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# SUP $\times$ INF INEQUALITY FOR AN ELLIPTIC EQUATION.

SAMY SKANDER BAHOURA

ABSTRACT. We give a sup  $\times$  inf inequality for an equation with nonlinear perturbation with Serrin exponent.

## 1. INTRODUCTION AND MAIN RESULTS

We are on Riemannian manifold  $(M, g)$  of dimension  $n \geq 3$ . In this paper we denote  $\Delta_g = -\nabla^j(\nabla_j)$  the Laplace-Beltrami operator and  $N = \frac{2n}{n-2}$ .

We consider the following equation

$$(1) \quad \Delta_g u = V u^{N-1} + W u^{n/(n-2)}, \quad u > 0.$$

Where  $V$  is a function.

For  $a, b, c, d, (A_i), (B_i) > 0$ , we consider a sequence  $(u_i, V_i, W_i)_i$  of solutions of the previous equation with the following conditions:

$$0 < a \leq V_i \leq b < +\infty,$$

$$\|\nabla V_i\|_\infty \leq A_i \rightarrow 0.$$

$$0 < c \leq W_i \leq d < +\infty,$$

$$\|\nabla W_i\|_\infty \leq B_i.$$

Here we study some properties of this nonlinear elliptic equation. We try to find some estimates of type sup  $\times$  inf. We denote by  $S_g$  the scalar curvature.

There are many existence and compactness results which concern this type of equations, see for example [1-21]. In particular in [1], we can find some results about the Yamabe equation and the Prescribed scalar curvature equation. Many methods were used to solve these problems, as a variational approach and some other topological methods. Note that the problems come from the nonlinearity of the critical Sobolev exponent. We can find in [1] some uniform estimates for various equations on the unit sphere or for the Monge-Ampere equation. Note that Tian and Siu proved uniform upper and lower bounds for the sup + inf for the Monge-Ampere equation under some condition on the Chern class, see [1]. In the case of the Scalar curvature equation and in dimension 2 Shafirir used the isoperimetric inequality of Alexandrov to prove an inequality of type sup + inf with only  $L^\infty$  assumption on the prescribed curvature, see [21]. The result of Shafirir is an extension of a result of Brezis and Merle, see [4] and later, Brezis-Li-Shafirir proved a sharp sup + inf inequality for the same equation with Lipschitzian assumption on the prescribed scalar curvature, see [3]. Li in [17] extend the previous last result to compact Riemannian surfaces. In the higher dimensional case, we can find in [15] a proof of the sup  $\times$  inf inequality in the constant case for the scalar curvature equation on open set of  $\mathbb{R}^n$ . We have various estimates in [2] when we consider the nonconstant case. To prove our result, we use a blow-up analysis and the moving-plane method, based on the maximum principle and the Hopf Lemma as showed in [2, 3, 15, 17], and a condition on the scalar curvature is sufficient to prove the estimate.

Our main result is:

**Theorem 1.1.** *Assume  $S_g > 0$  on  $M$ , then, for every compact  $K$  of  $M$ , there exist a positive constant  $c' = c'(a, b, c, d, (A_i), (B_i), K, M, n, g)$  such that:*

$$\sup_K u_i \times \inf_M u_i \leq c.$$

**Remark:** in the case where  $(M, g) = (\Omega \subset \mathbb{R}^n, \delta)$  an open set of the euclidean space with the flat metric, we have the same inequality on compact sets of  $\Omega$  in this case the scalar curvature  $S_\delta \equiv 0$ , see a previous print.

If we consider the Green function  $G$  of the Laplacian with Dirichlet condition on small balls of  $M$ , we can have a positive lower bound for  $G$  and we have the following corollary:

**Corollary 1.2.** *Assume  $S_g > 0$  on  $M$ , then, for every compact  $K$  of  $M$ , there exist a positive constant  $c'' = c''(a, b, c, d, (A_i), (B_i), K, M, n, g)$  such that:*

$$\int_K u_i^{\frac{2n}{n-2}} dv_g \leq c''.$$

## 2. PROOF OF THE THEOREM.

Let us consider  $x_0 \in M$ , by a conformal change of the metric  $\tilde{g} = \varphi^{4/(n-2)}g$  with  $\varphi > 0$  we can consider an equation of type:

$$(2) \quad \Delta_{\tilde{g}}u + R_{\tilde{g}}u = Vu^{N-1} + W\varphi^{-2/(n-2)}u^{n/(n-2)} + R_g\varphi^{2-N}u, \quad u > 0.$$

with,

$$\text{Ricci}_{\tilde{g}}(x_0) = 0.$$

$$\text{Here; } R_g = \frac{n-2}{4(n-1)}S_g \text{ and } R_{\tilde{g}} = \frac{n-2}{4(n-1)}S_{\tilde{g}}$$

### **Part I: The metric in polar coordinates.**

Let  $(M, g)$  a Riemannian manifold. We note  $g_{x,ij}$  the local expression of the metric  $g$  in the exponential map centered in  $x$ .

We are concerning by the polar coordinates expression of the metric. Using Gauss lemma, we can write:

$$g = ds^2 = dt^2 + g_{ij}^k(r, \theta)d\theta^i d\theta^j = dt^2 + r^2 \tilde{g}_{ij}^k(r, \theta)d\theta^i d\theta^j = g_{x,ij} dx^i dx^j,$$

in a polar chart with origin  $x$ ,  $]0, \epsilon_0[ \times U^k$ , with  $(U^k, \psi)$  a chart of  $\mathbb{S}_{n-1}$ . We can write the element volume:

$$dV_g = r^{n-1} \sqrt{|\tilde{g}^k|} dr d\theta^1 \dots d\theta^{n-1} = \sqrt{[\det(g_{x,ij})]} dx^1 \dots dx^n,$$

then,

$$dV_g = r^{n-1} \sqrt{[\det(g_{x,ij})]} [\exp_x(r\theta)] \alpha^k(\theta) dr d\theta^1 \dots d\theta^{n-1},$$

where,  $\alpha^k$  is such that,  $d\sigma_{\mathbb{S}_{n-1}} = \alpha^k(\theta) d\theta^1 \dots d\theta^{n-1}$ . (Riemannian volume element of the sphere in the chart  $(U^k, \psi)$ ).

Then,

$$\sqrt{|\tilde{g}^k|} = \alpha^k(\theta) \sqrt{[\det(g_{x,ij})]}.$$

Clearly, we have the following proposition:

**Proposition 2.1.** *Let  $x_0 \in M$ , there exist  $\epsilon_1 > 0$  and if we reduce  $U^k$ , we have:*

$$|\partial_r \tilde{g}_{ij}^k(x, r, \theta)| + |\partial_r \partial_{\theta^m} \tilde{g}_{ij}^k(x, r, \theta)| \leq Cr, \quad \forall x \in B(x_0, \epsilon_1) \quad \forall r \in [0, \epsilon_1], \quad \forall \theta \in U^k.$$

and,

$$|\partial_r |\tilde{g}^k|(x, r, \theta)| + \partial_r \partial_{\theta^m} |\tilde{g}^k|(x, r, \theta)| \leq Cr, \quad \forall x \in B(x_0, \epsilon_1) \quad \forall r \in [0, \epsilon_1], \quad \forall \theta \in U^k.$$

**Remark:**

$\partial_r[\log \sqrt{|\tilde{g}^k|}]$  is a local function of  $\theta$ , and the restriction of the global function on the sphere  $\mathbb{S}_{n-1}$ ,  $\partial_r[\log \sqrt{\det(g_{x,ij})}]$ . We will note,  $J(x, r, \theta) = \sqrt{\det(g_{x,ij})}$ .

**Part II: The laplacian in polar coordinates**

Let's write the laplacian in  $[0, \epsilon_1] \times U^k$ ,

$$-\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r[\log \sqrt{|\tilde{g}^k|}] \partial_r + \frac{1}{r^2 \sqrt{|\tilde{g}^k|}} \partial_{\theta^i} (\tilde{g}^{\theta^i \theta^j} \sqrt{|\tilde{g}^k|} \partial_{\theta^j}).$$

We have,

$$-\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r \log J(x, r, \theta) \partial_r + \frac{1}{r^2 \sqrt{|\tilde{g}^k|}} \partial_{\theta^i} (\tilde{g}^{\theta^i \theta^j} \sqrt{|\tilde{g}^k|} \partial_{\theta^j}).$$

We write the laplacian ( radial and angular decomposition),

$$-\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r[\log J(x, r, \theta)] \partial_r - \Delta_{S_r(x)},$$

where  $\Delta_{S_r(x)}$  is the laplacian on the sphere  $S_r(x)$ .

We set  $L_\theta(x, r)(\dots) = r^2 \Delta_{S_r(x)}(\dots)[\exp_x(r\theta)]$ , clearly, this operator is a laplacian on  $\mathbb{S}_{n-1}$  for particular metric. We write,

$$L_\theta(x, r) = \Delta_{g_{x,r,\mathbb{S}_{n-1}}},$$

and,

$$\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r[J(x, r, \theta)] \partial_r - \frac{1}{r^2} L_\theta(x, r).$$

If,  $u$  is function on  $M$ , then,  $\bar{u}(r, \theta) = u[\exp_x(r\theta)]$  is the corresponding function in polar coordinates centered in  $x$ . We have,

$$(3) \quad -\Delta u = \partial_{rr} \bar{u} + \frac{n-1}{r} \partial_r \bar{u} + \partial_r[J(x, r, \theta)] \partial_r \bar{u} - \frac{1}{r^2} L_\theta(x, r) \bar{u}.$$

**Part III: "Blow-up" and "Moving-plane" methods**

**The "blow-up" analysis**

Let,  $(u_i)_i$  a sequence of functions on  $M$  such that,

$$(4) \quad \Delta_{\tilde{g}} u_i + R_{\tilde{g}} u_i = V_i u_i^{N-1} + W_i \varphi^{-2/(n-2)} u_i^{n/(n-2)} + R_g \varphi^{2-N} u_i, \quad u_i > 0, \quad N = \frac{2n}{n-2},$$

We argue by contradiction and we suppose that  $\sup \times \inf$  is not bounded.

We assume that:

$\forall c, R > 0 \exists u_{c,R}$  solution of  $(E)$  such that:

$$R^{n-2} \sup_{B(x_0, R)} u_{c,R} \times \inf_M u_{c,R} \geq c. \quad (H)$$

**Proposition 2.2.** *There exist a sequence of points  $(y_i)_i$ ,  $y_i \rightarrow x_0$  and two sequences of positive real number  $(l_i)_i, (L_i)_i$ ,  $l_i \rightarrow 0$ ,  $L_i \rightarrow +\infty$ , such that if we consider  $v_i(y) = \frac{u_i[\exp_{y_i}(y)]}{u_i(y_i)}$ , we have:*

$$\begin{aligned} i) \quad & 0 < v_i(y) \leq \beta_i \leq 2^{(n-2)/2}, \beta_i \rightarrow 1. \\ ii) \quad & v_i(y) \rightarrow \left( \frac{1}{1+|y|^2} \right)^{(n-2)/2}, \text{ uniformly on every compact set of } \mathbb{R}^n. \\ iii) \quad & l_i^{(n-2)/2} [u_i(y_i)] \times \inf_M u_i \rightarrow +\infty \end{aligned}$$

**Proof:**

Without loss of generality, we can assume that:

$$V(x_0) = n(n-2).$$

We use the hypothesis (H). We can take two sequences  $R_i > 0$ ,  $R_i \rightarrow 0$  and  $c_i \rightarrow +\infty$ , such that,

$$R_i^{(n-2)} \sup_{B(x_0, R_i)} u_i \times \inf_M u_i \geq c_i \rightarrow +\infty.$$

Let,  $x_i \in B(x_0, R_i)$ , such that  $\sup_{B(x_0, R_i)} u_i = u_i(x_i)$  and  $s_i(x) = [R_i - d(x, x_i)]^{(n-2)/2} u_i(x)$ ,  $x \in B(x_i, R_i)$ . Then,  $x_i \rightarrow x_0$ .

We have,

$$\max_{B(x_i, R_i)} s_i(x) = s_i(y_i) \geq s_i(x_i) = R_i^{(n-2)/2} u_i(x_i) \geq \sqrt{c_i} \rightarrow +\infty.$$

Set :

$$l_i = R_i - d(y_i, x_i), \bar{u}_i(y) = u_i[\exp_{y_i}(y)], v_i(z) = \frac{u_i[\exp_{y_i}(z/[u_i(y_i)]^{2/(n-2)})]}{u_i(y_i)}.$$

Clearly,  $y_i \rightarrow x_0$ . We obtain:

$$L_i = \frac{l_i}{(c_i)^{1/2(n-2)}} [u_i(y_i)]^{2/(n-2)} = \frac{[s_i(y_i)]^{2/(n-2)}}{c_i^{1/2(n-2)}} \geq \frac{c_i^{1/(n-2)}}{c_i^{1/2(n-2)}} = c_i^{1/2(n-2)} \rightarrow +\infty.$$

If  $|z| \leq L_i$ , then  $y = \exp_{y_i}[z/[u_i(y_i)]^{2/(n-2)}] \in B(y_i, \delta_i l_i)$  with  $\delta_i = \frac{1}{(c_i)^{1/2(n-2)}}$  and  $d(y, y_i) < R_i - d(y_i, x_i)$ , thus,  $d(y, x_i) < R_i$  and,  $s_i(y) \leq s_i(y_i)$ , we can write,

$$u_i(y)[R_i - d(y, y_i)]^{(n-2)/2} \leq u_i(y_i)(l_i)^{(n-2)/2}.$$

But,  $d(y, y_i) \leq \delta_i l_i$ ,  $R_i > l_i$  and  $R_i - d(y, y_i) \geq R_i - \delta_i l_i > l_i - \delta_i l_i = l_i(1 - \delta_i)$ , we obtain,

$$0 < v_i(z) = \frac{u_i(y)}{u_i(y_i)} \leq \left[ \frac{l_i}{l_i(1 - \delta_i)} \right]^{(n-2)/2} \leq 2^{(n-2)/2}.$$

We set,  $\beta_i = \left( \frac{1}{1 - \delta_i} \right)^{(n-2)/2}$ , clearly  $\beta_i \rightarrow 1$ .

The function  $v_i$  is solution of:

$$-g^{jk}[\exp_{y_i}(y)] \partial_{jk} v_i - \partial_k \left[ g^{jk} \sqrt{|g|} \right] [\exp_{y_i}(y)] \partial_j v_i = \frac{W_i \varphi^{-2/(n-2)}}{[u_i(y_i)]^{2/(n-2)}} v_i^{n/(n-2)} + \frac{\mu \varphi^{-4/(n-2)}}{[u_i(y_i)]^{4/(n-2)}} v_i + V_i v_i^{N-1},$$

By elliptic estimates and Ascoli, Ladyzenskaya theorems,  $(v_i)_i$  converge uniformly on each compact to the function  $v$  solution on  $\mathbb{R}^n$  of,

$$(5) \quad \Delta v = n(n-2)v^{N-1}, \quad v(0) = 1, \quad 0 \leq v \leq 1 \leq 2^{(n-2)/2},$$

By using maximum principle, we have  $v > 0$  on  $\mathbb{R}^n$ , the result of Caffarelli-Gidas-Spruck (see [6]) give,  $v(y) = \left(\frac{1}{1+|y|^2}\right)^{(n-2)/2}$ . We have the same properties for  $v_i$  in the previous paper [2].

### **Polar coordinates and "moving-plane" method**

Let,

$$w_i(t, \theta) = e^{(n-2)/2} \bar{u}_i(e^t, \theta) = e^{(n-2)t/2} u_i \circ \exp_{y_i}(e^t \theta), \quad \text{et } a(y_i, t, \theta) = \log J(y_i, e^t, \theta).$$

**Lemma 2.3.** *The function  $w_i$  is solution of:*

$$(6) \quad -\partial_{tt} w_i - \partial_t a \partial_t w_i - L_\theta(y_i, e^t) + c w_i = V_i w_i^{N-1} + e^t W_i \varphi^{-2/(n-2)} w_i^{n/(n-2)} + \mu e^{2t} \varphi^{-4/(n-2)} w_i,$$

with,

$$c = c(y_i, t, \theta) = \left(\frac{n-2}{2}\right)^2 + \frac{n-2}{2} \partial_t a + R_g e^{2t}.$$

### **Proof:**

We write:

$$\partial_t w_i = e^{nt/2} \partial_r \bar{u}_i + \frac{n-2}{2} w_i, \quad \partial_{tt} w_i = e^{(n+2)t/2} \left[ \partial_{rr} \bar{u}_i + \frac{n-1}{e^t} \partial_r \bar{u}_i \right] + \left(\frac{n-2}{2}\right)^2 w_i.$$

$$\partial_t a = e^t \partial_r \log J(y_i, e^t, \theta), \quad \partial_t a \partial_t w_i = e^{(n+2)t/2} [\partial_r \log J \partial_r \bar{u}_i] + \frac{n-2}{2} \partial_t a w_i.$$

the lemma is proved.

$$\text{Now we have, } \partial_t a = \frac{\partial_t b_1}{b_1}, \quad b_1(y_i, t, \theta) = J(y_i, e^t, \theta) > 0,$$

We can write,

$$-\frac{1}{\sqrt{b_1}} \partial_{tt} (\sqrt{b_1} w_i) - L_\theta(y_i, e^t) w_i + [c(t) + b_1^{-1/2} b_2(t, \theta)] w_i = V_i w_i^{N-1} + e^t W_i \varphi^{-2/(n-2)} w_i^{n/(n-2)} + \mu e^{2t} \varphi^{-4/(n-2)} w_i,$$

$$\text{where, } b_2(t, \theta) = \partial_{tt} (\sqrt{b_1}) = \frac{1}{2\sqrt{b_1}} \partial_{tt} b_1 - \frac{1}{4(b_1)^{3/2}} (\partial_t b_1)^2.$$

Let,

$$\tilde{w}_i = \sqrt{b_1} w_i,$$

**Lemma 2.4.** *The function  $\tilde{w}_i$  is solution of:*

$$(7) \quad -\partial_{tt} \tilde{w}_i + \Delta_{g_{y_i, e^t, \mathbb{S}_{n-1}}}(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i) \cdot \nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2} b_2 - c_2) \tilde{w}_i =$$

$$V_i \left(\frac{1}{b_1}\right)^{(N-2)/2} \tilde{w}_i^{N-1} + e^t W_i \varphi^{-2/(n-2)} \left(\frac{1}{b_1}\right)^{1/(n-2)} \tilde{w}_i^{n/(n-2)} + \mu e^{2t} \varphi^{-4/(n-2)} \tilde{w}_i,$$

where,  $c_2 = \left[\frac{1}{\sqrt{b_1}} \Delta_{g_{y_i, e^t, \mathbb{S}_{n-1}}}(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2\right]$ .

**Proof:**

We have:

$$\begin{aligned} -\partial_{tt}\tilde{w}_i - \sqrt{b_1}\Delta_\theta w_i + (c + b_2)\tilde{w}_i &= V_i \left(\frac{1}{b_1}\right)^{(N-2)/2} \tilde{w}_i^{N-1} + \\ &+ e^t W_i \varphi^{-2/(n-2)} \left(\frac{1}{b_1}\right)^{1/(n-2)} \tilde{w}_i^{n/(n-2)} + \mu e^{2t} \varphi^{-4/(n-2)} \tilde{w}_i. \end{aligned}$$

But,

$$\Delta_\theta(\sqrt{b_1}w_i) = \sqrt{b_1}\Delta_{g_{y_i, e^t, \mathbb{S}_{n-1}}} w_i - 2\nabla_\theta w_i \cdot \nabla_\theta \sqrt{b_1} + w_i \Delta_{g_{y_i, e^t, \mathbb{S}_{n-1}}}(\sqrt{b_1}),$$

and,

$$\nabla_\theta(\sqrt{b_1}w_i) = w_i \nabla_\theta \sqrt{b_1} + \sqrt{b_1} \nabla_\theta w_i,$$

we deduce than,

$$\sqrt{b_1}\Delta_\theta w_i = \Delta_\theta(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i) \cdot \nabla_\theta \log(\sqrt{b_1}) - c_2 \tilde{w}_i,$$

with  $c_2 = [\frac{1}{\sqrt{b_1}}\Delta_{g_{y_i, e^t, \mathbb{S}_{n-1}}}(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2]$ . The lemma is proved.

**The "moving-plane" method:**

Let  $\xi_i$  a real number, and suppose  $\xi_i \leq t$ . We set  $t^{\xi_i} = 2\xi_i - t$  and  $\tilde{w}_i^{\xi_i}(t, \theta) = \tilde{w}_i(t^{\xi_i}, \theta)$ .

We have,

$$\begin{aligned} -\partial_{tt}\tilde{w}_i^{\xi_i} + \Delta_\theta(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i^{\xi_i}) \cdot \nabla_\theta \log(\sqrt{b_1})\tilde{w}_i^{\xi_i} + [c(t^{\xi_i}) + b_1^{-1/2}(t^{\xi_i}, \cdot)b_2(t^{\xi_i}) - c_2^{\xi_i}]\tilde{w}_i^{\xi_i} &= \\ &= V_i \left(\frac{1}{b_1^{\xi_i}}\right)^{(N-2)/2} (\tilde{w}_i^{\xi_i})^{N-1} + \\ &+ e^t W_i \varphi^{-2/(n-2)} \left(\frac{1}{b_1}\right)^{1/(n-2)} \tilde{w}_i^{n/(n-2)} + \mu e^{2t} \varphi^{-4/(n-2)} \tilde{w}_i. \end{aligned}$$

By using the same arguments than in [2], we have:

**Proposition 2.5.** *We have:*

$$1) \tilde{w}_i(\lambda_i, \theta) - \tilde{w}_i(\lambda_i + 4, \theta) \geq \tilde{k} > 0, \forall \theta \in \mathbb{S}_{n-1}.$$

For all  $\beta > 0$ , there exist  $c_\beta > 0$  such that:

$$2) \frac{1}{c_\beta} e^{(n-2)t/2} \leq \tilde{w}_i(\lambda_i + t, \theta) \leq c_\beta e^{(n-2)t/2}, \forall t \leq \beta, \forall \theta \in \mathbb{S}_{n-1}.$$

We set,

$$\bar{Z}_i = -\partial_{tt}(\dots) + \Delta_\theta(\dots) + 2\nabla_\theta(\dots) \cdot \nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2}b_2 - c_2)(\dots)$$

**Remark:** In the operator  $\bar{Z}_i$ , by using the proposition 3, the coefficient  $c + b_1^{-1/2}b_2 - c_2$  satisfies:

$$c + b_1^{-1/2}b_2 - c_2 \geq k' > 0, \text{ pour } t \ll 0,$$

it is fundamental if we want to apply Hopf maximum principle.

**Goal:**

Like in [2], we have elliptic second order operator. Here it is  $\bar{Z}_i$ , the goal is to use the "moving-plane" method to have a contradiction. For this, we must have:

$$\bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) \leq 0, \text{ if } \tilde{w}_i^{\xi_i} - \tilde{w}_i \leq 0.$$

We write:

$$\begin{aligned} \bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) &= (\Delta_\theta^{\xi_i} - \Delta_\theta)\tilde{w}_i^{\xi_i} + \\ &+ 2(\nabla_{\theta, e^{t\xi_i}} - \nabla_{\theta, e^t})(w_i^{\xi_i}) \cdot \nabla_{\theta, e^{t\xi_i}} \log(\sqrt{b_1^{\xi_i}}) + 2\nabla_{\theta, e^t}(\tilde{w}_i^{\xi_i}) \cdot \nabla_{\theta, e^{t\xi_i}} [\log(\sqrt{b_1^{\xi_i}}) - \log \sqrt{b_1}] + \\ &+ 2\nabla_{\theta, e^t} w_i^{\xi_i} \cdot (\nabla_{\theta, e^{t\xi_i}} - \nabla_{\theta, e^t}) \log \sqrt{b_1} - [(c + b_1^{-1/2}b_2 - c_2)^{\xi_i} - (c + b_1^{-1/2}b_2 - c_2)]\tilde{w}_i^{\xi_i} + \\ &\quad + V_i^{\xi_i} \left( \frac{1}{b_1^{\xi_i}} \right)^{(N-2)/2} (\tilde{w}_i^{\xi_i})^{N-1} - V_i \left( \frac{1}{b_1} \right)^{(N-2)/2} \tilde{w}_i^{N-1} + \\ &+ e^{t\xi_i} (W_i \varphi^{-2/(n-2)})^{\xi_i} b_1^{\xi_i^{-1/(n-2)}} (\tilde{w}_i^{\xi_i})^{n/(n-2)} - e^t (W_i \varphi^{-2/(n-2)}) b_1^{-1/(n-2)} (\tilde{w}_i)^{n/(n-2)} + \\ &\quad + \mu(e^{2t\xi_i} (\varphi^{-4/(n-2)})^{\xi_i} \tilde{w}_i^{\xi_i} - e^{2t} \varphi^{-4/(n-2)} \tilde{w}_i) \quad (***) \end{aligned}$$

Clearly, we have:

**Lemma 2.6.**

$$b_1(y_i, t, \theta) = 1 - \frac{1}{3} \text{Ricci}_{y_i}(\theta, \theta) e^{2t} + \dots,$$

$$R_g(e^t \theta) = R_g(y_i) + \langle \nabla R_g(y_i) | \theta \rangle e^t + \dots$$

According to proposition 1 and lemma 3,

**Proposition 2.7.**

$$\begin{aligned} \bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) &\leq \tilde{A}_i(e^t - e^{t\xi_i})(\tilde{w}_i^{\xi_i})^{N-1} + (1/2)(e^{t\xi_i} - e^t)(\tilde{w}_i^{\xi_i})^{n/(n-2)} + \\ &+ C[e^{2t} - e^{2t\xi_i}] \left[ |\nabla_\theta \tilde{w}_i^{\xi_i}| + |\nabla_\theta^2(\tilde{w}_i^{\xi_i})| + o(1)[\tilde{w}_i^{\xi_i} + (\tilde{w}_i^{\xi_i})^{N-1}] + \varphi^{-2/(n-2)} (\tilde{w}_i^{\xi_i})^{n/(n-2)} + \mu \tilde{w}_i^{\xi_i} \right]. \end{aligned}$$

**Proof:**

We use proposition 1, we have:

$$a(y_i, t, \theta) = \log J(y_i, e^t, \theta) = \log b_1, |\partial_t b_1(t)| + |\partial_{tt} b_1(t)| + |\partial_{tt} a(t)| \leq C e^{2t},$$

and,

$$|\partial_{\theta_j} b_1| + |\partial_{\theta_j, \theta_k} b_1| + |\partial_{t, \theta_j} b_1| + |\partial_{t, \theta_j, \theta_k} b_1| \leq C e^{2t},$$

then,

$$|\partial_t b_1(t^{\xi_i}) - \partial_t b_1(t)| \leq C' |e^{2t} - e^{2t\xi_i}|, \text{ on } ]-\infty, \log \epsilon_1] \times \mathbb{S}_{n-1}, \forall x \in B(x_0, \epsilon_1)$$

Locally,

$$\Delta_\theta = L_\theta(y_i, e^t) = -\frac{1}{\sqrt{|\tilde{g}^k(e^t, \theta)|}} \partial_{\theta^i} [\tilde{g}^{\theta^i \theta^j}(e^t, \theta) \sqrt{|\tilde{g}^k(e^t, \theta)|} |\partial_{\theta^j}|].$$

Thus, in  $[0, \epsilon_1] \times U^k$ , we have,

$$A_i = \left[ \left[ \frac{1}{\sqrt{|\tilde{g}^k|}} \partial_{\theta^i} (\tilde{g}^{\theta^i \theta^j} \sqrt{|\tilde{g}^k|} |\partial_{\theta^j}|) \right]^{\xi_i} - \frac{1}{\sqrt{|\tilde{g}^k|}} \partial_{\theta^i} (\tilde{g}^{\theta^i \theta^j} \sqrt{|\tilde{g}^k|} |\partial_{\theta^j}|) \right] (\tilde{w}_i^{\xi_i})$$

then,  $A_i = B_i + D_i$  with,



$$B_i = \left[ \tilde{g}^{\theta^i \theta^j}(e^{t^{\xi_i}}, \theta) - \tilde{g}^{\theta^i \theta^j}(e^t, \theta) \right] \partial_{\theta^i \theta^j} \tilde{w}_i^{\xi_i}(t, \theta),$$

and,

$$D_i = \left[ \frac{1}{\sqrt{|\tilde{g}^k|}(e^{t^{\xi_i}}, \theta)} \partial_{\theta^i} [\tilde{g}^{\theta^i \theta^j}(e^{t^{\xi_i}}, \theta) \sqrt{|\tilde{g}^k|}(e^{t^{\xi_i}}, \theta)] - \frac{1}{\sqrt{|\tilde{g}^k|}(e^t, \theta)} \partial_{\theta^i} [\tilde{g}^{\theta^i \theta^j}(e^t, \theta) \sqrt{|\tilde{g}^k|}(e^t, \theta)] \right] \partial_{\theta^i} \tilde{w}_i^{\xi_i}(t, \theta),$$

we deduce,

$$A_i \leq C_k |e^{2t} - e^{2t^{\xi_i}}| \left[ |\nabla_{\theta} \tilde{w}_i^{\xi_i}| + |\nabla_{\theta}^2(\tilde{w}_i^{\xi_i})| \right],$$

We take  $C = \max\{C_i, 1 \leq i \leq q\}$  and if we use (\*\*\*) , we obtain proposition 4.

We have:

$$\frac{\partial_{\theta_j} w_i^\lambda(t, \theta)}{w_i^\lambda} = \frac{e^{(n-2)[(\lambda-\lambda_i)+(\xi_i-t)]/2} e^{[(\lambda-\lambda_i)+(\xi_i-t)]} (\partial_{\theta_j} v_i)(e^{[(\lambda-\lambda_i)+(\lambda-t)]\theta})}{e^{(n-2)[(\lambda-\lambda_i)+(\lambda-t)]/2} v_i[e^{(\lambda-\lambda_i)+(\lambda-t)\theta}]} \leq \bar{C}_i,$$

and,

$$\frac{\partial_{\theta_j, \theta_i} w_i^\lambda(t, \theta)}{w_i^\lambda} = \frac{e^{(n-2)[(\lambda-\lambda_i)+(\xi_i-t)]/2} e^{2[(\lambda-\lambda_i)+(\xi_i-t)]} (\partial_{\theta_j, \theta_i} v_i)(e^{[(\lambda-\lambda_i)+(\lambda-t)]\theta})}{e^{(n-2)[(\lambda-\lambda_i)+(\lambda-t)]/2} v_i[e^{(\lambda-\lambda_i)+(\lambda-t)\theta}]} \leq \bar{C}_i,$$

with  $\bar{C}_i$  tending to 0 and does not depend on  $\lambda \leq \lambda_i + 2$ .

We have,

$$c(y_i, t, \theta) = \left( \frac{n-2}{2} \right)^2 + \frac{n-2}{2} \partial_t a + R_g e^{2t}, \quad (\alpha_1)$$

$$b_2(t, \theta) = \partial_{tt}(\sqrt{b_1}) = \frac{1}{2\sqrt{b_1}} \partial_{tt} b_1 - \frac{1}{4(b_1)^{3/2}} (\partial_t b_1)^2, \quad (\alpha_2)$$

$$c_2 = \left[ \frac{1}{\sqrt{b_1}} \Delta_{g_{y_i, e^t, s_{n-1}}}(\sqrt{b_1}) + |\nabla_{\theta} \log(\sqrt{b_1})|^2 \right], \quad (\alpha_3)$$

Then,

$$\partial_t c(y_i, t, \theta) = \frac{(n-2)}{2} \partial_{tt} a,$$

by proposition 1,

$$|\partial_t c_2| + |\partial_t b_1| + |\partial_t b_2| + |\partial_t c| \leq K_1 e^{2t}.$$

We have:

$$w_i(2\xi_i - t, \theta) = w_i[(\xi_i - t + \xi_i - \lambda_i - 2) + (\lambda_i + 2)],$$

Thus,

$$w_i(2\xi_i - t, \theta) = e^{[(n-2)(\xi_i - t + \xi_i - \lambda_i - 2)]/2} e^{n-2} v_i[\theta e^2 e^{(\xi_i - t) + (\xi_i - \lambda_i - 2)}] \leq 2^{(n-2)/2} e^{n-2} = \bar{c}.$$

$$\text{We set } \delta = \frac{(n+2) - (n-2)\alpha}{2}.$$

The left right side are denoted  $Z_1$  et  $Z_2$ , we can write:

$$Z_1 = (\bar{V}_i^{\xi_i} - \bar{V}_i)(\tilde{w}_i^{\xi_i})^{N-1} + \bar{V}_i[(\tilde{w}_i^{\xi_i})^{N-1} - \tilde{w}_i^{N-1}],$$

and,

$$Z_2 = e^t \varphi^{-2/(n-2)} [(\tilde{w}_i^{\xi_i})^{n/(n-2)} - (\tilde{w}_i)^{n/(n-2)}] + e^{t^{\xi_i}} (\tilde{w}_i^{\xi_i})^{n/(n-2)} [(W_i \varphi^{-2/(n-2)})^{\xi_i} - W_i \varphi^{-2/(n-2)}] + \\ + \varphi^{-2/(n-2)} (\tilde{w}_i^{\xi_i})^{n/(n-2)} (e^{t^{\xi_i}} - e^t).$$

We can write the part with nonlinear terms as:

$$(\tilde{w}_i^{\xi_i})^{n/(n-2)}[(A_i w_i^{\xi_i 2/(n-2)} + o(1))(e^t - e^{t^{\xi_i}}) + c(e^{t^{\xi_i}} - e^t)].$$

where  $o(1) = e^{t^{\xi_i}} \rightarrow 0$  come from the difference  $[W_i \varphi^{-2/(n-2)}]^{\xi_i} - [W_i \varphi^{-2/(n-2)}]$ .  
Because  $w_i^{\xi_i} \leq \bar{c}$ , we have:

$$-\bar{Z}_i((\tilde{w}_i)^{\xi_i} - (\tilde{w}_i)) \leq (\tilde{w}_i^{\xi_i})^{n/(n-2)}[(A_i \bar{c}^{2/(n-2)} + o(1))(e^t - e^{t^{\xi_i}}) + c(e^{t^{\xi_i}} - e^t)] + (\tilde{w}_i)^{\xi_i} (\mu/2)(e^{2t^{\xi_i}} - e^{2t})$$

where  $o(1) = e^{t^{\xi_i}} \rightarrow 0$ .

Finally:

$$-\bar{Z}_i((\tilde{w}_i)^{\xi_i} - (\tilde{w}_i)) \leq (\tilde{w}_i^{\xi_i})^{n/(n-2)}[-1 + A_i \bar{c}^{2/(n-2)} + o(1)](e^t - e^{t^{\xi_i}}) + (\tilde{w}_i)^{\xi_i} (\mu/2)(e^{2t^{\xi_i}} - e^{2t})$$

We apply proposition 3. We take  $t_i = \log \sqrt{l_i}$  with  $l_i$  like in proposition 2. The fact  $\sqrt{l_i}[u_i(y_i)]^{2/(n-2)} \rightarrow +\infty$  ( see proposition 2), implies  $t_i = \log \sqrt{l_i} > \frac{2}{n-2} \log u_i(y_i) + 2 = \lambda_i + 2$ . Finally, we can work on  $]-\infty, t_i]$ .

We define  $\xi_i$  by:

$$\xi_i = \sup\{\lambda \leq \lambda_i + 2, \tilde{w}_i(2\lambda - t, \theta) - \tilde{w}_i(t, \theta) \leq 0 \text{ on } [\lambda, t_i] \times \mathbb{S}_{n-1}\}.$$

If we use proposition 4 and the similar technics that in [2] we can deduce by Hopf maximum principle,

$$\min_{\mathbb{S}_{n-1}} \tilde{w}_i(t_i, \theta) \leq \max_{\mathbb{S}_{n-1}} \tilde{w}_i(2\xi_i - t_i, \theta),$$

which implies,

$$l_i^{(n-2)/2} u_i(y_i) \times \min_M u_i \leq c.$$

It is in contradiction with proposition 2.

Then we have,

$$\sup_K u \times \inf_M u \leq c = c(a, b, A, K, M, g, n).$$

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