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## **DYNAMICS OF ELASTIC OR RIGID BODIES WITH FRICTIONAL CONTACT: NUMERICAL METHODS.**

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**Abstract.** The mutual impenetrability of the parts of a mechanical system and their possible confinement by given external obstacles are unilateral constraints, the description of which results in joint inequalities and equalities. The system dynamics is thus governed by a set of very nonsmooth relations. The effect of dry friction at possible contacts makes evolution problems still less regular; velocity jumps may occur in this case, even in the absence of collision. In developing time-discretization procedures to solve these evolution problems numerically, the authors have chosen to face nonsmoothness without resorting to any regularization technique.

### **1. Introduction.**

Let a mechanical system involve several parts, referred to as *bodies*. This lecture is devoted to computing the motion, when the possible *contacts* between some of these bodies, or between some of them and external obstacles with given motion, are subject to discussion at every instant.

Friction, affecting contacts, is a very intricate phenomenon. By restricting ourselves to the "dry" sort, we exclude the consideration of lubricants. Even so, the tribological data needed for an accurate description of contact interaction, when attempts are made at investigating engineering situations, are practically always missing. Recall that, in usual instances, the alteration of contacting surfaces through wear and the accumulation of debris in the affected zone play an important role. Some authors also insist on the pressure-dependent crushing of micro-asperities and

introduce a specific nonlinear local compliance in the description of contact. An extensive discussion of dry friction, with a vast list of references, may be found in [1]. By representing friction through the traditional *law of Coulomb*, or a generalization of it to anisotropic situations, we choose to neglect these effects. Anyway, in our computing techniques of time-discretization, it is easy to make the parameters of the law of Coulomb depend, for each time-step, on some "status" variable determined by the past motion. More simply, at the beginning of each step, one may have the friction data depend, possibly in discontinuous way, on the available values of the system position and velocity.

Coulomb friction and *unilaterality*, i.e. the possibility of contact breaking at unknown instants, raise problems whose mathematical difficulty has long been recognized [2]-[6]. *Collisions*, i.e. the sudden occurrence of new contacts, causes still rougher irregularity.

A very accurate treatment of collisions would require data usually impossible to identify and to collect experimentally. The suddenly introduced contacts may persist in subsequent motion, or on the contrary, the colliding bodies may *bounce* against each other. The empirical *coefficient of restitution* bears little consistency beyond the special case of the collision of two otherwise unconstrained perfectly rigid bodies. An alternative concept has been proposed in [7], avoiding in particular the energetical paradoxes pointed out by some authors [8][9]; it is presented in Sec.6 below.

The traditional approach to the above problems consists of performing, at the first stage, some tentative calculation of the motion under the maximal regularity assumptions: persistence of existing contacts, no collisions, no occurrence of singular points in the differential equations. From this calculation, one tries to identify the instants at which some contradiction arises: contact forces with inadmissible direction, collision, dynamical locking due to friction,... At the first encountered of such signals, a special analysis of the instant behaviour (possibly with the help of infinitesimal time rescaling) is expected to yield the initial conditions for a possible subsequent phase of regular motion. Even in the no-friction case, this approach has long been recognized as requiring caution: the contacts which get loose after some of the critical instants are not necessarily those for which the previously calculated reaction exhibits inadmissible sign [10][11].

The mathematical tools allowing for a global formulation of the concerned evolution problems, not based on the division of time into intervals of smooth motion, have been made available only in recent years. The existential study [12]-[15] and the corresponding numerical procedures [16] are still under development.

For deformable bodies, the situation is still more complicated, since the contact status may, at every instant, vary from one region to another, delimited by unknown moving boundaries. So far, papers devoted to the computation of such dynamical problems are not very numerous [17]-[21]. In the event of a collision, shock waves are expected to propagate through the system; these are numerically

considered in [22]. In the applications we have in view, the time taken by waves to travel the whole system is short enough for these to be inapparent in computation.

Anyway, it should be kept in mind that the presented models can only carry a crude representation of physical reality. This is a fact of life in all the applications of Science, but in the present topics, the lack of physical information is so patent that, however sophisticated the computation methods might be, seeking high precision in their implementation could be regarded as a waste of effort.

## 2. Analytical setting.

In the first Sections of this lecture, a mechanical system with finite number of degrees of freedom will be considered. Naturally, the numerical treatment of continuous media also relies on finite sets of variables and we shall come to this case in Sec. 9.

Let the possible configurations of the system be parametrized, at least locally, through generalized coordinates, say  $q = (q^1, q^2, \dots, q^n)$ . Such a reduction to finite freedom is assumed here to result from (bilateral) ideal constraints, namely the rigorous internal rigidity of the various parts and the possible operation of frictionless linkages.

After constructing this parametrization, one additionally takes into account some *unilateral constraints* whose geometric effect is expressed by a finite set of inequalities

$$(2.1) \quad f_\alpha(q) \leq 0, \quad \alpha \in \{1, 2, \dots, \kappa\},$$

where  $f_1, f_2, \dots, f_\kappa$  are given functions of  $q$ . Usually, such inequalities describe the mutual impenetrability of some parts of the system or the confinement of some of them by given external boundaries. Assuming, as we are doing here for simplicity, that the functions  $f_\alpha$  are constant in time involves that the possible external boundaries are fixed relative to the reference frame in use.

For brevity, parametrization is supposed *scleronomic*, i.e. time-independent. Then, for every imagined motion  $t \rightarrow q(t)$  and for  $t$  such that the derivative  $\dot{q}(t) \in \mathbb{R}^n$  exists, the kinetic energy of the systems lets itself be expressed as a quadratic form in  $\dot{q}$ , say

$$(2.2) \quad \mathcal{E}_c(q, \dot{q}) = \frac{1}{2} A_{ij}(q) \dot{q}^i \dot{q}^j, \quad \text{with } A_{ij}(q) = A_{ji}(q).$$

Then, as far as *smooth*, i.e. twice differentiable, motions are concerned, the system Dynamics is governed by Lagrange equations, written as an equality in  $\mathbb{R}^n$

$$(2.3) \quad A(q) \ddot{q} = F(t, q, \dot{q}) + r.$$

Expression  $F$  here comprises the terms of first differential order in Lagrange equations and the generalized components, relative to the parametrization  $(q)$ , of

some applied forces supposed given as functions of time, position and velocity. The element  $\mathbf{r} = (r_1, r_2, \dots, r_n)$  of  $\mathbf{R}^n$  is made of the generalized components of the totality of the contact forces corresponding to unilateral constraints.

Connecting  $\mathbf{r}$  with the proper contact forces, which are vectors of physical space, resort to the standard machinery of Analytical Dynamics. Suppose that inequality  $f_\alpha(q) \leq 0$  expresses the mutual impenetrability of some pair of bodies  $\mathcal{B}$  and  $\mathcal{B}'$ . Let us consider a position  $q_1$  of the system such that equality  $f_\alpha(q_1) = 0$  holds, which means that in position  $q_1$  the two bodies touch each other at some point of space called  $M_\alpha$ . This we shall assume to be an isolated contact point, but other contacts, corresponding to different values of  $\alpha$ , may also be effective between the same bodies in the considered position. Let  $\mathcal{R}^\alpha$  denote the contact force that body  $\mathcal{B}$  experiences at this contact point from body  $\mathcal{B}'$ ; then  $\mathcal{B}'$  experiences from  $\mathcal{B}$  the force  $-\mathcal{R}^\alpha$ . Defining the covariant components of this pair of forces, relative to the parametrization  $(q)$ , rests on the kinematics of the respective particles of the bodies to which they are applied. One imagines a motion (usually qualified as *virtual*, nonnecessarily complying with the constraint inequalities)  $t \rightarrow q(t)$  such that  $q(t_1) = q_1$ . To every value, say  $u \in \mathbf{R}^n$ , that the derivative  $\dot{q}(t_1)$  may come to take, correspond some values  $\mathcal{V}_\alpha$  and  $\mathcal{V}'_\alpha$  of the respective velocity vectors of the particles of  $\mathcal{B}$  and  $\mathcal{B}'$  passing at point  $M_\alpha$ . The corresponding *power* (usually said *virtual*) of the pair of contact forces then equals

$$\mathcal{P}_\alpha = \mathcal{R}^\alpha \cdot \mathcal{V}_\alpha + (-\mathcal{R}^\alpha) \cdot \mathcal{V}'_\alpha = \mathcal{R}^\alpha \cdot (\mathcal{V}_\alpha - \mathcal{V}'_\alpha).$$

*In all the sequel, Greek indices, when repeated in an expression, shall not be understood as implying summation.*

$\mathcal{V}_\alpha$  and  $\mathcal{V}'_\alpha$  above are velocity vectors relative to the reference frame in use. The difference  $\mathcal{U}_\alpha = \mathcal{V}_\alpha - \mathcal{V}'_\alpha$  equals the *relative velocity* of body  $\mathcal{B}$  with regard to body  $\mathcal{B}'$  at point  $M_\alpha$ . In computation, this vector will be represented by its components relative to some chosen orthonormal base  $(\mathbf{n}_\alpha, \mathbf{t}_\alpha, \mathbf{t}'_\alpha)$ , depending on  $q$ . We shall assume in the sequel that the unit vector  $\mathbf{n}_\alpha$  is normal to the contacting bodies at point  $M_\alpha$  and directed toward  $\mathcal{B}$ . Through the geometric investigation of the system, one expresses  $\mathcal{U}_\alpha$  from  $u$  in the form

$$(2.4) \quad \mathcal{U}_\alpha = G_\alpha^q u$$

where  $G_\alpha^q : \mathbf{R}^n \rightarrow \mathbf{R}^3$  is a  $q$ -dependent *linear* mapping. Such at least are the findings under the assumption we have made of a *scleronomic* parametrization. In general,  $G_\alpha^q$  would also depend on  $t$  and a complementary term, constant in  $u$ , should be added to the right-hand side of (2.4).

By definition, the generalized components of the pair of contact forces make the element  $\mathbf{r}^\alpha$  of  $\mathbf{R}^n$  such that  $u \cdot \mathbf{r}^\alpha = \mathcal{P}_\alpha$  for every  $u$  in  $\mathbf{R}^n$ . Equivalently

$$(2.5) \quad \mathbf{r}^\alpha = G_\alpha^{q*} \mathcal{R}^\alpha.$$

This expression of  $r^\alpha$  through *the transpose of the linear mapping*  $G_\alpha^q$  is known to remain valid in the absence of the scleronomy assumption we have made here for simplicity. In fact, the definition of generalized components, in Analytical Dynamics, rests on *virtual motions at fixed time*, i.e. chains of positions, say  $\tau \rightarrow q(\tau)$ , depending on a real variable  $\tau$  in differentiable way, with subsequent calculations made at constant  $t$ .

Observe that, at the present stage, the considered values of  $u$  are not required to be compatible with the contact geometry. This will be discussed in Sec. 3.

What precedes implicitly concerned a pair of bodies  $\mathcal{B}$  and  $\mathcal{B}'$  which both were parts of the investigated mechanical system. The conclusion remains formally the same in the case where only  $\mathcal{B}$  belongs to the system, while  $\mathcal{B}'$  denotes an external obstacle with prescribed motion. Among the two contact forces, only the force  $\mathcal{R}^\alpha$  experienced by  $\mathcal{B}$  has to be kept into account in this case and one makes  $\mathcal{U}_\alpha$  simply stand for the velocity vector  $\mathcal{V}'_\alpha$ .

Summing up, under the smoothness assumptions made, the Dynamics of the system is governed by the differential equation (2.3) where the element  $r = \sum r^\alpha$  is involved in the problem through (2.4), (2.5),  $u = \dot{q}$ , and through a system of *contact laws*

$$(2.6) \quad \text{law}(\alpha, q, \mathcal{U}_\alpha, \mathcal{R}^\alpha) = \text{true}.$$

The latter describe what, in physical space, happens at every contact  $\alpha$ . In conditions (2.4) to (2.6), the index  $\alpha$  has to take all values such that  $f_\alpha(q) = 0$ , since  $\mathcal{U}_\alpha$ ,  $\mathcal{R}^\alpha$ , etc. only make sense in the case of effective contact. Actually, in existential studies, as well as in computation, it proves convenient to make  $\alpha$  range through a larger subset of  $\{1, 2, \dots, \kappa\}$  *provided the formulation (2.6) is contrived in such a way that*  $f_\alpha(q) < 0 \Rightarrow \mathcal{R}^\alpha = 0$ . In the same context, the matrix  $G_\alpha^q$  which, strictly speaking, was only defined for  $q$  lying in the hypersurface  $f_\alpha(q) = 0$  of  $\mathbb{R}^n$  has to be extended (in some smooth arbitrary way) to every  $q$  in a neighbourhood of this hypersurface.

One is looking for motions verifying the  $\kappa$  inequalities (2.1) for every  $t$ . Instead of explicitly adjoining these to (2.3)-(2.6), *we prefer to rely on some adequate formulation of the contact laws (2.6) for securing them.*

### 3. Complete contact law.

DEFINITION. A contact law, i.e. a relation of the form (2.6), is said *complete* if it involves the three following implications

$$(3.1) \quad f_\alpha(q) < 0 \Rightarrow \mathcal{R}^\alpha = 0.$$

$$(3.2) \quad f_\alpha(q) \geq 0 \Rightarrow \mathbf{n}_\alpha \cdot \mathcal{U}_\alpha \geq 0,$$



$$(3.3) \quad \mathbf{n}_\alpha \cdot \mathbf{u}_\alpha > 0 \Rightarrow \mathcal{R}_\alpha = 0.$$

The meaning of (3.1) is simply that the reaction force  $\mathcal{R}_\alpha$  vanishes when contact  $\alpha$  is not in effect. Let us comment on the importance of (3.2). Put

$$(3.4) \quad \mathcal{K}_\alpha(q) = \begin{cases} \{ \mathbf{u} \in \mathbf{R}^3 : \mathbf{n}_\alpha \cdot \mathbf{u} \geq 0 \} & \text{if } f_\alpha(q) \geq 0 \\ \mathbf{R}^3 & \text{if } f_\alpha(q) < 0, \end{cases}$$

called the set of the *right-admissible* values for the relative velocity of the two concerned bodies at point  $M_\alpha$ . As in previous Section, the definition of  $\mathbf{n}_\alpha$  is assumed extended, so as to keep this notation meaningful for  $f_\alpha(q) > 0$ , i.e. for system positions which violate the impenetrability of the considered bodies.

The following is easily established [7].

**PROPOSITION.** *Let  $I$  be a time-interval with origin  $t_0$  and let a motion  $q: I \rightarrow \mathbf{R}^n$  be defined through a locally integrable velocity function  $u: I \rightarrow \mathbf{R}^n$  by*

$$(3.5) \quad t \rightarrow q(t) = q(t_0) + \int_{t_0}^t u(s) ds.$$

*If  $\mathbf{u}_\alpha(t) = G_\alpha^q u(t)$  belongs to  $\mathcal{K}_\alpha(q(t))$  for almost every  $t$  and if inequality  $f_\alpha(q(t)) \leq 0$  holds at the initial instant  $t_0$ , then this inequality holds for every  $t \in I$ .*

In other words, provided the initial position is correct, the impenetrability condition  $f_\alpha \leq 0$  is automatically taken care of by (3.2). Observe that this proposition is sensitive to the ordering of time. In the symmetric statement, involving instead of the initial instant  $t_0$ , some *final* instant, one should replace  $\mathcal{K}_\alpha$  by  $-\mathcal{K}_\alpha$ , which may be viewed as the set of the *left-admissible* values of  $\mathbf{u}_\alpha$ .

The importance of (3.3) will only become apparent in further Sections, devoted to numerical algorithms and to the study of collisions.

From our standpoint, synthetic formulations of contact laws, possessing properties (3.1)-(3.3) without a priori statement of them, should be preferred. The theoretical or numerical developments based on such formulations offer a securizing consistency and should accept with better robustness the further adjustments needed to improve the agreement with physical reality. Such a synthetic formulation for the *frictionless* contact law has been presented in [7]. Several ways of doing the same for dry frictional contact will be developed in further publications.

#### 4. Coulomb contact.

In this lecture, the status of complete contact law will be given to Coulomb contact by asserting from start that properties (3.1) to (3.3) hold. Therefore  $\mathcal{R}_\alpha$  can be nonzero only if  $f_\alpha(q) \geq 0$  and  $\mathbf{n}_\alpha \cdot \mathbf{u}_\alpha = 0$ , and there just remains to develop this case.

In some early publications of one of the authors [23]-[25], the traditional statement of Coulomb law, and a plausible extension of it to possible *anisotropic* friction, were replaced by equivalent assertions involving some concepts now standard in Convex Analysis. These statements are listed below under the title of "First formulation". Their advantages were shown to lie in possible handling through minimization techniques and in the simplification brought to the treatment of mechanical systems combining several frictional contacts.

In what follows, the data for every contact point consist of a *closed convex cone*  $C$ , in the linear space of physical vectors, called the *Coulomb cone* of the considered contact. Let us use the base  $(\mathbf{n}, \mathbf{t}, \mathbf{t}')$  introduced in Sec. 2, omitting for brevity the contact index  $\alpha$ . Vectors  $\mathbf{t}$  and  $\mathbf{t}'$  generate the two-dimensional linear subspace  $T$ , called the *tangent plane* at the considered contact. The Coulomb cone is assumed to contain  $\mathbf{n}$  in its interior and to lie in totality on the same side of  $T$ .

Giving the cone  $C$  is equivalent to giving its section at unit distance of  $T$  or, more conveniently, the orthogonal projection of this section onto  $T$ , say  $\mathcal{D}_1$ . This is a closed convex subset of  $T$ , containing the origin as interior point. Giving  $\mathcal{D}_1$  is in turn equivalent to giving its *support function*. One preferably uses the image of this function under the symmetry relative to the origin, i.e.

$$(4.1) \quad \mathcal{U} \rightarrow \phi_1(\mathcal{U}) = \sup\{\mathcal{U} \cdot \mathbf{T} \in \mathbf{R} : \mathbf{T} \in -\mathcal{D}_1\}.$$

It will be seen in a moment that  $\phi_1$  may be called the *dissipation function at unit normal component*.

For the traditional *isotropic* Coulomb friction,  $\mathcal{D}_1$  consists of the disk centered at the origin, with radius equal to the *friction coefficient*  $\gamma$ , so  $\phi_1(\mathcal{U}) = \gamma \|\mathcal{U}\|$ .

**FIRST FORMULATION OF COULOMB LAW.** One decomposes the contact force in the form

$$(4.2) \quad \mathcal{R} = \mathcal{R}_T + \rho \mathbf{n}, \text{ with } \rho \in \mathbf{R} \text{ and } \mathbf{n} \cdot \mathcal{R}_T = 0.$$

It is first stipulated that *the normal component*  $\rho$  *is nonnegative*. As soon as this real number is treated as known, one may denote by  $\phi_\rho$  the sublinear function  $\rho\phi_1$ , by  $\mathcal{D}_\rho$  the set  $\rho\mathcal{D}_1$  and by  $\psi_\rho$  the *indicator function* of this set (namely  $\psi_\rho = 0$  in  $\mathcal{D}_\rho$  and  $\psi_\rho = +\infty$  elsewhere). Then Coulomb law lets itself be expressed by any of the *equivalent* following relations between the elements  $\mathcal{U}$  and  $\mathcal{R}_T$  of  $T$ :

$$(4.3) \quad -\mathcal{U} \in \partial\psi_\rho(\mathcal{R}_T)$$

$$(4.4) \quad -\mathcal{R}_T \in \partial\phi_\rho(\mathcal{U})$$

$$(4.5) \quad \mathcal{R}_T \in \mathcal{D}_\rho \text{ and } -\mathcal{U} \cdot \mathcal{R}_T = \phi_\rho(\mathcal{U})$$

$$(4.6) \quad \mathcal{R}_T \in \mathcal{D}_\rho \text{ and } \forall \mathbf{T} \in \mathcal{D}_\rho : -\mathcal{U} \cdot \mathbf{T} \leq -\mathcal{U} \cdot \mathcal{R}_T.$$



Recall that  $\partial\psi_\rho$ , the *subdifferential* of the indicator function of the set  $\mathcal{D}_\rho$  is nothing but the (outward) *normal cone* to this convex set at the considered point. In Convex Analysis, the normal cone to a convex set is defined at any point, say  $x$ , of the embedding linear space. This cone is empty if and only if  $x$  does not belong to the set; otherwise, it contains at least the origine and may reduce to this (in particular when  $x$  lies in the interior of the set). So (4.3) involves that  $\mathcal{R}_T \in \mathcal{D}_\rho$ .

As a point of vocabulary, one may express (4.4) by saying that, for fixed  $\rho$ , the (multivalued) relation between the velocity  $\mathcal{U}$  and the frictional force  $\mathcal{R}_T$  admits the convex real function  $\phi_\rho$  as *pseudo-potential*.

The equivalence of (4.3) to (4.4) is due to the fact that the function  $\psi_\rho$  (the indicator of  $\mathcal{D}_\rho$ ) and the function  $\mathcal{V} \rightarrow \phi_\rho(-\mathcal{V})$  (the support function of  $\mathcal{D}_\rho$ ) make a pair of *Fenchel conjugate functions*. Then  $\mathcal{R}_T$  and  $-\mathcal{U}$  are *conjugate points* relative to this pair of conjugate functions, a situation classically expressed by *Fenchel equality*. Conditions (4.5) are merely the special form that Fenchel equality takes in the present context; the equality on the right justifies the name of *dissipation function* (for the value  $\rho$  of the normal component) given to  $\phi_\rho$ . Conditions (4.6) amount to rewriting (4.5) in the form of a "principle" of *maximal dissipation*.

In short, when  $\rho$  is treated as known, Coulomb law is exactly similar to a *law of perfect plasticity*, well known to generate mathematical problems in the form of variational inequalities. In situations involving frictional contact, it is commonly found that the determination of  $\rho$  is in turn governed by a variational inequality involving  $\mathcal{R}_T$  as parameter, so one is in the presence of *two coupled variational inequalities*.

**SECOND FORMULATION OF COULOMB LAW.** A formulation avoiding the separation of the normal and tangential components of  $\mathcal{R}$  has more recently been proposed [7][26]. It involves the indicator function  $\psi_C$  of the Coulomb cone, a convex lower semicontinuous function defined on the three-dimensional space of physical vectors, with values 0 or  $+\infty$ . Assuming again that (3.1)-(3.3) hold, what precedes is found equivalent to

$$(4.7) \quad -\mathcal{U} \in \text{proj}_T \partial\psi_C(\mathcal{R}).$$

The precise proof of this equivalence may be based on standard arguments of Convex Analysis [7]. Let us only comment on (4.7) by observing that it exhibits the expected features. The subdifferential  $\partial\psi_C(\mathcal{R})$  is nothing but the normal cone to the convex set  $C$  at point  $\mathcal{R}$ . Since this cone is empty for  $\mathcal{R} \notin C$ , relation (4.7) forces  $\mathcal{R}$  to belong to  $C$  whichever is  $\mathcal{U}$  in  $T$ . If  $\mathcal{R}$  lies in the interior of  $C$ , the normal cone reduces to the set  $\{0\}$ ; hence it is only for  $\mathcal{U}=0$  that  $\mathcal{R}$  can take values interior to  $C$ . The normal cone to  $C$  at the origin consists of the totality of the *polar cone*  $C^\circ$  of  $C$ . In most usual situations  $\mathcal{D}_1$  is bounded; then through standard propositions of Convex Analysis,  $C^\circ$  admits  $-\mathbf{n}$  as an interior point. Consequently, the projection of  $C^\circ$  onto  $T$  equals the whole of  $T$ ; in other words, (4.7) lets the value  $\mathcal{R}=0$  compatible with any  $\mathcal{U} \in T$ .

An alternative statement to (4.7) is obtained by defining on the three-dimensional space of physical vectors, the function  $\theta$  with values in  $[0, +\infty]$

$$(4.8) \quad u \rightarrow \theta(u) = \frac{1}{2} \|u\|^2 + \psi_T(u).$$

Then (4.7) is found equivalent to

$$(4.9) \quad 0 \in \partial\psi_C(\mathcal{R}) + \partial\theta(u).$$

The latter has been successfully used in the numerical treatment of elementary three-dimensional contact problems [7]. Expressions based on standard Convex Analysis and involving, as (4.9) does, *two independent "pseudo-potential" functions* could also be applied to the formulation of other resistance laws, e.g. plasticity laws with flow rule "non associated" with the yield criterium. In the same line, a new type of numerical procedures of the variational sort has recently been proposed by G. De Saxcé and Z. Q. Feng [27] under the name of the "method of bipotentials". This method may also lead to alternative statements of Coulomb law.

## 5. Collisions.

When the unilateral constraints of impenetrability are taken into account, the treatment of dynamical problems cannot in general exclude the possibility of *collisions*. Unfortunately, in practical instances, the physical data required for accurately predicting the consequent motion are missing. The assumption of *elastic shocks* is commonly made, which means that the considered collisions involve no energy loss. This assumption yields a scalar equation, which anyway is not enough to predict the after-shock velocities of all system parts, if several unilateral contacts are present at the collision instant.

Energy preservation can rarely be justified. The concept of a collision between bodies which, in the rest of the motion, are treated as perfectly rigid can only be the schematic representation of a "very short" interaction involving "very large" contact forces. Even if it is safely established that, in spite of the large stresses involved, the material of which the system is made behaves elastically, the collision locus is sure to be the source of dynamical disturbances which propagate through the whole system. If the system is linked to supports, these disturbances will also affect the external world, carrying away an irrecoverable amount of energy. In the case of a bounce, various parts of the system will still be vibrating when the colliding bodies get loose from each other. By considering all parts as rigid in the subsequent evolution one treats vibration amplitudes as "microscopic", but this does not imply that the vibrational energy is negligible.

Of course, the possible inelastic deformation of the colliding bodies, however small its amplitude is estimated, may also cause energy loss.

What precedes holds true even if, during the "very short" duration of the collision, the contact of colliding bodies is considered as frictionless. Things become

still worse if friction is taken into account. Drawing the momentum balance (or performing the equivalent calculation in the setting of Analytical Dynamics) would require the knowledge of contact *percussions*, i.e. the integrals, over the "very short" duration of the collision, of "very large" contact forces. Even if one makes the hazardous assumption that, under such high contact pressures and rapidly varying conditions, friction obeys the same laws as in regular situations, there remains to observe that Coulomb law consists of a *nonlinear* relation between forces and velocities. Commuting it with time-integration, so as to obtain a relation between the contact percussion and the average contact velocity, is thus illegal.

A favourable special case should however be mentioned: if during the collision, the relative velocity  $\mathcal{U}$  of the contacting bodies (see Sec. 4 above) has constant direction, i.e.  $\mathcal{U}(t) = \sigma(t) \mathcal{U}_0$ , where  $\mathcal{U}_0$  is a constant vector and  $\sigma(t) > 0$ , Coulomb law yields that the contact force  $\mathcal{R}(t)$  remains in a fixed closed convex cone, namely the set of values of  $\mathcal{R}$  that (4.7) associates with  $\mathcal{U}_0$ . Therefore, the contact percussion also belongs to this cone, so it appears to be associated with  $\mathcal{U}_{\text{average}} = \sigma_{\text{average}} \mathcal{U}_0$ .

An early attempt at analysing the variation of  $\mathcal{U}$  during a frictional collision through *rescaling* in time, was made by G. Darboux [28] and later improved by J. Pérès [29]; see also [8]. This consisted in deriving from the equations of Dynamics, joined with Coulomb law, a system of first order differential equations governing the function  $t \rightarrow u(t)$  during the collision, while the position  $q$  was treated as a constant. Some assumption had to be added in order to decide the end of the process. In this approach, the bodies were considered as perfectly rigid. Physical situations likely exist, where the assumptions made are valid. The cruder treatment we propose in the sequel may be equally useful in practice.

## 6. The dissipation index.

A collision, occurring at some instant  $t_s$ , is expected to generate a discontinuity for the velocity function  $t \rightarrow u(t)$ . Traditionally, the *left-side limit*  $u^-(t_s)$  and the *right-side limit*  $u^+(t_s)$  are assumed to exist. A consistent mathematical framework where this existence is secured, is provided by assuming that  $u$  is a function of *bounded variation* [30]. Theoretical arguments in favour of this assumption are developed in [31].

Let us apply the usual trick of integrating both members of (2.3) over the "very short" duration of the shock. The position  $q(t)$  is treated as a constant and the term  $F$  remains bounded, so it yields a negligible integral. One thus obtains

$$(6.1) \quad A(q(t_s)) (u^+(t_s) - u^-(t_s)) = s,$$

where the element  $s$  of  $\mathbf{R}^n$  equals the time-integral of  $r$ . Force vectors of physical space are connected with their analytical representatives through relations (2.5). Like  $q$ , the matrices  $G_\alpha^{q*}$  are treated as constant during the shock, so one has

$$(6.2) \quad s = \sum_{\alpha} G_{\alpha}^{q*} S^{\alpha},$$

where  $S^{\alpha}$ , the time-integral of  $\mathcal{R}^{\alpha}$ , may be called the *shock percussion* at contact  $\alpha$ . During the shock process, the relative velocity  $\mathcal{U}_{\alpha}$  of the contacting bodies rapidly varies from  $\mathcal{U}_{\alpha}^{-} = G_{\alpha}^q u^{-}(t_s)$  to  $\mathcal{U}_{\alpha}^{+} = G_{\alpha}^q u^{+}(t_s)$ .

Now comes the embarrassing question of connecting the unknown vectors  $S_{\alpha}$  with the sought motion. Rather than attempting to calculate the evolution of  $\mathcal{U}_{\alpha}$  during the shock, we propose, as a first approximation, to accept that a contact law similar to (2.6) holds between  $S_{\alpha}$  and a certain *weighted mean*

$$(6.3) \quad \mathcal{U}_{\alpha}^m = \frac{1-\delta}{2} \mathcal{U}_{\alpha}^{-} + \frac{1+\delta}{2} \mathcal{U}_{\alpha}^{+},$$

where  $\delta$  denotes a real constant, called the *dissipation index* [7].

Some confidence in this procedure may be gained by drawing the *energy balance* of the collision. From the expression (2.2) of the kinetic energy, one readily derives the jump of this function in the form

$$\mathcal{E}_c^{+} - \mathcal{E}_c^{-} = \frac{1}{2} A_{ij} (u_i^{+} - u_i^{-})(u_j^{+} + u_j^{-}).$$

Here  $A_{ij}$  stands for the matrix  $A_{ij}(q(t_s))$ , so (6.1) yields

$$\mathcal{E}_c^{+} - \mathcal{E}_c^{-} = \frac{1}{2} (u^{+} + u^{-}) \cdot s.$$

By making use of (6.1), (6.2) and (6.3), one obtains the signed *decrement* of kinetic energy

$$(6.4) \quad \mathcal{E}_c^{-} - \mathcal{E}_c^{+} = \frac{1}{2} \delta A_{ij} (u_i^{+} - u_i^{-})(u_j^{+} - u_j^{-}) - \sum_{\alpha} \mathcal{U}_{\alpha}^m \cdot S^{\alpha}.$$

As far as  $\delta \geq 0$ , the first term on the right-hand member is nonnegative ; we propose to call it the *structural dissipation*. The nonnegativity of  $\delta$  may be viewed as asserting that no external supply of energy comes to enhance the possible bounces. The last term depends on the adopted contact law and may be called (with the minus sign included) the *frictional dissipation*.

The following shows the importance of including in contact laws the "completeness" assertion of Sec.3. In the present context, (3.3) becomes

$$S^{\alpha} \neq 0 \Rightarrow \mathbf{n}_{\alpha} \cdot \mathcal{U}_{\alpha}^m \leq 0.$$

In view of the definition (6.3) of  $\mathcal{U}_{\alpha}^m$ , the latter inequality yields

$$(6.5) \quad \frac{1+\delta}{2} \mathbf{n}_{\alpha} \cdot \mathcal{U}_{\alpha}^{+} \leq \frac{\delta-1}{2} \mathbf{n}_{\alpha} \cdot \mathcal{U}_{\alpha}^{-}.$$

For an effective collision,  $\mathbf{n}_{\alpha} \cdot \mathcal{U}_{\alpha}^{-}$  is strictly negative. If  $\delta$  was strictly greater than one,  $\mathbf{n}_{\alpha} \cdot \mathcal{U}_{\alpha}^{+}$  would therefore be strictly negative, in contradiction with the impenetrability constraint.

In short, the parameter  $\delta$  must be assigned a value in the interval  $[0, 1]$ .

Making now use of (3.2), one concludes that, *provided*  $S^\alpha \neq 0$ , (6.5) actually holds as an equality, hence

$$\mathbf{n}_\alpha \cdot \mathbf{u}_\alpha^+ = -\frac{1-\delta}{1+\delta} \mathbf{n}_\alpha \cdot \mathbf{u}_\alpha^-,$$

which allows one to identify the expression  $1-\delta/1+\delta$  with the *coefficient of restitution*, traditionally introduced for the description of two-body collisions. But when studying cases where several contacts are in effect at the collision instant, it is essential to observe that this equality *does not necessarily hold if*  $S^\alpha = 0$ . An elementary example of such a situation is provided by the rocking oscillations of a column in contact with a fixed horizontal plane.

## 7. Principle of algorithms.

Let  $[t_I, t_F]$ ,  $t_F = t_I + h$ , denote one of the intervals of a time-discretization. Some approximate values  $q_I, u_I$  of the functions  $q$  and  $u$  at instant  $t_I$  result from the preceding step. In order to calculate an approximate value of  $u$  at  $t_F$ , say  $u_F$ , one discretizes (2.3) into

$$(7.1) \quad u_F = u_L + A^{-1} \sum_{\alpha} G_{\alpha}^{q*} S^{\alpha}$$

where  $u_L = u_I + A^{-1} F h$ , the "loose velocity", is the value one would find for  $u_F$  in the absence of contact force. Here,  $A, F, G_{\alpha}^{q*}$  denote the values calculated for the corresponding functions at a point  $(t, q, u)$ , chosen as typical of the time-step and computed in some *explicit* way. In contrast, the procedure we shall use to calculate  $u_F$  may be qualified as *implicit*. In harmony with the notations of the preceding Section,  $S^{\alpha}$  denotes the integral of the contact force  $\mathcal{K}^{\alpha}$  on  $[t_I, t_F]$ . Therefore the algorithm will be able to treat possible shocks on the same footing as smooth motions.

The decision of applying a computing policy of the implicit type, induces to invoke the contact law, for every selected  $\alpha$ , as a relation of the form (2.6) holding between  $S^{\alpha}$  and the *final* local velocity, namely in view of (2.4)

$$(7.2) \quad u_{\alpha F} = G_{\alpha}^q u_F.$$

After applying  $G_{\beta}^q$  to both members of (7.1) and (7.2), one obtains

$$(7.3) \quad u_{\beta F} = u_{\beta L} + \sum_{\alpha} G_{\beta}^q A^{-1} G_{\alpha}^{q*} S^{\alpha}$$

with  $u_{\beta L} = G_{\beta}^q u_L$ , to be joined to the contact law

$$(7.4) \quad \text{law}(\alpha, q, u_{\alpha F}, S^{\alpha}) = \text{true}.$$

Of course  $\alpha$  and  $\beta$  will be restricted to range in the (possibly empty) subset ACT of  $\{1, 2, \dots, \kappa\}$  corresponding to the contacts considered as active in the time-



step. The simplest criterium for determining this set is based on the signs that the functions  $f_\alpha$  take at point  $q_I$  (or at some other test position).

Solving (7.3)-(7.4) will yield the vectors  $S^\alpha$ , then  $u_F$  through (7.1). After that, the computation step may be concluded by taking

$$q_F = q_I + \frac{1}{2} h (u_I + u_F).$$

As soon as several contacts are active, (7.3)-(7.4) can only be treated by successive approximations. *Let us stress that, in general, one cannot expect uniqueness for the solution vectors  $S^\alpha$ .* Dry friction commonly generates hyperstatic problems. In favourable cases, the calculation may however result in a unique  $u_F$ .

The technique we have applied in all recent experiments is a relaxation method, based on the solution of the *single contact problem*. In particular, for the two-dimensional case, the case of a single contact can readily be solved, yielding for every  $\alpha$  an easily programmed *piecewise affine* mapping  $u_{\alpha L} \rightarrow S^\alpha$ . Starting with a tentative set of values for the vectors  $S^\alpha$ ,  $\alpha \in \text{ACT}$ , one makes  $\alpha$  cyclically run through ACT. Each time, one updates the corresponding approximate value of  $S^\alpha$  by solving a single contact problem, where the previously obtained approximates  $S^\beta$ ,  $\beta \neq \alpha$ , are treated as known forces applied to the system.

The above computation is valid even if a collision occurs in the time interval  $[t_I, t_F]$ , a situation revealed by the appearance in ACT of some contacts which were not active at the antecedent time-step. Clearly, by invoking  $u_{\alpha F}$  as value for the local velocity in (7.4), we make the procedure treat possible collisions as *inelastic*, i.e. the dissipation index equals one. By introducing into (7.4) the same weighted mean as in (6.3), one could impose a different value of  $\delta$ . It proves more expedient to program the algorithm for the case  $\delta = 1$  only, getting a result one may denote by  $u_F = \text{solver}(u_L)$ . Due to the fact that relations (7.2)-(7.4) are *positively homogeneous with degree one*, the time-step result for arbitrary  $\delta$ , is found to be  $u_F = [2 \text{solver}(u_L) - (1-\delta) u_L] / (1+\delta)$ .

It has long been recognized [2] that in dynamical problems involving Coulomb friction, some "catastrophes" may occur, generating velocity jumps in the absence of any collision. The above algorithms work all right in such situations [7].

In most our applications, the end of the iteration procedure is not decided upon the quasi-stationarity of the sequence of approximants, but on testing the precision at which the required relations are satisfied. In all experiments, the process has behaved properly, even for unrealistically large friction, but we have no proof of its convergence.

## 8. Deformable bodies.

The deformability of the considered bodies is accounted for through the introduction of a finite number of additional variables. In some instances these



variables may arise from a modal representation. More frequently, the displacement fields, with respect to some reference position, are discretized by means of finite element meshes. Let the real variables  $q^1, q^2, \dots, q^n$  refer to the corresponding degrees of freedom. Then the displacement at time  $t$ , say  $Q(t, x) \in \mathbf{R}^3$ , of a particle  $x$  belonging to one of the bodies, admits the approximate representation

$$Q(t, x) = q^i(t) E_i(x),$$

where  $E_1, \dots, E_n$  denote  $\mathbf{R}^3$ -valued functions associated with finite elements. In the course of a motion  $t \rightarrow q(t)$ , the approximate velocity vector of  $x$  is obtained by time derivation, as a linear expression in the derivatives  $\dot{q}^i$ .

Let  $\partial_c \Omega$  denote a material boundary part, that preliminary investigation has detected as possibly entering into contact with another body of the system or with some external obstacle. For every particle  $x \in \partial_c \Omega$ , one may express the relative velocity of  $x$ , with regard to its potential antagonist, in the form

$$(8.1) \quad U(t, x) = \mathcal{U}_i(t) E_i(x) = G_{ij}^q \dot{q}^j(t) E_i(x).$$

Here  $\mathcal{U}_i$  and  $G_{ij}^q$  have been constructed in such a way that the value  $U \in \mathbf{R}^3$  consists of the components of the said velocity with respect to a  $x$ -dependent orthonormal base made of vectors normal and tangent to the antagonist body at the expected contact point. Index  $i$  has only to run through a subset that we shall suppose to be  $\{1, \dots, p\}$ , corresponding to the finite elements involved in the representation of  $\partial_c \Omega$ .

In the principle, a contact law of the form (2.6) should be stipulated pointwise throughout  $\partial_c \Omega$ , enforcing a relation between the local relative velocity and the density of reaction force  $R(t, x)$ . Actually, one has to rely on an approximate representation of this force density, using an adequate set  $(F_j)$  of  $\mathbf{R}^3$ -valued base functions, in the form  $R(t, x) = \mathcal{R}_j(t) \cdot F_j(x)$ , with  $j$  running in  $\{1, \dots, p\}$  and  $\mathcal{R}_j \in \mathbf{R}^3$ . One constructs the matrix

$$P_{ij} = \int_{\partial_c \Omega} E_i(x) \cdot F_j(x) \, da \quad (da = \text{area}),$$

so the power of a distribution of reaction forces is expressed as  $(P G^q \dot{q}(t)) \cdot \mathcal{R}(t)$  or also  $(P \mathcal{U}(t)) \cdot \mathcal{R}(t)$ , where  $\mathcal{R}$  and  $\mathcal{U}$  denote two collections of  $p$  elements of  $\mathbf{R}^3$ .

Contact is discussed by introducing, instead of the functions  $f_\alpha$ , the *normal gap*  $q_N(t, x)$ , between particle  $x$  and the antagonist boundary. Then the contact laws adopted in the preceding Sections yield the classical *Signorini condition* which, for every  $x \in \partial_c \Omega$ , may be written as

$$(8.2) \quad R_N(t, x) \geq 0 \quad \text{and} \quad \forall s_N \in \mathbf{R}^+ : q_N(t, x)(s_N - R_N(t, x)) \geq 0.$$

Now  $q_N(t, x)$  lets itself be represented in the form  $Q_{Ni}(t)E_i(x)$ . Let us denote by  $P_T$  and  $P_N$  the blocks of the matrix  $P$  referring to tangential and normal components. By integrating both members of this inequality on  $\partial_c \Omega$ , one obtains

$$\mathcal{R}_N \in (\mathbb{R}^+)^p \quad \text{and} \quad \forall \mathcal{S}_N \in (\mathbb{R}^+)^p : P_N Q_N (\mathcal{S}_N - \mathcal{R}_N) \geq 0.$$

The same technique may be applied to Coulomb law, invoked in the form (4.6). For every  $j$  in  $\{1, \dots, p\}$ , a convex subset  $\mathcal{D}$  of the tangent plane is introduced as in Sec.4. The multiplication of these convex sets by the respective nonnegative normal reactions, yields a convex subset of the product space, denoted by  $\mathcal{R}_N \mathcal{D}$ . Then (4.6) becomes

$$\mathcal{R}_T = \mathcal{R}_N \mathcal{D} \quad \text{and} \quad \forall \mathcal{S}_T \in \mathcal{R}_N \mathcal{D} : P_T \mathcal{U}_T (\mathcal{S}_T - \mathcal{R}_T) \geq 0.$$

The numerical treatment of the problem is thus quite similar to what has been presented in the foregoing for the case of a system of rigid bodies.

The base vector functions  $F_j$  through which one represents the contact force densities are always chosen with nonnegative components. If the finite elements used in representing displacements are, for instance, of type T3 or Q4 the components of functions  $E_i$  are nonnegative, as well as the elements of the matrix  $P$ . In such a case, the approximation of the contact forces distributed on  $\partial_c \Omega$  by forces applied to the nodes proves to be a valid computation technique. On the contrary, the use of finite elements of type T6 or Q8 yields components of  $E_i$  with no definite sign and then the above machinery provides much better results than the approximation by forces concentrated at the nodes [32] [33].

In the event of a *collision*, an important difference should be stressed between the treatment of perfectly rigid bodies and that of deformable ones. In the latter case, *the total mass of the material which directly takes part in the contact is zero*. Since this material has by itself no inertia, any condition distinguishing between its right- and left-velocities, like those stipulated in Sec. 6 for systems of rigid bodies, is dynamically meaningless. Signorini conditions (8.2), joined with friction laws and, of course, the dynamical equations satisfied throughout the deformable bodies, are all what governs collisions.

## 9. Examples.

The first figure presents computed successive positions for a two-dimensional system of 18 rectangular rigid bodies submitted to gravity. This is meant to model a pillar made of blocks superposed without mortar. As a crude simulation of *earthquake*, the horizontal ground is given a motion of sinusoidal horizontal translation. The same value 0.3 has been adopted for the Coulomb friction coefficient between blocks and with ground. The dissipation index equals 1.

When two blocks touch each other along a line segment, interaction is treated as concentrated at vertices (as if block edges were slightly concave). Even so, the

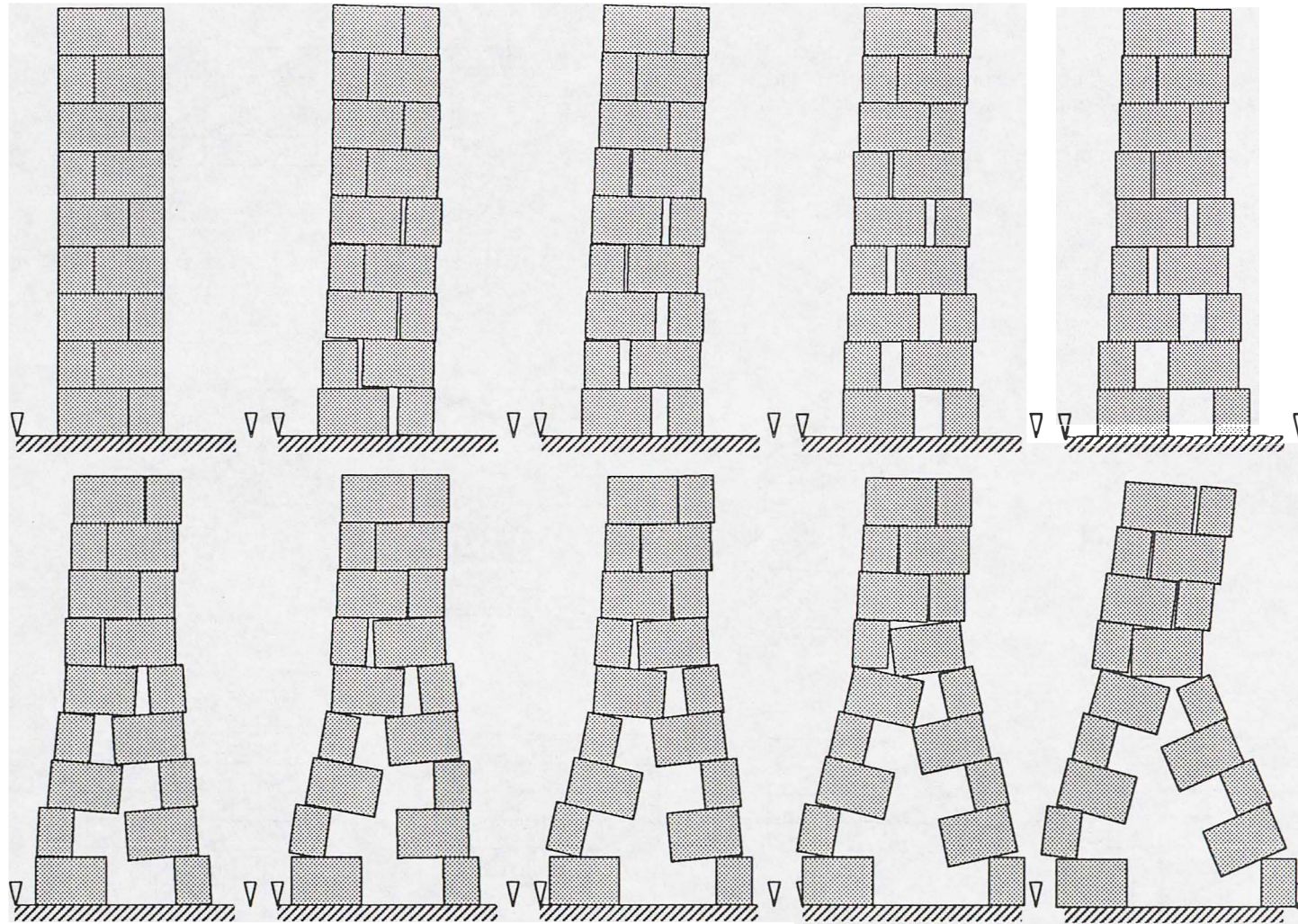
problem of calculating contact forces at every instant is likely to be "hyperstatic" and the motion consequent to initial data is not expected to be uniquely determined. Calculation is also very sensitive to constants: if the height of some block is slightly different from that of its neighbours, the motion is visibly altered. However, all simulations agree with a consistent ruin pattern: *the lower blocks progressively get apart, creating a gap.*

The second figure shows successive positions of two elastic blocks, the deformation of which is accounted for through two-dimensional finite element meshes, as developed in Sec. 9 above. The two blocks are made of the same material ( $E = 1300 \text{ MPa}$ ,  $\nu = 0.2$ , density  $= 7.8 \cdot 10^3 \text{ Kg/m}^3$ ).

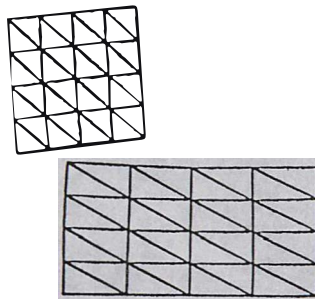
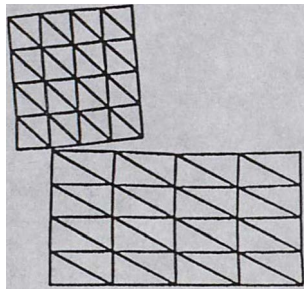
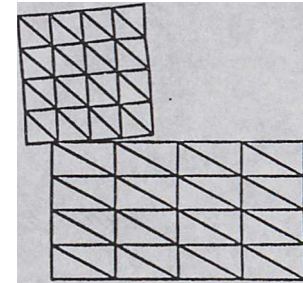
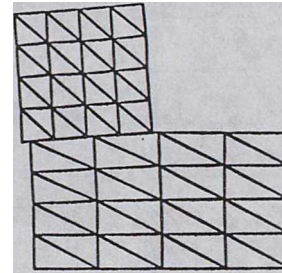
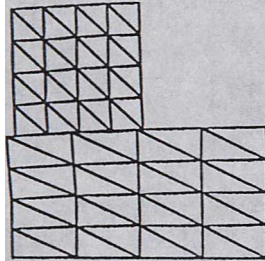
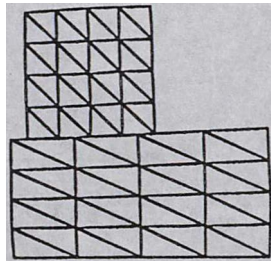
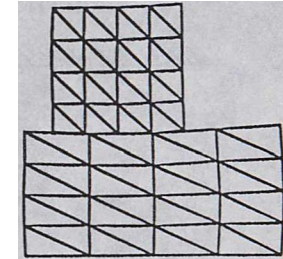
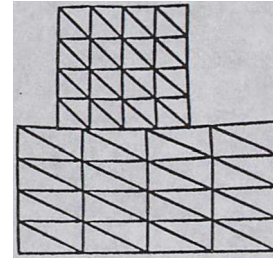
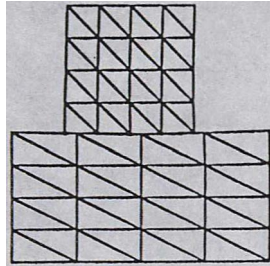
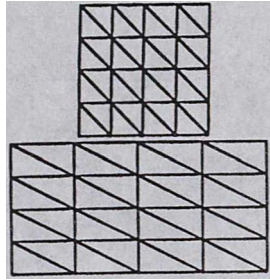
The larger block is fixed along its lower edge and initially at rest. Before hitting it, the upper block, with natural size of  $0.04 \text{ m} \times 0.04 \text{ m}$ , has a motion of translation with velocity components  $(-75 \text{ m/s}, -50 \text{ m/s})$ . Friction coefficient equals 0.1. Ten successive positions are shown, from  $t = 0$  to  $t = 0.75 \cdot 10^{-3} \text{ s}$ . After a phase of sliding contact, the overall motion results in a bounce, with persistent vibrations of each body.

The two series of drawings are extracted from microcomputer animations which allow for a more vivid perception of the effect of contacts.





Horizontal ground oscillation: period 0.4 s., range 15 cm. Pillar initial size: width 90 cm, height 360 cm.  
 Successive states shown at one period intervals. Friction coefficient between blocks and with ground: 0.3



### FRICTIONAL COLLISION OF TWO ELASTIC BLOCKS

Larger block initially at rest and fixed along its lower edge.

Smaller block collides with zero angular velocity  
and velocity components  $-75 \text{ m/s}$ ,  $-50 \text{ m/s}$ .



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