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VISCOSITY SOLUTION OF SECOND ORDER
INTEGRO-DIFFERENTIAL EQUATION WITHOUT
MONOTONICITY CONDITION: CASE WITH THE
MEASURE OF LÉVY INFINITE L. SYLLA**

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► **To cite this version:**

Lamine Sylla. REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATION WITH JUMPS AND VISCOSITY SOLUTION OF SECOND ORDER INTEGRO-DIFFERENTIAL EQUATION WITHOUT MONOTONICITY CONDITION: CASE WITH THE MEASURE OF LÉVY INFINITE L. SYLLA . 2018. hal-01863755

HAL Id: hal-01863755

<https://hal.science/hal-01863755v1>

Preprint submitted on 5 Sep 2018

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REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATION WITH
JUMPS AND VISCOSITY SOLUTION OF SECOND ORDER
INTEGRO-DIFFERENTIAL EQUATION WITHOUT MONOTONICITY
CONDITION: CASE WITH THE MEASURE OF LÉVY INFINITE

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December, 21, 2017

Abstract

We consider the problem of viscosity solution of integro-partial differential equation(IPDE in short) with one obstacle via the solution of reflected backward stochastic differential equations(RBSDE in short) with jumps. We show existence and uniqueness of a continuous viscosity solution of equation with non local terms, in case the generator is not monotonous and Levy's measure is infinite.

Keywords: Integro-partial differential equation; Reflected stochastic differential equations with jumps; Viscosity solution; Non-local operator.

MSC 2010 subject classifications: 35D40, 35R09, 60H30.

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1 Introduction

We consider the following system of integro-partial differential equation with one-obstacle ℓ , which is a function of (t, x) : $\forall i \in \{1, \dots, m\}$,

$$\begin{cases} \min \left\{ u^i(t, x) - \ell(t, x); -\partial_t u^i(t, x) - b(t, x)^\top D_x u^i(t, x) - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 u^i(t, x)) \right. \\ \quad \left. - K_i u^i(t, x) - h^{(i)}(t, x, u^i(t, x), (\sigma^\top D_x u^i)(t, x), B_i u^i(t, x)) \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^k; \\ u^i(T, x) = g^i(x); \end{cases} \quad (1.1)$$

where the operators B_i and K_i are defined as follows:

$$\begin{aligned} B_i u^i(t, x) &= \int_{\mathbf{E}} \gamma^i(t, x, e) (u^i(t, x + \beta(t, x, e)) - u^i(t, x)) \lambda(de); \\ K_i u^i(t, x) &= \int_{\mathbf{E}} (u^i(t, x + \beta(t, x, e)) - u^i(t, x) - \beta(t, x, e)^\top D_x u^i(t, x)) \lambda(de). \end{aligned} \quad (1.2)$$

The resolution of (1.1) is in connection with the following system of backward stochastic differential equations with jumps and one-obstacle ℓ :

$$\begin{cases} (i) \quad dY_s^{i;t,x} = -f^{(i)}(s, X_s^{t,x}, (Y_s^{i;t,x})_{i=1,m}, Z_s^{i;t,x}, U_s^{i;t,x}) ds - dK_s^{i;t,x} \\ \quad \quad \quad + Z_s^{i;t,x} dB_s + \int_{\mathbf{E}} U_s^{i;t,x}(e) \tilde{\mu}(ds, de), \quad s \leq T; \\ (ii) \quad Y_s^{i;t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{i;t,x} - \ell(s, X_s^{t,x})) dK_s^{i;t,x} = 0; \end{cases} \quad (1.3)$$

and

the following standard stochastic differential equation of diffusion-jump type:

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dB_r + \int_t^s \int_{\mathbf{E}} \beta(r, X_{r-}^{t,x}, e) \tilde{\mu}(dr, de), \quad (1.4)$$

for $s \in [t, T]$ and $X_s^{t,x} = x$ if $s \leq t$.

It is recalled that pioneering work was done for the resolution of (1.1), among these works we can mention those of Barles and al. [1] in case without obstacle, Harraj and al. [6] in the case with two obstacles; with as common point the hypothesis of monotony on the generator and $\gamma \geq 0$. But recently Hamadène and Morlais relaxed these conditions with $\lambda(\cdot)$ finite [4].

In this work we propose to solve (1.1) by relaxing the monotonicity of the generator and the positivity of γ and assuming that $\lambda = \infty$.

Our paper is organized as follows: in the next section we give the notations and the assumptions of our objects; in section 3 we recall a number of existing results; in section 4 we build estimates and properties for a good resolution of our problem; section 5 is reserved to give our main result

and the section 6 for doing an extension of our result.

And in the end, classical definition of the concept of viscosity solution is put in appendix.

2 Notations and assumptions

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, \mathbb{P})$ be a stochastic basis such that \mathcal{F}_0 contains all \mathbb{P} -null sets of \mathcal{F} , and $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}$, $t \geq 0$, and we suppose that the filtration is generated by the two mutually independents processes:

- (i) $B := (B_t)_{t \geq 0}$ a d -dimensional Brownian motion and,
- (ii) a Poisson random measure μ on $\mathbb{R}^+ \times E$ where $E := \mathbb{R}^\ell - \{0\}$ is equipped with its Borel field \mathcal{E} ($\ell \geq 1$). The compensator $\nu(dt, de) = dt\lambda(de)$ is such that $\{\tilde{\mu}([0, t] \times A) = (\mu - \lambda)([0, t] \times A)\}_{t \geq 0}$ is a martingale for all $A \in \mathcal{E}$ satisfying $\lambda(A) < \infty$. We also assume that λ is a σ -finite measure on (E, \mathcal{E}) , integrates the function $(1 \wedge |e|^2)$ and $\lambda(E) = \infty$.

Let's now introduce the following spaces:

- (iii) \mathcal{P} (resp. \mathbf{P}) the field on $[0, T] \times \Omega$ of $\mathcal{F}_{t \leq T}$ -progressively measurable (resp. predictable) sets.
 - (iv) For $\kappa \geq 1$, $\mathbb{L}_\kappa^2(\lambda)$ the space of Borel measurable functions $\varphi := (\varphi(e))_{e \in E}$ from E into \mathbb{R}^κ such that $\|\varphi\|_{\mathbb{L}_\kappa^2(\lambda)}^2 = \int_E |\varphi(e)|_\kappa^2 \lambda(de) < \infty$; $\mathbb{L}_1^2(\lambda)$ will be simply denoted by $\mathbb{L}^2(\lambda)$;
 - (v) $\mathcal{S}^2(\mathbb{R}^\kappa)$ the space of RCLL (for right continuous with left limits) \mathcal{P} -measurable and \mathbb{R}^κ -valued processes such that $\mathbb{E}[\sup_{s \leq T} |Y_s|^2] < \infty$; \mathcal{A}_c^2 is its subspace of continuous non-decreasing processes $(K_t)_{t \leq T}$ such that $K_0 = 0$ and $\mathbb{E}[(K_T)^2] < \infty$;
 - (vi) $\mathbb{H}^2(\mathbb{R}^{\kappa \times d})$ the space of processes $Z := (Z_s)_{s \leq T}$ which are \mathcal{P} -measurable, $\mathbb{R}^{\kappa \times d}$ -valued and satisfying $\mathbb{E} \left[\int_0^T |Z_s|^2 ds \right] < \infty$;
 - (vii) $\mathbb{H}^2(\mathbb{L}_\kappa^2(\lambda))$ the space of processes $U := (U_s)_{s \leq T}$ which are \mathbf{P} -measurable, $\mathbb{L}_\kappa^2(\lambda)$ -valued and satisfying $\mathbb{E} \left[\int_0^T \|U_s(\omega)\|_{\mathbb{L}_\kappa^2(\lambda)}^2 ds \right] < \infty$;
 - (viii) Π_g the set of deterministic functions
- $\varpi : (t, x) \in [0, T] \times \mathbb{R}^\kappa \mapsto \varpi(t, x) \in \mathbb{R}$ of polynomial growth, i.e., for which there exists two non-negative constants C and p such that for any $(t, x) \in [0, T] \times \mathbb{R}^\kappa$,

$$|\varpi(t, x)| \leq C(1 + |x|^p).$$

The subspace of Π_g of continuous functions will be denoted by Π_g^c ;

- (ix) \mathcal{U} the subclass of Π_g^c which consists of functions

$\Phi : (t, x) \in [0, T] \times \mathbb{R}^\kappa \mapsto \mathbb{R}$ such that for some non-negative constants C and p we have

$$|\Phi(t, x) - \Phi(t, x')| \leq C(1 + |x|^p + |x'|^p) |x - x'|, \text{ for any } t, x, x'.$$

(x) For any process $\theta := (\theta_s)_{s \leq T}$ and $t \in (0, T]$, $\theta_{t-} = \lim_{s \nearrow t} \theta_s$ and

$$\Delta_t \theta = \theta_t - \theta_{t-}.$$

Now let b and σ be the following functions:

$$b : (t, x) \in [0, T] \times \mathbb{R}^k \mapsto b(t, x) \in \mathbb{R}^k;$$

$$\sigma : (t, x) \in [0, T] \times \mathbb{R}^k \mapsto \sigma(t, x) \in \mathbb{R}^{k \times d}.$$

We assume that they are jointly continuous in (t, x) and Lipschitz continuous w.r.t. x uniformly in t , i.e., there exists a constant C such that,

$$\forall (t, x, x') \in [0, T] \times \mathbb{R}^{k+k}, \quad |b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq C |x - x'|. \quad (2.5)$$

Let us notice that by (2.5) and continuity, the functions b and σ are of linear growth, i.e., there exists a constant C such that

$$\forall (t, x, x') \in [0, T] \times \mathbb{R}^{k+k}, \quad |b(t, x)| + |\sigma(t, x)| \leq C |1 + x|. \quad (2.6)$$

Let $\beta : (t, x, e) \in [0, T] \times \mathbb{R}^k \times E \mapsto \beta(t, x, e) \in \mathbb{R}^k$ be a measurable function such that for some real constant C , and for all $e \in E$,

$$(i) \quad |\beta(t, x, e)| \leq C(1 \wedge |e|); \quad (2.7)$$

$$(ii) \quad |\beta(t, x, e) - \beta(t, x', e)| \leq C |x - x'| (1 \wedge |e|);$$

$$(iii) \quad \text{the mapping } (t, x) \in [0, T] \times \mathbb{R}^k \mapsto \beta(t, x, e) \in \mathbb{R}^k \text{ is continuous for any } e \in E.$$

We are now going to introduce the objects which are specifically connected to the RBSDE with jumps we will deal with. Let ℓ the barrier of (1.3); $(g^i)_{i=1, m}$ and $(h^{(i)})_{i=1, m}$ be two functions defined as follows: for $i = 1, \dots, m$,

$$\begin{aligned} g^i : \mathbb{R}^k &\longrightarrow \mathbb{R}^m \\ x &\longmapsto g^i(x) \end{aligned}$$

and

$$\begin{aligned} h^{(i)} : [0, T] \times \mathbb{R}^{k+m+d+1} &\longrightarrow \mathbb{R} \\ (t, x, y, z, q) &\longmapsto h^{(i)}(t, x, y, z, q). \end{aligned}$$

Moreover we assume they satisfy:

(H1): The reflecting barrier ℓ is real valued and \mathcal{P} -measurable process satisfying, $\ell \in \mathcal{U}$ i.e., it

is continuous and there exists constants C and p such that,

$$|\ell(t, x) - \ell(t, x')| \leq C(1 + |x|^p + |x'|^p) |x - x'|, \text{ for any } t \geq 0, x, x'.$$

(H2): For any $i \in \{1, \dots, m\}$, the function g^i belongs to \mathcal{U} .

(H3): For any $i \in \{1, \dots, m\}$,

- (i) the function $h^{(i)}$ is Lipschitz in (y, z, q) uniformly in (t, x) , i.e., there exists a real constant C such that for any $(t, x) \in [0, T] \times \mathbb{R}^k$, (y, z, q) and (y', z', q') elements of \mathbb{R}^{m+d+1} ,
- $$\left| h^{(i)}(t, x, y, z, q) - h^{(i)}(t, x, y', z', q') \right| \leq C(|y - y'| + |z - z'| + |q - q'|); \quad (2.8)$$

- (ii) the $(t, x) \mapsto h^{(i)}(t, x, y, z, q)$, for fixed $(y, z, q) \in \mathbb{R}^{m+d+1}$, belongs uniformly to \mathcal{U} , i.e., it is continuous and there exists constants C and p (which do not depend on (y, z, q)) such that,
- $$\left| h^{(i)}(t, x, y, z, q) - h^{(i)}(t, x', y, z, q) \right| \leq C(1 + |x|^p + |x'|^p) |x - x'|, \text{ for any } t \geq 0, x, x'. \quad (2.9)$$

Next let γ^i , $i = 1, \dots, m$ be Borel measurable functions defined from $[0, T] \times \mathbb{R}^k \times E$ into \mathbb{R} and satisfying:

- (i) $|\gamma^i(t, x, e)| \leq C(1 \wedge |e|)$;
- (ii) $|\gamma^i(t, x, e) - \gamma^i(t, x', e)| \leq C(1 \wedge |e|) |x - x'| (1 + |x|^p + |x'|^p)$;
- (iii) the mapping $t \in [0, T] \mapsto \gamma^i(t, x, e) \in \mathbb{R}$ is continuous for any $(x, e) \in \mathbb{R}^k \times E$.

Finally we introduce the following functions $(f^{(i)})_{i=1, m}$ defined by:

$$\forall (t, x, y, z, \zeta) \in [0, T] \times \mathbb{R}^{k+m+d} \times \mathbb{L}^2(\lambda), f^{(i)}(t, x, y, z, \zeta) := h^{(i)} \left(t, x, y, z, \int_E \gamma^i(t, x, e) \zeta(e) \lambda(de) \right). \quad (2.11)$$

The functions $(f^{(i)})_{i=1, m}$, enjoy the two following properties:

- (a) The function $f^{(i)}$ is Lipschitz in (y, z, ζ) uniformly in (t, x) , i.e., there exists a real constant C such that

$$\left| f^{(i)}(t, x, y, z, \zeta) - f^{(i)}(t, x, y', z', \zeta') \right| \leq C(|y - y'| + |z - z'| + \|\zeta - \zeta'\|_{\mathbb{L}^2(\lambda)}); \quad (2.12)$$

since $h^{(i)}$ is uniformly Lipschitz in (y, z, q) and γ^i verifies (2.8)-(i);

- (b) The function $(t, x) \in [0, T] \times \mathbb{R}^k \mapsto f^{(i)}(t, x, 0, 0, 0)$ belongs to Π_g^c ;

$$\text{and then } \mathbb{E} \left[\int_0^T \left| f^{(i)}(r, X_r^{t, x}, 0, 0, 0) \right|^2 dr \right] < \infty.$$

Having defined our data and put our assumptions, we can look at the state of the art.

3 Preliminaires

3.1 A class of diffusion processes with jumps

Let $(t, x) \in [0, T] \times \mathbb{R}^d$ and $(X_s^{t,x})_{s \leq T}$ be the stochastic process solution of (1.4). Under assumptions (2.5)-(2.7) the solution of Equation (1.4) exists and is unique (see [2] for more details). We state some properties of the process $\{(X_s^{t,x}), s \in [0, T]\}$ which can be found in [2].

Proposition 3.1 *For each $t \geq 0$, there exists a version of $\{(X_s^{t,x}), s \in [t, T]\}$ such that $s \rightarrow X_s^t$ is a $C^2(\mathbb{R}^d)$ -valued rcll process. Moreover it satisfies the following estimates: $\forall p \geq 2, x, x' \in \mathbb{R}^d$ and $s \geq t$,*

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq r \leq s} |X_r^{t,x} - x|^p \right] &\leq M_p (s-t) (1 + |x|^p); \\ \mathbb{E} \left[\sup_{t \leq r \leq s} |X_r^{t,x} - X_r^{t,x'} - (x - x')|^p \right] &\leq M_p (s-t) (|x - x'|^p); \end{aligned} \quad (3.13)$$

for some constant M_p .

3.2 Existence and uniqueness for a RBSDE with jumps

Let $(t, x) \in [0, T] \times \mathbb{R}^d$ and we consider the following m -dimensional RBSDE with jumps:

$$\left\{ \begin{array}{l} (i) \ \vec{Y}^{t,x} := (Y_s^{i,t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \ Z^{t,x} := (Z_s^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ \quad K^{t,x} := (K_s^{i,t,x})_{i=1,m} \in \mathcal{A}_c^2, \ U^{t,x} := (U_s^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ (ii) \ dY_s^{i,t,x} = -f^{(i)}(s, X_s^{t,x}, (Y_s^{i,t,x})_{i=1,m}, Z_s^{i,t,x}, U_s^{i,t,x}) ds - dK_s^{i,t,x} \\ \quad \quad \quad + Z_s^{i,t,x} dB_s + \int_{\mathbb{E}} U_s^{i,t,x}(e) \tilde{\mu}(ds, de), \quad s \leq T; \\ (iii) \ Y_s^{i,t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{i,t,x} - \ell(s, X_s^{t,x})) dK_s^{i,t,x} = 0; \end{array} \right. \quad (3.14)$$

where for any $i \in \{1, \dots, m\}$, $Z_s^{i,t,x}$ is the i th row of $Z_s^{t,x}$, $K_s^{i,t,x}$ is the i th component of $K_s^{t,x}$ and $U_s^{i,t,x}$ is the i th component of $U_s^{t,x}$.

The following result is related to existence and uniqueness of a solution for the RBSDE with jumps (3.14).

Its proof is given in [5] by using the penalization method (see p. 5-12) and the Snell envelope method (see p. 14-16).

Proposition 3.2 *Assume that assumptions **(H1)**, **(H2)** and **(H3)** hold. Then for any $(t, x) \in [0, T] \times \mathbb{R}^d$, the RBSDE (3.14) has a unique solution $(\vec{Y}^{t,x}, Z^{t,x}, U^{t,x}, K^{t,x})$.*

Remark 1 *The solution of this RBSDE with jumps exist and is unique since:*

$$(i) \quad \mathbb{E} \left[\left| g(X_T^{t,x}) \right|^2 \right] < \infty, \text{ due to polynomial growth of } g \text{ and estimate (3.13) on } X^{t,x};$$

- (ii) for any $i = 1, \dots, m$, $f^{(i)}$ verifies the properties (a)-(b) related to uniform Lipschitz w.r.t. (y, z, ζ) and $ds \otimes d\mathbb{P}$ – square integrability of the process $(f^{(i)}(s, X_s^{t,x}, 0, 0, 0))_{s \leq T}$.

3.3 Viscosity solutions of integro-differential partial equation with one obstacle

The following result on one obstacle is proved with two obstacles in Harraj and al. (see. [6], Theorem 4.6. p. 47 by using (4.1) p. 44), establishes the relationship between the solution of (3.14) and the one of system (1.1).

Proposition 3.3 *Assume that (H1), (H2), (H3) are fulfilled. Then there exists deterministic continuous functions $(u^i(t, x))_{i=1, m}$ which belong to Π_g such that for any $(t, x) \in [0, T] \times \mathbb{R}^k$, the solution of the RBSDE (3.14) verifies:*

$$\forall i \in \{1, \dots, m\}, \forall s \in [t, T], Y_s^{i; t, x} = u^i(s, X_s^{t, x}). \quad (3.15)$$

Moreover if for any $i \in \{1, \dots, m\}$,

- (i) $\gamma^i \geq 0$;
- (ii) for any fixed $(t, x, \vec{y}, z) \in [0, T] \times \mathbb{R}^{k+m+d}$, the mapping $(q \in \mathbb{R}) \mapsto h^{(i)}(t, x, \vec{y}, z, q) \in \mathbb{R}$ is non-decreasing.

The function $(u^i)_{i=1, m}$ is a continuous viscosity solution (in Barles and al. 's sense, see Definition 6.1 in the Appendix) of (1.1).

For the proof see [6] for the same way.

Finally, the solution $(u^i)_{i=1, m}$ of (1.1) is unique in the class Π_g^c .

Remark 2 (see [6]) *Under the assumptions (H1), (H2), (H3), there exists a unique viscosity solution of (1.1) in the class of functions satisfying*

$$\lim_{|x| \rightarrow +\infty} |u(t, x)| e^{-\tilde{A}[\log(|x|)]^2} = 0 \quad (3.16)$$

uniformly for $t \in [0, T]$, for some $\tilde{A} > 0$.

4 Estimates and properties

In this section we provide estimates for the functions $(u^i)_{i=1, m}$ defined in (3.15). Recall that, $(\vec{Y}^{t,x}, Z^{t,x}, U^{t,x}, K^{t,x}) := ((Y^{i;t,x})_{i=1, m}, (Z^{i;t,x})_{i=1, m}, (U^{i;t,x})_{i=1, m}, (K^{i;t,x})_{i=1, m})$ is the unique solution of the RBSDE with jumps (3.14).

Lemma 4.1 Under assumption **(H1)**, **(H2)**, **(H3)**, for any $p \geq 2$ there exists two non-negative constants C and ρ such that,

$$\mathbb{E} \left[\left\{ \int_0^T ds \left(\int_E |U_s^{t,x}(e)|^2 \lambda(de) \right) \right\}^{\frac{p}{2}} \right] = \mathbb{E} \left[\left\{ \int_0^T ds \|U_s^{t,x}\|_{\mathbb{L}_m^2(\lambda)}^2 \right\}^{\frac{p}{2}} \right] \leq C(1 + |x|^\rho). \quad (4.17)$$

Proof. First let us point out that since $X_s^{t,x} = x$ for $s \in [0, t]$ then, the uniqueness of the solution of RBSDE of (3.14) implies that,

$$Z_s^{t,x} = U_s^{t,x} = K_s^{t,x} = 0, \quad ds \otimes d\mathbf{P} - \text{a.e. on } [0, t] \times \Omega. \quad (4.18)$$

Next let $p \geq 2$ be fixed. Using the representation (3.15), for any $i \in \{1, \dots, m\}$ and $s \in [t, T]$ we have,

$$\begin{aligned} Y_s^{i;t,x} &= g^i(X_T^{t,x}) + \int_s^T f^{(i)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i;t,x}, U_s^{i;t,x}) dr + K_T^{i;t,x} - K_s^{i;t,x} \\ &\quad - \int_s^T Z_r^{i;t,x} dB_r - \int_s^T \int_E U_r^{i;t,x}(e) \tilde{\mu}(dr, de). \end{aligned} \quad (4.19)$$

This implies that the system of RBSDEs with jumps (3.14) turns into a decoupled one since the equations in (4.19) are not related each other.

Next for any $i = 1, \dots, m$, the functions u^i , g^i and $(t, x) \mapsto f^{(i)}(t, x, 0, 0, 0)$ are of polynomial growth and finally $y \mapsto f^{(i)}(t, x, y, 0, 0)$ is Lipschitz uniformly w.r.t. (t, x) . Then for some C and $\rho \geq 0$,

$$\mathbb{E} \left[\left| g^i(X_T^{t,x}) \right|^p + \left(\int_s^T \left| f^{(i)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, 0, 0) \right|^2 dr \right)^{\frac{p}{2}} \right] \leq C(1 + |x|^\rho). \quad (4.20)$$

Let us now fix $i_0 \in \{1, \dots, m\}$, $\forall s \in [t, T]$,

$$\left\{ \begin{array}{l} (i) \ Y_s^{i_0,t,x} \in \mathcal{S}^2(\mathbb{R}), \ Z_s^{i_0,t,x} \in \mathbb{H}^2(\mathbb{R}^d), \ K_s^{i_0,t,x} \in \mathcal{A}_c^2, \ U_s^{i_0,t,x} \in \mathbb{H}^2(\mathbb{L}^2(\lambda)); \\ (ii) \ Y_s^{i_0,t,x} = g^{i_0}(X_T^{t,x}) + \int_s^T f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x}) dr + K_T^{i_0,t,x} - K_s^{i_0,t,x} \\ \quad - \int_s^T Z_r^{i_0,t,x} dB_r - \int_s^T \int_E U_r^{i_0,t,x}(e) \tilde{\mu}(dr, de). \\ (iii) \ Y_s^{i_0,t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{i_0,t,x} - \ell(s, X_s^{t,x})) dK_s^{i_0,t,x} = 0. \end{array} \right.$$

Applying Itô formula to $|Y_s^{i_0,t,x}|^2$ between s and T , we have

$$|Y_s^{i_0,t,x}|^2 + \int_s^T |Z_r^{i_0,t,x}|^2 dr + \sum_{s \leq r \leq T} (\Delta Y_r^{i_0,t,x})^2 \quad (4.21)$$

$$\begin{aligned}
&= \left| g^{i_0}(X_T^{t,x}) \right|^2 + 2 \int_s^T Y_r^{i_0,t,x} f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x}) dr + 2 \int_s^T Y_r^{i_0,t,x} dK_r^{i_0,t,x} \\
&\quad - 2 \int_s^T \int_{\mathbb{E}} Y_r^{i_0,t,x} U_r^{i_0,t,x} \tilde{\mu}(dr, de) - 2 \int_s^T Y_r^{i_0,t,x} Z_r^{i_0,t,x} dB_r.
\end{aligned}$$

Notice that $Y_r^{i_0,t,x} = u^{i_0}(r, X_r^{t,x})$ and we have that $|u^{i_0}(r, X_r^{t,x})| \leq C(1 + |X_r^{t,x}|^q)$. Next let us set $\Sigma = 1 + \sup_{s \leq T} |X_s^{t,x}|$. Therefore $|Y_r^{i_0,t,x}| \leq C_q \Sigma^q$ and $|g^{i_0}(X_T^{t,x})| \leq C_q \Sigma^q$.

By raising to the power $\frac{p}{2}$ and then taking expectation, it follows from (4.21) and the fact of, there exists $C > 0$ such that,

$$\begin{aligned}
\mathbb{E} \left[\left(\int_s^T \|U_r^{i_0,t,x}\|_{\mathbb{L}^2(\lambda)}^2 dr \right)^{\frac{p}{2}} \right] &\leq C \mathbb{E} \left[\left(\sum_{s \leq r \leq T} (\Delta Y_r^{i_0,t,x})^2 \right)^{\frac{p}{2}} \right], \text{ (see [7] p. 28-45),} \\
\mathbb{E} \left[|Y_s^{i_0,t,x}|^p \right] + \mathbb{E} \left[\left(\int_s^T |Z_r^{i_0,t,x}|^2 dr \right)^{\frac{p}{2}} \right] + \mathbb{E} \left[\left(\int_s^T \|U_r^{i_0,t,x}\|_{\mathbb{L}^2(\lambda)}^2 dr \right)^{\frac{p}{2}} \right] \\
&\leq 5^{\frac{p}{2}-1} \mathbb{E} \left[|g^{i_0}(X_T^{t,x})|^p \right] + 5^{\frac{p}{2}-1} \mathbb{E} \left[\left(\int_s^T 2 \left| Y_r^{i_0,t,x} f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x}) \right| dr \right)^{\frac{p}{2}} \right] \\
&\quad + 5^{\frac{p}{2}-1} \mathbb{E} \left[\left| \int_s^T Y_r^{i_0,t,x} dK_r^{i_0,t,x} \right|^{\frac{p}{2}} \right] \\
&\quad + 5^{\frac{p}{2}-1} \mathbb{E} \left[\left| \int_s^T \int_{\mathbb{E}} 2Y_r^{i_0,t,x} U_r^{i_0,t,x} \tilde{\mu}(dr, de) \right|^{\frac{p}{2}} \right] + 5^{\frac{p}{2}-1} \mathbb{E} \left[\left| \int_s^T 2Y_r^{i_0,t,x} Z_r^{i_0,t,x} dB_r \right|^{\frac{p}{2}} \right]. \tag{4.22}
\end{aligned}$$

For more comprehension, we adopt the following scripture for inequality (4.22);

$$\begin{aligned}
\mathbb{E} \left[|Y_s^{i_0,t,x}|^p \right] + \mathbb{E} \left[\left(\int_s^T |Z_r^{i_0,t,x}|^2 dr \right)^{\frac{p}{2}} \right] + \mathbb{E} \left[\left(\int_s^T \|U_r^{i_0,t,x}\|_{\mathbb{L}^2(\lambda)}^2 dr \right)^{\frac{p}{2}} \right] &\leq 5^{\frac{p}{2}-1} \mathbb{E} \left[|g^{i_0}(X_T^{t,x})|^p \right] \\
+ 5^{\frac{p}{2}-1} T_1(s) + 5^{\frac{p}{2}-1} T_2(s) + 5^{\frac{p}{2}-1} T_3(s) + 5^{\frac{p}{2}-1} T_4(s) &\tag{4.23}
\end{aligned}$$

where,

$$\begin{aligned}
T_1(s) &= \mathbb{E} \left[\left(\int_s^T 2 \left| Y_r^{i_0,t,x} f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x}) \right| dr \right)^{\frac{p}{2}} \right]; \\
T_2(s) &= \mathbb{E} \left[\left| \int_s^T Y_r^{i_0,t,x} dK_r^{i_0,t,x} \right|^{\frac{p}{2}} \right]; \\
T_3(s) &= \mathbb{E} \left[\left| \int_s^T \int_{\mathbb{E}} 2Y_r^{i_0,t,x} U_r^{i_0,t,x} \tilde{\mu}(dr, de) \right|^{\frac{p}{2}} \right]; \\
T_4(s) &= \mathbb{E} \left[\left| \int_s^T 2Y_r^{i_0,t,x} Z_r^{i_0,t,x} dB_r \right|^{\frac{p}{2}} \right].
\end{aligned}$$

We will estimate $T_1(s)$, $T_2(s)$, $T_3(s)$ and $T_4(s)$, $\forall s \in [t, T]$.

(a) Before starting our estimations, let's linearize f with respect to $(u^j(X_r^{t,x}))_{j=1,m}$ and $Z_r^{i_0,t,x}$ i.e.

$$f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x}) = a_r^{t,x} Z_r^{i_0,t,x} + b_r^{t,x} (u^j(X_r^{t,x}))_{j=1,m} + f^{(i_0)}(r, X_r^{t,x}, 0, 0, U_r^{i_0,t,x});$$

where $a_r^{t,x}$ and $b_r^{t,x}$ are progressively measurable processes respectively bounded by the Lipschitz constants of f in $Z_r^{t,x}$ and $(u^j(X_r^{t,x}))_{j=1,m}$ i.e $|a_r^{t,x}| \leq C_Z$ and $|b_r^{t,x}| \leq \lambda_1$.

(b) We also take the fact that f is Lipschitz in $U_r^{i_0,t,x}$ i.e there exists a constant Lipschitz λ_2 such that $|f^{(i_0)}(r, X_r^{t,x}, 0, 0, U_r^{i_0,t,x})| \leq |f^{(i_0)}(r, X_r^{t,x}, 0, 0, 0)| + \lambda_2 \|U_r^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}$.

By combining (a) and (b) we have

$$\begin{aligned} |f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x})| &\leq |a_r^{t,x} Z_r^{i_0,t,x}| + |b_r^{t,x} (u^j(X_r^{t,x}))_{j=1,m}| + |f^{(i_0)}(r, X_r^{t,x}, 0, 0, 0)| \\ &\quad + \lambda_2 \|U_r^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}. \end{aligned} \quad (4.24)$$

Let's start our estimates.

For $T_1(s)$

By using (4.24), it follows that;

$$\begin{aligned} &\int_s^T 2 \left| Y_r^{i_0,t,x} f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x}) \right| dr \\ &\leq \int_s^T 2 |Y_r^{i_0,t,x} (a_r^{t,x} Z_r^{i_0,t,x})| dr + \int_s^T 2 |Y_r^{i_0,t,x} (b_r^{t,x} (u^j(X_r^{t,x}))_{j=1,m})| dr + \lambda_2 \int_s^T 2 |Y_r^{i_0,t,x} (U_r^{i_0,t,x})| dr \\ &\quad + \int_s^T 2 |Y_r^{i_0,t,x} f^{(i_0)}(r, X_r^{t,x}, 0, 0, 0)| dr \\ &\leq C_q^2 C_Z T \epsilon_1^{-1} \Sigma^{2q} + \epsilon_1 C_Z \int_s^T |Z_r^{i_0,t,x}|^2 dr + C_q^2 \lambda_1 T \epsilon_2^{-1} \Sigma^{2q} + \epsilon_2 \lambda_1 C_q^2 T \Sigma^{2q} + C_q^2 T \lambda_2 \epsilon_3^{-1} \Sigma^{2q} \\ &\quad + \lambda_2 \epsilon_3 \int_s^T \|U_r^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}^2 dr + C_q^2 T \epsilon_4^{-1} \Sigma^{2q} + C_q^2 T \epsilon_4 \Sigma^{2q}. \end{aligned}$$

By raising to the power $\frac{p}{2}$ and then taking expectation it follows that,

$$\begin{aligned}
T_1(s) &\leq 8^{\frac{p}{2}-1} C C_q^p \left\{ (C_Z T \epsilon_1^{-1})^{\frac{p}{2}} + (\lambda_1 T \epsilon_2^{-1})^{\frac{p}{2}} + C_q^p (\epsilon_2 \lambda_1 T)^{\frac{p}{2}} + (T \lambda_2 \epsilon_3^{-1})^{\frac{p}{2}} + (T \epsilon_4^{-1})^{\frac{p}{2}} \right. \\
&\quad \left. + (T \epsilon_4)^{\frac{p}{2}} \right\} |1 + |x|^{pq}| + 8^{\frac{p}{2}-1} (\epsilon_1 C_Z)^{\frac{p}{2}} \mathbb{E} \left[\left(\int_s^T |Z_r^{i_0, t, x}|^2 dr \right)^{\frac{p}{2}} \right] \\
&\quad + 8^{\frac{p}{2}-1} (\lambda_2 \epsilon_3)^{\frac{p}{2}} \mathbb{E} \left[\left(\int_s^T \|U_r^{i_0, t, x}\|_{\mathbb{L}_m^2(\lambda)}^2 dr \right)^{\frac{p}{2}} \right]. \tag{4.25}
\end{aligned}$$

Before estimating $T_2(s)$, let us first give an estimate of $\mathbb{E} \left[|K_T^{i_0, t, x} - K_s^{i_0, t, x}|^p \right]$ which will serve us in that of $T_2(s)$.

$$\begin{aligned}
K_T^{i_0, t, x} - K_s^{i_0, t, x} &= Y_s^{i_0, t, x} - g^{i_0}(X_T^{t, x}) - \int_s^T f^{(i_0)}(r, X_r^{t, x}, (u^j(X_r^{t, x}))_{j=1, m}, Z_r^{i_0, t, x}, U_r^{i_0, t, x}) dr \\
&\quad + \int_s^T Z_r^{i_0, t, x} dB_r + \int_s^T \int_E U_r^{i_0, t, x}(e) \tilde{\mu}(dr, de).
\end{aligned}$$

By (4.24) and Cauchy-Schwartz inequality; it follows that,

$$\begin{aligned}
|K_T^{i_0, t, x} - K_s^{i_0, t, x}| &\leq 2C_q \Sigma^q + C_q \lambda_1 T \Sigma^q + C_q T \Sigma^q + C_Z \left(\int_s^T |Z_r^{i_0, t, x}|^2 dr \right)^{\frac{1}{2}} \\
&\quad + \lambda_2 \left(\int_s^T \|U_r^{i_0, t, x}\|^2 dr \right)^{\frac{1}{2}} + \int_s^T Z_r^{i_0, t, x} dB_r + \int_s^T \int_E U_r^{i_0, t, x}(e) \tilde{\mu}(dr, de).
\end{aligned}$$

By raising to the power p , taking expectation and BDG inequality we have,

$$\begin{aligned}
\mathbb{E} \left[|K_T^{i_0, t, x} - K_s^{i_0, t, x}|^p \right] &\leq 5^{p-1} C C_q^p \{2^p + (T \lambda_1)^p + T^p\} |1 + |x|^{pq}| + 5^{p-1} (C_Z^p + C_p) \mathbb{E} \left(\int_0^T |Z_s^{i_0, t, x}|^2 ds \right)^{\frac{p}{2}} \\
&\quad + 5^{p-1} (\lambda_2^p + C_p) \mathbb{E} \left\{ \int_0^T ds \|U_s^{i_0, t, x}\|_{\mathbb{L}_m^2(\lambda)}^2 \right\}^{\frac{p}{2}}. \tag{4.26}
\end{aligned}$$

For $T_2(s)$

$$\begin{aligned}
\int_s^T |Y_s^{i_0, t, x}| dK_s^{i_0, t, x} &\leq \int_s^T |(Y_s^{i_0, t, x} - \ell(s, X_s^{t, x}))| dK_s^{i_0, t, x} + \int_s^T |\ell(s, X_s^{t, x})| dK_s^{i_0, t, x} \\
&\leq \sup_{s \leq T} |\ell(s, X_s^{t, x})| K_T^{i_0, t, x} \\
&\leq \epsilon_5^{-1} \sup_{s \leq T} |\ell(s, X_s^{t, x})|^2 + \epsilon_5 (K_T^{i_0, t, x})^2 \\
&\leq \epsilon_5^{-1} C_q^2 \Sigma^{2q} + \epsilon_5 (K_T^{i_0, t, x})^2. \tag{4.27}
\end{aligned}$$

By using (4.26) and (4.27); it follows that

$$T_2(s) \leq 2^{\frac{p}{2}-1} C C_q^p (\epsilon_5^{-1})^{\frac{p}{2}} |1 + |x|^{pq}| + 2^{\frac{p}{2}-1} (\epsilon_5)^{\frac{p}{2}} \mathbb{E} \left[(K_T^{i_0, t, x})^p \right]$$

$$\begin{aligned}
&\leq 2^{\frac{p}{2}-1} C C_q^p (\epsilon_5^{-1})^{\frac{p}{2}} |1 + |x|^{pq}| + (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} C C_q^p \{2^p + (T\lambda_1)^p + T^p\} |1 + |x|^{pq}| + \\
&\quad (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (C_Z^p + C_p) \mathbb{E} \left(\int_0^T |Z_s^{i_0,t,x}|^2 ds \right)^{\frac{p}{2}} \\
&\quad + (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (\lambda_2^p + C_p) \mathbb{E} \left\{ \int_0^T ds \|U_s^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}^2 \right\}^{\frac{p}{2}} \\
T_2(s) &\leq \left\{ 2^{\frac{p}{2}-1} C C_q^p (\epsilon_5^{-1})^{\frac{p}{2}} + (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} C C_q^p (2^p + (T\lambda_1)^p + T^p) \right\} (1 + |x|^{pq}) \\
&\quad + (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (C_Z^p + C_p) \mathbb{E} \left(\int_0^T |Z_s^{i_0,t,x}|^2 ds \right)^{\frac{p}{2}} \\
&\quad + (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (\lambda_2^p + C_p) \mathbb{E} \left\{ \int_0^T ds \|U_s^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}^2 \right\}^{\frac{p}{2}}.
\end{aligned}$$

For $T_3(s)$

By BDG inequality,

$$\begin{aligned}
T_3(s) &\leq C_p \mathbb{E} \left(\int_0^T |Y_s^{i_0,t,x}|^2 |Z_s^{i_0,t,x}|^2 ds \right)^{\frac{p}{4}} \\
&\leq C_p \mathbb{E} \left(\sup_{s \leq T} |Y_s^{i_0,t,x}|^2 \int_0^T |Z_s^{i_0,t,x}|^2 ds \right)^{\frac{p}{4}} \\
&\leq C_p C_q^p \epsilon_6^{-1} (1 + |x|^{pq}) + C_p \epsilon_6 \mathbb{E} \left(\int_0^T |Z_s^{i_0,t,x}|^2 ds \right)^{\frac{p}{2}}. \tag{4.28}
\end{aligned}$$

For $T_4(s)$

By BDG inequality,

$$\begin{aligned}
T_4(s) &\leq C_p \mathbb{E} \left(\int_0^T |Y_s^{i_0,t,x}|^2 \|U_s^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}^2 ds \right)^{\frac{p}{4}} \\
&\leq C_p \mathbb{E} \left(\sup_{s \leq T} |Y_s^{i_0,t,x}|^2 \int_0^T \|U_s^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}^2 ds \right)^{\frac{p}{4}} \\
&\leq C_p C_q^p \epsilon_7^{-1} (1 + |x|^{pq}) + C_p \epsilon_7 \mathbb{E} \left(\int_0^T \|U_s^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}^2 ds \right)^{\frac{p}{2}}. \tag{4.29}
\end{aligned}$$

Finally by taking estimation of $T_1(s)$, $T_2(s)$, $T_3(s)$, $T_4(s)$ and choosing ϵ_1 , ϵ_2 , ϵ_3 , ϵ_4 , ϵ_5 , ϵ_6 , ϵ_7 such that;

$$\{(\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (C_Z^p + C_p) + C_p \epsilon_6\} < 1,$$

$$\{(\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (\lambda_2^p + C_p) + C_p \epsilon_7\} < 1,$$

and the sum of all coefficients of $(1 + |x|^{pq})$ was small than 1.

It follows then

$$\mathbb{E} \left[\left\{ \int_0^T ds \|U_s^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}^2 \right\}^{\frac{p}{2}} \right] \leq C (1 + |x|^\rho). \tag{4.30}$$

Where $\rho = pq$.

Finally since $i_0 \in \{1, \dots, m\}$ is arbitrary we then obtain the estimate (4.17).

Proposition 4.2 *For any $i = 1, \dots, m$, u^i belongs to \mathcal{U} .*

Proof. Let x and x' be elements of \mathbb{R}^k . Let $(\vec{Y}^{t,x}, Z^{t,x}, U^{t,x}, K^{t,x})$ (resp. $(\vec{Y}^{t,x'}, Z^{t,x'}, U^{t,x'}, K^{t,x'})$) be the solution of the RBSDE with jumps (3.14) associated with $f(s, X_s^{t,x}, y, \eta, \zeta, g(X_T^{t,x}))$ (resp. $f(s, X_s^{t,x'}, y, \eta, \zeta, g(X_T^{t,x'}))$). Applying Itô formula to $|\vec{Y}^{t,x} - \vec{Y}^{t,x'}|^2$ between s and T , we have

$$\begin{aligned} & \left| \vec{Y}_s^{t,x} - \vec{Y}_s^{t,x'} \right|^2 + \int_s^T |\Delta Z_r|^2 dr + \sum_{s \leq r \leq T} (\Delta_r \vec{Y}_r^{t,x})^2 \\ &= \left| g(X_T^{t,x}) - g(X_T^{t,x'}) \right|^2 + 2 \int_s^T \langle (\vec{Y}_s^{t,x} - \vec{Y}_s^{t,x'}), \Delta f(r) \rangle dr + 2 \int_s^T (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}) d(\Delta K_r) \\ & \quad - 2 \int_s^T \int_{\mathbb{E}} (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}) (\Delta U_r(e)) \tilde{\mu}(dr, de) - 2 \int_s^T (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}) (\Delta Z_r) dB_r; \end{aligned} \quad (4.31)$$

and taking expectation we obtain: $\forall s \in [t, T]$,

$$\begin{aligned} & \mathbb{E} \left[\left| \vec{Y}_s^{t,x} - \vec{Y}_s^{t,x'} \right|^2 + \int_s^T |\Delta Z_r|^2 dr + \int_s^T \|\Delta U_r\|_{\mathbb{L}^2(\lambda)}^2 dr \right] \\ & \leq \mathbb{E} \left[\left| g(X_T^{t,x}) - g(X_T^{t,x'}) \right|^2 + 2 \int_s^T \langle (\vec{Y}_s^{t,x} - \vec{Y}_s^{t,x'}), \Delta f(r) \rangle dr \right] \\ & \quad + \mathbb{E} \left[2 \int_s^T (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}) d(\Delta K_r) \right], \end{aligned} \quad (4.32)$$

where the processes $\Delta X_r, \Delta Y_r, \Delta f(r), \Delta K_r, \Delta Z_r, \Delta U_r$ and $\Delta \ell_r$ are defined as follows: $\forall r \in [t, T]$,

$$\Delta f(r) := ((\Delta f^{(i)}(r))_{i=1,m} = (f^{(i)}(r, X_r^{i;t,x}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, U_r^{i;t,x}) - f^{(i)}(r, X_r^{i;t,x'}, \vec{Y}_r^{t,x'}, Z_r^{i;t,x'}, U_r^{i;t,x'}))_{i=1,m},$$

$$\Delta X_r = X_r^{t,x} - X_r^{t,x'}, \Delta Y(r) = \vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'} = (Y_r^{j;t,x} - Y_r^{j;t,x'})_{j=1,m},$$

$$\Delta K_r = K_r^{t,x} - K_r^{t,x'}, \Delta Z_r = Z_r^{t,x} - Z_r^{t,x'}, \Delta U_r = U_r^{t,x} - U_r^{t,x'} \text{ and } \Delta \ell_r = (\ell(r, X_r^{t,x}) - \ell(r, X_r^{t,x'}))$$

($\langle \cdot, \cdot \rangle$ is the usual scalar product on \mathbb{R}^m). Now we will give an estimation of each three terms of the second member of inequality (4.32).

• As for any $i \in \{1, \dots, m\}$ g^i belongs to \mathcal{U} ; therefore

$$\begin{aligned} \mathbb{E} \left[\left| g(X_T^{t,x}) - g(X_T^{t,x'}) \right|^2 \right] & \leq C \left| X_T^{t,x} - X_T^{t,x'} \right|^2 (1 + \left| X_T^{t,x} \right|^{2p} + \left| X_T^{t,x'} \right|^{2p}) \\ & \leq \mathbb{E} \left[\left| x - x' \right|^2 (1 + \left| (X_T^{t,x} - x) + x \right|^{2p} + \left| (X_T^{t,x'} - x') + x' \right|^{2p}) \right], \end{aligned}$$

and by subsequently using the triangle inequality, the relation of proposition 3.1 and the fact that

$$(a+b)^p \leq 2^{p-1}(a^p + b^p).$$

$$\mathbb{E} \left[\left| g(X_T^{t,x}) - g(X_T^{t,x'}) \right|^2 \right] \leq C |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}), \quad (4.33)$$

• using (iii) of (3.14): $\mathbb{E} \left[2 \int_s^T \left(\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'} \right) d(\Delta K_r) \right]$ can be replaced by

$$\mathbb{E} \left[2 \int_s^T \left(\ell(r, X_r^{t,x}) - \ell(r, X_r^{t,x'}) \right) d(\Delta K_r) \right].$$

Now by **(H1)** and Cauchy-Schwartz inequality we obtain:

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} (\Delta \ell_t)^2 \right] \times \mathbb{E} \left[(\Delta K_T)^2 \right] \leq 2CC' |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}); \quad (4.34)$$

where $C' = \mathbb{E} \left[(\Delta K_T)^2 \right]$.

• To complete our estimation of (4.32) we need to deal with $\mathbb{E} \left[2 \int_s^T \left(\vec{Y}_s^{t,x} - \vec{Y}_s^{t,x'} \right), \Delta f(r) > dr \right]$.

Taking into account the expression of $f^{(i)}$ given by (2.11) we then split $\Delta f(r)$ in the follows way: for $r \leq T$,

$$\Delta f(r) = (\Delta f(r))_{i=1,m} = \Delta_1(r) + \Delta_2(r) + \Delta_3(r) + \Delta_4(r) = (\Delta_1^i(r) + \Delta_2^i(r) + \Delta_3^i(r) + \Delta_4^i(r))_{i=1,m},$$

where for any $i = 1, \dots, m$,

$$\begin{aligned} \Delta_1^i(r) &= h^{(i)} \left(r, X_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right) \\ &\quad - h^{(i)} \left(r, X_r^{t,x'}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right); \\ \Delta_2^i(r) &= h^{(i)} \left(r, X_r^{t,x'}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right) \\ &\quad - h^{(i)} \left(r, X_r^{t,x'}, \vec{Y}_r^{t,x'}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right); \\ \Delta_3^i(r) &= h^{(i)} \left(r, X_r^{t,x'}, \vec{Y}_r^{t,x'}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right) \\ &\quad - h^{(i)} \left(r, X_r^{t,x'}, \vec{Y}_r^{t,x'}, Z_r^{i;t,x'}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right); \\ \Delta_4^i(r) &= h^{(i)} \left(r, X_r^{t,x'}, \vec{Y}_r^{t,x'}, Z_r^{i;t,x'}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right) \\ &\quad - h^{(i)} \left(r, X_r^{t,x'}, \vec{Y}_r^{t,x'}, Z_r^{i;t,x'}, \int_E \gamma^i(r, X_r^{t,x'}, e) U_r^{i;t,x'}(e) \lambda(de) \right). \end{aligned}$$

By Cauchy-Schwartz inequality, the inequality $2ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2$, the relation (2.11) and the estimate (3.13) we have:

$$\begin{aligned} \mathbb{E} \left[2 \int_s^T \langle \Delta Y(r), \Delta_1(r) \rangle dr \right] &\leq \mathbb{E} \left[\frac{1}{\epsilon} \int_s^T |\Delta Y(r)|^2 dr + C^2 \epsilon \int_s^T |X_r^{t,x} - X_r^{t,x'}|^2 (1 + |X_r^{t,x}|^p + |X_r^{t,x'}|^p)^2 dr \right] \\ &\leq \mathbb{E} \left[\frac{1}{\epsilon} \int_s^T |\Delta Y(r)|^2 dr \right] + C^2 \epsilon |x - x'|^2 (1 + |x|^p + |x'|^p)^2. \end{aligned} \quad (4.35)$$

Besides since $h^{(i)}$ is Lipschitz w.r.t. (y, z, q) then,

$$\mathbb{E} \left[2 \int_s^T \langle \Delta Y(r), \Delta_2(r) \rangle dr \right] \leq 2C \mathbb{E} \left[\int_s^T |\Delta Y(r)|^2 dr \right], \quad (4.36)$$

and

$$\mathbb{E} \left[2 \int_s^T \langle \Delta Y(r), \Delta_3(r) \rangle dr \right] \leq \mathbb{E} \left[\frac{1}{\epsilon} \int_s^T |\Delta Y(r)|^2 dr + C^2 \epsilon \int_s^T |\Delta Z(r)|^2 dr \right]. \quad (4.37)$$

It remains to obtain a control of the last term. But for any $s \in [t, T]$ we have,

$$\begin{aligned} &\mathbb{E} \left[2 \int_s^T \langle \Delta Y(r), \Delta_4(r) \rangle dr \right] \quad (4.38) \\ &\leq 2C \mathbb{E} \left[\int_s^T |\Delta Y(r)| dr \times \left| \int_E \left(\gamma(r, X_r^{t,x}, e) U_r^{t,x}(e) - \gamma(r, X_r^{t,x'}, e) U_r^{t,x'}(e) \right) \lambda(de) \right| \right]. \end{aligned}$$

Next by splitting the crossing terms as follows $\gamma(r, X_r^{t,x}, e) U_r^{t,x}(e) - \gamma(r, X_r^{t,x'}, e) U_r^{t,x'}(e) = \Delta U_s(e) \gamma(s, X_s^{t,x}, e) + U_s^{t,x'} \left(\gamma(s, X_s^{t,x}, e) - \gamma(s, X_s^{t,x'}, e) \right)$ and setting $\Delta \gamma_s(e) := \left(\gamma(s, X_s^{t,x}, e) - \gamma(s, X_s^{t,x'}, e) \right)$,

we obtain,

$$\begin{aligned} \mathbb{E} \left[2 \int_s^T \langle \Delta Y(r), \Delta_4(r) \rangle dr \right] &\leq 2C \mathbb{E} \left[\int_s^T |\Delta Y(r)| \times \left(\int_E (|U_r^{t,x'}(e) \Delta \gamma_r(e)| + |\Delta U_r(e) \gamma(r, X_r^{t,x}, e)|) \lambda(de) \right) dr \right] \\ &\leq \frac{2}{\epsilon} \mathbb{E} \left[\int_s^T |\Delta Y(r)|^2 dr \right] + C^2 \epsilon \mathbb{E} \left[\int_s^T \left(\int_E (|U_r^{t,x'}(e) \Delta \gamma_r(e)| \lambda(de)) \right)^2 dr \right] \\ &\quad + C^2 \epsilon \mathbb{E} \left[\int_s^T \left(\int_E (|\Delta U_r(e) \gamma(r, X_r^{t,x}, e)| \lambda(de)) \right)^2 dr \right]. \end{aligned} \quad (4.39)$$

By Cauchy-Schwartz inequality, (2.10) and (3.13), and the result of Lemma 4.1 it holds:

$$\begin{aligned} \mathbb{E} \left[\int_s^T \left(\int_E (|U_r^{t,x'}(e) \Delta \gamma_r(e)| \lambda(de)) \right)^2 dr \right] &\leq \mathbb{E} \left[\int_s^T dr \left(\int_E |U_r^{t,x'}(e)|^2 \lambda(de) \right) \left(\int_E |\Delta \gamma_r(e)|^2 \lambda(de) \right) \right] \\ &\leq C \mathbb{E} \left[\left\{ \sup_{r \in [t, T]} |X_r^{t,x} - X_r^{t,x'}|^2 (1 + \sup_{r \in [t, T]} |X_r^{t,x}|^p + |X_r^{t,x'}|^p)^2 \right\} dr \right] \\ &\quad \times \mathbb{E} \left[\int_s^T dr \left(\int_E |U_r^{t,x'}(e)|^2 \lambda(de) \right) \right] \\ &\leq C \sqrt{\mathbb{E} \left[\left\{ \sup_{r \in [t, T]} |X_r^{t,x} - X_r^{t,x'}|^4 (1 + \sup_{r \in [t, T]} |X_r^{t,x}|^p + |X_r^{t,x'}|^p)^4 \right\} \right]} \end{aligned}$$

$$\begin{aligned}
& \times \sqrt{\mathbb{E} \left[\left\{ \int_s^T dr \left(\int_E |U_r^{t,x'}(e)|^2 \lambda(de) \right) \right\}^2 \right]} \\
& \leq C |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}). \tag{4.40}
\end{aligned}$$

For some exponent p . On the other hand using once more Cauchy-Schwartz inequality and (2.10)-(i) we get

$$\begin{aligned}
\mathbb{E} \left[\int_s^T \left(\int_E (|\Delta U_r(e) \gamma(r, X_r^{t,x}, e)| \lambda(de)) \right)^2 dr \right] & \leq \mathbb{E} \left[\int_s^T dr \left(\int_E (|\Delta U_r(e)|^2 \lambda(de)) \right) \left(\int_E |\gamma(r, X_r^{t,x}, e)|^2 \lambda(de) \right) \right] \\
& \leq C \mathbb{E} \left[\int_s^T dr \left(\int_E (|\Delta U_r(e)|^2 \lambda(de)) \right) \right]. \tag{4.41}
\end{aligned}$$

Taking now into account inequalities (4.35)-(4.41) we obtain:

$$\begin{aligned}
& \mathbb{E} \left[\left| \vec{Y}_s^{t,x} - \vec{Y}_s^{t,x'} \right|^2 + \int_s^T |\Delta Z_r|^2 dr + \int_s^T \|\Delta U_r\|_{\mathbb{L}^2(\lambda)}^2 dr \right] \\
& \leq \mathbb{E} \left[\left| g(X_T^{t,x}) - g(X_T^{t,x'}) \right|^2 + 2 \int_s^T \langle (\vec{Y}_s^{t,x} - \vec{Y}_s^{t,x'}), \Delta f(r) \rangle dr \right] \\
& \quad + \mathbb{E} \left[2 \int_s^T (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}) d(\Delta K_r) \right] \\
& \leq |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}) (C + 2CC' + C^2\epsilon + C^3\epsilon) + \left(\frac{3}{\epsilon} + 2C \right) \mathbb{E} \left[\int_s^T |\Delta Y(r)|^2 dr \right] \\
& \quad + C^2\epsilon \mathbb{E} \left[\int_s^T |\Delta Z(r)|^2 dr \right] + C^3\epsilon \mathbb{E} \left[\int_s^T dr \left(\int_E (|\Delta U_r(e)|^2 \lambda(de)) \right) \right].
\end{aligned}$$

Choosing now ϵ small enough we deduce the existence of a constant $C \geq 0$ such that for any $s \in [t, T]$,

$$\mathbb{E} [|\Delta Y(s)|^2] \leq C |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}) + \mathbb{E} \left[\int_s^T |\Delta Y(r)|^2 dr \right]$$

and by Gronwall lemma this implies that for any $s \in [t, T]$,

$$\mathbb{E} [|\Delta Y(s)|^2] \leq C |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}).$$

Finally in taking $s = t$ and considering (3.15) we obtain the desired result.

Remark 3 This result give also estimate of U where we use the function $h^{(i)} \forall i = 1, \dots, m$ contrary in estimate (4.17).

Corollary 4.3 For $u^i \in \mathcal{U} \forall i = 1, \dots, m$ $B_i u^i$ defined in (1.2) is well posed since the functions β and $(\gamma_i)_{i=1,m}$ verify (2.7) and (2.10) respectively.

Proof. The main point to notice is that λ integrates $(1 \wedge |e|^p) \forall p \geq 2$.

We have that

$$\begin{aligned} |\mathbb{B}_i u^i(t, x)| &\leq \int_{\mathbb{E}} |\gamma^i(t, x, e)| \times |(u^i(t, x + \beta(t, x, e)) - u^i(t, x))| \lambda(de) \\ &\leq \int_{\mathbb{E}} C(1 \wedge |e|) |\beta(t, x, e)| (1 + |x + \beta(t, x, e)|^p + |x|^p) \lambda(de) \\ &\leq C^2(1 + |x|^p(1 + 2^{p-1})) \int_{\mathbb{E}} C(1 \wedge |e|^2) \lambda(de) + (2^{\frac{p-1}{p}} C^{\frac{p+2}{p}})^p \int_{\mathbb{E}} C(1 \wedge |e|^p) \lambda(de). \end{aligned}$$

Which finish the proof.

Now by remark 3, the last estimate of U confirm the following result;

Proposition 4.4 For any $i = 1, \dots, m$, $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$U_s^{i;t,x}(e) = u^i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u^i(s, X_{s-}^{t,x}), \quad d\mathbb{P} \otimes ds \otimes d\lambda - a.e. \text{ on } \Omega \times [t, T] \times E. \quad (4.42)$$

Proof. First note that since the measure λ is not finite, then we cannot use the same technique as in [4] where the authors use the jumps of processes and (3.15).

In our case $U^{i;t,x}$ is only square integrable and not necessarily integrable w.r.t. $d\mathbb{P} \otimes ds \otimes d\lambda$.

Therefore we first begin by truncating the Lévy measure as the same way in [3].

Step 1: Truncation of the Lévy measure

For any $k \geq 1$, let us first introduce a new Poisson random measure μ_k (obtained from the truncation of μ) and its associated compensator ν_k as follows:

$$\mu_k(ds, de) = 1_{\{|e| \geq \frac{1}{k}\}} \mu(ds, de) \quad \text{and} \quad \nu_k(ds, de) = 1_{\{|e| \geq \frac{1}{k}\}} \nu(ds, de).$$

Which means that, as usual, $\widetilde{\mu}_k(ds, de) := (\mu_k - \nu_k)(ds, de)$, is the associated random martingale measure.

The main point to notice is that

$$\begin{aligned} \lambda_k(E) &= \int_E \lambda_k(de) = \int_E 1_{\{|e| \geq \frac{1}{k}\}} \lambda(de) \\ &= \int_{\{|e| \geq \frac{1}{k}\}} \lambda(de) \\ &= \lambda(\{|e| \geq \frac{1}{k}\}) < \infty. \end{aligned} \quad (4.43)$$

As in [3], let us introduce the process ${}^k X^{t,x}$ solving the following standard SDE of jump-diffusion type:

$${}^k X_s^{t,x} = x + \int_t^s b(r, {}^k X_r^{t,x}) dr + \int_t^s \sigma(r, {}^k X_r^{t,x}) dB_r$$

$$+ \int_t^s \int_E \beta(r, {}^k X_{r-}^{t,x}, e) \tilde{\mu}_k(dr, de), \quad t \leq s \leq T; \quad {}^k X_r^{t,x} = x \text{ if } s \leq t. \quad (4.44)$$

Note that thanks to the assumptions on b, σ, β the process ${}^k X^{t,x}$ exists and is unique. Moreover it satisfies the same estimates as in (3.13) since λ_k is just a truncation at the origin of λ which integrates $(1 \wedge |e|^2)_{e \in E}$.

On the other hand let us consider the following Markovian RBSDE with jumps

$$\left\{ \begin{array}{l} (i) \mathbb{E} \left[\sup_{s \leq T} |{}^k Y_s^{t,x}|^2 + \int_s^T |{}^k Z_r^{t,x}|^2 dr + \int_s^T \|{}^k U_r^{t,x}\|_{\mathbb{L}^2(\lambda_k)}^2 dr \right] < \infty \\ (ii) {}^k Y^{t,x} := ({}^k Y^{i,t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad {}^k Z^{t,x} := ({}^k Z^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ {}^k K^{t,x} := ({}^k K^{i,t,x})_{i=1,m} \in \mathcal{A}_c^2, \quad {}^k U^{t,x} := ({}^k U^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda_k)); \\ (iii) {}^k Y_s^{t,x} = g({}^k X_T^{t,x}) + \int_s^T f_{\mu_k}(r, {}^k X_r^{t,x}, {}^k Y_r^{t,x}, {}^k Z_r^{t,x}, {}^k U_r^{t,x}) dr + {}^k K_T^{t,x} - {}^k K_s^{t,x} \\ \quad - \int_s^T \left\{ {}^k Z_r^{t,x} dB_r + \int_E U_r^{t,x}(e) \tilde{\mu}_k(dr, de) \right\}, \quad s \leq T; \\ (iv) {}^k Y_s^{i;t,x} \geq \ell(s, {}^k X_s^{t,x}) \text{ and } \int_0^T ({}^k Y_s^{i;t,x} - \ell(s, {}^k X_s^{t,x})) d({}^k K_s^{i;t,x}) = 0. \end{array} \right. \quad (4.45)$$

Finally let us introduce the following functions $(f^{(i)})_{i=1,m}$ defined by: $\forall (t, x, y, z, \zeta) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{L}_m^2(\lambda_k)$,

$$f_{\mu_k}(t, x, y, z, \zeta) = (f_{\mu_k}^{(i)}(t, x, y, z_i, \zeta_i))_{i=1,m} := \left(h^{(i)} \left(t, x, y, z, \int_E \gamma^i(t, x, e) \zeta_i(e) \lambda_k(de) \right) \right)_{i=1,m}.$$

First let us emphasize that this latter RBSDE is related to the filtration $(\mathcal{F}_s^k)_{s \leq T}$ generated by the Brownian motion and the independent random measure μ_k . However this point does not raise major issues since for any $s \leq T$, $\mathcal{F}_s^k \subset \mathcal{F}_s$ and thanks to the relationship between μ and μ_k .

Next by the properties of the functions b, σ, β and by the same opinions of proposition 3.2 and proposition 3.3, there exists a unique quadruple $({}^k Y^{t,x}, {}^k K^{t,x}, {}^k Z^{t,x}, {}^k U^{t,x})$ solving (4.45) and there also exists a function u^k from $[0, T] \times \mathbb{R}^k$ into \mathbb{R}^m of Π_g^c such that

$$\forall s \in [t, T], \quad {}^k Y^{t,x} := u^k(s, {}^k X_s^{t,x}), \quad \mathbb{P} - a.s. \quad (4.46)$$

Moreover as in proposition 4.2, there exists positive constants C and p which do not depend on k such that:

$$\forall t, x, x', \quad |u^k(t, x) - u^k(t, x')| \leq C |x - x'| (1 + |x|^p + |x'|^p). \quad (4.47)$$

Finally as λ_k is finite then we have the following relationship between the process ${}^k U^{t,x} := ({}^k U^{i;t,x})_{i=1,m}$ and the deterministic functions $u^k := (u_i^k)_{i=1,m}$ (see [4]): $\forall i = 1, \dots, m$;

$${}^k U_s^{i;t,x}(e) = u_i^k(s, {}^k X_{s-}^{t,x} + \beta(s, {}^k X_{s-}^{t,x}, e)) - u_i^k(s, {}^k X_{s-}^{t,x}), \quad d\mathbb{P} \otimes ds \otimes d\lambda_k - a.e. \text{ on } \Omega \times [t, T] \times E.$$

This is mainly due to the fact that ${}^k U^{t,x}$ belongs to $\mathbb{L}^1 \cap \mathbb{L}^2(ds \otimes d\mathbb{P} \otimes d\lambda_k)$ since $\lambda_k(E) < \infty$ and then we can split the stochastic integral w.r.t. $\tilde{\mu}_k$ in (4.45). Therefore for all $i = 1, \dots, m$,

$${}^k U_s^{i;t,x}(e) 1_{\{|e| \geq \frac{1}{k}\}} = (u_i^k(s, {}^k X_{s-}^{t,x} + \beta(s, {}^k X_{s-}^{t,x}, e)) - u_i^k(s, {}^k X_{s-}^{t,x})) 1_{\{|e| \geq \frac{1}{k}\}}, \quad d\mathbb{P} \otimes ds \otimes d\lambda_k - a.e. \text{ on } \Omega \times [t, T] \times E. \quad (4.48)$$

Step 2: Convergence of the auxiliary processes

Let's now prove the following convergence result;

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \leq T} \left| Y_s^{t,x} - {}^k Y_s^{t,x} \right|^2 + (K_T^{t,x} - {}^k K_T^{t,x})^2 + \int_0^T \left| Z_s^{t,x} - {}^k Z_s^{t,x} \right|^2 ds \right. \\ & \left. + \int_0^T ds \int_E \lambda(de) \left| U_s^{t,x}(e) - {}^k U_s^{t,x}(e) 1_{\{|e| \geq \frac{1}{k}\}} \right|^2 \right] \xrightarrow{k \rightarrow +\infty} 0; \end{aligned} \quad (4.49)$$

where $(Y^{t,x}, K^{t,x}, Z^{t,x}, U^{t,x})$ is solution of the RBSDE with jumps (3.14).

First note that the following convergence result was established in [3]

$$\mathbb{E} \left[\sup_{s \leq T} \left| X_s^{t,x} - {}^k X_s^{t,x} \right|^2 \right] \xrightarrow{k \rightarrow +\infty} 0. \quad (4.50)$$

We now focus on (4.49). Note that we can apply Ito's formula, even if the RBSDEs are related to filtrations and Poisson random measures which are not the same, since:

(i) $\mathcal{F}_s^k \subset \mathcal{F}_s, \forall s \leq T$;

(ii) for any $s \leq T$, $\int_0^s \int_E U^{i;t,x}(e) \tilde{\mu}_k(dr, de) = \int_0^s \int_E U^{i;t,x}(e) 1_{\{|e| \geq \frac{1}{k}\}} \tilde{\mu}(dr, de)$ and then the first $(\mathcal{F}_s^k)_{s \leq T}$ -martingale is also an $(\mathcal{F}_s)_{s \leq T}$ -martingale. $\forall s \in [0, T]$,

$$\begin{aligned} & \left| \bar{Y}_s^{t,x} - {}^k Y_s^{t,x} \right|^2 + \int_0^T \left| Z_s^{t,x} - {}^k Z_s^{t,x} \right|^2 ds + \sum_{s \leq r \leq T} ({}^k \Delta_r \bar{Y}_r^{t,x})^2 \\ & = \left| g(X_T^{t,x}) - g({}^k X_T^{t,x}) \right|^2 + 2 \int_s^T \left(\bar{Y}_r^{t,x} - {}^k Y_r^{t,x} \right) \times {}^k \Delta f(r) dr + 2 \int_s^T \left(\bar{Y}_r^{t,x} - {}^k Y_r^{t,x} \right) d({}^k \Delta K_r) \\ & - 2 \int_s^T \int_E \left(\bar{Y}_r^{t,x} - {}^k Y_r^{t,x} \right) ({}^k \Delta U_r(e)) \tilde{\mu}(dr, de) - 2 \int_s^T \left(\bar{Y}_r^{t,x} - {}^k \bar{Y}_r^{t,x} \right) ({}^k \Delta Z_r) dB_r; \end{aligned}$$

and taking expectation we obtain: $\forall s \in [t, T]$,

$$\begin{aligned} & \mathbb{E} \left[\left| \bar{Y}_s^{t,x} - {}^k Y_s^{t,x} \right|^2 + \left| {}^k \Delta K_T \right|^2 + \int_0^T \left\{ \left| Z_s^{t,x} - {}^k Z_s^{t,x} \right|^2 + \int_E \left| U_s^{t,x} - {}^k U_s^{t,x} 1_{\{|e| \geq \frac{1}{k}\}} \right|^2 \lambda(de) \right\} ds \right] \\ & \leq \mathbb{E} \left[\left| g(X_T^{t,x}) - g({}^k X_T^{t,x}) \right|^2 + 2 \int_s^T \left(\bar{Y}_r^{t,x} - {}^k Y_r^{t,x} \right) \times {}^k \Delta f(r) dr \right] + \mathbb{E} \left[\sup_{s \leq T} \left| {}^k \Delta \ell_s \right|^2 \right]; \end{aligned} \quad (4.51)$$

where the processes ${}^k \Delta X_r, {}^k \Delta Y_r, {}^k \Delta f(r), {}^k \Delta K_r, {}^k \Delta Z_r, {}^k \Delta U_r$ and ${}^k \Delta \ell_r$ are defined as follows: $\forall r \in [0, T]$,

$$\begin{aligned}
{}^k\Delta f(r) &:= (({}^k\Delta f^{(i)}(r))_{i=1,m} = (f^{(i)}(r, X_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, U_r^{i;t,x}) - f_k^{(i)}(r, {}^kX_r^{t,x}, {}^kY_r^{t,x}, {}^kZ_r^{t,x}, {}^kU_r^{t,x}))_{i=1,m}, \\
{}^k\Delta X_r &= X_r^{t,x} - {}^kX_r^{t,x}, \quad {}^k\Delta Y(r) = \vec{Y}_r^{t,x} - {}^kY_r^{t,x} = (Y_r^{j;t,x} - {}^kY_r^{j;t,x})_{j=1,m}, \\
{}^k\Delta K_r &= K_r^{t,x} - {}^kK_r^{t,x}, \quad {}^k\Delta Z_r = Z_r^{t,x} - {}^kZ_r^{t,x}, \quad {}^k\Delta U_r = U_r^{t,x} - {}^kU_r^{t,x} \mathbf{1}_{\{|e| \geq \frac{1}{k}\}} \text{ and } {}^k\Delta \ell_r = \left(\ell(r, X_r^{t,x}) - \ell(r, {}^kX_r^{t,x}) \right).
\end{aligned}$$

Next let us set for $r \leq T$,

$${}^k\Delta f(r) = (f(r, X_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{t,x}, U_r^{t,x}) - f_k(r, {}^kX_r^{t,x}, {}^kY_r^{t,x}, {}^kZ_r^{t,x}, {}^kU_r^{t,x})) = A(r) + B(r) + C(r) + D(r);$$

where for any $i = 1, \dots, m$,

$$\begin{aligned}
A(r) &= \left(h^{(i)} \left(r, X_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right) \right. \\
&\quad \left. - h^{(i)} \left(r, {}^kX_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right) \right)_{i=1,m}; \\
B(r) &= \left(h^{(i)} \left(r, {}^kX_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right) \right. \\
&\quad \left. - h^{(i)} \left(r, {}^kX_r^{t,x}, {}^kY_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right) \right)_{i=1,m}; \\
C(r) &= \left(h^{(i)} \left(r, {}^kX_r^{t,x}, {}^kY_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right) \right. \\
&\quad \left. - h^{(i)} \left(r, {}^kX_r^{t,x}, {}^kY_r^{t,x}, {}^kZ_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right) \right)_{i=1,m}; \\
D(r) &= \left(h^{(i)} \left(r, {}^kX_r^{t,x}, {}^kY_r^{t,x}, {}^kZ_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right) \right. \\
&\quad \left. - h^{(i)} \left(r, {}^kX_r^{t,x}, {}^kY_r^{t,x}, {}^kZ_r^{i;t,x}, \int_E \gamma^i(r, {}^kX_r^{t,x}, e) {}^kU_r^{i;t,x}(e) \lambda_k(de) \right) \right)_{i=1,m}.
\end{aligned}$$

By (4.50) and the of $g \in \mathcal{U}$ and $\ell \in \mathcal{U}$ we have,

$$\mathbb{E} \left[\left| g(X_T^{t,x}) - g({}^kX_T^{t,x}) \right|^2 \right] \xrightarrow[k \rightarrow +\infty]{} 0 \quad (4.52)$$

and

$$\mathbb{E} \left[\sup_{s \leq T} \left| \ell(X_s^{t,x}) - \ell({}^kX_s^{t,x}) \right|^2 \right] \xrightarrow[k \rightarrow +\infty]{} 0. \quad (4.53)$$

Now we will interest to $\mathbb{E} \left[\int_s^T \left(\vec{Y}_r^{t,x} - {}^kY_r^{t,x} \right) \times {}^k\Delta f(r) dr \right]$ for found (4.49).

By (2) and (2.8), we have: $\forall r \in [0, T]$

$$\begin{aligned}
|A(r)| &\leq C \left| X_r^{t,x} - {}^kX_r^{t,x} \right| (1 + |X_r^{t,x}|^p + |{}^kX_r^{t,x}|^p); \\
|B(r)| &\leq C \left| \vec{Y}_r^{t,x} - {}^kY_r^{t,x} \right| \text{ and } |C(r)| \leq \left| Z_r^{t,x} - {}^kZ_r^{t,x} \right|;
\end{aligned} \quad (4.54)$$

where C is a constant. Finally let us deal with $D(r)$ which is more involved. First note that $D(r) = (D_i(r))_{i=1,m}$ where

$$\begin{aligned} D_i(r) &= h^{(i)} \left(r, {}^k X_r^{t,x}, {}^k Y_r^{t,x}, {}^k Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right) \\ &\quad - h^{(i)} \left(r, {}^k X_r^{t,x}, {}^k Y_r^{t,x}, {}^k Z_r^{i;t,x}, \int_E \gamma^i(r, {}^k X_r^{t,x}, e) {}^k U_r^{i;t,x}(e) \lambda_k(de) \right). \end{aligned}$$

But as $h^{(i)}$ is Lipschitz w.r.t to the last component q then,

$$\begin{aligned} |D(r)|^2 &\leq C \left\{ \int_E \left| \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) - \gamma^i(r, {}^k X_r^{t,x}, e) {}^k U_r^{i;t,x}(e) 1_{\{|e| \geq \frac{1}{k}\}} \right|^2 \lambda(de) \right\} \\ &\leq C \left\{ \left\{ \int_E \left| \gamma^i(r, X_r^{t,x}, e) - \gamma^i(r, {}^k X_r^{t,x}, e) \right| |U_r^{i;t,x}(e)| \lambda(de) \right\}^2 \right. \\ &\quad \left. + \left\{ \int_E \left| \gamma^i(r, X_r^{t,x}, e) \right| \left| U_r^{i;t,x}(e) - {}^k U_r^{i;t,x}(e) 1_{\{|e| \geq \frac{1}{k}\}} \right| \lambda(de) \right\}^2 \right\} \\ &\leq C \left\{ \left| X_r^{t,x} - {}^k X_r^{t,x} \right| (1 + |X_r^{t,x}|^p + |{}^k X_r^{t,x}|^p) \left| \int_E U_r^{i;t,x}(e) \lambda(de) \right|^2 \right. \\ &\quad \left. + C \int_E (1 \wedge |e|) \left| U_r^{i;t,x}(e) - {}^k U_r^{i;t,x}(e) 1_{\{|e| \geq \frac{1}{k}\}} \right|^2 \lambda(de), \right. \end{aligned} \quad (4.55)$$

and (4.51) become by using the majorations obtain in (4.54) and in (4.55);

$$\begin{aligned} &\mathbb{E} \left[\left| \vec{Y}_s^{t,x} - {}^k Y_s^{t,x} \right|^2 + \left| {}^k \Delta K_T \right|^2 + \int_0^T \left\{ \left| Z_s^{t,x} - {}^k Z_s^{t,x} \right|^2 + \int_E \left| U_s^{t,x} - {}^k U_s^{t,x} 1_{\{|e| \geq \frac{1}{k}\}} \right|^2 \lambda(de) \right\} ds \right] \\ &\leq \mathbb{E} \left[\left| g(X_T^{t,x}) - g({}^k X_T^{t,x}) \right|^2 \right] + \mathbb{E} \left[\sup_{s \leq T} \left| \ell(X_s^{t,x}) - \ell({}^k X_s^{t,x}) \right|^2 \right] + C \mathbb{E} \left[\int_s^T \left| \vec{Y}_s^{t,x} - {}^k Y_s^{t,x} \right|^2 \right] \\ &+ C \mathbb{E} \left[\int_0^T \left| X_r^{t,x} - {}^k X_r^{t,x} \right|^2 (1 + |X_r^{t,x}|^p + |{}^k X_r^{t,x}|^p)^2 dr \right] \\ &+ C \mathbb{E} \left[\int_0^T dr \left\{ \left| X_r^{t,x} - {}^k X_r^{t,x} \right| (1 + |X_r^{t,x}|^p + |{}^k X_r^{t,x}|^p) \int_E U_r^{i;t,x}(e) \lambda(de) \right\}^2 \right]. \end{aligned} \quad (4.56)$$

The two first terms converge to 0 by (4.52) and (4.53).

For the fourth term we have:

$$\begin{aligned} &\mathbb{E} \left[\int_0^T \left| X_r^{t,x} - {}^k X_r^{t,x} \right|^2 (1 + |X_r^{t,x}|^p + |{}^k X_r^{t,x}|^p)^2 dr \right] \\ &\leq \mathbb{E} \left[\sup_{r \leq T} \left| X_r^{t,x} - {}^k X_r^{t,x} \right|^2 \int_0^T (1 + |X_r^{t,x}|^p + |{}^k X_r^{t,x}|^p)^2 dr \right] \\ &\leq \left\{ \mathbb{E} \left[\sup_{r \leq T} \left| X_r^{t,x} - {}^k X_r^{t,x} \right|^2 \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left[\left(\int_0^T (1 + |X_r^{t,x}|^p + |{}^k X_r^{t,x}|^p)^2 \left| X_r^{t,x} - {}^k X_r^{t,x} \right| \right)^2 dr \right] \right\}^{\frac{1}{2}}. \end{aligned}$$

The first factor in the right-hand side of this inequality goes to 0 when $k \rightarrow \infty$ due to (4.50) and the second factor is uniformly bounded by the uniform estimates (3.13) of $X^{t,x}$ and ${}^k X^{t,x}$.

Note also the last term converge to 0 when $k \rightarrow \infty$, it is a consequence of (4.50), the fact that ${}^k X^{t,x}$ verifies estimates (3.13) uniformly, the Cauchy-Schwartz inequality (used twice) and finally (4.17) of lemma 4.1. Then by Gronwall's lemma we deduce first that for any $s \leq T$,

$$\mathbb{E} \left[\left| \vec{Y}_s^{t,x} - {}^k Y_s^{t,x} \right|^2 \right] \xrightarrow[k \rightarrow +\infty]{} 0 \quad (4.57)$$

and in taking $s = t$ we obtain $u^k(t, x) \xrightarrow[k \rightarrow +\infty]{} u(t, x)$. As $(t, x) \in [0, T] \times \mathbb{R}^k$ is arbitrary then $u^k \xrightarrow[k \rightarrow +\infty]{} u$ pointwisely.

Next going back to (4.56) take the limit w.r.t k and using the uniform polynomial growth of u^k and the Lebesgue dominated convergence theorem as well, to obtain:

$$\mathbb{E} \left[\int_t^T \int_E \left| U_s^{t,x} - {}^k U_s^{t,x} 1_{\{|e| \geq \frac{1}{k}\}} \right|^2 \lambda(de) ds \right] \xrightarrow[k \rightarrow +\infty]{} 0. \quad (4.58)$$

Step 3: Conclusion

First note that by (4.47) and the pointwise convergence of $(u^k)_k$ to u , if $(x_k)_k$ is a sequence of \mathbb{R}^k which converge to x then $((u^k(t, x_k))_k)$ converge to $u(t, x)$.

Now let us consider a subsequence which we still denote by $\{k\}$ such that $\sup_{s \leq T} \left| X_s^{t,x} - {}^k X_s^{t,x} \right|^2 \xrightarrow[k \rightarrow +\infty]{} 0$, \mathbb{P} -a.s. (and then $\left| X_{s-}^{t,x} - {}^k X_{s-}^{t,x} \right| \xrightarrow[k \rightarrow +\infty]{} 0$ since $\left| X_{s-}^{t,x} - {}^k X_{s-}^{t,x} \right| \leq \sup_{s \leq T} \left| X_s^{t,x} - {}^k X_s^{t,x} \right|^2$). By (4.50), this subsequence exists. As the mapping $x \mapsto \beta(t, x, e)$ is Lipschitz then the sequence

$$\begin{aligned} \left({}^k U_s^{t,x}(e) 1_{\{|e| \geq \frac{1}{k}\}} \right)_k &= \left((u_i^k(s, {}^k X_{s-}^{t,x} + \beta(s, {}^k X_{s-}^{t,x}, e)) - u_i^k(s, {}^k X_{s-}^{t,x})) 1_{\{|e| \geq \frac{1}{k}\}} \right)_{k \geq 1} \xrightarrow[k \rightarrow +\infty]{} \\ (u_i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u_i(s, X_{s-}^{t,x})), & \quad d\mathbb{P} \otimes ds \otimes d\lambda - a.e. \quad \text{on } \Omega \times [t, T] \times E \end{aligned} \quad (4.59)$$

for any $i = 1, \dots, m$. Finally from (4.58) we deduce that

$$U_s^{t,x}(e) = (u_i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u_i(s, X_{s-}^{t,x})), \quad \text{on } \Omega \times [t, T] \times E \quad (4.60)$$

which is the desired result.

5 The main result

First we give the definition of viscosity solution of IPDEs as given in [3] and [4]. Our main result deal with this definition.

Definition 5.1 *We say that a family of deterministic functions $u = (u^i)_{i=1,m}$ which belongs to $\mathcal{U} \quad \forall i \in \{1, \dots, m\}$ is a viscosity sub-solution (resp. super-solution) of the IPDE (1.1) if:*

(i) $\forall x \in \mathbb{R}^k$, $u^i(x, T) \leq g^i(x)$ (resp. $u^i(x, T) \geq g^i(x)$);

(ii) For any $(t, x) \in [0, T] \times \mathbb{R}^k$ and any function ϕ of class $C^{1,2}([0, T] \times \mathbb{R}^k)$ such that (t, x) is a global maximum point of $u^i - \phi$ (resp. global minimum point of $u^i - \phi$) and $(u^i - \phi)(t, x) = 0$ one has

$$\min \left\{ u^i(t, x) - \ell(t, x); -\partial_t \phi(t, x) - \mathcal{L}^X \phi(t, x) - h^i(t, x, (u^j(t, x))_{j=1, m}, \sigma^\top(t, x)) D_x \phi(t, x), B_i u^i(t, x) \right\} \leq 0 \quad (5.61)$$

(resp.

$$\min \left\{ u^i(t, x) - \ell(t, x); -\partial_t \phi(t, x) - \mathcal{L}^X \phi(t, x) - h^i(t, x, (u^j(t, x))_{j=1, m}, \sigma^\top(t, x)) D_x \phi(t, x), B_i u^i(t, x) \right\} \geq 0 \Big). \quad (5.62)$$

The family $u = (u^i)_{i=1, m}$ is a viscosity solution of (1.1) if it is both a viscosity sub-solution and viscosity super-solution.

Note that $\mathcal{L}^X \phi(t, x) = b(t, x)^\top D_x \phi(t, x) + \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 \phi(t, x)) + K \phi(t, x)$;
where $K \phi(t, x) = \int_{\mathbb{E}} (\phi(t, x + \beta(t, x, e)) - \phi(t, x) - \beta(t, x, e)^\top D_x \phi(t, x)) \lambda(de)$.

Theorem 5.2 Under assumptions **(H1)**, **(H2)** and **(H3)**, the IPDE (1.1) has unique solution which is the m -tuple of functions $(u^i)_{i=1, m}$ defined in proposition 3.3 by (3.15).

Proof. Step 1: Existence

Assume that assumptions **(H1)**, **(H2)** and **(H3)** are fulfilled, then the following multi-dimensional RBSDEs with jumps

$$\left\{ \begin{array}{l} (i) \ \underline{Y}^{t,x} := (\underline{Y}^{i;t,x})_{i=1, m} \in \mathcal{S}^2(\mathbb{R}^m), \ \underline{Z}^{t,x} := (\underline{Z}^{i;t,x})_{i=1, m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \ \underline{K}^{t,x} := (\underline{K}^{i;t,x})_{i=1, m} \in \mathcal{A}_c^2, \\ \underline{U}^{t,x} := (\underline{U}^{i;t,x})_{i=1, m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ (ii) \ \underline{Y}_s^{i;t,x} = g^i(X_T^{t,x}) + \underline{K}_T^{i;t,x} - \underline{K}_s^{i;t,x} - \int_s^T \underline{Z}^{i;t,x} dB_r - \int_s^T \int_{\mathbb{E}} \underline{U}_r^{i;t,x}(e) \tilde{\mu}(dr, de) \\ \quad + \int_s^T h^{(i)}(r, X_r^{t,x}, \underline{Y}_r^{i;t,x}, \underline{Z}_r^{i;t,x}, \int_{\mathbb{E}} \gamma^i(t, X_r^{t,x}, e) \{ (u^i(t, X_r^{t,x} + \beta(t, X_r^{t,x}, e)) - u^i(t, X_r^{t,x})) \} \lambda(de)) dr \\ (iii) \ \underline{Y}_s^{i;t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (\underline{Y}_s^{i;t,x} - \ell(s, X_s^{t,x})) d\underline{K}_s^{i;t,x} = 0; \end{array} \right. \quad (5.63)$$

has unique solution $(\underline{Y}, \underline{Z}, \underline{K}, \underline{U})$. Next as for any $i = 1, \dots, m$, u^i belongs to \mathcal{U} , then by proposition 3.3 the (3.15), there exists a family of deterministic continuous functions of polynomial growth $(\underline{u}^i)_{i=1, m}$ that fact for any $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$\forall s \in [t, T], \quad \underline{Y}_s^{i;t,x} = \underline{u}^i(s, X_s^{t,x}).$$

Such that by the same proposition, the family $(\underline{u}^i)_{i=1,m}$ is a viscosity solution of the following system:

$$\begin{cases} \min \left\{ \underline{u}^i(t, x) - \ell(t, x); -\partial_t \underline{u}^i(t, x) - b(t, x)^\top D_x \underline{u}^i(t, x) - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 \underline{u}^i(t, x)) \right. \\ \quad \left. - K_i \underline{u}^i(t, x) - h^{(i)}(t, x, (\underline{u}^j(t, x))_{j=1,m}, (\sigma^\top D_x \underline{u}^i)(t, x), B_i \underline{u}^i(t, x)) \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^k; \\ \underline{u}^i(T, x) = g^i(x). \end{cases} \quad (5.64)$$

Now we have the family $(\underline{u}^i)_{i=1,m}$ is a viscosity solution, our main objective is to found relation between $(\underline{u}^i)_{i=1,m}$ and $(u^i)_{i=1,m}$ which is defined in (3.15).

For this, let us consider the system of RBSDE with jumps

$$\begin{cases} (i) \bar{Y}^{t,x} := (Y^{i;t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \bar{Z}^{t,x} := (Z^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \bar{K}^{t,x} := (K^{i;t,x})_{i=1,m} \in \mathcal{A}_c^2, \\ \bar{U}^{t,x} := (U^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ (ii) Y_s^{i;t,x} = g^i(X_T^{t,x}) + K_T^{i;t,x} - K_s^{i;t,x} - \int_s^T Z_r^{i;t,x} dB_r - \int_s^T \int_E U_r^{i;t,x}(e) \tilde{\mu}(dr, de) \\ \quad + \int_s^T h^{(i)}(r, X_r^{t,x}, Y_r^{i;t,x}, \bar{Z}_r^{i;t,x}, \int_E \gamma^i(t, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de)) dr; \\ (iii) Y_s^{i;t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{i;t,x} - \ell(s, X_s^{t,x})) dK_s^{i;t,x} = 0. \end{cases} \quad (5.65)$$

By uniqueness of the solution of the RBSDEs with jumps (5.62), that for any $s \in [t, T]$ and $\forall i \in \{1, \dots, m\}$, $\underline{Y}_s^{i;t,x} = Y_s^{i;t,x}$.

Therefore $\underline{u}^i = u^i$, such that by (4.58) we obtain $U_s^{t,x}(e) = (u_i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u_i(s, X_{s-}^{t,x}))$, on $\Omega \times [t, T] \times E$, which give the viscosity solution in the sense of definition 5.1 (see [3]) by plugging (4.59) in $h^{(i)}$ of (5.64).

Step 2: Uniqueness

For uniqueness, let $(\bar{u}^i)_{i=1,m}$ be another family of \mathcal{U} which is solution viscosity of the system (1.1) in the sense of definition 5.1 and we consider RBSDE with jumps defined with \bar{u}^i .

$$\begin{cases} (i) \bar{Y}^{t,x} := (\bar{Y}^{i;t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \bar{Z}^{t,x} := (\bar{Z}^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \bar{K}^{t,x} := (\bar{K}^{i;t,x})_{i=1,m} \in \mathcal{A}_c^2, \\ \bar{U}^{t,x} := (\bar{U}^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ (ii) \bar{Y}_s^{i;t,x} = g^i(X_T^{t,x}) + \bar{K}_T^{i;t,x} - \bar{K}_s^{i;t,x} - \int_s^T \bar{Z}_r^{i;t,x} dB_r - \int_s^T \int_E \bar{U}_r^{i;t,x}(e) \tilde{\mu}(dr, de) \\ \quad + \int_s^T h^{(i)}(r, X_r^{t,x}, \bar{Y}_r^{i;t,x}, \bar{Z}_r^{i;t,x}, \int_E \gamma^i(t, X_r^{t,x}, e) (\bar{u}_i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - \bar{u}_i(s, X_{s-}^{t,x})) \lambda(de)) dr; \\ (iii) \bar{Y}_s^{i;t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (\bar{Y}_s^{i;t,x} - \ell(s, X_s^{t,x})) d\bar{K}_s^{i;t,x} = 0. \end{cases} \quad (5.66)$$

By Feynman Kac formula $\bar{u}^i(s, X_s^{t,x}) = Y_s^{i;t,x}$ where $Y_s^{i;t,x}$ satisfies the RBSDE with jumps (1.3)

associated to IPDE (1.1).

Since that the RBSDE with jumps (5.64) has solution and it is unique by assumed that **(H1)**, **(H2)** and **(H3)** are verified. By proposition 3.3 the (3.15), there exists a family of deterministic continuous functions of polynomial growth $(v^i)_{i=1,m}$ that fact for any $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$\forall s \in [t, T], \quad \bar{Y}_s^{i;t,x} = v^i(s, X_s^{t,x}).$$

Such that by the same proposition, the family $(v^i)_{i=1,m}$ is a viscosity solution of the following system:

$$\begin{cases} \min \left\{ v^i(t, x) - \ell(t, x); -\partial_t v^i(t, x) - b(t, x)^\top D_x v^i(t, x) - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 v^i(t, x)) \right. \\ \quad \left. - K_i v^i(t, x) - h^{(i)}(t, x, (v^j(t, x))_{j=1,m}, (\sigma^\top D_x v^i)(t, x), B_i \bar{u}^i(t, x)) \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^k; \\ u^i(T, x) = g^i(x). \end{cases} \quad (5.67)$$

By uniqueness of solution of (5.65) \bar{u}^i is viscosity solution of (5.66); and by proposition 3.3 $v^i = \bar{u}^i \forall i \in \{1, \dots, m\}$.

Now for completing our proof we show that on $\Omega \times [t, T] \times E$, $ds \otimes d\mathbb{P} \otimes d\lambda - \text{a.e.} \quad \forall i \in \{1, \dots, m\}$;

$$\begin{aligned} \bar{U}_s^{i;t,x}(e) &= (v^i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - v^i(s, X_{s-}^{t,x})) \\ &= (\bar{u}_i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - \bar{u}_i(s, X_{s-}^{t,x})). \end{aligned} \quad (5.68)$$

By Remark 3.4 in [3]; let us considere $(x_k)_{k \geq 1}$ a sequence of \mathbb{R}^k which converges to $x \in \mathbb{R}^k$ and the two following RBSDE with jumps (adaptation is w.r.t. \mathcal{F}^k):

$$\begin{cases} (i) \bar{Y}^{\bar{k},t,x} := (\bar{Y}^{i;k,t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad \bar{Z}^{k,t,x} := (\bar{Z}^{i;k,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ \bar{K}^{k,t,x} := (\bar{K}^{i;k,t,x})_{i=1,m} \in \mathcal{A}_c^2, \quad \bar{U}^{k,t,x} := (\bar{U}^{i;k,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ (ii) \bar{Y}_s^{i;k,t,x} = g^i(X_T^{k,t,x}) + \bar{K}_T^{i;k,t,x} - \bar{K}_s^{i;k,t,x} - \int_s^T \bar{Z}^{i;k,t,x} dB_r - \int_s^T \int_E \bar{U}_r^{i;k,t,x}(e) \tilde{\mu}(dr, de) \\ \quad + \int_s^T h^{(i)}(r, X_r^{k,t,x}, \bar{Y}_r^{i;k,t,x}, \bar{Z}_r^{i;k,t,x}, \\ \quad \quad \quad \int_E \gamma^i(t, X_r^{k,t,x_k}, e) (\bar{u}_i(s, X_{s-}^{k,t,x_k} + \beta(s, X_{s-}^{k,t,x_k}, e)) - \bar{u}_i(s, X_{s-}^{k,t,x_k})) \lambda(de) \Big) dr; \\ (iii) \bar{Y}_s^{i;k,t,x_k} \geq \ell(s, X_s^{k,t,x_k}) \text{ and } \int_0^T (\bar{Y}_s^{i;k,t,x_k} - \ell(s, X_s^{k,t,x_k})) d\bar{K}_s^{i;k,t,x_k} = 0; \end{cases} \quad (5.69)$$

and

$$\left\{ \begin{array}{l}
(i) \bar{Y}^{k,t,x_k} := (\bar{Y}^{i;k,t,x_k})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \bar{Z}^{k,t,x_k} := (\bar{Z}^{i;k,t,x_k})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\
\bar{K}^{k,t,x_k} := (\bar{K}^{i;k,t,x_k})_{i=1,m} \in \mathcal{A}_c^2, \bar{U}^{k,t,x_k} := (\bar{U}^{i;k,t,x_k})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\
(ii) \bar{Y}_s^{i;k,t,x_k} = g^i(X_T^{k,t,x_k}) + \bar{K}_T^{i;k,t,x_k} - \bar{K}_s^{i;k,t,x_k} - \int_s^T \bar{Z}^{i;k,t,x_k} dB_r - \int_s^T \int_E \bar{U}_r^{i;k,t,x_k}(e) \tilde{\mu}(dr, de) \\
\quad + \int_s^T h^{(i)}(r, X_r^{k,t,x_k}, \bar{Y}_r^{i;k,t,x_k}, \bar{Z}_r^{i;k,t,x_k}, \\
\quad \quad \quad \int_E \gamma^i(t, X_r^{k,t,x_k}, e)(\bar{u}_i(s, X_{s-}^{k,t,x_k} + \beta(s, X_{s-}^{k,t,x_k}, e)) - \bar{u}_i(s, X_{s-}^{k,t,x_k})) \lambda(de) dr; \\
(iii) \bar{Y}_s^{i;k,t,x_k} \geq \ell(s, X_s^{k,t,x_k}) \text{ and } \int_0^T (\bar{Y}_s^{i;k,t,x_k} - \ell(s, X_s^{k,t,x_k})) d\bar{K}_s^{i;k,t,x_k} = 0.
\end{array} \right. \quad (5.70)$$

By proof of step 2 of proposition 4.4, $(\bar{Y}^{i;k,t,x}, \bar{K}^{i;k,t,x}, \bar{Z}^{i;k,t,x}, \bar{U}^{i;k,t,x} 1_{\{|e| \geq \frac{1}{k}\}})_k$ converge to $(\bar{Y}^{i;t,x}, \bar{K}^{i;t,x}, \bar{Z}^{i;t,x}, \bar{U}^{i;t,x})$ in $\mathcal{S}^2(\mathbb{R}) \times \mathcal{A}_c^2 \times \mathbb{H}^2(\mathbb{R}^{\kappa \times d}) \times \mathbb{H}^2(\mathbb{L}^2(\lambda))$.

Let $((v_{i=1,m}^k))_{k \geq 1}$ be the sequence of continuous deterministic functions such that for any $t \leq T$ and $s \in [t, T]$,

$$\bar{Y}_s^{i;k,t,x} = \bar{v}_i^k(s, X_s^{k,t,x}) \text{ and } \bar{Y}_s^{i;k,t,x_k} = \bar{v}_i^k(s, X_s^{k,t,x_k}) \quad \forall i = 1, \dots, m.$$

Such that we have respectively by proof of proposition 4.4 in step 1 and step 2:

- (i) $\bar{U}_s^{i;k,t,x}(e) = (v^i(s, X_{s-}^{k,t,x} + \beta(s, X_{s-}^{k,t,x}, e)) - v^i(s, X_{s-}^{k,t,x}))$, $ds \otimes d\mathbb{P} \otimes d\lambda_k$ -a.e on $[t, T] \times \Omega \times E$;
- (ii) the sequence $((v_{i=1,m}^k))_{k \geq 1}$ converge to $v^i(t, x)$ by using (4.57).

So that $x_k \rightarrow_k x$ we take the following estimation which is obtaining by Ito's formula and by the properties of $h^{(i)}$.

$$\begin{aligned}
& \mathbb{E} \left[\left| \bar{Y}_s^{k,t,x_k} - Y_s^{k,t,x} \right|^2 + \left| K_T^{k,t,x_k} - K_T^{k,t,x} \right|^2 + \int_0^T \left\{ \left| Z_s^{k,t,x_k} - Z_s^{k,t,x} \right|^2 \right. \right. \\
& \quad \left. \left. + \int_E \left| U_s^{k,t,x_k} - U_s^{k,t,x} \right|^2 \lambda_k(de) \right\} ds \right] \\
& \leq \mathbb{E} \left[\left| g(k X_T^{t,x_k}) - g(k X_T^{t,x}) \right|^2 \right] + \mathbb{E} \left[\sup_{s \leq T} \left| \ell(k X_s^{t,x_k}) - \ell(k X_s^{t,x}) \right|^2 \right] + C \mathbb{E} \left[\int_s^T \left| \bar{Y}_r^{k,t,x_k} - \bar{Y}_r^{k,t,x} \right|^2 dr \right] \\
& + C \mathbb{E} \left[\int_0^T \left| {}^k X_r^{t,x_k} - {}^k X_r^{t,x} \right|^2 (1 + \left| {}^k X_r^{t,x_k} \right|^p + \left| {}^k X_r^{t,x} \right|^p)^2 dr \right] \\
& + C \sum_{i=1,m} \mathbb{E} \left[\int_s^T \left| B_i \bar{u}^i(r, X_r^{k,t,x_k}) - B_i \bar{u}^i(r, X_r^{k,t,x}) \right|^2 dr \right].
\end{aligned} \quad (5.71)$$

Next using (4.52) and (4.53), the continuity of the function $(t, x) \mapsto B_i \bar{u}^i(t, x)$ and the fact of it belong to Π_g and in the other hand the majoration of the fourth term of (4.56); we can use

Gronwall's lemma for $s = t \forall i = 1, \dots, m$,

$$v_i^k(t, x_k) \xrightarrow{k} v_i^k(t, x).$$

Therefore by (i)-(ii) we have, for any $i = 1, \dots, m$,

$$\overline{U}_s^{i;t,x}(e) = (v^i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - v^i(s, X_{s-}^{t,x})) \quad ds \otimes d\mathbb{P} \otimes d\lambda - \text{a.e. in } [t, T] \times \Omega \times E, \quad \forall i \in \{1, \dots, m\}. \quad (5.72)$$

By this result we can replace $(\overline{u}_i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - \overline{u}_i(s, X_{s-}^{t,x}))$ by $\overline{U}_s^{i;t,x}(e)$ in (5.68), we deduce that the quadruple $(\overline{Y}^{t,x}, \overline{K}^{t,x}, \overline{Z}^{t,x}, \overline{U}^{t,x})$ verifies: $\forall i \in \{1, \dots, m\}$

$$\left\{ \begin{array}{l} (i) \overline{Y}^{t,x} := (\overline{Y}^{i;t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad \overline{Z}^{t,x} := (\overline{Z}^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \quad \overline{K}^{t,x} := (\overline{K}^{i;t,x})_{i=1,m} \in \mathcal{A}_c^2, \\ \overline{U}^{t,x} := (\overline{U}^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ (ii) \overline{Y}_s^{i;t,x} = g^i(X_T^{t,x}) + \overline{K}_T^{i;t,x} - \overline{K}_s^{i;t,x} - \int_s^T \overline{Z}_r^{i;t,x} dB_r - \int_s^T \int_E \overline{U}_r^{i;t,x}(e) \tilde{\mu}(dr, de) \\ \quad + \int_s^T h^{(i)}(r, X_r^{t,x}, \overline{Y}_r^{i;t,x}, \overline{Z}_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) \overline{U}_r^{i;t,x} \lambda(de)) dr; \\ (iii) \overline{Y}_s^{i;t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (\overline{Y}_s^{i;t,x} - \ell(s, X_s^{t,x})) d\overline{K}_s^{i;t,x} = 0. \end{array} \right. \quad (5.73)$$

It follows that

$$\forall i \in \{1, \dots, m\}, \quad \overline{Y}^{i;t,x} = Y^{i;t,x}.$$

With the uniqueness of solution (5.66), we have $u^i = \overline{u}^i = v^i$ which means that the solution of (1.1) in the sense of Definition 5.1 is unique inside the class \mathcal{U} .

6 Extension

In this section, we will redefine the function $h^{(i)}$ as a function of $\|U^{i,t,x}\|_{\mathbb{L}^2(\lambda)} \forall i \in \{1, \dots, m\}$.

And to show that the results of the previous section remain valid.

Let us consider for any $i \in \{1, \dots, m\}$ the functions $f^{(i)}$, defined by

$$\forall (t, x, y, z, \zeta) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{m+d} \times \mathbb{L}^2(\lambda); \quad f^{(i)}(t, x, y, z, \zeta) = h^{(i)}(t, x, y, z, \|\zeta\|_{\mathbb{L}^2(\lambda)});$$

where the functions $(h^{(i)})_{i=1,m}$ are the same defined in section 2.

We recall that the result of Theorem 5.2 is obtained by having mainly $U_s^{t,x}(e) = (u^i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u^i(s, X_{s-}^{t,x}))$; this makes it possible to have the definition 4.1 by passing through a modification of the expression of $B_i u^i \forall i \in \{1, \dots, m\}$.

We show that $\|U_s^{i,t,x}(e)\|_{\mathbb{L}^2(\lambda)}^2 = \|(u^i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u^i(s, X_{s-}^{t,x}))\|_{\mathbb{L}^2(\lambda)}^2$ and that in this

case $B_i w^i$ is well $\forall i \in \{1, \dots, m\}$.

Let now $(t, x) \in [0, T] \times \mathbb{R}^d$ and let us consider the following m -dimensional RBSDE with jumps:

$$\left\{ \begin{array}{l} (i) \vec{Y}^{t,x} := (Y^{i,t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), Z^{t,x} := (Z^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ K^{t,x} := (K^{i,t,x})_{i=1,m} \in \mathcal{A}_c^2, U^{t,x} := (U^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ \forall i \in \{1, \dots, m\} Y_T^{i,t,x} = g^i(X_T^{t,x}) \text{ and;} \\ (ii) dY_s^{i,t,x} = -f^{(i)}(s, X_s^{t,x}, (Y_s^{i,t,x})_{i=1,m}, Z_s^{i,t,x}, \|U_s^{t,x}(e)\|_{\mathbb{L}^2(\lambda)}) ds - dK_s^{i,t,x} \\ \quad + Z_s^{i,t,x} dB_s + \int_{\mathbb{E}} U_s^{i,t,x}(e) \tilde{\mu}(ds, de), \quad s \leq T; \\ (iii) Y_s^{i,t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{i,t,x} - \ell(s, X_s^{t,x})) dK_s^{i,t,x} = 0. \end{array} \right. \quad (6.74)$$

By assumed that **(H1)**, **(H2)** and **(H3)** are verified and by proposition 3.3 the (3.15), there exists a family of deterministic continuous functions of polynomial growth $(w^i)_{i=1,m}$ that fact for any $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$\forall s \in [t, T], \quad Y_s^{i,t,x} = w^i(s, X_s^{t,x}).$$

Such that by the same proposition, the family $(w^i)_{i=1,m}$ is a viscosity solution of the following system:

$$\left\{ \begin{array}{l} \min \left\{ w^i(t, x) - \ell(t, x); -\partial_t w^i(t, x) - b(t, x)^\top D_x w^i(t, x) - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 w^i(t, x)) \right. \\ \quad \left. - K_i w^i(t, x) - h^{(i)}(t, x, (w^j(t, x))_{j=1,m}, (\sigma^\top D_x w^i)(t, x), B_i w^i(t, x)) \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^k; \\ w^i(T, x) = g^i(x). \end{array} \right. \quad (6.75)$$

Indeed, using Lemma 4.1 and the fact that U^{i,t,x_k} converges to $U^{i,t,x} \forall i \in \{1, \dots, m\}$ when $x_k \rightarrow_k x$, we deduce that $\|U_s^{i,t,x_k}(e)\|_{\mathbb{L}^2(\lambda)} \rightarrow_k \|U_s^{i,t,x}(e)\|_{\mathbb{L}^2(\lambda)}$.

Moreover, from property of $h^{(i)}$ and the proof of theorem 5.2 step 2 (viscosity solution uniqueness), $\|U_s^{i,t,x}(e)\|_{\mathbb{L}^2(\lambda)}^2 = \|(w^i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - w^i(s, X_{s-}^{t,x}))\|_{\mathbb{L}^2(\lambda)}^2$; from where

$$B_i w^i = \left\{ \int_{\mathbb{E}} |(w^i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - w^i(s, X_{s-}^{t,x}))|^2 \lambda(de) \right\}^{\frac{1}{2}}.$$

Thanks to corollary 4.3, we deduce that $B_i w^i$ is well defined $\forall i \in \{1, \dots, m\}$.

Appendix. Barles et al.'s definition for viscosity solution of IPDE

(1.1)

In the paper by Barles et al. [1], the definition of the viscosity solution of the system (1.1) is given as follows.

Definition 6.1 *We say that a family of deterministic functions $u = (u^i)_{i=1,m}$ which is continuous $\forall i \in \{1, \dots, m\}$, is a viscosity sub-solution (resp. super-solution) of the IPDE (1.1) if:*

- (i) $\forall x \in \mathbb{R}^k$, $u^i(x, T) \leq g^i(x)$ (resp. $u^i(x, T) \geq g^i(x)$);
- (ii) *For any $(t, x) \in [0, T] \times \mathbb{R}^k$ and any function ϕ of class $C^{1,2}([0, T] \times \mathbb{R}^k)$ such that (t, x) is a global maximum point of $u^i - \phi$ (resp. global minimum point of $u^i - \phi$) and $(u^i - \phi)(t, x) = 0$, one has*

$$\min \left\{ u^i(t, x) - \ell(t, x); -\partial_t \phi(t, x) - \mathcal{L}^X \phi(t, x) - h^i(t, x, (u^j(t, x))_{j=1,m}, \sigma^\top(t, x)) D_x \phi(t, x), B_i \phi(t, x) \right\} \leq 0$$

(resp.

$$\min \left\{ u^i(t, x) - \ell(t, x); -\partial_t \phi(t, x) - \mathcal{L}^X \phi(t, x) - h^i(t, x, (u^j(t, x))_{j=1,m}, \sigma^\top(t, x)) D_x \phi(t, x), B_i \phi(t, x)(t, x) \right\} \geq 0).$$

The family $u = (u^i)_{i=1,m}$ is a viscosity solution of (1.1) if it is both a viscosity sub-solution and viscosity super-solution.

Note that $\mathcal{L}^X \phi(t, x) = b(t, x)^\top D_x \phi(t, x) + \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 \phi(t, x)) + K \phi(t, x)$;

where $K \phi(t, x) = \int_{\mathbf{E}} (\phi(t, x + \beta(t, x, e)) - \phi(t, x) - \beta(t, x, e)^\top D_x \phi(t, x)) \lambda(de)$.

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