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► **To cite this version:**

Claude Jauffret, Annie-Claude Perez. Efficient Pseudolinear Estimation for Bearings and Frequencies Target Motion Analysis. International Conference on Information Fusion, Jul 2018, cambridge, United Kingdom. hal-01863573

**HAL Id: hal-01863573**

**<https://hal.science/hal-01863573v1>**

Submitted on 22 Sep 2021

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# Efficient Pseudolinear Estimation for Bearings and Frequencies Target Motion Analysis

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**Abstract**—In this paper, we present a pseudolinear estimate for bearings and frequencies target motion analysis. We propose two ways to reduce its inherent bias and evaluate its performance with Monte Carlo simulations, with reference to the Cramér-Rao lower bound.

**Keywords**—bearings and frequencies target motion analysis, observability, maximum likelihood estimate, pseudolinear estimate, Cramér-Rao lower bound, bias reduction.

## I. INTRODUCTION

Target motion analysis (TMA) is now an old problem well-known by our community. The most common variants are probably bearings-only TMA (BOTMA) [1], and bearings and frequency (or Doppler) TMA (BFTMA) [2]. The first needs the platform to maneuver properly, whereas the second has no specific requirement to make the task feasible (observability condition). The solutions proposed in the literature can be classified into two main groups: the batch methods devoted to the scenarios during which the source does not maneuver, and those devoted to possibly maneuvering sources. For the first situation, the maximum likelihood estimate is most often proposed, computed via a numerical routine [1]. For the second situation, a recursive (compatible with real-time) filter is most appropriate: extended and unscented Kalman filters (with convenient coordinate systems) can be run, but with a risk of divergence of the filter [3] [4], particle filters with low risk of divergence but demanding a strong computational resource [5]. All these methods are iterative (or time-recursive, which is a kind of iteration). In this paper, we give the proof of observability in BFTMA in discrete time, the analysis of observability having been made in continuous time in the past [6]. Then, we propose two ways to compute directly two almost unbiased estimates in BFTMA by a straightforward computation. This is based upon the so-called modified instrumental method, initially introduced in [7].

The paper is organized as follows:

In section II, the problem formulation and our notations are presented. Section III is devoted to observability analysis in discrete time. We present in section IV a pseudolinear estimate, which is biased. Section V offers two ways to reduce the bias, and presents some simulations, illustrated with typical scenarios. A conclusion and an appendix, containing some computational details, follow.

## II. PROBLEM FORMULATION AND NOTATIONS

We consider the common situation where a **motionless**

observer acquires sequentially a couple of measurements composed of an azimuth (or bearing) and a frequency of a single tone emitted by a source moving at constant velocity. Due to the source's motion, the detected frequency is Doppler-shifted. The plane in which the two actors play is given a Cartesian coordinate system. In this system, the position vector at time  $t$  and the velocity vector of the source are  $P_s(t) = [x_s(t) \ y_s(t)]^T$  and  $V_s = [\dot{x}_s \ \dot{y}_s]^T$ , respectively. The vectors  $P_s(t)$  and  $V_s$  are also described by their polar coordinates:  $P_s(t) = r(t)[\sin \theta(t) \ \cos \theta(t)]^T$  and  $V_s = v[\sin h \ \cos h]^T$ . The source is assumed to be emitting a pure single tone at the unknown frequency  $\gamma_0$ . The observer is located at  $P_o = [0 \ 0]^T$ . At time  $t_k$ , the noiseless azimuth and frequency are respectively  $\theta_k = \theta(t_k) = \text{Atan}[x_s(t_k), y_s(t_k)]$  and  $f_k = \gamma_0 \left[ 1 - \frac{\dot{r}_k}{c} \right] = \gamma_0 \left[ 1 - \frac{\dot{x}_s \sin \theta_k + \dot{y}_s \cos \theta_k}{c} \right]$ , where  $\dot{r}_k$  stands for  $\frac{dr}{dt}(t_k)$ . Note that in these two expressions the propagation

delay is neglected. The letter  $c$  denotes the speed of sound in the medium ( $c=1500$  m/s in water). At this time, the observer acquires the pair of measurements  $(\theta_{m,k}, f_{m,k})$  defined by  $\theta_{m,k} = \theta_k + \varepsilon_k$  and  $f_{m,k} = f_k + \eta_k$ . The terms  $\varepsilon_k$  and  $\eta_k$  are respectively the measurement noises corrupting the bearing and the frequency acquired at time  $t_k$ . The noise vector  $\left[ [\varepsilon_1 \ \eta_1]^T \ \dots \ [\varepsilon_k \ \eta_k]^T \ \dots \ [\varepsilon_K \ \eta_K]^T \right]^T$  is assumed to be a zero-mean Gaussian vector whose covariance matrix is diagonal. With no loss of generality, we will assume that  $\text{Var}(\varepsilon_k)$  and  $\text{Var}(\eta_k)$  are constant, *i.e.*  $\text{Var}(\varepsilon_k) = \sigma_\theta^2$  and  $\text{Var}(\eta_k) = \sigma_f^2$ .

The purpose of BFTMA is to estimate the trajectory of the source from a set of collected measurements  $\{(\theta_{m,k}, f_{m,k}), k=1, \dots, K\}$ . Because  $\gamma_0$  is unknown, it must be estimated, as well.

The “natural” state vector that characterizes the source (*i.e.* its motion and its “music”, that is  $\gamma_0$ ) should be  $[x_s(t_1) \ y_s(t_1) \ \dot{x}_s \ \dot{y}_s \ \gamma_0]^T$ . However, we make another choice: the state vector defined as  $X = [\gamma_0 x_s(t_1) \ \gamma_0 y_s(t_1) \ \gamma_0 \dot{x}_s \ \gamma_0 \dot{y}_s \ \gamma_0]^T$  is more convenient to

establish the observability of the trajectory from the set of noiseless measurements  $\{(\theta_k, f_k), k=1, \dots, K\}$  whose proof is given in the next section, and to construct a pseudolinear estimate. This state vector was proposed first in [1], then later used in [2].

### III. OBSERVABILITY ANALYSIS IN DISCRETE TIME

The analysis of the observability is conducted in discrete time. This is thus the counterpart of the proof in continuous time given in [6]. Before giving the main result, we recall the following property (the proof is omitted):

#### Lemma

The three statements are equivalent:

- (i) The source's trajectory is aligned with the observer's position,
- (ii) Two bearings are equal,
- (iii) All the bearings are equal.

The conclusion about observability is summarized by the:

#### Proposition

If the source's trajectory is not aligned with the observer's position, then its trajectory is observable from three pairs of measurements  $\{(\theta_k, f_k), k=1, 2, 3\}$ .

Proof:

At time  $t_k$ , the noiseless measurements  $(\theta_k, f_k)$  and the state vector satisfy the following linear system:

$$\begin{bmatrix} \cos \theta_k & -\sin \theta_k & t_k \cos \theta_k & -t_k \sin \theta_k & 0 \\ 0 & 0 & -\frac{\sin \theta_k}{c} & -\frac{\cos \theta_k}{c} & 1 \end{bmatrix} X = \begin{bmatrix} 0 \\ f_k \end{bmatrix}. \quad (1)$$

With three pairs of measurements  $\{(\theta_k, f_k), k=1, 2, 3\}$ , the state vector  $X$  must satisfy the linear system (we implicitly assume that  $t_1 = 0$ )

$$\underbrace{\begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sin \theta_1}{c} & -\frac{\cos \theta_1}{c} & 1 \\ \cos \theta_2 & -\sin \theta_2 & t_2 \cos \theta_2 & -t_2 \sin \theta_2 & 0 \\ 0 & 0 & -\frac{\sin \theta_2}{c} & -\frac{\cos \theta_2}{c} & 1 \\ \cos \theta_3 & -\sin \theta_3 & t_3 \cos \theta_3 & -t_3 \sin \theta_3 & 0 \\ 0 & 0 & -\frac{\sin \theta_3}{c} & -\frac{\cos \theta_3}{c} & 1 \end{bmatrix}}_{=A} X = \underbrace{\begin{bmatrix} 0 \\ f_1 \\ 0 \\ f_2 \\ 0 \\ f_3 \end{bmatrix}}_{y_3}$$

which is equivalent to

$$\underbrace{\begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & 0 & 0 \\ \cos \theta_2 & -\sin \theta_2 & t_2 \cos \theta_2 & -t_2 \sin \theta_2 & 0 \\ \cos \theta_3 & -\sin \theta_3 & t_3 \cos \theta_3 & -t_3 \sin \theta_3 & 0 \\ 0 & 0 & -\frac{\sin \theta_1}{c} & -\frac{\cos \theta_1}{c} & 1 \\ 0 & 0 & -\frac{\sin \theta_2}{c} & -\frac{\cos \theta_2}{c} & 1 \\ 0 & 0 & -\frac{\sin \theta_3}{c} & -\frac{\cos \theta_3}{c} & 1 \end{bmatrix}}_{=B} X = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}}_Z$$

We are going to prove that  $X$  is unique, i.e. if another

vector  $X'$  satisfies the equation  $\mathbf{B}X' = Z$  then  $X' = X$  or equivalently, the kernel of  $\mathbf{B}$  is the null vector of  $R^5$ .

#### First step:

$$\text{Let us consider the submatrix } \begin{bmatrix} -\frac{\sin \theta_1}{c} & -\frac{\cos \theta_1}{c} & 1 \\ -\frac{\sin \theta_2}{c} & -\frac{\cos \theta_2}{c} & 1 \\ -\frac{\sin \theta_3}{c} & -\frac{\cos \theta_3}{c} & 1 \end{bmatrix}.$$

Obviously, its determinant is proportional to the determinant

$$\text{of } \mathbf{C} = \begin{bmatrix} -\sin \theta_1 & -\cos \theta_1 & 1 \\ -\sin \theta_2 & -\cos \theta_2 & 1 \\ -\sin \theta_3 & -\cos \theta_3 & 1 \end{bmatrix} \text{ which is equal to}$$

$$\det(\mathbf{C}) = \sin(\theta_1 - \theta_2) + \sin(\theta_2 - \theta_3) + \sin(\theta_3 - \theta_1).$$

We thus have

$$\det(\mathbf{C}) = 0 \Leftrightarrow \sin(\theta_1 - \theta_2) + \sin(\theta_2 - \theta_3) = \sin(\theta_1 - \theta_3).$$

$$\Leftrightarrow 2 \sin\left(\frac{\theta_1 - \theta_3}{2}\right) \cos\left(\frac{\theta_1 + \theta_3}{2} - \theta_2\right) = 2 \sin\left(\frac{\theta_1 - \theta_3}{2}\right) \cos\left(\frac{\theta_1 - \theta_3}{2}\right)$$

$$\Leftrightarrow \sin\left(\frac{\theta_1 - \theta_3}{2}\right) \left[ \cos\left(\frac{\theta_1 - \theta_3}{2}\right) - \cos\left(\frac{\theta_1 + \theta_3}{2} - \theta_2\right) \right] = 0$$

$$\Leftrightarrow \sin\left(\frac{\theta_1 - \theta_3}{2}\right) \sin\left(\frac{\theta_1 - \theta_2}{2}\right) \sin\left(\frac{\theta_2 - \theta_3}{2}\right) = 0$$

So  $\det(\mathbf{C}) = 0 \Leftrightarrow$  one of the last three conditions is verified:

$$\theta_1 = \theta_3 \pmod{2\pi}$$

$$\theta_1 = \theta_2 \pmod{2\pi}$$

$$\theta_2 = \theta_3 \pmod{2\pi}$$

The conclusion of this first step is that the matrix  $\mathbf{C}$  is nonsingular since all the bearings are different; that is, the source's trajectory is not aligned with the observer's location.

#### Second step:

Let us identify the kernel of  $\mathbf{B}$ . We have to solve the equation  $\mathbf{B}X = [0 \ 0 \ 0 \ 0 \ 0]^T$ .

The  $i^{\text{th}}$  component of  $X$  being denoted  $x_i$ , the last three rows of the above equation can be written as

$\mathbf{C}[x_3 \ x_4 \ x_5]^T = [0 \ 0 \ 0]^T$ . Since  $\mathbf{C}$  is nonsingular, this yields  $[x_3 \ x_4 \ x_5]^T = [0 \ 0 \ 0]^T$ .

Now reporting this result in the first two rows, we get the system

$$\begin{bmatrix} \cos \theta_1 & -\sin \theta_1 \\ \cos \theta_2 & -\sin \theta_2 \\ \cos \theta_3 & -\sin \theta_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ whose solution is } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The conclusion of step 2 is that the kernel of  $\mathbf{B}$  is the null vector of  $R^5$ . As a consequence, the equation  $\mathbf{B}X' = Z$  has a unique solution if and only if the source's trajectory is not aligned with the observer's position.

QED.

In the sequel, we will assume that  $t_k = (k-1)\Delta t$  and  $\Delta t = \frac{T}{K-1}$ ,  $T$  being the duration of the scenario.

### IV. THE PSEUDOLINEAR ESTIMATE

The basic idea of the so-called "pseudolinear" estimate comes from the possibility of transforming a nonlinear measurement

equation into a linear form, the price being that the noise is no longer additive. From our knowledge, the first paper invoking this transformation is [7], for BOTMA. Since then, this technique has been widely re-used for BFTMA as well [2, 9, 10].

#### A. Construction of the estimate

Suppose now that  $K$  pairs of measurements are available. Let us apply the linear transformation that allowed us to get (1) to the data:

#### Proposition

$$\begin{bmatrix} 0 \\ f_{m,k} \end{bmatrix} = \mathbf{M}_k X + \xi_k$$

with

$$\xi_k = -\gamma_0 \left[ \frac{v}{c} [\cos(\theta_k - h)(1 - \cos \varepsilon_k) + \sin(\theta_k - h)\sin \varepsilon_k] \right] + \begin{bmatrix} 0 \\ \eta_k \end{bmatrix}$$

$$\text{and } \mathbf{M}_k = \begin{bmatrix} \cos \theta_{m,k} & -\sin \theta_{m,k} & t_k \cos \theta_{m,k} & -t_k \sin \theta_{m,k} & 0 \\ 0 & 0 & -\frac{\sin \theta_{m,k}}{c} & -\frac{\cos \theta_{m,k}}{c} & 1 \end{bmatrix}$$

where  $r_k$  stands for  $r(t_k)$ .

Proof:

- The bearings:

We compute the expression  $x_k \cos \theta_{m,k} - y_k \sin \theta_{m,k}$  (similar to  $x_k \cos \theta_k - y_k \sin \theta_k = 0$ ):

$$\begin{aligned} x_k \cos \theta_{m,k} - y_k \sin \theta_{m,k} &= x_k \cos(\theta_k + \varepsilon_k) - y_k \sin(\theta_k + \varepsilon_k) \\ &= r_k \sin \theta_k \cos(\theta_k + \varepsilon_k) - r_k \cos \theta_k \sin(\theta_k + \varepsilon_k) \\ &= r_k \sin(\theta_k - \theta_k - \varepsilon_k) \\ &= -r_k \sin \varepsilon_k. \end{aligned}$$

That is,

$$x_k \cos \theta_{m,k} - y_k \sin \theta_{m,k} = -r_k \sin \varepsilon_k. \quad (2)$$

By using the state vector  $X$ , Eq. (2) becomes:

$$\begin{bmatrix} \cos \theta_{m,k} & -\sin \theta_{m,k} & t_k \cos \theta_{m,k} & -t_k \sin \theta_{m,k} & 0 \end{bmatrix} X = -\gamma_0 r_k \sin \varepsilon_k.$$

- The frequency

Now, we compute  $\dot{x} \sin \theta_{m,k} + \dot{y} \cos \theta_{m,k}$ :

$$\begin{aligned} \dot{x} \sin \theta_{m,k} + \dot{y} \cos \theta_{m,k} &= v \sin h \sin \theta_{m,k} + v \cos h \cos \theta_{m,k} \\ &= v \cos(\theta_{m,k} - h) \\ &= v \cos(\theta_k - h) \cos \varepsilon_k - v \sin(\theta_k - h) \sin \varepsilon_k \end{aligned}$$

As a consequence,

$$\begin{aligned} &\gamma_0 \left( 1 - \frac{\dot{x}}{c} \sin \theta_{m,k} - \frac{\dot{y}}{c} \cos \theta_{m,k} \right) = \\ &\gamma_0 \left[ 1 - \frac{v}{c} \cos(\theta_k - h) \cos \varepsilon_k + \frac{v}{c} \sin(\theta_k - h) \sin \varepsilon_k \right] = \\ &f_k + \gamma_0 \frac{v}{c} [\cos(\theta_k - h)(1 - \cos \varepsilon_k) + \sin(\theta_k - h)\sin \varepsilon_k] = \\ &f_{k,m} + \gamma_0 \frac{v}{c} [\cos(\theta_k - h)(1 - \cos \varepsilon_k) + \sin(\theta_k - h)\sin \varepsilon_k] - \eta_k \end{aligned}$$

We end up with the following pseudolinear measurement equation:

$$\underbrace{\begin{bmatrix} 0 \\ f_{m,k} \end{bmatrix}}_{=y_k} = \mathbf{M}_k X - \gamma_0 \underbrace{\left[ \frac{v}{c} [\cos(\theta_k - h)(1 - \cos \varepsilon_k) + \sin(\theta_k - h)\sin \varepsilon_k] \right]}_{=\xi_k} + \begin{bmatrix} 0 \\ \eta_k \end{bmatrix}$$

for  $k=1, \dots, K$ .

QED.

Now, we construct the following pseudolinear estimate

$$\hat{X} = \left( \sum_{k=1}^K \mathbf{M}_k^T \mathbf{M}_k \right)^{-1} \left( \sum_{k=1}^K \mathbf{M}_k^T y_k \right), \text{ designated by PSLE.}$$

Due to the poor conditioning of  $\sum_{k=1}^K \mathbf{M}_k^T \mathbf{M}_k$ , it is convenient<sup>1</sup>

to weight the pseudo measurements vector  $y_k$  with the inverse of the covariance matrix  $\mathbf{R}_k$  of the pseudo noise  $\xi_k$ :

$$\begin{aligned} \hat{X} &= \left( \sum_{k=1}^K \mathbf{M}_k^T \mathbf{R}_k^{-1} \mathbf{M}_k \right)^{-1} \left( \sum_{k=1}^K \mathbf{M}_k^T \mathbf{R}_k^{-1} y_k \right) \\ &= X + \left( \sum_{k=1}^K \mathbf{M}_k^T \mathbf{R}_k^{-1} \mathbf{M}_k \right)^{-1} \left( \sum_{k=1}^K \mathbf{M}_k^T \mathbf{R}_k^{-1} \xi_k \right) \end{aligned}$$

We deduce that

$$\mathbf{E}(\hat{X}) = X + \mathbf{E} \left\{ \left( \sum_{k=1}^K \mathbf{M}_k^T \mathbf{R}_k^{-1} \mathbf{M}_k \right)^{-1} \left( \sum_{k=1}^K \mathbf{M}_k^T \mathbf{R}_k^{-1} \xi_k \right) \right\}.$$

Because of the presence of the noise in  $\mathbf{M}_k$ ,

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<sup>1</sup>  $\sum_{k=1}^K \mathbf{M}_k^T \mathbf{M}_k$  is nonsingular, but a standard numerical procedure of inversion can fail.

$\mathbf{E}\left\{\left(\sum_{k=1}^K \mathbf{M}_k^T \mathbf{R}_k^{-1} \mathbf{M}_k\right)^{-1} \left(\sum_{k=1}^K \mathbf{M}_k^T \mathbf{R}_k^{-1} \xi_k\right)\right\}$  is non zero. This is the

bias of  $\hat{X}$ , denoted  $\mathbf{b}$ . We propose to approximate  $\mathbf{R}_k$  as follows:

When  $\sigma_\theta \ll 1$ , then  $\xi_k \cong \gamma_0 \sin \varepsilon_k \begin{bmatrix} -r_k \\ \frac{v}{c} \sin(\theta_k - h) \end{bmatrix} + \begin{bmatrix} 0 \\ -\eta_k \end{bmatrix}$  and

its covariance matrix can be readily computed:

$$\mathbf{R}_k \cong \begin{bmatrix} \gamma_0^2 r_k^2 \sigma_\theta^2 & -\gamma_0^2 r_k \frac{v}{c} \sin(\theta_k - h) \sigma_\theta^2 \\ -\gamma_0^2 r_k \frac{v}{c} \sin(\theta_k - h) \sigma_\theta^2 & \gamma_0^2 \frac{v^2}{c^2} \sin^2(\theta_k - h) \sigma_\theta^2 + \sigma_f^2 \end{bmatrix}$$

$$\Rightarrow \mathbf{R}_k^{-1} \cong \frac{1}{\sigma_f^2} \begin{bmatrix} \frac{v^2}{c^2 r_k^2} \sin^2(\theta_k - h) + \frac{1}{\gamma_0^2 r_k^2} \frac{\sigma_f^2}{\sigma_\theta^2} & \frac{v}{c r_k} \sin(\theta_k - h) \\ \frac{v}{c r_k} \sin(\theta_k - h) & 1 \end{bmatrix}.$$

Note that the multiplicative term  $\frac{1}{\sigma_f^2}$  does not intervene in the weighting. Three elements of  $\mathbf{R}_k^{-1}$  are unknown. Fortunately, their orders of magnitude can be easily given: for a vessel or a submarine, the speed is about 10 m/s, and  $\frac{\sigma_f}{\sigma_\theta} \approx 10^{-3}$ ; the range is between 5 and 20 km and  $\sigma_\theta$  is close to  $1^\circ$  ( $\frac{\pi}{180}$  rad). These values allow us to approximate (even very roughly)  $\mathbf{R}_k^{-1}$  by  $\begin{bmatrix} 10^{-11} & 10^{-6} \\ 10^{-6} & 1 \end{bmatrix} = \Omega$ . The elements of  $\Omega$  are denoted  $\omega_{ij}$ .

### B. The problem of the bias

Computing the bias analytically is hopeless: indeed, if such a computation was tractable, we would end up with an unbiased estimator; based on a fundamental theorem of mathematical statistics, this estimator would be the MLE and the model would be linear, which is not the case. The only achievable aim is to approximate or to estimate the bias  $\mathbf{b}$ .

In the open literature, two major ways to reduce this bias are proposed:

-The so-called modified instrumental variable estimate which consists in iterating the formula up to convergence

$$\hat{X}_p = \left( \sum_{k=1}^K \hat{\mathbf{M}}_{k,p-1}^T \mathbf{R}_k^{-1} \mathbf{M}_k \right)^{-1} \left( \sum_{k=1}^K \hat{\mathbf{M}}_{k,p-1}^T \mathbf{R}_k^{-1} \mathbf{y}_k \right),$$

where  $\hat{\mathbf{M}}_{k,p-1}$  is the matrix  $\mathbf{M}_k$  constructed with the estimated bearings computed from  $\hat{X}_{p-1}$ . The reduction of the bias is justified by invoking Slutski's theorem [11].

- The estimation of the bias: once  $\hat{\mathbf{b}}$  is obtained, we compute  $\hat{X} - \hat{\mathbf{b}}$ . This is our choice and we present two estimates in the next section.

## V. BIAS REDUCTION

### A. Bias estimation: first solution

To estimate the bias, we will employ the following trick: we artificially introduce the factor  $K$  up and down in the expression of  $\mathbf{b}$ :

$$\mathbf{b} = \mathbf{E}\left\{\left(\frac{1}{K} \sum_{k=1}^K \mathbf{M}_k^T \mathbf{R}_k^{-1} \mathbf{M}_k\right)^{-1} \left(\frac{1}{K} \sum_{k=1}^K \mathbf{M}_k^T \mathbf{R}_k^{-1} \xi_k\right)\right\}.$$

So, the two terms appear as an average of two quantities. As a consequence,  $\mathbf{b}$  is approximated by

$$\mathbf{E}\left\{\left[\frac{1}{K} \sum_{k=1}^K \mathbf{M}_k^T \Omega \mathbf{M}_k\right]^{-1}\right\} \mathbf{E}\left[\frac{1}{K} \sum_{k=1}^K \mathbf{M}_k^T \Omega \xi_k\right]$$

provided  $K$  is large enough. The justification of this technique, pioneered by Dogancay [8], requires that the random matrix

$\frac{1}{K} \sum_{k=1}^K \mathbf{M}_k^T \Omega \mathbf{M}_k$  converges in distribution to a deterministic

nonsingular matrix and that the random vector

$\frac{1}{K} \sum_{k=1}^K \mathbf{M}_k^T \Omega \xi_k$  converges in distribution to a deterministic

vector.

This needs us to compute analytically  $\mathbf{E}\left[\sum_{k=1}^K \mathbf{M}_k^T \Omega \xi_k\right]$ .

A simple calculation ends up with

$$\sum_{k=1}^K \mathbf{M}_k^T \Omega \xi_k = - \sum_{k=1}^K \left\{ \gamma_0 \sin \varepsilon_k \begin{bmatrix} a_k \cos \theta_{m,k} \\ -a_k \sin \theta_{m,k} \\ t_k a_k \cos \theta_{m,k} - \frac{1}{c} b_k \sin \theta_{m,k} \\ -t_k a_k \sin \theta_{m,k} - \frac{1}{c} b_k \cos \theta_{m,k} \\ b_k \end{bmatrix} \right.$$

$$\left. + \eta_k \begin{bmatrix} -\omega_{12} \cos \theta_{m,k} \\ \omega_{12} \sin \theta_{m,k} \\ -\omega_{12} t_k \cos \theta_{m,k} + \frac{\omega_{22}}{c} \sin \theta_{m,k} \\ \omega_{12} t_k \sin \theta_{m,k} + \frac{\omega_{22}}{c} \cos \theta_{m,k} \\ -\omega_{22} \end{bmatrix} \right\}$$

where  $a_k = -\omega_{11} r_k + \omega_{12} \frac{v}{c} \sin(\theta_k - h)$  and

$b_k = -\omega_{12} r_k + \omega_{22} \frac{v}{c} \sin(\theta_k - h)$ . Its mathematical expectation is

$$\mathbf{E} \left( \sum_{k=1}^K \mathbf{M}_k^T \boldsymbol{\Omega} \boldsymbol{\xi}_k \right) \equiv \sigma_\theta^2 \gamma_0 \begin{bmatrix} \sum_{k=1}^K a_k \sin \theta_k \\ \sum_{k=1}^K a_k \cos \theta_k \\ \sum_{k=1}^K t_k a_k \sin \theta_k + \frac{1}{c} b_k \cos \theta_k \\ \sum_{k=1}^K t_k a_k \cos \theta_k - \frac{1}{c} b_k \sin \theta_k \\ 0 \end{bmatrix} \quad \text{since}$$

$\varepsilon_k$  and  $\eta_k$  are independent (see the lemma given in the Appendix).

In practice, its value is estimated according to the method of moments [12] by

$$\sigma_\theta^2 \hat{\gamma}_0 \sum_{k=1}^K \begin{bmatrix} \hat{a}_k \sin \theta_{m,k} \\ \hat{a}_k \cos \theta_{m,k} \\ t_k \hat{a}_k \sin \theta_{m,k} + \frac{1}{c} \hat{b}_k \cos \theta_{m,k} \\ t_k \hat{a}_k \cos \theta_{m,k} - \frac{1}{c} \hat{b}_k \sin \theta_{m,k} \\ 0 \end{bmatrix}, \quad \text{with}$$

$$\hat{a}_k = -\omega_{11} \hat{r}_k + \omega_{12} \frac{\hat{v}}{c} \sin(\theta_{m,k} - \hat{h}) \quad \text{and} \quad \hat{b}_k = -\omega_{12} \hat{r}_k + \omega_{22} \frac{\hat{v}}{c} \sin(\theta_{m,k} - \hat{h}).$$

The computation of  $\mathbf{E} \left[ \frac{1}{K} \sum_{k=1}^K \mathbf{M}_k^T \boldsymbol{\Omega} \mathbf{M}_k \right]^{-1}$  being

impossible, it is estimated by  $\left[ \frac{1}{K} \sum_{k=1}^K \mathbf{M}_k^T \boldsymbol{\Omega} \mathbf{M}_k \right]^{-1}$ . Finally we

get an estimate  $\hat{\mathbf{b}}$  of  $\mathbf{b}$  and we replace  $\hat{X}$  by  $\hat{X} - \hat{\mathbf{b}}$ . We call this an *unbiased pseudolinear estimate* (UPSLE).

### B. Bias estimation: second solution

Another way to estimate the bias is to compute the value  $\mathbf{b}$  for which the least squares criterion is the least:

$$C(\mathbf{b}) = \sum_{k=1}^K \frac{1}{\sigma_\theta^2} [\theta_{m,k} - \theta_k(\hat{X} - \mathbf{b})]^2 + \frac{1}{\sigma_f^2} [f_{m,k} - f_k(\hat{X} - \mathbf{b})]^2,$$

approximated by

$$\begin{aligned} & \sum_{k=1}^K \frac{1}{\sigma_\theta^2} [\theta_{m,k} - \theta_k(\hat{X}) + \nabla_X^T \theta_k(\hat{X}) \mathbf{b}]^2 \\ & + \frac{1}{\sigma_f^2} [f_{m,k} - f_k(\hat{X}) + \nabla_X^T f_k(\hat{X}) \mathbf{b}]^2 \end{aligned}$$

We end up with

$$\hat{\mathbf{b}} = - \left[ \sum_{k=1}^K \frac{1}{\sigma_\theta^2} \nabla_X \theta_k(\hat{X}) \nabla_X^T \theta_k(\hat{X}) + \frac{1}{\sigma_f^2} \nabla_X f_k(\hat{X}) \nabla_X^T f_k(\hat{X}) \right]^{-1} \sum_{k=1}^K \frac{1}{\sigma_\theta^2} \nabla_X \theta_k(\hat{X}) [\theta_{m,k} - \theta_k(\hat{X})] + \frac{1}{\sigma_f^2} \nabla_X f_k(\hat{X}) [f_{m,k} - f_k(\hat{X})].$$

In this expression,  $\theta_k(X)$  and  $f_k(X)$  are the mathematical

bearing and frequency corresponding to a trajectory defined by  $X$ . We recognize the inverse of the Fisher information matrix (*i.e.* the CRLB) evaluated at  $\hat{X}$ . Again, we replace  $\hat{X}$  by  $\hat{X} - \hat{\mathbf{b}}$ . Note that  $\hat{X} - \hat{\mathbf{b}}$  is the approximate maximum likelihood estimate (AMLE).

We give in the next subsection some examples to evaluate the respective performance of each method for some scenarios.

### C. Examples

In the two following examples, the observer is located at  $[0 \ 0]^T$ , the source is moving with a constant speed of 6 m/s and a heading equal to  $90^\circ$ . Its initial position is  $[0 \ 9000]^T$  (m) in the first example and  $[-3576 \ 9000]^T$  (m) in the second one. The number of measurements is 150, with a sampling period  $\Delta t$  equal to 4 s; the standard deviations are  $\sigma_\theta = 0.5^\circ$  and  $\sigma_f = 0.5$  Hz, while  $\gamma_0 = 500$  Hz.

The performance of the three estimates is evaluated by their respective bias and their respective standard deviation of each component. The minimum reachable standard deviation given by the CRLB is also computed. All these quantities are presented in Table I for scenario 1 and in Table II for scenario 2.

For each case, three figures have been plotted: the first one depicts the set of PSLE; in the second figure, the set of the UPSLE is plotted; the third one shows the AMLE.

Scenario 1

TABLE I

$X$	Bias			Standard deviation			
	PSLE	UPSLE	AMLE	PSLE	UPSLE	AMLE	CRLB
0 (m)	14.8	4.1	6.4	20.6	23.9	20.6	19.2
9000 (m)	-2396.1	-471.4	177.4	1265.1	1616.9	1689.1	1707.7
6 (m/s)	-1.65	-0.33	0.09	0.85	1.09	1.12	1.14
0 (m/s)	-0.02	-0.01	-0.01	0.47	0.54	0.42	0.38
500 (Hz)	-0.111	-0.023	0.006	0.176	0.201	0.166	0.153

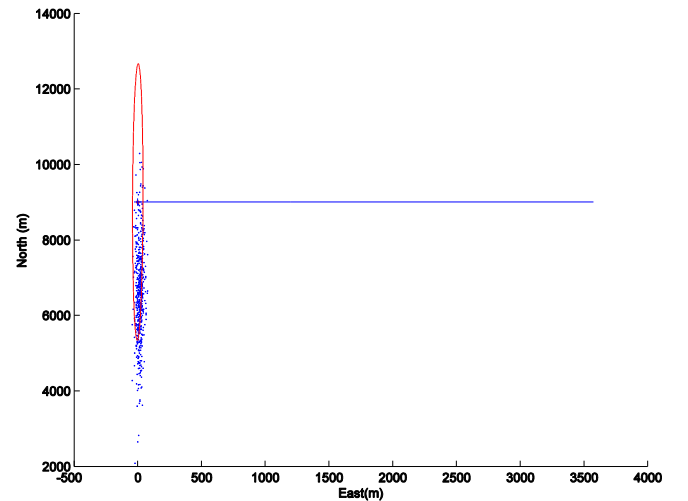


Fig. 1. Pseudolinear estimate in Scenario 1.

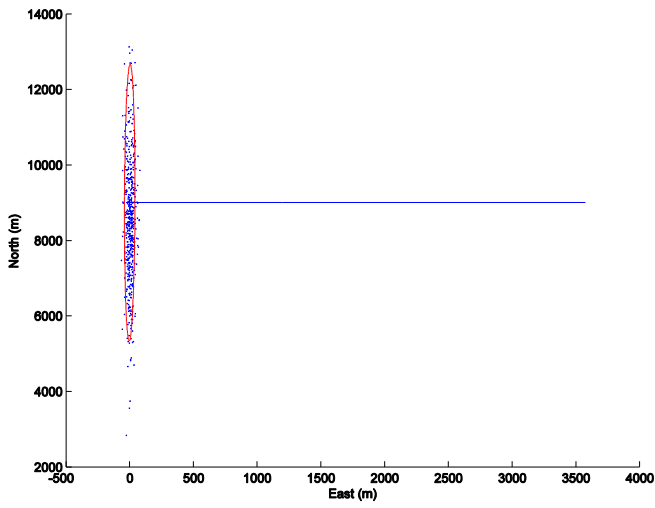


Fig. 2. Unbiased pseudolinear estimate for Scenario 1.

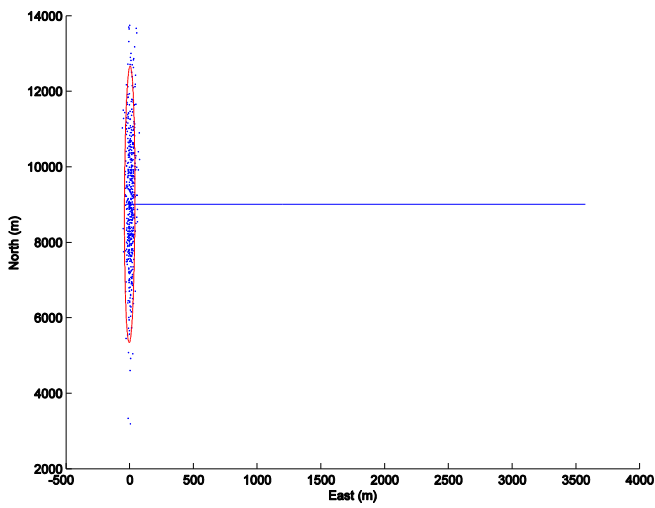


Fig. 3. Approximate maximum likelihood estimate for Scenario 1.

### Scenario 2

TABLE II

$X$	Bias			Standard deviation			
	<i>PSLE</i>	<i>UPSLE</i>	<i>AMLE</i>	<i>PSLE</i>	<i>UPSLE</i>	<i>AMLE</i>	<i>CRLB</i>
-3576 (m)	1065.5	311.9	52.0	537.4	691.6	722.8	678.0
9000 (m)	-2646.2	-774.3	-115.2	1357.7	1741.3	1825.8	1710.4
6 (m/s)	-1.81	-0.53	-0.10	0.90	1.16	1.21	1.14
0 (m/s)	0.01	-0.00	0.01	0.47	0.53	0.42	0.38
500 (Hz)	0.119	0.034	0.009	0.166	0.191	0.163	0.153

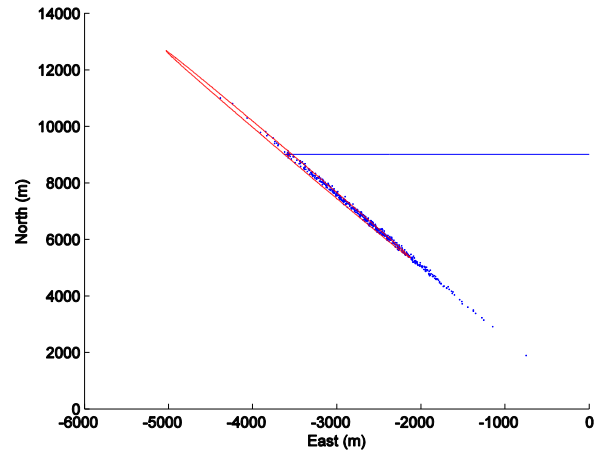


Fig. 4. Pseudolinear estimate in Scenario 2.

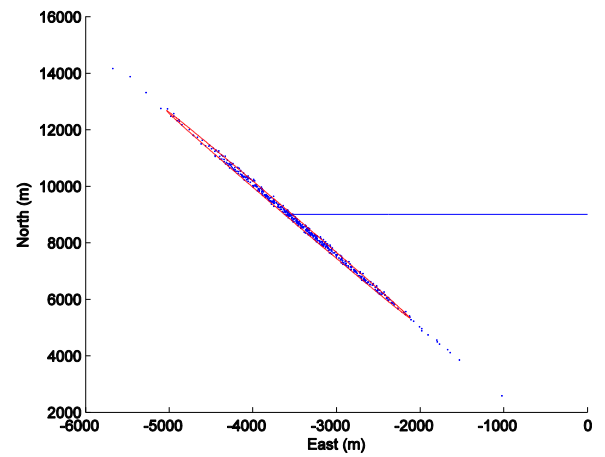


Fig. 5. Unbiased pseudolinear estimate for Scenario 2.

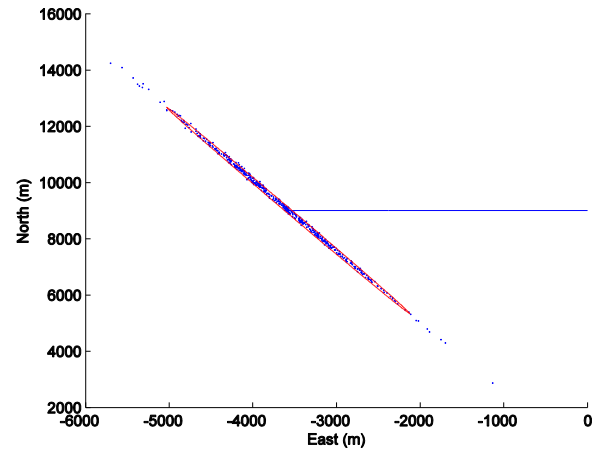


Fig. 6. Approximate maximum likelihood estimate for Scenario 2.

We can conclude that the two approaches to reduce the bias work properly and the obtained estimates (*AMLE* and *UPSLE*) are practically efficient.

## VI. CONCLUSION

In this paper, we presented a proof of observability of a source (emitting a single tone) moving at constant velocity in a plane from a motionless observer which acquired simultaneously a measured bearing and a measured frequency.

We proposed to estimate the state vector by using a linear representation of the measurement, yielding to the so-called pseudolinear estimate. Based on the statistical assumptions commonly used, we proposed to estimate the bias of this estimate (which is by essence biased because of the presence of the bearing noise in the state matrix and in the measurements). Two ways were explored: one yielding the maximum likelihood estimate and the second based upon the method of moments. The estimated bias is removed. The two final estimates are practically efficient. This technique could be adapted to the case of multi-frequency and bearings TMA [13].

## APPENDIX

### COMPUTATION OF THE COVARIANCE MATRIX OF THE NOISE

If the random variable  $Z$  has a normal (or Gaussian) distribution with parameter  $\mu$  (mean) and  $\sigma$  (standard deviation), then  $\mathbf{E}(\cos aZ) = \cos(a\mu) \exp\left(-\frac{a^2\sigma^2}{2}\right)$ .

If  $\mu = 0$ , then  $\mathbf{E}(\cos Z) = \exp\left(-\frac{\sigma^2}{2}\right)$  and

$$\mathbf{Var}(\cos Z) = \frac{1}{2} + \frac{1}{2} \exp(-2\sigma^2) - \exp(-\sigma^2).$$

We have also  $\mathbf{E}(\sin Z) = 0$  and  $\mathbf{Var}(\sin Z) = \frac{1}{2} - \frac{1}{2} \exp(-2\sigma^2)$

When  $\sigma \ll 1$ , then  $\mathbf{Var}(\cos Z) \cong \frac{\sigma^4}{2}$ . The random variable  $\cos Z$  can be considered as the deterministic value equal to  $\exp\left(-\frac{\sigma^2}{2}\right) \cong 1$ . And  $\mathbf{Var}(\sin Z) \cong \sigma^2$ .

### Lemma

Let us consider a deterministic quantity  $\theta$  and a Gaussian 0-mean noise  $\varepsilon$  of variance  $\sigma^2$ , then

$$\mathbf{E}[\cos \varepsilon \sin \varepsilon] = 0$$

$$\mathbf{E}[\cos(\theta + \varepsilon)] = \cos \theta \exp\left(-\frac{\sigma^2}{2}\right) \stackrel{\sigma \ll 1}{\cong} \cos \theta$$

$$\mathbf{E}[\sin(\theta + \varepsilon)] = \sin \theta \exp\left(-\frac{\sigma^2}{2}\right) \stackrel{\sigma \ll 1}{\cong} \sin \theta$$

$$\mathbf{E}[\cos^2(\theta + \varepsilon)] = \frac{1 + \cos 2\theta \exp(-2\sigma^2)}{2} \stackrel{\sigma \ll 1}{\cong} \cos^2 \theta$$

$$\mathbf{E}[\sin^2(\theta + \varepsilon)] = 1 - \mathbf{E}[\cos^2(\theta + \varepsilon)] = \frac{1 - \cos 2\theta \exp(-2\sigma^2)}{2}$$

$$\stackrel{\sigma \ll 1}{\cong} \sin^2 \theta$$

$$\mathbf{E}[\cos(\theta + \varepsilon) \sin \varepsilon] \stackrel{\sigma \ll 1}{\cong} -\sigma^2 \sin \theta,$$

$$\mathbf{E}[\sin(\theta + \varepsilon) \sin \varepsilon] \stackrel{\sigma \ll 1}{\cong} \sigma^2 \cos \theta.$$

Computation of the first two moments of  $\xi_k$ .

$$\text{Recall that } \xi_k = \begin{bmatrix} -f_0 r_k \sin \varepsilon_k \\ f_0 \frac{v}{c} \sin(\theta_k - h) \sin \varepsilon_k - \eta_k \end{bmatrix}.$$

$$\text{Obviously, } \mathbf{E}(\xi_k) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The statistical independence between  $\varepsilon_k$  and  $\nu_k$  allows us to simplify the expression of its covariance matrix:

$$\mathbf{Cov}(\xi_k) = \mathbf{E} \begin{bmatrix} \gamma_0^2 r_k^2 \sin^2 \varepsilon_k & -\gamma_0^2 r_k \frac{v}{c} \sin(\theta_k - h) \sin^2 \varepsilon_k \\ -\gamma_0^2 r_k \frac{v}{c} \sin(\theta_k - h) \sin^2 \varepsilon_k & \gamma_0^2 \frac{v^2}{c^2} \sin^2(\theta_k - h) \sin^2 \varepsilon_k + \eta_k^2 \end{bmatrix}$$

Using the approximation,

$$\mathbf{E}(\sin^2 \varepsilon_k) = \frac{1}{2} - \frac{1}{2} \mathbf{E}[\cos(2\varepsilon_k)]$$

$$= \frac{1}{2} - \frac{1}{2} \exp(-2\sigma_\theta^2)$$

$$\cong \sigma_\theta^2,$$

we get

$$\mathbf{Cov}(\xi_k) = \begin{bmatrix} \gamma_0^2 r_k^2 \sigma_\theta^2 & -\gamma_0^2 r_k \frac{v}{c} \sin(\theta_k - h) \sigma_\theta^2 \\ -\gamma_0^2 r_k \frac{v}{c} \sin(\theta_k - h) \sigma_\theta^2 & \gamma_0^2 \frac{v^2}{c^2} \sin^2(\theta_k - h) \sigma_\theta^2 + \sigma_f^2 \end{bmatrix}$$

QED.

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