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# Application of convex analysis to some problems of dry friction

## 1 Introduction

The core of what is meant today by 'convex analysis' consists in studying convex subsets of linear spaces, convex real functions defined on such spaces, the extremal problems involving them and their minimax counterpart. This subject has received considerable attention during recent decades, stimulated by the frequent occurrence of convexity assumptions in optimization, economics and the related numerical analysis. Such modern developments as variational inequalities or monotone operators are closely interrelated with convex analysis.

To the author's historical knowledge, mechanics was the first domain of science to make a precise use of the concept of a convex set (17th century): the equilibrium positions of a solid body lying on a horizontal plane and subject to gravity are characterized by the condition that the vertical line drawn through its centre of mass meets the convex hull of the points of support. Investigating the statics or the dynamics of systems with unilateral constraints in this spirit constituted the author's primary motivation for taking some part in the recent development of convex analysis (*see, for example, Moreau [1, 2]*).

On the other hand, the importance of convexity assumptions regarding the potential function of a force law has been repeatedly stressed, mainly after Hill [3], in relation to stability, sometimes with hints at thermodynamics.

The concepts of the subdifferential of a convex function, now of general use in many domains, and of the superpotential of a force law, were defined by the author in order to include classical force laws and perfect (possibly unilateral) constraints in a unified treatment (*see, for example, Moreau [4–6]*).

The duality of linear spaces plays a prominent role in modern convex analysis. Regarding this concept also, mechanists acted as forerunners; in fact, the mathematical structure of a pair of linear spaces, placed in

duality by a bilinear form, constitutes the essence of the traditional method of virtual work or virtual power.

The present paper, attemptedly self-consistent, develops an example of the application of modern convex analysis to dry friction and ends with two general theorems. This theme was first presented at a small symposium on convexity in 1970 (Moreau [7]); subsequently, modern convex analysis has been widely applied to resistance laws of various sorts (Moreau [8]), mainly to *plasticity theory* (Moreau [9–11], Nayroles [12–15], Debordes and Nayroles [16]) possibly with strain hardening (Nguyen and Halphen [17], Nguyen [18]). The concrete and elementary situation taken as an example in what follows may be studied as an introduction to these more elaborate topics.

## 2 The classical formulation of Coulomb's law

Let  $\mathcal{S}_0$  denote a perfectly rigid body, assumed to be fixed; another perfectly rigid body  $\mathcal{S}_1$  moves in contact with  $\mathcal{S}_0$ . This means that the respective boundary surfaces  $\Sigma_0$  and  $\Sigma_1$ , supposed geometrically smooth, remain tangent at a point  $M$ , *a priori* moving in both of them. All the following is relative to some definite instant; in the corresponding configuration, let  $\mathbf{v}$  denote the unit normal vector to  $\Sigma_0$  and  $\Sigma_1$  at the point  $M$ . By definition, the *sliding velocity*  $\mathbf{V}$  of  $\mathcal{S}_1$  on  $\mathcal{S}_0$  is the velocity vector relative to  $\mathcal{S}_0$  of the element  $M_1$  of  $\mathcal{S}_1$  which happens to be in  $M$  at the instant under consideration. Under the usual geometrical and kinematical smoothness assumptions, it is elementarily proved that  $\mathbf{v} \cdot \mathbf{V} = 0$ , i.e.,  $\mathbf{V}$  belongs to the two-dimensional linear spaces  $\Pi$  consisting of the vectors tangent at  $M$  to  $\Sigma_0$  and  $\Sigma_1$ .

The contact forces exerted by  $\mathcal{S}_0$  on  $\mathcal{S}_1$  are supposed to reduce to a single force  $\mathbf{R}$  acting on the element  $M_1$ ; let us decompose this vector into

$$\mathbf{R} = \mathbf{F} + N\mathbf{v} \quad \text{with} \quad \mathbf{F} \in \Pi.$$

The Coulomb law of dry friction, *when the normal component*  $N \geq 0$  *is treated as known*, states a relation between  $\mathbf{F}$  and  $\mathbf{V}$  traditionally formulated as follows.

*There exists*  $f \geq 0$ , *the friction coefficient, such that*

$$\text{if } \mathbf{V} = 0 : |\mathbf{F}| \leq fN, \quad (1)$$

$$\text{if } \mathbf{V} \neq 0 : |\mathbf{F}| = fN \quad (2)$$

*and the vectors*  $\mathbf{F}$  *and*  $\mathbf{V}$  *are parallel with opposite directions. (Here*  $|\cdot|$  *represents the Euclidean norm.)*

Such a juxtaposition of two apparently heterogeneous statements concerning the events  $\mathbf{V} = 0$  and  $\mathbf{V} \neq 0$  might look purely empirical. Actually,

the use of some elementary concepts of convex analysis will emphasize their strong consistency.

### 3 Generalization

In the same situation as above, let us consider the closed disc

$$D = \{\Phi \in \Pi : |\Phi| \leq fN\}.$$

The Coulomb law is equivalently expressed by

$$\left. \begin{array}{l} \mathbf{F} \in D, \\ \forall \Phi \in D : \mathbf{V} \cdot (\Phi - \mathbf{F}) \geq 0. \end{array} \right\} \quad (3)$$

In fact, when  $\mathbf{V} \neq 0$ , this means that the set

$$\{\Phi \in \Pi : \mathbf{V} \cdot (\Phi - \mathbf{F}) \leq 0\},$$

i.e., the closed half-plane having  $\mathbf{F}$  as a boundary point and  $\mathbf{V}$  as an outward normal vector is a *supporting half-plane* of the set  $D$  at the point  $\mathbf{F}$  (i.e., this half-plane has only boundary points in common with  $D$  and  $\mathbf{F}$  is one of them). By the elementary properties of the circle, this is equivalent to (2). In the case  $\mathbf{V} = 0$ , the equivalence of (3) to (1) is trivial.

From this stage it is quite natural to generalize the formulation into a law of *anisotropic friction*, as it may physically result from the directional structure of the material surfaces in contact: the disc  $D$  will be replaced by some subset  $C$  of  $\Pi$ , containing the origin.

On the other hand, one aim of this paper is to emphasize the consideration of many-dimensional pairs of linear spaces. Generally speaking, a linear space of *velocities*  $\mathcal{V}$  and a linear space of *forces*  $\mathcal{F}$  will be introduced. These spaces are *placed in duality* by the bilinear form 'power': for  $\mathbf{V} \in \mathcal{V}$  and  $\mathbf{F} \in \mathcal{F}$ , we shall denote by  $\langle \mathbf{V}, \mathbf{F} \rangle$  the power of the force  $\mathbf{F}$  if the motion has the velocity  $\mathbf{V}$ . In the preceding example,  $\mathcal{V}$  and  $\mathcal{F}$  were two copies of the same two-dimensional Euclidean space  $\Pi$ , placed in duality with itself by the Euclidean scalar product.

Similarly to (3), let us define a *friction law* as the relation between  $\mathbf{V} \in \mathcal{V}$  and  $\mathbf{F} \in \mathcal{F}$  formulated as follows:

$$\left. \begin{array}{l} \mathbf{F} \in C, \\ \forall \Phi \in C : \langle \mathbf{V}, \Phi - \mathbf{F} \rangle \geq 0, \end{array} \right\} \quad (4)$$

where  $C$  is a given subset of  $\mathcal{F}$ . This subset is assumed to contain the origin of  $\mathcal{F}$ , i.e., zero is a possible value of  $\mathbf{F}$ , compatible in particular with the value zero of  $\mathbf{V}$ . Then, by setting  $\Phi = 0$ , it turns out that for every pair  $\mathbf{F}$ ,  $\mathbf{V}$  satisfying (4), the power  $\langle \mathbf{V}, \mathbf{F} \rangle$  is non-positive: friction, as described by relation (4), is a *dissipative* phenomenon.

## 4 The principle of maximum dissipation

Relation (4) is obviously equivalent to the following statement: *the set of the elements  $\mathbf{F} \in \mathcal{F}$  which the relation associates with a given  $\mathbf{V} \in \mathcal{V}$  is identical with the set of the points of  $C$  where the function  $\Phi \mapsto \langle \mathbf{V}, \Phi \rangle$  attains its infimum relative to  $C$ .*

In most of the mechanical situations which a relation of the form (4) is meant to describe, it is required that every value of  $\mathbf{V}$  be feasible, i.e., for every  $\mathbf{V} \in \mathcal{V}$  the above set is non-empty. Here is the most usual mathematical assumption ensuring that: there exists a topology on the space  $\mathcal{F}$  relative to which the set  $C$  is compact, while the real function  $\Phi \mapsto \langle \mathbf{V}, \Phi \rangle$  is continuous for every  $\mathbf{V}$  in  $\mathcal{V}$ . In the case of finite-dimensional  $\mathcal{V}$  and  $\mathcal{F}$ , this will naturally be the topology defined by the use of components in these linear spaces; then, the continuity of linear functions is automatic and we only have to make the assumption that  $C$  is *closed and bounded*. For infinite-dimensional cases, such as those arising in the mechanics of continua, it is necessary to specify some topology on  $\mathcal{F}$  among those which are said to be *compatible with the duality defined by the bilinear form  $\langle \cdot, \cdot \rangle$  or topologies of the dual pair  $(\mathcal{V}, \mathcal{F}, \langle \cdot, \cdot \rangle)$* ; in order that compactness involve the mildest restriction about  $C$ , this should be the coarsest of these topologies, i.e., the *weak topology* usually denoted by  $\sigma(\mathcal{F}, \mathcal{V})$  (see, for example, Robertson and Robertson [23]).

In the proper friction phenomenon, the non-negative expression  $-\langle \mathbf{V}, \mathbf{F} \rangle$  is equal to the power transformed into heat and is called the *dissipated power*: thus, relation (4) may be entitled ‘the principle of maximum dissipation’.

## 5 Indicator functions and subdifferentials

Every subset  $C$  of  $\mathcal{F}$  may be described by giving its *indicator function*  $\psi_C$

$$\psi_C(\Phi) = \begin{cases} 0 & \text{if } \Phi \in C, \\ +\infty & \text{if } \Phi \notin C. \end{cases}$$

Using this, one writes relation (4) equivalently in the form

$$\mathbf{V}\Phi \in \mathcal{F} : \langle -\mathbf{V}, \Phi - \mathbf{F} \rangle + \psi_C(\mathbf{F}) \leq \psi_C(\Phi)$$

which states that  $\psi_C(\mathbf{F})$  is finite, that the *affine function*

$$\Phi \mapsto \langle -\mathbf{V}, \Phi - \mathbf{F} \rangle + \psi_C(\mathbf{F})$$

is a *minorant* of the function  $\psi_C$  and that this minorant is *exact* at the point  $\mathbf{F}$ , i.e., it takes the same value at this point as  $\psi_C$  (namely zero). The

element  $-\mathbf{V}$  of  $\mathcal{V}$  constitutes the *slope* or *gradient* of the considered affine function in the sense of the duality  $\langle \cdot, \cdot \rangle$ . According to the terminology introduced by the author [4] and now usual in the whole field of convex analysis, the gradient of an affine minorant of a function  $g : \mathcal{F} \rightarrow ]-\infty, +\infty]$ ; if this minorant is exact at the point  $\mathbf{F}$ , is called a *subgradient* of  $g$  at the point  $\mathbf{F}$ . The set, denoted by  $\partial g(\mathbf{F})$ , of the subgradients of  $g$  at the point  $\mathbf{F}$  is a (possibly empty) convex subset of  $\mathcal{V}$  called the *subdifferential* of  $g$  at  $\mathbf{F}$ .

In this notation, the relation (4) is equivalently written as

$$-\mathbf{V} \in \partial \psi_C(\mathbf{F}). \quad (5)$$

Usually,  $C$  is a closed convex set (*see* section 7 below); (5) means that  $\mathbf{F} \in C$  and that  $\mathbf{V}$  is, in a classical generalized sense, a *normal inward vector* to this set at the point  $\mathbf{F}$  (in particular,  $\mathbf{V} = 0$  if  $\mathbf{F}$  is internal to  $C$ ).

## 6 Dissipation function

Let  $g$  be a function with values in  $]-\infty, +\infty]$ , defined, for instance, on the member  $\mathcal{F}$  of the considered dual pair of linear spaces. The handling of the affine minorants of  $g$  induces us to construct the function  $g^*$ , with values in  $]-\infty, +\infty]$ , defined on  $\mathcal{V}$  by

$$g^*(\mathbf{W}) = \sup_{\Phi \in \mathcal{F}} [\langle \mathbf{W}, \Phi \rangle - g(\Phi)]. \quad (6)$$

It is called the *conjugate* or *polar function* of  $g$ . An affine function  $\Phi \rightarrow \langle \mathbf{W}, \Phi \rangle - \alpha$  is a minorant of  $g$  if and only if the real number  $\alpha$  satisfies  $\alpha \geq g^*(\mathbf{W})$ .

In the special case,  $g = \psi_C$ , the expression  $\langle \mathbf{W}, \Phi \rangle - \psi_C(\Phi)$  takes the value  $-\infty$  when  $\Phi \notin C$ ; therefore,

$$\psi_C^*(\mathbf{W}) = \sup_{\Phi \in C} \langle \mathbf{W}, \Phi \rangle.$$

This function is classically known under the (rather improper) name of the *support function* of the set  $C$ , relative to the considered duality. It is evidently *sublinear*, i.e., convex and positively homogeneous of degree 1.

In the present situation, it will prove more convenient to introduce the function  $\varphi$

$$\varphi(\mathbf{W}) = \psi_C^*(-\mathbf{W}) = -\inf_{\Phi \in C} \langle \mathbf{W}, \Phi \rangle, \quad (7)$$

i.e., the support function of the set  $-C$ . Using  $\varphi$  yields an equivalent

formulation of the relations (4) or (5)

$$\left. \begin{array}{l} \mathbf{F} \in C, \\ -\langle \mathbf{V}, \mathbf{F} \rangle = \varphi(\mathbf{V}). \end{array} \right\} \quad (8)$$

In other words, the values of  $\mathbf{F} \in \mathcal{F}$  that the relation associates with a given  $\mathbf{V} \in \mathcal{V}$  are the elements of  $C$  such that the dissipated power  $-\langle \mathbf{V}, \mathbf{F} \rangle$  is equal to  $\varphi(\mathbf{V})$ .

Hence the name of *the dissipation function*, given to  $\varphi$ .

For instance, in the case of Coulomb's law (3),

$$\varphi(\mathbf{V}) = fN |\mathbf{V}|.$$

## 7 The convexity of $C$

Returning to the definition (6) of  $g^*$ , one immediately finds that the relation  $\mathbf{W} \in \partial g(\mathbf{F})$  is equivalent to

$$g^*(\mathbf{W}) + g(\mathbf{F}) - \langle \mathbf{W}, \mathbf{F} \rangle = 0 \quad (9)$$

where the  $=$  sign may be replaced by  $\leq$  because the left-hand side is essentially non-negative.

The above does not, in general, involve the symmetry between the spaces  $\mathcal{V}$  and  $\mathcal{F}$ . In fact, a polar function is, by construction, the supremum of a collection of continuous affine functions; therefore, it is convex and lower semi-continuous (l.s.c.) (relative to every topology compatible with the considered duality). As we started with an arbitrary  $g: \mathcal{F} \rightarrow ]-\infty, +\infty]$ , it cannot be expected, in general, that  $g$  would in turn be the polar function of  $g^*$ . However, standard separation arguments (i.e., the Hahn–Banach theorem) may be used to prove that  $g$  is equal to  $g^*$  if and only if  $g$  is convex and lower semi-continuous (for some of the topologies of the dual pair  $(\mathcal{V}, \mathcal{F}, \langle \cdot, \cdot \rangle)$ , consequently for all of them). *If such is the case, the symmetry of (9) implies that  $\mathbf{W} \in \partial g(\mathbf{F})$  is equivalent to  $\mathbf{F} \in \partial g^*(\mathbf{W})$ .*

In the following, we shall deal with the special case  $g = \psi_C$ ; this function is convex and l.s.c. if and only if *the subset  $C$  of  $\mathcal{F}$  is convex and closed*. If such is the case, the friction law, as expressed equivalently by (4), (5) or (8), is also equivalent to

$$\mathbf{F} \in \partial \psi_C^*(-\mathbf{V}),$$

i.e., in view of the definition (7) of  $\varphi$ ,

$$-\mathbf{F} \in \partial \varphi(\mathbf{V}). \quad (10)$$

**Remark 1** A relation of this form between some velocity  $\mathbf{V}$  and some force  $\mathbf{F}$  may be called a *resistance law*, admitting the (convex and l.s.c.)

function  $\varphi$  as *superpotential* or *pseudopotential*. The more general case, where  $\varphi$  is not necessarily sublinear, was studied in Moreau [8], where the connection of the superpotential with the dissipated power was also investigated.

**Remark 2** In all the following, the set  $C$  is assumed to be convex. As far as the contact friction is concerned, the contrary would seem unrealistic. In fact, one must keep in mind that the point contact between two bodies is only a schematic representation of some contact which takes place on a very small area  $\mathcal{A}$ . We may imagine, this area to be arbitrarily divided into two others,  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , in which the sliding velocity has the same value, namely zero in what follows. Let  $\mathbf{R}_1$  and  $\mathbf{R}_2$  be the resultant forces experienced by  $\mathcal{S}_1$  through  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , respectively. Then

$$\mathbf{R} = \mathbf{R}_1 + \mathbf{R}_2 \quad (11)$$

and, concerning the normal components, one has  $N = N_1 + N_2$ . The values of  $N_1$  and  $N_2$  in the last equation depend on some ‘micro-information’ about the distribution of pressure in  $\mathcal{A}$ . In order to obtain a law which does not depend on the microscopic pressure distribution in  $\mathcal{A}$  for the global reaction  $\mathbf{R}$ , one must admit the following as the law of friction in every subarea such as  $\mathcal{A}_1$  (or  $\mathcal{A}_2$ , or  $\mathcal{A}$  itself): for zero sliding velocity and an arbitrary non-negative pressure component, the set of the possible values of  $\mathbf{R}_1$  (or  $\mathbf{R}_2$  or  $\mathbf{R}$ ) is a conic subset  $\Gamma$ , with vertex at the origin in the space of the three-dimensional vectors. And (11) entails the inclusion  $\Gamma + \Gamma \subset \Gamma$ , which means that  $\Gamma$  is convex. Returning to the formulation (4), one finds that  $\Gamma$  is the cone generated in the space of three-dimensional vectors by the set  $C + N\mathbf{v}$ ; hence, the convexity of  $C$ .

## 8 Product spaces

Let  $(\mathcal{V}_1, \mathcal{F}_1, \langle \cdot, \cdot \rangle_1)$  and  $(\mathcal{V}_2, \mathcal{F}_2, \langle \cdot, \cdot \rangle_2)$  be two dual pairs of linear spaces. The two product spaces  $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$  and  $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$  are placed in duality by the bilinear form  $\langle \cdot, \cdot \rangle$ , defined as follows: for  $v = (v_1, v_2) \in \mathcal{V}_1 \times \mathcal{V}_2$  and  $f = (f_1, f_2) \in \mathcal{F}_1 \times \mathcal{F}_2$ , set

$$\langle v, f \rangle = \langle v_1, f_1 \rangle_1 + \langle v_2, f_2 \rangle_2. \quad (12)$$

For instance, if  $v_1, v_2$  are some independent velocity parameters of a mechanical system and  $f_1, f_2$  are the associated force parameters such that the terms on the right in (12) represent their respective powers, the bilinear form  $\langle v, f \rangle$  represents the power of the whole.

In this framework, the following is easily established. Let  $g_1: \mathcal{F}_1 \rightarrow ]-\infty, +\infty]$  and  $g_2: \mathcal{F}_2 \rightarrow ]-\infty, +\infty]$  and let  $g$  be the function defined for



every  $f = (f_1, f_2)$  in  $\mathcal{F}$  by

$$g(f) = g_1(f_1) + g_2(f_2).$$

If  $g_1^*$  and  $g_2^*$  are the respective polar functions of  $g_1$  and  $g_2$ , the polar function  $g^*$  in the sense of the duality (12) is defined for every  $v = (v_1, v_2)$  in  $\mathcal{V}$  by

$$g^*(v) = g_1^*(v_1) + g_2^*(v_2).$$

Concerning the subdifferential sets, on the other hand, one has, in the sense of the three respective dualities, the equivalence

$$v \in \partial g(f) \Leftrightarrow v_1 \in \partial g_1(f_1) \quad \text{and} \quad v_2 \in \partial g_2(f_2). \quad (13)$$

## 9 An example of composite friction law

Let  $\mathcal{S}_1$  be one of the wheels, with radius  $a$ , by which some vehicle  $\mathcal{S}_2$  is supported, possibly with skidding, upon the horizontal ground  $\mathcal{S}_0$ . We shall treat this wheel as a perfectly rigid body presenting a single point of contact  $M$  with the plane surface of the ground. Let us describe the friction at this point by means of the notations of section 2. The reaction  $\mathbf{R}$  exerted by the ground on the wheel is written as

$$\mathbf{R} = \mathbf{F} + N\mathbf{v}$$

and, in view of (5), the Coulomb law takes the form

$$-\mathbf{V} \in \partial \psi_D(\mathbf{F}), \quad (14)$$

where  $D$  denotes the closed disc with radius  $fN$ , with centre at the origin in the two-dimensional Euclidean linear space  $\Pi$ ; recall that  $\mathbf{V}$  is the sliding velocity of  $\mathcal{S}_1$  relative to  $\mathcal{S}_0$ .

On the other hand a *brake* is supposed to act on the wheel. Let  $\mathbf{i}$  denote a unit vector of the wheel axis, assumed to be parallel to the ground. Let  $h$  be the moment, relative to this oriented axis, of the forces that the wheel  $\mathcal{S}_1$  experiences from the vehicle body  $\mathcal{S}_2$ . Neither the driving torque nor the friction in the bearings are taken into account, so that  $h$  is, in fact, the braking torque. Let  $\omega$  be the angular velocity of  $\mathcal{S}_1$  relative to  $\mathcal{S}_2$ . We assume that the operation of the brake involves a dry friction with a given normal component. This is expressed by a relation between the real numbers  $h$  and  $\omega$ , namely the one-dimensional case of the general formalism presented in the foregoing. Here the space of velocities and the space of forces are two copies of the real line  $\mathbb{R}$  and the bilinear form ‘power’ reduces to the ordinary product. In the sense of this duality, the brake law is written as

$$-\omega \in \partial \psi_I(h) \quad (15)$$

where  $I$  denotes a given interval  $[-b, +b]$ ; this summarizes the familiar relations:  $h = -b \operatorname{sgn} \omega$  if  $\omega \neq 0$  and  $h$  arbitrary in  $[-b, +b]$  if  $\omega = 0$ .

In view of the preceding section, taking into account (14) and (15), we note that, equivalently, we have the relation

$$-(\omega, \mathbf{V}) \in \partial \psi_{I \times D}(h, \mathbf{F}) \quad (16)$$

in the sense of the duality  $(\mathcal{V}, \mathcal{F}, \langle \cdot, \cdot \rangle)$ . Here,  $\mathcal{V}$  and  $\mathcal{F}$  are two copies of the three-dimensional linear space  $\mathbb{R} \times \Pi$ ; by definition,

$$\langle (\omega, \mathbf{V}), (h, \mathbf{F}) \rangle = \omega h + \mathbf{V} \cdot \mathbf{F} \quad (17)$$

is the total power of the torque  $h$  and of the ground reaction  $\mathbf{R} = \mathbf{F} + N\mathbf{v}$  acting simultaneously on the wheel.

Here is the first problem we are to deal with in the following. The wheel load, i.e., the normal component  $N$  of  $\mathbf{R}$ , will be treated as known. Let  $\mathbf{G} \in \Pi$  denote the horizontal component of the resultant force experienced by the vehicle  $\mathcal{S}_2$  from the wheel  $\mathcal{S}_1$ . Let  $\mathbf{W} \in \Pi$  denote the velocity of the wheel centre, which is also the velocity of the corresponding point of  $\mathcal{S}_2$  or the velocity of  $M$ , the 'geometrical' point of contact with the ground. Under the assumption that the wheel is sufficiently light and the motion sufficiently slow for the inertia of the wheel to be negligible, we are to summarize the combination of possible skidding on the ground and of possible brake action into a simple relation between  $\mathbf{W}$  and  $\mathbf{G}$ . It will turn out, under the above simplifying assumptions, that the wheel may be forgotten and the interaction between the vehicle and the ground be described as the anisotropic friction related to a certain convex set  $C$ .

This consists in the *elimination* of the variables  $\omega$ ,  $\mathbf{V}$ ,  $h$ ,  $\mathbf{F}$  from the following set of relations.

(i) The kinematical relation

$$\mathbf{W} = \mathbf{V} - \omega a \mathbf{j} \quad (18)$$

expressing the fact that the wheel is a rigid body; here  $\mathbf{j}$  is the unit vector  $\mathbf{v} \times \mathbf{i}$ .

(ii) The quasi-equilibrium equations of the wheel:

$$\mathbf{G} - \mathbf{F} = 0, \quad (19)$$

$$h + a \mathbf{j} \cdot \mathbf{F} = 0. \quad (20)$$

(iii) The composite friction law (16).

An adequate use of various rules of the 'subdifferential calculus' would do the job, but it will be more instructive to place the reasoning in a general setting.

## 10 Subdifferentials and linear mappings

Let  $(\mathcal{V}, \mathcal{F})$  and  $(\mathcal{V}', \mathcal{F}')$  be two dual pairs of linear spaces; both corresponding bilinear forms will be denoted by  $\langle \cdot, \cdot \rangle$ . Let  $L: \mathcal{F}' \rightarrow \mathcal{F}$  be a linear mapping; in infinite-dimensional cases, it will be assumed that  $L$  is continuous in the weak topologies of the dual pairs. Let  $g: \mathcal{F} \rightarrow ]-\infty, +\infty]$  be convex and l.s.c.; then, the composite function  $g' = g \circ L: \mathcal{F}' \rightarrow ]-\infty, +\infty]$  is convex and l.s.c. A classical rule of the 'subdifferential calculus' is the following (Rockafellar [24]):

*If there exists a point in the range of  $L$ , where the function  $g$  is finite and continuous (in some topology of the dual pair  $(\mathcal{V}, \mathcal{F})$ ), one has, for every  $F'$  in  $\mathcal{F}'$ ,*

$$\partial(g \circ L)(F') = L^*(\partial g(L(F'))), \quad (21)$$

where  $L^*$  denotes the transpose of  $L$ .

**Application** Returning to our mechanical problem, let us set  $\mathcal{V} = \mathcal{F} = \mathbb{R} \times \Pi$  with the duality defined in (17). Moreover, let  $\mathcal{V}' = \mathcal{F}' = \Pi$  with the duality defined by the Euclidean scalar product and define  $L: \mathcal{F}' \rightarrow \mathcal{F}$  by

$$L(\mathbf{G}) = (-a\mathbf{j} \cdot \mathbf{G}, \mathbf{G}).$$

Elementary computation yields the transpose  $L^*: \mathcal{V}' \rightarrow \mathcal{V}'$ , namely

$$L^*(\omega, \mathbf{V}) = \mathbf{V} - \omega a\mathbf{j}.$$

Then, (18) amounts to

$$\mathbf{W} = L^*(\omega, \mathbf{V}), \quad (22)$$

while (19) and (20) are condensed into

$$(h, \mathbf{F}) = L(\mathbf{G}). \quad (23)$$

Take  $g = \psi_{I \times D}$ , so that (16) takes the form

$$-(\omega, \mathbf{V}) \in \partial g(h, \mathbf{F}). \quad (24)$$

The elimination of  $\omega$ ,  $\mathbf{V}$ ,  $h$ ,  $\mathbf{F}$  from (22), (23), (24) yields

$$-\mathbf{W} \in L^*(\partial g(L(\mathbf{G}))). \quad (25)$$

This, indeed, is the necessary and sufficient condition to be satisfied by  $\mathbf{W}$  and  $\mathbf{G}$  in order that there exist  $\omega$ ,  $\mathbf{V}$ ,  $h$ ,  $\mathbf{F}$  which also satisfy the above conditions.

Suppose now that the interval  $I$  and the disc  $D$  do not degenerate into single points; then the zero of  $\mathcal{F}$  constitutes a point in the range of  $L$  where  $g$  is continuous; thus (21) holds, making (25) equivalent to

$$-\mathbf{W} \in \partial(g \circ L)(\mathbf{G}).$$

Here is the expression for the function  $g \circ L$ :

$$\begin{aligned}(g \circ L)(\mathbf{G}) &= \psi_{I \times D}(-a\mathbf{j} \cdot \mathbf{G}, \mathbf{G}) \\ &= \begin{cases} 0 & \text{if } -a\mathbf{j} \cdot \mathbf{G} \in I \text{ and } \mathbf{G} \in D, \\ +\infty & \text{otherwise.} \end{cases}\end{aligned}$$

In other words  $g \circ L$  is the indicator function of the closed convex subset  $C = B \cap D$  of  $\Pi$ , where  $B$  denotes the strip

$$B = \left\{ \mathbf{G} \in \Pi : -\frac{b}{a} \leq \mathbf{j} \cdot \mathbf{G} \leq \frac{b}{a} \right\}.$$

Hence, the final form of (25),

$$-\mathbf{W} \in \partial\psi_C(\mathbf{G}), \quad (26)$$

which constitutes a *friction law* in the general sense of section 3. As announced, this presents the interaction between the vehicle and the ground by forgetting the wheel. Recall that  $N$ , the load supported by the wheel, is treated as known.

Let us now discuss the various cases.

(i) If  $b \geq afN$ , the width of the strip  $B$  is greater than or equal to the diameter of the disc  $D$ ; hence,  $C = D$ . This means that the brake is so tightly applied that the wheel stays always locked; thus, the interaction between the vehicle and the ground amounts to the simple Coulomb friction.

(ii) If  $b < afN$  the shape of  $C$  is shown in Fig. 1. Recall that (26) expresses that  $\mathbf{G} \in C$  and that, in the classical generalized sense,  $\mathbf{W}$  is an *inward normal vector* to  $C$  at the point  $\mathbf{G}$  (in particular,  $\mathbf{W}$  is necessarily zero if  $\mathbf{G}$  is internal to  $C$ ). The presence of the rectilinear parts in the boundary of  $C$  implies that a value of  $\mathbf{W}$  parallel to  $\mathbf{j}$  (i.e., normal to the wheel plane) corresponds to an infinity of possible values of the force  $\mathbf{G}$ . The presence of corner points in the boundary implies that these corner

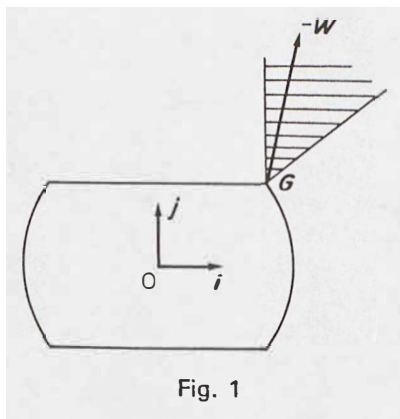


Fig. 1

values of  $\mathbf{G}$  correspond to an infinity of values for  $\mathbf{W}$ , the set of them form a closed convex angular region in  $\Pi$ .

Note that some more refined arguments of convex analysis (see, for example, Rockafellar [20], Theorem 23.8) allow us to remove the assumption  $b \neq 0$  made in the foregoing. The case  $b = 0$  is that where no brake is applied; then,  $C$  reduces to a line segment and (26) describes an extreme case of anisotropic friction. Such a side-slipping free wheel is the key device of the Amsler planimeter and of some other ancient integrating instruments.

**Remark** The above computation involving a pair of mutually transpose linear mappings  $L$  and  $L^*$  is more than an occasional mathematical trick. It is based on the fact that the rigidity of the wheel constitutes a *perfect* mechanical constraint; in fact, the external forces applied to the wheel may be summarized as:

- (i) the force  $\mathbf{F}$  applied to the contact point (as the normal component  $N\mathbf{v}$  does not matter here);
- (ii) the force  $-\mathbf{G}$  applied to the wheel centre;
- (iii) the axial torque  $h$ .

By the definition of a perfect constraint (see more developments and examples in Moreau [6, 10]) the (quasi-)equilibrium condition of the wheel may be expressed by the fact that the above system of forces should yield a zero power for every set of values of the velocity parameters  $\mathbf{V}$ ,  $\mathbf{W}$ ,  $\omega$  satisfying the kinematical equations of constraint, namely (22). In other words,

$$-\mathbf{G} \cdot L^*(\omega, \mathbf{V}) + \mathbf{F} \cdot \mathbf{V} + h\omega = 0$$

must hold for every  $(\omega, \mathbf{V})$ ; by the definition of transpose mappings, this is precisely (23).

## 11 The principle of minimum dissipation

The purpose of the preceding section was the elimination of  $\mathbf{V}$ , the sliding velocity of the wheel on the ground, and  $\omega$ , the sliding angular velocity in the brake. A complementary problem now is the determination of these sliding velocities when the motion of the vehicle is treated as known, i.e., when  $\mathbf{W}$  is given. The existence of at least one solution  $\omega$ ,  $\mathbf{V}$  for every  $\mathbf{W} \in \Pi$  results from the preceding section, in view of the compactness of  $C$ .

For a given  $\mathbf{W}$ , the set of the pairs  $(\omega, \mathbf{V})$  satisfying the kinematical condition of rigidity of the wheel (22) is an affine submanifold  $\mathcal{E}$  of  $\mathbb{R} \times \Pi$ ,

namely

$$\mathcal{E} = \{(\omega, \mathbf{V}) \in \mathbb{R} \times \Pi : \mathbf{V} - \omega \mathbf{a} \mathbf{j} = \mathbf{W}\}.$$

Introducing its indicator function  $\psi_{\mathcal{E}}$ , one observes that the subdifferential set  $\partial\psi_{\mathcal{E}}(\omega, \mathbf{V})$ , empty if  $(\omega, \mathbf{V}) \notin \mathcal{E}$ , consists otherwise in the subspace of  $\mathbb{R} \times \Pi$  orthogonal to  $\mathcal{E}$ ; this is precisely the set of the pairs  $(h, \mathbf{F})$  satisfying the quasi-equilibrium condition (23) for some  $\mathbf{G}$ . Thus, (22) and (23) are equivalently condensed into

$$(h, \mathbf{F}) \in \partial\psi_{\mathcal{E}}(\omega, \mathbf{V}). \quad (27)$$

(Regarding such an interpretation of perfect constraints as ‘resistance laws’ with pseudopotentials, see Moreau [8, 10].)

On the other hand, using section 6, the composite friction law (16) is equivalently written as

$$(-h, -\mathbf{F}) \in \partial\varphi(\omega, \mathbf{V}), \quad (28)$$

where  $\varphi : \mathbb{R} \times \Pi \rightarrow \mathbb{R}$  denotes the total dissipation function

$$\begin{aligned} \varphi(\omega, \mathbf{V}) &= \psi_{I \times D}^*(-\omega, -\mathbf{V}) = \psi_I^*(-\omega) + \psi_D^*(-\mathbf{V}) \\ &= b |\omega| + fN |\mathbf{V}|. \end{aligned} \quad (29)$$

The elimination of  $h$  and  $\mathbf{F}$  from (27) and (28) yields

$$(0, 0) \in \partial\psi_{\mathcal{E}}(\omega, \mathbf{V}) + \partial\varphi(\omega, \mathbf{V}). \quad (30)$$

Since the function  $\varphi$  is continuous, the *addition rule for subdifferentials* (cf., Moreau [4] or the books [19–22]) may be applied so that (30) exactly expresses that  $(\omega, \mathbf{V})$  is a point of  $\mathbb{R} \times \Pi$ , where the function  $\psi_{\mathcal{E}} + \varphi$  attains its minimum. Recalling that the function  $\psi_{\mathcal{E}}$  takes the value zero on  $\mathcal{E}$  and  $+\infty$  elsewhere, one concludes as follows.

*If  $\mathbf{W}$  is given, the values of  $\omega, \mathbf{V}$  solving the problem minimize the dissipation function (29) under the kinematical condition (18).*

Consequently, if one is only interested in the unknown  $\omega$ , it turns out that the values of it which solve the problems are exactly the points of  $\mathbb{R}$  where the function

$$\omega \mapsto fN |\mathbf{W} + \omega \mathbf{a} \mathbf{j}| + b |\omega|$$

attains its minimum.

**Remark** The preceding expresses a minimization ‘principle’ for the dissipation function restricted to the set of the ‘kinematically admissible’ velocities. The same is a classical feature in plasticity theory. (Concerning the use of convex analysis in treating the variational properties of elastoplastic systems, see Moreau [25].) Another interesting example, involving a continuous system, is that of a heavy perfectly flexible inextensible rope lying with dry friction on a horizontal plane. Here, the

velocity distribution entailed in quasistatic evolution by some imposed motion of the rope extremities, is characterized by minimizing the dissipation function on the set of the velocity distributions agreeing with these end conditions and with inextensibility.

## 12 Quotient spaces

We present now an abstract structure in which the reader will recognize a generalization of the foregoing.

Let us consider a mechanical system in a given configuration. Let  $\mathcal{V}$  denote a linear space, the elements of which constitute, in some general sense, the possible values of the *velocity* of the system if it passes through the considered configuration. Let  $\mathcal{F}$  denote a linear space the elements of which are, in a general sense, the possible values of the various forces the system may experience in this configuration. These two spaces are placed in duality by the bilinear-form ‘power’ noted  $\langle \cdot, \cdot \rangle$ .

The spaces  $\mathcal{V}$  and  $\mathcal{F}$  may have very diverse functional realizations, namely spaces of vector fields, of tensor fields, etc. and the considered mathematical procedure can usually be applied to a given mechanical situation in several different ways (*see*, for example Moreau [10], Nayroles [14].)

Let us suppose that the system is subject to a friction force  $F$  obeying the law

$$-V \in \partial\psi_C(F), \quad (31)$$

where  $C$  is a given convex subset of  $\mathcal{F}$ , closed in the topologies of the dual pair  $(\mathcal{V}, \mathcal{F}, \langle \cdot, \cdot \rangle)$ .

In addition, the system experiences a *moving constraint* or ‘driving’ which implies that its velocity  $V$  belongs to a certain affine submanifold  $\mathcal{E}$  of  $\mathcal{V}$ , a translate of some given linear subspace  $\mathcal{W}$  of  $\mathcal{V}$ , closed in the topologies of the dual pair  $(\mathcal{V}, \mathcal{F}, \langle \cdot, \cdot \rangle)$ . Specifying  $\mathcal{E}$  among the various translates of  $\mathcal{W}$  amounts to prescribing an element  $W$  of the quotient space  $\mathcal{V}/\mathcal{W}$ : we shall refer to this element as the *driving velocity* (mathematically,  $\mathcal{E}$  and  $W$  are the same thing but using two notations seems clearer in mechanical applications).

The moving constraint is assumed to be perfect, i.e., the ‘force’  $R \in \mathcal{F}$  exerted on the system by the driving device is orthogonal to the affine manifold  $\mathcal{E}$ , in the sense of the duality  $(\mathcal{V}, \mathcal{F}, \langle \cdot, \cdot \rangle)$

$$R \in \mathcal{W}^\perp. \quad (32)$$

The constraint is also assumed to be *firm*, i.e., the driving device is strong enough to provide any value of  $R$  satisfying (32). (Concerning the concept of the firmness of a mechanical constraint, *see* Moreau [26].) The

opposite,  $G = -R$ , may be interpreted as the *resistance of the system to the driving*.

Inertia is neglected, so that the motion is characterized by the quasi-static equation

$$G = F. \quad (33)$$

The duality  $(\mathcal{V}, \mathcal{F}, \langle \cdot, \cdot \rangle)$  classically induces a duality between the pair  $\mathcal{V}/\mathcal{W}$  and  $\mathcal{W}^\perp$ . Our purpose is the elimination of  $V$  and  $F$ , yielding a relation between the elements  $W$  and  $G$  of the latter dual pair.

**Proposition 1** *If the above conditions are satisfied, the elements  $W$  and  $G$  satisfy*

$$-W \in \partial\psi_D(G) \quad (34)$$

*in the sense of the dual pair of spaces  $\mathcal{V}/\mathcal{W}$  and  $\mathcal{W}^\perp$ , with  $D = C \cap \mathcal{W}^\perp$ .*

*Conversely, if  $\mathcal{W}^\perp$  meets the interior of  $C$  (in some topology of the dual pair  $\mathcal{V}, \mathcal{F}$ ), every  $(W, G)$  satisfying (34) corresponds to at least one pair  $(V, F)$  agreeing with above conditions.*

*Under the same topological assumption, the dissipation function  $\gamma$  of the friction law (34) is defined on  $\mathcal{V}/\mathcal{W}$  by*

$$\gamma(W) = \min_{U \in \mathcal{W}} \varphi(U), \quad (35)$$

*where  $\varphi$  denotes the dissipation function of the friction law (31).*

For the proof, we may call  $L$  the natural injection of  $\mathcal{W}^\perp$  into  $\mathcal{F}$  and  $L^*$  its transpose, namely the natural surjection of  $\mathcal{V}$  onto  $\mathcal{V}/\mathcal{W}$ . This allows us to write the conditions of the problem in the form

$$F = L(G) \quad W = L^*(V)$$

by which the elimination of  $V$  and  $F$  from (31) leads to

$$-W \in L^*(\partial\psi_C(L(G))). \quad (36)$$

Since  $\psi_C \circ L = \psi_D$ , the computation rule of section 10 yields the equivalence of (34) to (36), because the assumption that  $\mathcal{W}^\perp$  meets the interior of  $C$  means the existence of a point of the range of  $L$  where  $\psi_C$  is finite and continuous. Without this assumption, however, (36) is easily proved to entail (34); this is the first statement of the proposition.

Finally, (35) results from the computation rule for  $(\psi_C \circ L)^*$  (see Rockafellar [24], Theorem 3).

The above proposition, involving the quotient space  $\mathcal{V}/\mathcal{W}$ , may be described as a way of processing some *partial information* about the considered mechanical system. Actually, science is always dealing with



partial information about nature; thus, it could be said that similar constructions of quotient spaces implicitly underlie every scientific act.

As in Section 11, let us turn now to the complementary question of characterizing the value of  $V$  corresponding to a given  $W$ , which, incidentally, will throw some light on the expression (35) of  $\gamma$ .

**Proposition 2** *In the affine manifold  $\mathcal{E}$ , the set of the elements  $V$  satisfying the conditions of the problem, if not empty, is equal to the set of the points where the restriction  $\varphi|_{\mathcal{E}}$  of the dissipation function  $\varphi$  attains a finite minimum.*

*If moreover, there exists a point in  $\mathcal{E}$  where  $\varphi$  is finite and continuous (in some topology of the dual pair  $\mathcal{V}, \mathcal{F}$ ) the two above sets are equal, even if the first one is empty.*

In fact, the conditions  $V \in \mathcal{E}$  and  $F \in \mathcal{W}^\perp$  are condensed into

$$F \in \partial\psi_{\mathcal{E}}(V),$$

while (31) is, by section 7, equivalent to

$$-F \in \partial\varphi(V).$$

Therefore, the values of  $V$  satisfying the conditions of the problem are characterized by

$$0 \in \partial\psi_{\mathcal{E}}(V) + \partial\varphi(V). \quad (37)$$

In view of the trivial inclusion

$$\partial\psi_{\mathcal{E}}(V) + \partial\varphi(V) \subset \partial(\psi_{\mathcal{E}} + \varphi)(V), \quad (38)$$

this property implies that the function  $\psi_{\mathcal{E}} + \varphi$  (equal to  $\varphi$  on  $\mathcal{E}$  and taking the value  $+\infty$  elsewhere) achieves a finite minimum at the point  $V$ . Suppose the existence of at least one pair  $V, F$  satisfying the conditions of the problem; then, in view of (8),

$$\varphi(V) = -\langle V, F \rangle.$$

Let  $V'$  denote another point where the restriction of  $\varphi$  to  $\mathcal{E}$  attains its minimum; then,  $\varphi(V) = \varphi(V')$ . As  $V - V' \in \mathcal{W}$  and  $F \in \mathcal{W}^\perp$ , one has

$$\langle V, F \rangle = \langle V', F \rangle;$$

thus,

$$\varphi(V') = -\langle V', F \rangle$$

which proves that  $V'$  satisfies, with the same  $F$ , the conditions of the problem. This proves the first part of the proposition.

Finally the existence of a point in  $\mathcal{E}$  where  $\varphi$  is finite and continuous implies that the inclusion (38) is actually an equality of sets (*see*, for

example, Moreau [4]; then, the fact that  $\varphi|_{\mathcal{G}}$  attains a finite minimum at the point  $V$  is equivalent to (37).

**Remark** All the preceding could be adapted to the case where the system experiences, in addition, some given constant *load*, namely  $G \in \mathcal{F}$ . This amounts to replacing  $C$  by its translate  $C + G$ . If this set meets  $W^\perp$ , there may exist, for some given driving velocity  $W \in \mathcal{V}/W$ , a quasi-static evolution of the system.

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