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VARIATIONAL CHARACTERIZATION OF INVISCID INCOMPRESSIBLE FLOWS WITH ABRUPT INHOMOGENEITY OR FREE SURFACE

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1. INTRODUCTION.

The "method of horizontal variations" has been proposed by the author as a way of characterizing variationally the solutions to some field equations. It consists in having the investigated fields (scalar or tensor) transported by an imagined continuous medium, in motion over the concerned region of the space of independent variables. The solutions are then characterized as yielding in the process a zero rate of change for some integral functional.

The imagined continuous medium, called a *carrier* is defined by its velocity field $\underset{\sim}{\varphi}$ and the parameter ordering its evolution is denoted by τ, not to be mistaken for the physical

time t when the latter figures among independent variables. Associating a transport process with some vector field is well known in Differential Geometry as generating *Lie derivatives* (see e.g. [1]) . The vocabulary and procedures we propose seem better adapted to applications.

To date, we have only used the device in analyzing the mathematical structure of the considered equations (see e.g. [2] , which tends to explain in that context the preeminence of measures over other sorts of distributions). Numerical experimentation is planned : the transport will be approximated by making some finite element mesh move, with a velocity field $\underset{\sim}{\varphi}$ possibly dependent on the formal time τ . This is similar to what is currently done in the computation of large deformations of materials (see e.g. [3][4]) and also in problems of optimization of domains [5][6] .

Horizontal variations may also be used in constructing perturbations of known solutions.

In other words, the vector field $\underset{\sim}{\varphi}$ constitutes a *test field* and, when applied to Mechanics, the method may be viewed as an extension of the traditional procedure of "virtual work" or "virtual power".

Application to the dynamics of compressible fluids is presented in [7] .

The material of this lecture is extracted from a paper to appear [8] , devoted to stationary flows of incompressible inviscid fluids, with possible sharp inhomogeneity hence displaying *free boundaries*. Proofs of the statements made in the sequel may be found in it.

2. THE BASIC DEVICE.

Let Ω denote an open subset of the space $\mathbb{R}^n$ of the independent variables. Let $\underset{\sim}{\varphi} \in \mathcal{D}^k(\Omega, \mathbb{R}^n)$, i.e. $\underset{\sim}{\varphi}$ is a vector field, with continuous partial derivatives up to the order k at least and compact support in Ω. This vector field may also depend on the real variable τ, in which case continuous differentiability up to order k is supposed to hold in the $n+1$ variables jointly. With τ playing the role of time, $\underset{\sim}{\varphi}$ may be seen as the Eulerian velocity field of some n-dimensional fluid Λ, called a *carrier*, in motion over Ω. If K is a compact subset of Ω containing (for every τ) the support of $\underset{\sim}{\varphi}$, clearly the portion of fluid contained in $\Omega \setminus K$ remains at rest. For short we shall say that Λ is *a compact* C^k *carrier in* Λ.

Standard theorems on the differentiability of solutions to differential equations with respect to initial data readily yield the following : the carrier Λ, the elements of which are naturally called *particles*, may be equipped with the structure of a C^k-differential manifold, in such a way that the *placement* mapping $\lambda \to \pi_\tau(\lambda)$ is a τ-dependent C^k-diffeomorphism of Λ onto Ω.

In view of this structure, Λ may house various mathematical objects ; every placement π_τ gives of every such object an image, which is an object of the same nature in Ω : as τ ranges through some real interval, the τ-dependent thing so obtained in Ω will be called *a moving object transported by the carrier* Λ.

As first example, consider a τ-dependent C^h real function (h integer, $0 \leqslant h \leqslant k$), say $f^\tau : \Omega \longrightarrow \mathbb{R}$. It is classically said *convected by* Λ if it equals the image under π_τ of some τ-constant real function in Λ, i.e. for every $\lambda \in \Lambda$, the real number $f^\tau(\pi_\tau(\lambda))$ does not depend on τ. Equivalently the *follow-up derivative* $\delta_{\underset{\sim}{\varphi}} f^\tau$ exists, with zero value ; this is the τ-derivative of $f^\tau(\pi_\tau(\lambda))$, with fixed λ (observe that existence of this derivative does not require of f^τ to be differentiable in its proper arguments). When writing in the sequel such expression as $\delta_{\underset{\sim}{\varphi}} f^\tau$, we shall implicitely refer to some value of τ and some point x of Ω ; then $\delta_{\underset{\sim}{\varphi}} f^\tau$ is precisely the τ-derivative of $f^\tau(\pi_\tau(\lambda))$, with λ fixed at the value $\pi_\tau^{-1}(x)$; this complies with common habits in the Mechanics of Continua.

As usual, we equip $\mathbb{R}^n$ with its natural Euclidean structure. Suppose now that f^τ is at least C^1 in Ω, thus possesses a *Euclidean gradient* $\underset{\sim}{g}^\tau = \operatorname{grad} f^\tau$, a τ-dependent vector field in Ω. Equivalently, the function $f^\tau \circ \pi_\tau : \Lambda \longrightarrow \mathbb{R}$ is C^1 in Λ, thus possesses a gradient (or "differential"), a *covector field* of the differential manifold Λ. Clearly, if f^τ is convected, the real function $f^\tau \circ \pi_\tau$ does not depend on τ ; thus the Euclidean vector field $\underset{\sim}{g}^\tau$ in Ω equals the image under π_τ of some τ-constant covector field of Λ : in such a case we shall say that the moving Euclidean vector field is *transvected* by the carrier. Denoting by g_i the components of $\underset{\sim}{g}^\tau$, one may obtain a characterization of such a transport in terms of follow-up derivatives :

(2.1) $$\delta_{\varphi}\, g_i = - \varphi^j{}_{,i}\, g_j$$

where φ^j are the Cartesian components of $\underset{\sim}{\varphi}$ and $\varphi^j{}_{,i}$ the partial derivatives of these components.

Another mode of transport of a Euclidean vector by a moving medium is familiar in Continuum Mechanics : A vector field $\underset{\sim}{v}^\tau$ in Ω is said *convected* by Λ if, for every $\lambda \in \Lambda$, the moving vector $\underset{\sim}{v}^\tau(\pi_\tau(\lambda))$ may be represented, up to the multiplication by some τ-constant infinitely large scalar, as connecting the images under π_τ of a pair of particles infinitely close to λ. If v^i denote the components of $\underset{\sim}{v}^\tau$, this is classically expressed by

(2.2) $$\delta_{\varphi}\, v^i = \varphi^i{}_{,j}\, v^j$$

Clearly if $\underset{\sim}{g}^\tau$ and $\underset{\sim}{v}^\tau$ are respectively a transvected and a convected vector field, the Euclidean scalar product $\underset{\sim}{g}^\tau \cdot \underset{\sim}{v}^\tau$ constitutes a convected real function.

Let us illustrate the above concepts by computational representations. Suppose $n = 2$ and the region Ω of $\mathbb{R}^2$ paved by triangles. In the case of transport by a carrier, the vertices A , B , C of such a triangle move . Suppose the vector field $\underset{\sim}{u}^\tau$ approximated by $\underset{\sim}{U}^\tau$, a single value for the whole triangle. The fact that $\underset{\sim}{u}^\tau$ is a *convected* vector field is approximated by making $\underset{\sim}{U}^\tau$ change with τ in such a way that this vector equals a constant linear combination of vectors $\underset{\sim}{AB}$ and $\underset{\sim}{AC}$ (or, equivalently, of vectors $\underset{\sim}{BA}$ and $\underset{\sim}{BC}$, etc...). In contrast, when

a *transvected* vector field $\underset{\sim}{g}^{\tau}$ is approximated by some vector $\underset{\sim}{G}^{\tau}$ on the considered triangle, this vector value has to vary with τ in such a way that the Euclidean scalar products $\underset{\sim}{G}^{\tau} \cdot \underset{\sim}{AB}$ and $\underset{\sim}{G}^{\tau} \cdot \underset{\sim}{AC}$ remain constant (hence also $\underset{\sim}{G}^{\tau} \cdot \underset{\sim}{BC}$).

3. VECTOR MEASURES.

Scalar, vector or tensor *measures* on a differential manifold such as Λ above, are readily introduced by the *duality* procedure, quite similarly to De Rham's theory of *currents* [9].

For instance, a *vector measure* $\underset{\sim}{\sigma}$ on Λ is, by definition, a real linear functional (satisfying certain continuity requirements ; see e.g. [7], Sect. 4) on the linear space $\mathcal{D}^{\circ}(\Lambda,\Lambda'^{\star})$ of the *continuous covector fields*, with compact support in Λ. The value that such a functional assigns to some element $\underset{\sim}{\theta}$ of $\mathcal{D}^{\circ}(\Lambda,\Lambda'^{\star})$ may be denoted by $\int <\underset{\sim}{\theta}, d\underset{\sim}{\sigma}>$, or also $\int <\underset{\sim}{\theta}, \underset{\sim}{\sigma}'_r> dr$ if dr represents some nonnegative real measure on Λ, relatively to which $\underset{\sim}{\sigma}$ admits a vector *density* $\underset{\sim}{\sigma}'_r \in \mathcal{L}^1_{loc}(\Lambda, dr ; \Lambda')$. Here the angle bracket $<.,.>$ refers to the duality scalar product between covectors and vectors at a point of Λ.

A very usual example is provided by a C^1 curve, i.e. a C^1 mapping $r \longrightarrow \gamma(r)$ of some real interval into Λ (more generally, one could assume only that such a mapping has a bounded

variation). For every r, the derivative $\underset{\sim}{\gamma}'(r)$ is an element of the tangent space to Λ at the point $\gamma(r)$. By assigning to every $\underset{\sim}{\theta} \in \mathcal{D}^0(\Lambda, \Lambda'^{\star})$ the real number $\int \langle \underset{\sim}{\theta}(\gamma(r)), \underset{\sim}{\gamma}'(r) \rangle dr$ one defines a vector measure.

A vector measure $\underset{\sim}{\sigma}$ will be said *divergence-free* if it yields a zero integral for every covector field $\underset{\sim}{\theta}$ which equals the gradient of an element of $\mathcal{D}^1(\Lambda, \mathbb{R})$ (the C^1 real functions in Λ, with compact support). For instance, the vector measure defined as above by a curve is divergence-free in Λ if and only if the curve possesses no end point in Λ, in particular when the curve is a circuit.

As an open subset of $\mathbb{R}^n$, Ω is naturally a differential manifold so that all what precedes applies. In addition, due to Euclidean structure, the tangent and cotangent spaces at every point $x \in \Omega$ merge into a single one, the same for every x, with duality defined by the Euclidean scalar product.

Since every placement π_τ is a C^k diffeomorphism of Λ onto Ω, it induces an isomorphism between the spaces of continuous covector fields, with compact supports, in Λ and Ω respectively ; hence, by transposition, the image under π_τ of a vector measure in Λ is defined, a vector measure in Ω.

A τ-dependent vector measure in Ω will be said *convected* by the carrier Λ if it equals for every τ the image under π_τ of some τ-constant vector measure in Λ. For instance a moving

curve in Ω *dragged-along* by the carrier Λ equals for every τ the image under π_τ of some τ-constant curve of Λ. Thus the vector measure associated with it is convected (for the application of this remark to a generalization of Kelvin- Helmholtz theory in Fluid Dynamics, see [10]) .

Clearly, the image of a divergence-free vector measure under any diffeomorphism is also divergence-free. Hence, for a convected vector measure, vanishing of divergence is a τ-independent property.

The concept of vanishing divergence, for a *vector measure*, is visibly free from any metric structure. Such is not the case, in classical Vector Analysis, for the divergence of a *vector field*. Here is the connection between both concepts : let us denote by ℓ the *Lebesgue measure* in the open subset Ω of the Euclidean space $\mathbb{R}^n$; suppose that some vector measure $\underset{\sim}{m}$ in Ω admits, relatively to ℓ , a *density*, say $\underset{\sim}{u} \in \mathcal{L}^1_{loc}(\Omega, \ell; \mathbb{R}^n)$. The vector measure $\underset{\sim}{m}$ is divergence-free if and only if $\underset{\sim}{u}$ has a zero divergence in the sense of Schwartz distributions ; this in turn reduces to the classical condition $\operatorname{div} \underset{\sim}{u} = 0$ if $\underset{\sim}{u}$ happens to be C^1 .

The following will also be of use in the sequel : suppose that $\underset{\sim}{u}$ depends on τ thus also the vector measure $\underset{\sim}{m} = \underset{\sim}{u}\,\ell$. The latter is found *convected* by the carrier Λ if and only if

$$\delta_\varphi u^i = \varphi^i{}_{,j}\, u^j - \varphi^j{}_{,j}\, u^i \tag{3.1}$$

a condition to distinguish from (2.1) .

Computational approximation of vector measures is quite feasible. For instance, a divergence-free vector measure may be approximated by a finite collection of vector measures associated with curves in the way described in the preceding. Curves in turn will be approximated by polygones. Having all this convected by a carrier is easy.

Another way of handling divergence-free vector measures, numerically, operates through stream functions or vectors potential : see Sect. 6 below.

4. THE INVESTIGATED EQUATIONS.

In what follows, the dimension n of the considered Euclidean space equals 2 or 3. Over the open subset Ω the *stationary motion of an incompressible inviscid fluid* is supposed to take place.

If $\underset{\sim}{u}$ is the velocity field, one classically expresses incompressibility by $\operatorname{div} \underset{\sim}{u} = 0$ and mass conservation by $\operatorname{div} \rho \underset{\sim}{u} = 0$, with ρ denoting the volume mass. As we mean to treat nonsmooth flows, these conditions have to be understood in a weak sense. Denoting, by ℓ, as in Sect. 3, the Lebesgue measure, we assume from the beginning $\underset{\sim}{u} \in \mathcal{L}^1_{loc}(\Omega, \ell, ; \mathbb{R}^n)$ and $\rho \in \mathcal{L}^\infty_{loc}(\Omega, \ell; \mathbb{R})$. Then

$\underset{\sim}{u}\ell$ and $\rho\,\underset{\sim}{u}\ell$ are *vector measures* in Ω ; we require of them to be *divergence-free* in the sense of Sect. 3 .

The equations of Dynamics, i.e. the balance of momentum, will be expressed in the same style, under assumption $\underset{\sim}{u} \in \mathcal{L}^2_{loc}(\Omega, \ell; \mathbb{R}^n)$. We suppose that the pressure p belongs to $\mathcal{L}^1_{loc}(\Omega, \ell; \mathbb{R})$ and that extraneous forces, such as gravity, possess, relatively to the mass measure, a density of the form grad U , with U a time-independent smooth real function. Then

$$(4.1) \qquad (\rho\, u_i\, u_j\, \ell)_{,j} = -\,(p\ell)_{,i} + \rho\, U_{,i}\,\ell$$

Subscripts preceded by a comma refer to partial derivation of the considered measures, in the sense of Distributions. Since coordinates are essentially orthonormal, no superscript are used to distinguish between variances.

Such an expression of Classical Dynamics, involving the Euclidean divergence of tensor measures, should not be seen as a mere convenience trick. It is in fact the stationary version of a synthetic formulation in the event space of Classical Dynamics, which applies, as well as to continua, to such singular systems as collections of mass-points. Arguments tending to explain, in that connection, the precedence of measures over Distributions of higher order are presented in [2] . Derivation in the sense of Distributions is connected, via transposition, to the elementary derivation applied to *test fields* such as $\underset{\sim}{\varphi}$ in the present paper. So the above is nothing but an extension of the traditional method of *virtual power* (or "virtual work").

The fields $\underset{\sim}{u}, \rho, p$ defining the flow may be discontinuous. For instance, one may suppose that ρ vanishes in some open subdomain Ω_a (atmosphere) of Ω, separated from the proper fluid domain Ω_f by a C^1-surface S. Then (4.1) readily implies that p *equals a constant in* Ω_a *and presents no jump across* S. The definition of $\underset{\sim}{u}$ in Ω_a is immaterial (provided $(\operatorname{div}(\underset{\sim}{u}\ell) = 0$, for consistency); one may take $\underset{\sim}{u} = 0$ in Ω_a. If ρ and $\underset{\sim}{u}$ are smooth in Ω_f, with well defined limits on the fluid side of S, condition $\operatorname{div}(\rho\,\underset{\sim}{u}\ell) = 0$ elementarily implies that $\underset{\sim}{u}$ is tangential to S. *Hence all classical conditions on the free surface of a liquid are automatically involved.*

Let us end this Section by recalling that, in stationary hydrodynamics, it often proves convenient to introduce, instead of the pressure p, the "total head" or "Bernoulli function"

$$b = p + \frac{1}{2}\rho\, u_i\, u_i - \rho U \tag{4.2}$$

Equations of hydrodynamics are found to imply that *the vector measure* $b\,\underset{\sim}{u}\ell$ *is divergence free*. This vector measure may be seen as the "energy current" in the considered stationary flow (while the vector measures $\underset{\sim}{u}\ell$ and $\rho\,\underset{\sim}{u}\ell$ respectively constitute the "volume current" and the "mass current"). In the special case mentioned above, where ρ vanishes in the atmosphere Ω_a, b equals the atmospheric pressure p_a throughout this subdomain.

The sequel will precisely put forward b instead of p.

5. MAIN VARIATIONAL STATEMENT.

For simplicity, it will be assumed in the sequel that $\int_\Omega |U|\, d\ell < \infty$; otherwise, some covering argument should be associated with the forthcoming.

PROPOSITION. *Let a vector field* $\underset{\sim}{u} \in \mathcal{L}^2(\Omega, \ell; \mathbb{R}^n)$ *and two scalar fields* $b \in \mathcal{L}^1(\Omega, \ell; \mathbb{R})$, $\rho \in \mathcal{L}^\infty(\Omega, \ell; \mathbb{R})$ *be defined in the open subset* Ω *of* $\mathbb{R}^n$ *with* ℓ *denoting as before the Lebesgue measure. These elements satisfy the dynamical equation* (4.1) , *with* (4.2) *used for expressing* p , *if and only if, for every compact* C^1 *carrier in* Ω , *the* τ*-derivative of the following functional vanishes at* $\tau = 0$

$$\mathcal{B}(\tau) = \int_\Omega \left(\frac{1}{2}\, \rho^\tau\, u_i^\tau\, u_i^\tau + \rho^\tau U + b^\tau\right) d\ell$$

where b^τ , ρ^τ , *reducing to* b *and* ρ *for* $\tau = 0$, *are scalar fields convected by the carrier and* $\underset{\sim}{u}^\tau$, *reducing to* $\underset{\sim}{u}$ *for* $\tau = 0$, *is a vector field such that the vector measure* $\underset{\sim}{u}^\tau \ell$ *is convected.*

Proof may be found in [7] ; actually for (4.1) to be satisfied, it suffices that the above variational property holds for every carrier whose velocity field $\underset{\sim}{\varphi}$ is a τ-constant C^∞ vector field, with compact support in Ω .

Conditions div $(\underset{\sim}{u}\ell) = 0$ and div $(\rho\, \underset{\sim}{u}\, \ell) = 0$ are not involved in this variational property. They have to be additionnally imposed to the unknown fields (a practical way of doing this is

presented in Sect. 6 below). Let us stress that *these conditions do not constitute constraints* in the customary sense of the Calculus of Variations. In fact, the assumptions involve that the vector measures $\underset{\sim}{u}^\tau \ell$ and $\rho^\tau \underset{\sim}{u}^\tau \ell$ are *convected* by the considered carriers ; hence (cf. Sect. 3 above) their divergences vanish for every τ , as soon as the same is true for $\tau = 0$. In other words the zero divergence condition is automatically satisfied by all competing elements. The situation may be depicted as follows : starting with some triplet $(\underset{\sim}{u}^1 , b^1 , \rho^1)$ verifying $\operatorname{div} (\underset{\sim}{u}^1 \ell) = 0$ and $\operatorname{div} (\rho^1 \underset{\sim}{u}^1 \ell) = 0$ in Ω , imagine the totality of the triplets resulting from this one under the transport, in the way specified by the Proposition, by arbitrary C^1 (resp. C^∞) compact carriers in Ω . This constitutes, roughly speaking, an infinite-dimensional manifold and the Proposition characterizes the elements of this manifold satisfying the balance of momentum (if any) as the *critical points* of the real functional $\mathcal{B}$.

Proving the *existence* of such critical points is still out of sight ; the situation is very similar to the existence problem in large deformation elasticity, a problem essentially unsolved to-date, though numerical methods are effective.

In contrast with the above non constrained variation procedure, one may alternatively restrict the considered transport to *volume-preserving carriers* ; this amounts to impose to their respective velocity fields the condition $\operatorname{div} \underset{\sim}{\varphi} = 0$. Then the contribution of b^τ in $B(\tau)$ is τ-constant ; hence one obtains

a variational characterization of the pairs $(\underset{\sim}{u}, \rho)$ compatible with the balance of momentum, which eliminates the unknown b, i.e. eliminates the pressure p (cf. [8], Prop. 6.d). Of course, when the carrier motion preserves volume, the vector measure $\underset{\sim}{u}\ell$ is convected if and only if the vector field $\underset{\sim}{u}$ is convected.

6. STREAM FUNCTIONS AND VECTOR POTENTIALS.

Let us first consider the two-dimensional case. The classical concept of the stream function of a divergence-free vector field may be generalized as follows : if a vector measure $\underset{\sim}{m}$ is divergence-free in a simply connected open subset Ω of $\mathbb{R}^2$, there exists a scalar distribution $\psi \in \mathcal{D}'(\Omega, \mathbb{R})$ such that the scalar measures m_1 and m_2 which constitute the components of $\underset{\sim}{m}$ verify

$$m_1 = \psi_{,2} \quad , \qquad m_2 = -\psi_{,1}$$

In the case in view, $\underset{\sim}{m} = \underset{\sim}{u}\ell$, with $\underset{\sim}{u} \in \mathcal{L}^2_{loc}(\Omega, \ell\,; \mathbb{R}^2)$; then one establishes $\psi = \psi\ell$, where ψ is a scalar field belonging to $\mathcal{L}^p_{loc}(\Omega, \ell\,; \mathbb{R})$ for every $p \in [1, \infty]$.

For the vector measure $\underset{\sim}{m}$ to be convected by a carrier Λ it is found sufficient that ψ be a convected scalar field.

When Ω is multiply connected, some flux conditions are classically needed for the existence of a single valued stream function ; they can easily be adapted when divergence-free vector measures are considered, instead of vector fields.

Let us pass on now to the three-dimensional case, by considering first a C^1 vector field $\underset{\sim}{\Pi}$. Let $\underset{\sim}{\Pi}^\tau$ denote the moving vector field *transvected* by the carrier Λ and reducing to $\underset{\sim}{\Pi}$ for $\tau = 0$. It is easily checked that *the vector measure* $(\text{curl } \underset{\sim}{\Pi}^\tau)\ell$ *is divergence-free and convected*. The vector field $\underset{\sim}{\Pi}^\tau$ is called a *vector potential* of the vector measure and this can be extended to cases where $\underset{\sim}{\Pi}^\tau$ is no more C^1.

If Ω is not topologically simple enough, some flux conditions are required in order that a divergence-free vector measure admit a vector potential.

Of course, adding to $\underset{\sim}{\Pi}^\tau$ the gradient of some convected scalar field yields another transvected vector potential of the same convected vector measure.

It may also be convenient to specialize $\underset{\sim}{\Pi}^\tau$ under the form $\psi^\tau \text{ grad } \theta^\tau$, where ψ^τ and θ^τ are convected scalar fields ; this is a (transvected) vector potential of the divergence-free convected vector measure $(\text{grad } \psi^\tau) \times (\text{grad } \theta^\tau)\ell$. Since θ^τ appears only through its gradient, this scalar field need not be single-valued. By taking as θ^τ the azimutal angle, which in fact is convected by every *axissymmetric carrier* one obtains a way of treating *axissymmetric incompressible flows* by performing horizontal

variations in some fixed meridian half-plane. Then ψ^T is nothing but *Stoke's stream function* of the flow.

All this generates convenient alternatives to the variational statement of Sect. 5 .

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