frac{|\det S|}{|\det V|} e^{\sigma t_1} \sqrt{\frac{|\mathbf{v}_2 \times \mathbf{b}|^2 + |\mathbf{v}_1 \times \mathbf{b}|^2}{|\mathbf{s}_1 \times \mathbf{v}_1|^2 + |\mathbf{s}_1 \times \mathbf{v}_2|^2}} \mathbf{s}_1, \end{aligned}$$

which can be expressed as in (7) since $\det S > 0$, $-\det V > 0$. In case of real, distinct eigenvalues, substituting the expression for $e^{\lambda_2 t_1} \mathbf{w}_2^T \mathbf{b} \mathbf{v}_2$ obtained from (8) into (3), we have

$$\begin{aligned} \mathbf{x}(t_1, \mathbf{b}) &= e^{\lambda_1 t_1} \frac{\mathbf{w}_1^T \mathbf{b}}{\mathbf{c}_2^T \mathbf{v}_2} [\mathbf{v}_1(\mathbf{c}_2^T \mathbf{v}_2) - \mathbf{v}_2(\mathbf{c}_2^T \mathbf{v}_1)] \\ &= e^{\lambda_1 t_1} \mathbf{w}_1^T \mathbf{b} (\mathbf{v}_1 + \mathbf{v}_2 \frac{\mathbf{w}_2^T \mathbf{s}_1}{\mathbf{w}_1^T \mathbf{s}_1}) = e^{\lambda_1 t_1} \frac{(\mathbf{v}_2 \times \mathbf{b}) \cdot \mathbf{k}}{(\mathbf{v}_2 \times \mathbf{s}_1) \cdot \mathbf{k}} \mathbf{s}_1, \end{aligned}$$

which again gives (7). Finally, in case of repeated eigenvalues, substituting the expression for t_1 obtained from (10) into (3), we have

$$\mathbf{x}(t_1, \mathbf{b}) = e^{\lambda t_1} \frac{\mathbf{w}_1^T \mathbf{b}}{\mathbf{c}_2^T \mathbf{v}_2} [\mathbf{v}_1(\mathbf{c}_2^T \mathbf{v}_2) - \mathbf{v}_2(\mathbf{c}_2^T \mathbf{v}_1)] = e^{\lambda t_1} \frac{(\mathbf{v}_2 \times \mathbf{b}) \cdot \mathbf{k}}{(\mathbf{v}_2 \times \mathbf{s}_1) \cdot \mathbf{k}} \mathbf{s}_1, \quad (14)$$

giving (7). The derivation for the expression $\mathbf{x}(t_2, \mathbf{b})$ in (7) is along the same lines. \square

We note that (8), (10), and (12) provide explicit expressions for t_1 that occur in (7). These, together with the dual expressions for t_2 , will be used in obtaining our main result. The classification of initial conditions made possible by conditions (i) and (ii) of Fact 2 are given in Figures 1-3, where basins leading to t_1 , t_2 , or neither are indicated relative to typical positions of the eigenvectors with respect to the region \mathbf{S} .

Definition 1: An eigenvector \mathbf{v} is *interior* to \mathbf{S} if \mathbf{v} or $-\mathbf{v}$ is in $\text{int}(\mathbf{S})$; it is *exterior* to \mathbf{S} if neither \mathbf{v} nor $-\mathbf{v}$ is in \mathbf{S} .

Note that eigenvectors of non-real eigenvalues are exterior since they are non-real.

Definition 2: A mode like (2) is called *transitive* (Iwatani & Hara, 2006) if either the trajectory intersects B_1 (at some finite time) for all $\mathbf{b} \in \mathbf{S}$ or it intersects B_2 for all $\mathbf{b} \in \mathbf{S}$. The mode will be called *negative-transitive* in the former, and *positive-transitive* in the latter case. A mode is a *source* if there exists $\mathbf{n} \in \text{int}(\mathbf{S})$ such that for all $\mathbf{b} = \alpha\mathbf{n} + \beta\mathbf{s}_1$ with $\alpha \geq 0, \beta > 0$, the trajectory intersects B_1 and for all $\mathbf{b} = \alpha\mathbf{n} + \beta\mathbf{s}_2$ with $\alpha \geq 0, \beta > 0$, the trajectory intersects B_2 .

It is clear from Figures 1-3 that if a mode is neither transitive nor a source, then it is either a *sink* and all trajectories starting in (or entering into) \mathbf{S} stay in \mathbf{S} or it is a *half-sink*, that is there is a sector of \mathbf{S} that is a sink. By Fact 2 (or, by Figures 1-3), it is also easy to see that

Fact 3: A mode (2) is transitive if and only if the eigenvector(s) are exterior. It is a source (resp., sink) if and only if there are two eigenvectors such that the one associated with the larger (resp., smaller) eigenvalue is exterior and the other interior. It is a half-sink if and only if the eigenvector(s) are interior.

Definition 3: If a mode i is transitive, then its *factor of expansion* is

$$F_i := \begin{cases} \ln \frac{|\mathbf{v} \times \mathbf{s}_1|}{|\mathbf{v} \times \mathbf{s}_2|} + \mu t_2 & \text{if it is positive-transitive,} \\ \ln \frac{|\mathbf{v} \times \mathbf{s}_2|}{|\mathbf{v} \times \mathbf{s}_1|} + \mu t_1 & \text{if it is negative-transitive,} \end{cases}$$

where t_1, t_2, \mathbf{v} , and μ are as in Fact 2 associated with mode (2). In view of Fact 2.iii, the factor of expansion is the natural logarithm of the gain $\frac{\mathbf{x}}{\mathbf{s}_k}$ a trajectory goes through when it starts at a border B_l and traverses the whole sector hitting the other border B_k .

In (Nishiyama & Hayakawa, 2008) some regions of initial conditions of Fact 2 and in both (Nishiyama & Hayakawa, 2008) and (Liu et al. , 2006), an integral expression for F_i in the non-real case have also been obtained. The “visible eigenvector” of (Araposthasis & Broucke, 2007) is one that lies *inside* the cone and serves the same purpose as the interior eigenvector of our Definition 1. A similar expression to that of F_i above also figures in the main condition of Theorem 6 of (Araposthasis & Broucke, 2007).

2.2 Condition for Stability

The planar system (1) is well-posed in the sense of Carathéodory if there exist a unique solution of the form

$$\mathbf{x}(t, \mathbf{b}) = \mathbf{b} + \int_{t_0}^t f(x(\tau))d\tau, \quad (15)$$

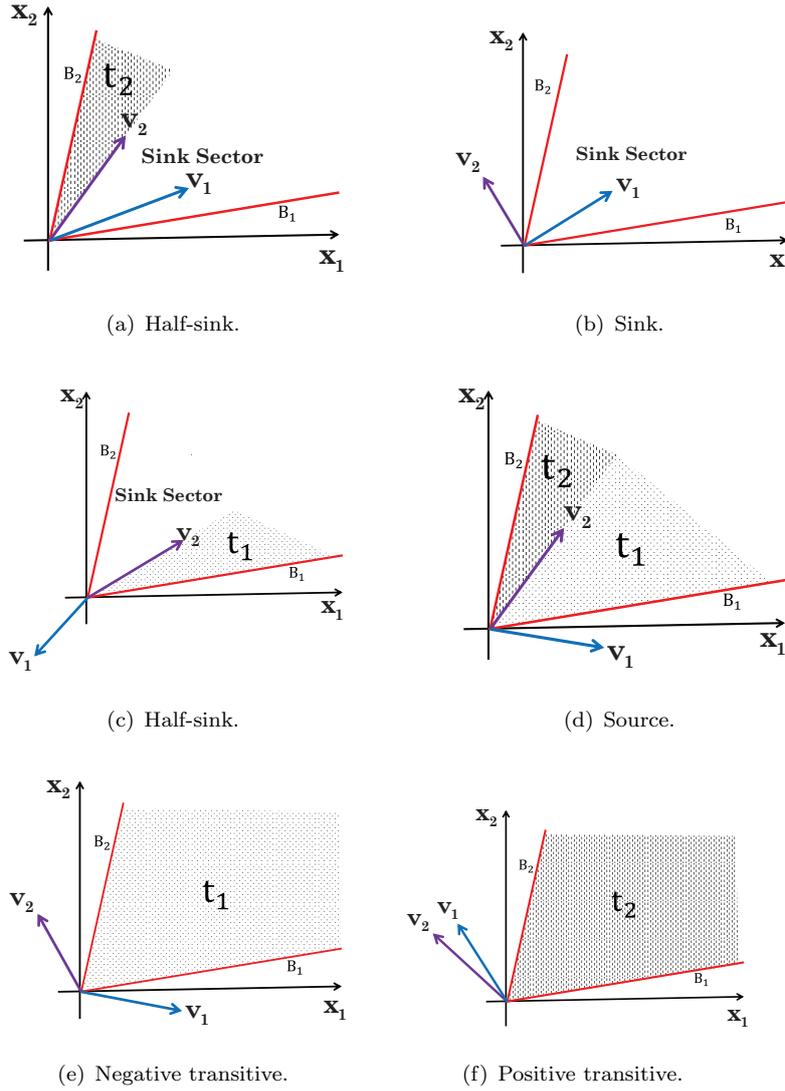


Figure 1. Basins for distinct eigenvalues, $\mathbf{v}_1 \times \mathbf{v}_2$ is positively oriented.

without any sliding mode, where $f(x(\tau))$ is the discontinuous vector field given by the right hand side of (1) and $\mathbf{x}(t_0) = \mathbf{b}$ (Imura & van der Schaft, 2000). Fact 1 implies a geometric condition for these. The condition (ii) below says, in effect, that trajectories of every pair of adjacent modes have the same direction on their common border.

Fact 4: *The system (1) is well-posed if and only if*

i) for every real eigenvector \mathbf{v}_{il} of mode i , it holds that

$$\mathbf{v}_{il} \times \mathbf{s}_{il} \neq 0, \text{ for } l = 1, 2 \text{ and for all } i = 1, \dots, m$$

and

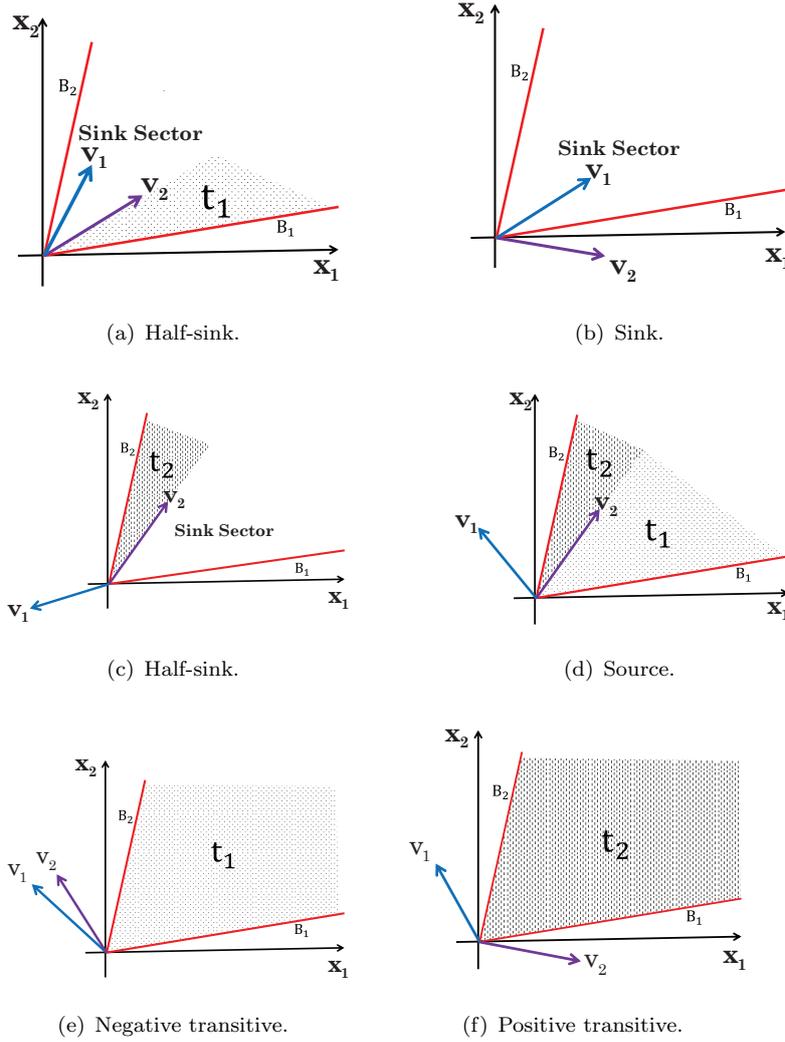


Figure 2. Basins for distinct eigenvalues, $\mathbf{v}_1 \times \mathbf{v}_2$ is negatively oriented.

ii) for every pair of adjacent modes (i, k) with common border B_{il} it holds that

$$\begin{cases} \det V_i(\mathbf{v}_{i1} \times \mathbf{s}_{il} \cdot \mathbf{v}_{i2} \times \mathbf{s}_{il}) \det V_k(\mathbf{v}_{k1} \times \mathbf{s}_{il} \cdot \mathbf{v}_{k2} \times \mathbf{s}_{il}) > 0 \\ \det V_i(\mathbf{v}_{i1} \times \mathbf{s}_{il} \cdot \mathbf{v}_{i2} \times \mathbf{s}_{il}) \det V_k > 0 \\ \det V_i \det V_k > 0. \end{cases} \quad (16)$$

respectively, if eigenvalues of both modes are distinct, if eigenvalues of mode i are distinct and of k are repeated or non-real, and otherwise.

We also remark here that in splitting a mode of dynamics A into two modes (to satisfy the assumption that all modes are defined on cones), one should take care to choose the common border *not* to coincide with any eigenvectors of A , since otherwise there will be a sliding mode by Fact 4. The assumption of well-posedness actually puts some serious constraint on the set of systems considered since any trajectory would have to evolve in one direction only. A thorough study of systems, like (Utkin, Guldner, & Shi, 1999), in which sliding modes and chattering is allowed is thus highly desirable, as pointed out in (Imura & van der Schaft, 2000).

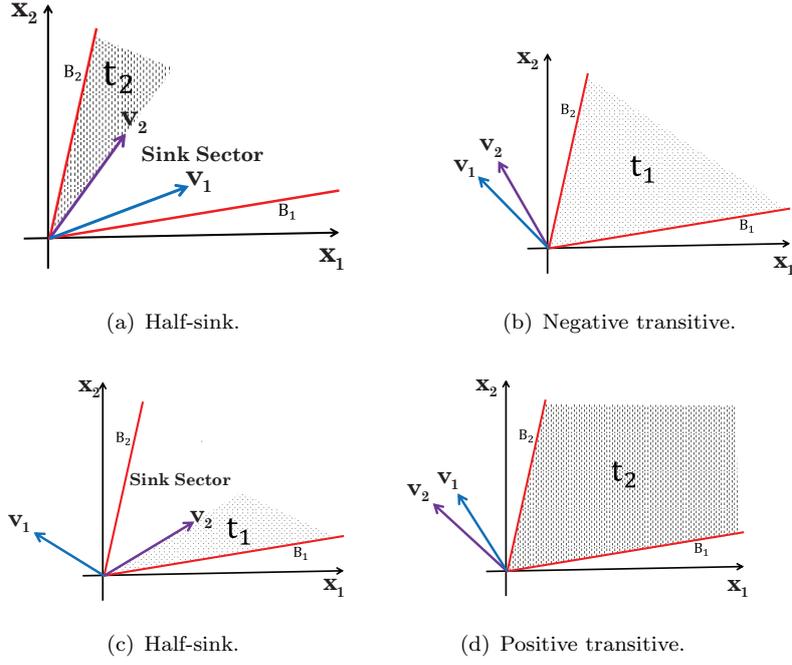


Figure 3. Basins for repeated eigenvalues, $\mathbf{v}_1 \times \mathbf{v}_2$ is positively and negatively oriented.

Fact 5: Let \log denote the complex principal logarithm. Define, for $i = 1, \dots, m$,

$$E_i := \frac{\lambda_{i1}}{\lambda_{i1} - \lambda_{i2}} \log \frac{(\hat{\mathbf{v}}^i \times \mathbf{s}_{i1}) \cdot \mathbf{k}}{(\hat{\mathbf{v}}^i \times \mathbf{s}_{i2}) \cdot \mathbf{k}} - \frac{\lambda_{i2}}{\lambda_{i1} - \lambda_{i2}} \log \frac{(\mathbf{v}^i \times \mathbf{s}_{i1}) \cdot \mathbf{k}}{(\mathbf{v}^i \times \mathbf{s}_{i2}) \cdot \mathbf{k}},$$

where \mathbf{v}^i is \mathbf{v}_{i2} or $\mathbf{v}_{i1} + j\mathbf{v}_{i2}$ and $\hat{\mathbf{v}}^i$ is \mathbf{v}_{i1} or $\mathbf{v}_{i1} - j\mathbf{v}_{i2}$ in case of real or non-real eigenvalues, respectively, and the right hand side is computed as $\lim(\lambda_{i1} - \lambda_{i2}) \rightarrow 0$ in case of repeated eigenvalues. Then, $F_i = E_i$ when mode i is positive-transitive and $F_i = -E_i$, when negative-transitive.

Proof. We omit ‘ i ’ whenever it is clear from the context. Let us consider the negative-transitive case. Suppose the eigenvalues are real and distinct so that

$$\begin{aligned} F_i &= \frac{\lambda_2}{\lambda_1 - \lambda_2} \log \frac{(\mathbf{v}_2 \times \mathbf{s}_1) \cdot \mathbf{k}}{(\mathbf{v}_2 \times \mathbf{s}_2) \cdot \mathbf{k}} - \frac{\lambda_1}{\lambda_1 - \lambda_2} \log \frac{(\mathbf{v}_1 \times \mathbf{s}_1) \cdot \mathbf{k}}{(\mathbf{v}_1 \times \mathbf{s}_2) \cdot \mathbf{k}} \\ &= \frac{\lambda_2}{\lambda_1 - \lambda_2} \ln \frac{|\mathbf{v}_2 \times \mathbf{s}_1|}{|\mathbf{v}_2 \times \mathbf{s}_2|} - \frac{\lambda_1}{\lambda_1 - \lambda_2} \ln \frac{|\mathbf{v}_1 \times \mathbf{s}_1|}{|\mathbf{v}_1 \times \mathbf{s}_2|} \\ &= \ln \frac{|\mathbf{v}_2 \times \mathbf{s}_2|}{|\mathbf{v}_2 \times \mathbf{s}_1|} + \frac{\lambda_1}{\lambda_1 - \lambda_2} \ln \frac{|\mathbf{v}_1 \times \mathbf{s}_2|}{|\mathbf{v}_1 \times \mathbf{s}_1|} \frac{|\mathbf{v}_2 \times \mathbf{s}_1|}{|\mathbf{v}_2 \times \mathbf{s}_2|} \\ &= \ln \frac{|\mathbf{v}_2 \times \mathbf{s}_2|}{|\mathbf{v}_2 \times \mathbf{s}_1|} + \lambda_1 t_1, \end{aligned} \tag{17}$$

where the first equality is by $\mathbf{v} = \mathbf{v}_2$, $\hat{\mathbf{v}} = \mathbf{v}_1$. The second follows by noting, for any transitive

mode, that

$$z_l := \frac{(\mathbf{v}_l \times \mathbf{s}_1) \cdot \mathbf{k}}{(\mathbf{v}_l \times \mathbf{s}_2) \cdot \mathbf{k}} > 0 \text{ for } l = 1, 2$$

since eigenvectors are exterior to \mathbf{S} and that $\log z_i = \ln z_i$ for any real, positive z_i . The last equality follows by the expression for t_1 in (8) since $\mathbf{b} = \mathbf{s}_2$ and $(\mathbf{c}_2^T \mathbf{v}_2) \mathbf{k} = -\mathbf{v}_2 \times \mathbf{s}_1$, $\det V(\mathbf{w}_1^T \mathbf{s}_2) \mathbf{k} = -\mathbf{v}_2 \times \mathbf{s}_2$, $\det V(\mathbf{w}_2^T \mathbf{s}_2) \mathbf{k} = \mathbf{v}_1 \times \mathbf{s}_2$.

To derive (7) as the limit of $F_i = -E_i$ in the case of repeated eigenvalues, consider the third expression in (14) with $\delta := \lambda_1 - \lambda_2$. By (8),

$$\lim_{\delta \rightarrow 0} \frac{\lambda_1}{\delta} \ln \frac{|\mathbf{v}_1 \times \mathbf{s}_2| |\mathbf{v}_2 \times \mathbf{s}_1|}{|\mathbf{v}_1 \times \mathbf{s}_1| |\mathbf{v}_2 \times \mathbf{s}_2|} = \lim_{\delta \rightarrow 0} \frac{\lambda_1}{\delta} \ln e^{\delta t_1} = \lambda t_1.$$

Thus,

$$\lim_{\delta \rightarrow 0} F_i = \ln \frac{|\mathbf{v}_2 \times \mathbf{s}_2|}{|\mathbf{v}_2 \times \mathbf{s}_1|} + \lambda t_1,$$

where t_1 , we interpret, is given by (10) with $\mathbf{b} = \mathbf{s}_2$. Finally, suppose the eigenvalues are non-real with $\lambda_1 = \sigma + j\omega = \bar{\lambda}_2$. With $\mathbf{v} = \mathbf{v}_1 + j\mathbf{v}_2$ and $\hat{\mathbf{v}} = \mathbf{v}_1 - j\mathbf{v}_2$, let us first note that

$$\log \frac{(\mathbf{v} \times \mathbf{s}_1) \cdot \mathbf{k}}{(\mathbf{v} \times \mathbf{s}_2) \cdot \mathbf{k}} = \ln \frac{|\mathbf{v} \times \mathbf{s}_1|}{|\mathbf{v} \times \mathbf{s}_2|} + j\theta, \log \frac{(\hat{\mathbf{v}} \times \mathbf{s}_1) \cdot \mathbf{k}}{(\hat{\mathbf{v}} \times \mathbf{s}_2) \cdot \mathbf{k}} = \ln \frac{|\mathbf{v} \times \mathbf{s}_1|}{|\mathbf{v} \times \mathbf{s}_2|} - j\theta,$$

where

$$\begin{aligned} \theta &:= \angle \frac{(\mathbf{v} \times \mathbf{s}_1) \cdot \mathbf{k}}{(\mathbf{v} \times \mathbf{s}_2) \cdot \mathbf{k}} = \angle \frac{(\mathbf{v}_1 \times \mathbf{s}_1 + j\mathbf{v}_2 \times \mathbf{s}_1) \cdot \mathbf{k}}{(\mathbf{v}_1 \times \mathbf{s}_2 + j\mathbf{v}_2 \times \mathbf{s}_2) \cdot \mathbf{k}} = \angle \frac{-\mathbf{c}_2^T \mathbf{v}_1 - j\mathbf{c}_2^T \mathbf{v}_2}{\mathbf{c}_1^T \mathbf{v}_1 + j\mathbf{c}_1^T \mathbf{v}_2} \\ &= \arctan \left\{ \frac{(\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{c}_1^T \mathbf{v}_1) - (\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{c}_1^T \mathbf{v}_2)}{(\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{c}_1^T \mathbf{v}_1) + (\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{c}_1^T \mathbf{v}_2)} \right\} = \omega t_1. \end{aligned} \tag{18}$$

The last equality follows upon setting $\mathbf{b} = \mathbf{s}_2$ in (12) and noting that $\mathbf{c}_1^T \mathbf{v}_2 = -\frac{\det V}{\det S} \mathbf{w}_1^T \mathbf{s}_2$, $\mathbf{c}_1^T \mathbf{v}_1 = \frac{\det V}{\det S} \mathbf{w}_2^T \mathbf{s}_2$. Therefore,

$$\begin{aligned} F_i &= \frac{\lambda_2}{\lambda_1 - \lambda_2} \log \frac{(\mathbf{v} \times \mathbf{s}_1) \cdot \mathbf{k}}{(\mathbf{v} \times \mathbf{s}_2) \cdot \mathbf{k}} - \frac{\lambda_1}{\lambda_1 - \lambda_2} \log \frac{(\hat{\mathbf{v}} \times \mathbf{s}_1) \cdot \mathbf{k}}{(\hat{\mathbf{v}} \times \mathbf{s}_2) \cdot \mathbf{k}} \\ &= \frac{\sigma - j\omega}{2j\omega} \left(\ln \frac{|\mathbf{v} \times \mathbf{s}_1|}{|\mathbf{v} \times \mathbf{s}_2|} + j\theta \right) - \frac{\sigma + j\omega}{2j\omega} \left(\ln \frac{|\mathbf{v} \times \mathbf{s}_1|}{|\mathbf{v} \times \mathbf{s}_2|} - j\theta \right) \\ &= -\ln \frac{|\mathbf{v} \times \mathbf{s}_1|}{|\mathbf{v} \times \mathbf{s}_2|} + \frac{\sigma}{\omega} \theta = \ln \frac{|\mathbf{v} \times \mathbf{s}_2|}{|\mathbf{v} \times \mathbf{s}_1|} + \sigma t_1. \end{aligned}$$

It follows that F_i is as in Definition 3 for the case of non-real eigenvalues as well. The expression for positive-transitive case is similarly derived. \square

Admittedly, the limit argument in case of repeated eigenvalues is heuristic and needs to be done more rigorously. Nevertheless, the mere fact that the factor of expansion in all three cases can be expressed by a single formula is very appealing.

We can now state and prove an alternative to the algebraic condition of (Iwatani & Hara, 2006) for stability of (1). The condition is “geometric” since a mode being transitive, source, or (half-)sink

is characterized solely in terms of the eigenvectors associated with the mode and how they stand in the phase-plane relative to the sector on which the mode is defined.

Theorem 1: *A well-posed system (1) is globally asymptotically stable if and only if*

$$\begin{aligned} \text{when all modes } i = 1, \dots, m \text{ are transitive} &\Rightarrow \sum_{i=1}^m F_i < 0, \\ \text{when a source } i \text{ exists} &\Rightarrow \lambda_{i2} < 0, \\ \text{when a sink or half-sink } i \text{ exists} &\Rightarrow \lambda_{i1} < 0. \end{aligned}$$

Proof. Suppose all modes are transitive. Then, by well-posedness, all are positive or all are negative transitive, since otherwise there will be chattering. In either case, for any $\mathbf{b} \in \mathbf{R}^2$, we must have $\mathbf{x}(t(\mathbf{b}), \mathbf{b}) = \gamma(\mathbf{b})\mathbf{b}$ for some $t(\mathbf{b}) > 0$ and $\gamma(\mathbf{b}) > 0$, i.e., the trajectory comes back to the ray passing through \mathbf{b} after going through an expansion or contraction of size $\gamma(\mathbf{b})$. We might as well consider the case $\mathbf{b} = \mathbf{s}_{12}$, which is taking the initial state to be on the left-hand border of the first mode, without loss of generality. If the modes are negative transitive, then by (7) of Fact 2,

$$\gamma(\mathbf{s}_{12}) = \prod_{i=1}^m \frac{|\mathbf{v}^i \times \mathbf{s}_{i2}|}{|\mathbf{v}^i \times \mathbf{s}_{i1}|} e^{\mu_i t(\mathbf{s}_{i2})},$$

where μ_i denotes λ_{i1} or σ_i and \mathbf{v}^i denotes \mathbf{v}_{i2} or $\mathbf{v}_{i1} + j\mathbf{v}_{i2}$ in the cases of mode i having real or non-real eigenvalues. The system is globally asymptotically stable if and only if $\gamma(\mathbf{s}_{12}) < 1$, which is equivalent to $F_1 + \dots + F_m < 0$ in view of $\ln[\gamma(\mathbf{s}_{12})] < 0$ and the expressions for $t(\mathbf{s}_{i2})$. Note that if all modes are positive transitive, then starting the trajectory at $\mathbf{b} = \mathbf{s}_{11}$, we have $\gamma(\mathbf{s}_{11}) = 1/\gamma(\mathbf{s}_{12})$, and the same condition is again obtained. Suppose now that not all modes are transitive and a mode i with $\mathbf{n} = \mathbf{v}_{i2}$ is a source. In this case there must be a mode k that is a sink or half-sink since otherwise a sliding mode or chattering would exist. If the system is stable, then $\lambda_{i2} < 0$ must clearly hold for trajectories starting along that eigenvector to converge. Conversely, $\lambda_{i2} < 0$ implies that trajectories starting in mode i converge to the origin if they start along \mathbf{v}_{i2} or they go outside \mathbf{S}_i and enter mode k . For such trajectories to converge to the origin, it is necessary that $\lambda_{k1}, \lambda_{k2} < 0$. The necessity of $\lambda_{k1}, \lambda_{k2} < 0$ for any sink or half-sink mode k is also clear by considering trajectories that start inside the sink-sector of k . Conversely, the sufficiency of the condition follows by the fact that any trajectory starting in a transitive mode or a non-sink sector of a half-sink must end up in either a sink or in the sink sector of a half-sink. \square

Corollary 1: *Let $B_1, B_2 \in \mathbf{R}^{2 \times 2}$ and $\mathbf{c} \in \mathbf{R}^2$ be given. A well-posed bimodal system*

$$\dot{\mathbf{x}} = \begin{cases} B_1 \mathbf{x} & \text{if } \mathbf{c}^T \mathbf{x} \geq 0, \\ B_2 \mathbf{x} & \text{if } \mathbf{c}^T \mathbf{x} \leq 0, \end{cases} \quad (19)$$

is globally asymptotically stable if and only if

$$\begin{aligned} \text{when both modes have non-real eigenvalues} &\Rightarrow \frac{\sigma_1}{\omega_1} + \frac{\sigma_2}{\omega_2} < 0, \\ \text{when a mode, say } i, \text{ has real eigenvalues } \lambda_{i1} \geq \lambda_{i2} &\Rightarrow \lambda_{i1} < 0. \end{aligned}$$

Proof. In order to be able to apply Theorem 1, we let \mathbf{c}_0 be any vector that is not perpendicular to any of the real eigenvectors of B_1 and B_2 , if any, and such that $d := \det[\mathbf{c} \ \mathbf{c}_0]$ is positive. Let $C_1^T := [\mathbf{c} \ \mathbf{c}_0]$. Note that $S_1 := C_1^{-1}$ satisfies $\det S_1 > 0$ and neither columns are in the direction of the real eigenvectors of B_1 or B_2 , if any. The four modal system $A_1 = A_2 = B_1$, $A_3 = A_4 = B_2$, $C_2^T := [-\mathbf{c}_0 \ \mathbf{c}]$, $C_3 = -C_1$, $C_4 = -C_2$ is well-posed, is in the framework of (1), and is equivalent to (19). Suppose, first that, say, B_1 has real eigenvalues so that modes 1 and 2 both have those

eigenvalues with the same corresponding eigenvector(s). If, say, mode 1 is a source, then eigenvalues must be distinct and \mathbf{v}_1 of mode 1 must be exterior. This implies, since mode 2 complements 1 in a half plane, that \mathbf{v}_1 is interior to mode 2 and \mathbf{v}_2 is exterior, that is mode 2 is a sink. By Theorem 1, the system is stable if and only if both eigenvalues of B_1 are negative. If mode 1 is transitive, then both eigenvectors are exterior to mode 1. Thus, the eigenvector(s) are interior to mode 2 so that mode 2 is a half-sink, which again implies that eigenvalues being negative is necessary and sufficient for stability. The other possibilities for mode 1 clearly give the same result. Suppose, second, that both B_1 and B_2 have non-real eigenvalues. It must be that all four modes are transitive in the same direction, say negative, by $A_1 = A_2 = B_1$, $A_3 = A_4 = B_2$ and by well-posedness of (19). It is easy to compute, by Fact 5 and by the expression in (18), that

$$F_1 + F_2 = \frac{\sigma_1}{\omega_1}\pi, F_3 + F_4 = \frac{\sigma_2}{\omega_2}\pi,$$

which implies by Theorem 1 that the four-modal system, and therefore (19), is stable if and only if $\frac{\sigma_1}{\omega_1} + \frac{\sigma_2}{\omega_2} < 0$. □

Example 1: Consider the bimodal system

$$A_1 = \begin{bmatrix} 6 & 3 \\ -3 & 6 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, c^T = [0 \quad 1]$$

This system is unstable by Corollary 1 since $\frac{\sigma_1}{\omega_1} + \frac{\sigma_2}{\omega_2} = \frac{6}{3} + \frac{-0.5}{\sqrt{3/4}} > 0$. Stability can be achieved, for instance, by simply adding a half-sink mode of arbitrarily narrow sector. Let $a \neq 0$. Consider the three-modal system, $A_3 = \begin{bmatrix} -2 & 1/a \\ a & -2 \end{bmatrix}$, with A_1, A_2 as given above and $C_1 = \begin{bmatrix} 1 & 0.5/a \\ 0 & 0.5/a \end{bmatrix}$, $C_2 = \begin{bmatrix} 0 & -0.5/a \\ 1 & -0.5/a \end{bmatrix}$, $C_3 = \begin{bmatrix} -0.5 & 0.25/a \\ -0.5 & -0.25/a \end{bmatrix}$. Then, $\lambda_{31} = -1$, $\lambda_{32} = -3$, $\mathbf{v}_{31}^T = [-1 \quad -a]$, and $\mathbf{v}_{32}^T = [-1 \quad a]$. By Fact 3, Mode 3 is a half-sink since the eigenvectors are both interior to \mathbf{S}_3 . Since $\lambda_{31} < 0$, it follows, by Theorem 1, that the three modal system is stable for all $a > 0$. Figure 4 illustrates the sectors of such a system and a trajectory for the value $a = 0.05$.

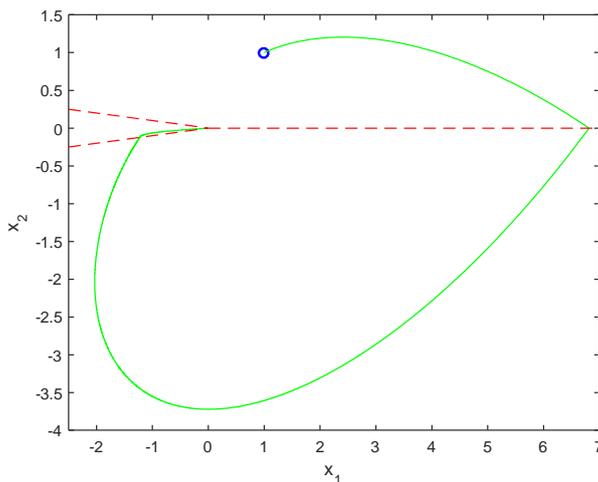


Figure 4. A typical trajectory of a three-modal system of Example 1

3. Conclusions

We have derived a new set of necessary and sufficient conditions for the stability of a planar piecewise linear system. An obvious question is whether this helps in addressing the stability question in higher dimensions, like (1) in which $A_i, C_i \in \mathbf{R}^{n \times n}$, with $n \geq 3$. It is by now known that in, e.g. 3D, the condition of stability is highly “parameter dependent,” which makes a progress difficult (Eldem & Öner, 2015; Şahan& Eldem, 2015). Nevertheless, we have been able to obtain some encouraging results using the approach presented in this paper.

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