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# Evaluation of the Fair Credit Risk Premium in Commercial Lending

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**Abstract:** We consider the problem of characterizing and computing the fair credit risk premium that a firm should pay when borrowing money from a bank. Using a risk-neutral approach, we show that there is a unique credit risk premium for a commercial loan depending on the firm's strategy such that the expected discounted value of the bank's payoff coincides with the loan's par value. We then propose a numerical procedure to estimate the premium.

**Keywords and phrases:** Corporate Loans, Credit Risk, Investment Decisions, Portfolio Choice, Contingent Pricing  
*2000 MSC: 60G44, G11-G13, G32-G33.*

## 1. Introduction

This paper discusses the pricing of corporate loans in the case where the risk premium is explicitly related to the firm's investment strategy.

A corporate loan is seen as a bilateral financial agreement issued, in the form of periodic payments, by a bank for its corporate non-financial firms. The main difference between loans and bonds is that banks are senior to bondholders in bankruptcy (Schwert, 2018). Loans are typically unsecured, which places the bank on equal level with unsecured bondholders, although it is unusual for a firm to default on an unsecured loan<sup>1</sup>. Of course, a firm can also borrow money by issuing unsecured corporate bonds in the loan market. To that end, Santos (2011) finds that after the sub-prime crisis, banks increased the interest rates on their loans to bank-dependent borrowers by more than they did on their loans to borrowers that have access to the bond market. This result is also confirmed by Schwert (2018) who shows that firms are willing to pay a higher cost to borrow from a bank due to the bank specialness. In clear, it seems there exists a preference for banks despite a relatively higher cost. The overall cost for a loan can be seen as the basis point spread over the money market rate (the main reference rate for corporate loans is Libor) plus the annual fees over the life of the loan.

The pricing of loan contracts has attracted considerable research interest<sup>2</sup>. The pricing structure can depend on the type of loans (Credit Lines versus Term Loans) as explained by Berg et al. (2017)<sup>3</sup> as some loan contracts are set with repayment on maturity (zero-coupon plan), while others are designed

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<sup>1</sup>Schwert (2018) gives the example of the average recovery rate for loans that is 80% whereas it is 40% for bonds, which implies that loan credit spreads should be smaller than bond credit spreads. From these aggregated statistics, he argues that the Duffie and Singleton (1999) model predicts that bond spreads should be approximately three times as large as loan spreads. More precisely, when the firm is close to default, bond spreads are significantly higher than loan spreads, but when it is far from default, the loan and bond spreads are similar.

<sup>2</sup>See Papin (2013)'s doctoral dissertation for an interesting discussion on that topic.

<sup>3</sup>In term loans, firms receive generally the full loan amount upfront and repay the loan at maturity, 5-8 years after loan origination, while other term loans are amortized until maturity. Credit lines are more frequent and more complex as they provide contingent liquidity, i.e. they do not draw down the committed loan amount, but keep the credit line as an insurance. The pricing structure of credit lines is more complex as it consists of various fees in addition to the loan spread.

with amortization plans (periodical payments plan).

An important problem with loan pricing comes from the assessment of the default probability. Credit risk is generally calculated implicitly from the price of loans, as it includes the investors expectation about future possible losses. But in practice, this information is provided by credit-rating agencies that compute long-term historical default rates for various horizons of time. These corporate credit ratings reduce the gap in terms of information asymmetry between lenders and borrowers to find an equilibrium price (Blöchlinger, 2018). The corresponding credit scoring is an important instrument in setting an appropriate default premium when determining the rate of interest charged to the borrower company with a past credit history. Hence, the loan spreads reflect the ex-ante credit risk as they are strongly correlated with probabilities of borrower default and capture the loan's credit risk smoothed through the cycle (Lee, Liu and Stebunovs, 2017). However, the credit rating does not incorporate the forward looking market consensus of the default risk and overall, it seems that past corporate bond spreads appear to have poor predictive power over default rates (Giesecke, Longstaff, Schaefer, and Strebulaev, 2011).

A related prominent problem with loan contracts is the determination of the fair risk premium that reflects the level of risk of the borrower's project. Each risky loan carries a risk premium for which the current price is the expected value of the pay-off discounted at the risk-free interest rate augmented by a risk premium. Loan risk premiums correspond to the loan rates less the risk-free rates and lower risk premiums arise when they carry lower losses due to default. In case of high risk profile for the borrowing firm, the bank can limit its exposure by requiring the riskier borrowers to select only short-term loans or imposing a collateral asset for hedging purpose. Interestingly, Allen, Scott and Vasso (2016) explain that in fact lenders offer a menu of contract terms such that applicants with higher-quality projects choose secured debt with lower risk premiums, while applicants with lower-quality projects self-select into unsecured debt with higher risk premiums. The valuation problem becomes relying on the firm's investment strategy given the relation between the quality of a project and the associated risk premium.

To that end, the present paper considers credit risk from the perspective of a bank willing to lend money according to the strategy of a non-financial firm. The very question is how large the minimum credit interest rate the bank re-

quires should be for it to compensate for the risk not to be reimbursed given the firm's strategy? This paper considers the realistic case where the bank has to fix the fair risk premium depending on the firm's strategy expressed in terms of quality of investment and level of consumption.

The article is organized as follows. Section 2 displays the literature review. Section 3 outlines the model. In Section 4, the main results are presented with their mathematical proofs. Section 5 concludes.

## 2. Review of literature

In financial theory, credit risk is defined as the case where the asset value ends up below the level of debts driving the firm to be insolvent. The fundamental contribution of the Merton (1974)'s structural model is that firm' risky debt can be viewed as a pseudo-bond equal to a portfolio composed of a risk-free debt minus a short put option on the value of the firm' assets. Structural models jointly capture the capital structure along with the evolution of firm asset value (Lee, Liu and Stebunovs, 2017). The structural model gives rational restrictions on the pricing of corporate debt and equity claims on a levered firm through classical option pricing theory (Black and Scholes, 1973; Merton, 1973) and classical corporate finance theory (Modigliani and Miller, 1958). This derivatives-based approach gives rational for pricing loans under a risk-neutral framework.

Important extensions of the Merton (1974)'s structural model has been done so far (Geske, 1977; Ingersoll, 1977) and particularly for credit risk because Merton model credit spreads are too low regarding what market prices. The literature refers to this phenomenon as the "credit spread puzzle" to explain why actual credit spreads are well above the theoretical spreads predicted by the debt valuation models beginning from the seminal Merton (1974) model.

There is an extensive theoretical literature examining this credit spread puzzle through the pricing of bonds in the presence of credit risk. Among the main pieces of work, study of the effects of safety covenants giving the bondholders the right to bankrupt and restrictions on the financing of interest and dividend payments (Black and Cox, 1976); pricing of contingent claims on the term structure of interest rates (Ho and Lee, 1986; Heath, Jarrow, Morton, 1992); derivation of solution for risky debts for finite maturity and

stochastic risk-free interest rate (Longstaff and Schwartz, 1992); connection of debt value to firm risk, taxes, bankruptcy cost and bond covenants (Leland, 1994); valuation of risky corporate debt that incorporates both default and interest rate risk (Longstaff and Schwartz, 1995); pricing of derivatives securities involving credit risk (Jarrow and Turnbull, 1995); optimal default (Leland and Toft, 1996); pricing of interest-rate swap yields with a standard term structure model incorporating default risk (Duffie and Singleton, 1997); agency costs (Leland, 1998); asymmetric information and uncertainty (Duffie and Lando, 2001); relation between risk premium in corporate bond prices and firms' investment and financing policies (Kuehn and Schmid, 2014); complex debt structures with multiple bonds and various covenants (Liu, Dai and Wang, 2016); idiosyncratic asset uncertainty measure as the residual volatility from a market model on equity (Culp, Nozawa and Veronesi, 2018).

However, Collin-Dufresne, Goldstein, and Martin (2001) find that credit spreads are driven by factors difficult to explain by standard credit models, which has been confirmed by Giesecke, Longstaff, Schaefer, and Strebulaev (2011) who study corporate bond default rates using an extensive data set (1866-2008). In fact, several assumptions behind these structural models tend to deviate the results from real world observations (Liu, Dai and Wang, 2016). For instance, the structural models tend to simplify the capital structure to build mathematical models that are tractable but fail to capture the complexity of debt structure (Eom, Helwege and Huang, 2004). In addition, structural models have difficulty in simultaneously explaining both the default probabilities and true credit spreads particularly for investment-grade bonds (Huang and Huang, 2012).

This paper adds to the broad literature on structural models of firm by deriving the fair value of the credit risk premium as being explicitly related to the firm's strategy. The novelty is to define the firm's strategy as a combination of the investment strategy (portfolio of risky assets) along with the financial policy (deleveraging ratio) and the dividend policy (pay-out ratio). Whatsoever, this fair pricing relies on the ability of the bank to share the same level of information with the borrowing non-financial firm.

### 3. The model

Let  $T > 0$  be an horizon date. We consider a financial market model defined on a complete stochastic basis  $(\Omega, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  satisfying the usual assumptions. Let  $(r_t^0)_{t \in [0, T]}$  be a stochastic process adapted to  $(\mathcal{F}_t)_{t \in [0, T]}$  interpreted as the risk-less interest rate of the market, i.e. such that it is possible for any firm to borrow money on the financial market at the interest rate  $r^0$  if and only if the firm is in position to reimburse its debt with probability 1. We denote by  $S^0$  the risk-less asset  $S^0$  defined by  $S_0^0 = 1$  and the dynamics  $dS_t^0 = r_t^0 S_t^0 dt$ , i.e.:

$$S_t^0 = \exp\left(\int_0^t r_u^0 du\right), \quad t \in [0, T]. \quad (3.1)$$

Let us consider a firm characterised (as in the Merton model) by its debt  $(D_t)_{t \in [0, T]}$  and the asset  $(A_t)_{t \in [0, T]}$  so that the equity is given by  $(E_t)_{t \in [0, T]}$  such that  $E = (A - D)^+$ . We suppose that the stochastic process  $D$  is adapted to  $(\mathcal{F}_t)_{t \in [0, T]}$  and it is defined by its initial value  $D_0$ , i.e. the amount of capital the firm needs to borrow at time 0, and the stochastic differential equation

$$dD_t = r_t D_t dt - K_t dt, \quad t \in [0, T]. \quad (3.2)$$

where  $r_t$  is the instantaneous interest rate of the credit at time  $t$  and  $K_t$  is the instantaneous amount of money that the firm reimburses at time  $t$  per unit of time. In the following, we set  $K_t := k_t D_t$  so that  $k_t$  is the ratio of the debt  $D_t$  with is reimbursed at time  $t$ .

The asset  $A$  of the firm satisfies by definition  $A_t = A_t^{(\theta^0, \theta)} := \theta_t^0 S_t^0 + \theta_t S_t$  where  $\theta^0$  is the quantity of asset  $S^0$  held by the firm. Similarly,  $\theta_t$  is the vector of all quantities of risky assets  $S = (S^1, \dots, S^d)$  invested by the firm at time  $t$ . Notice that the quantity  $\theta_t S_t$  is the usual scalar product between the vectors  $\theta_t$  and  $S_t$ . We suppose that  $S$  is a semi-martingale and the processes  $\theta^0$  and  $\theta$  are predictable. Moreover,  $\theta$  is supposed to be integrable with respect to  $S$ . In the following, we suppose the self-financing condition

$$dA_t = \theta_t^0 dS_t^0 + \theta_t dS_t - c_t dt, \quad t \in [0, T], \quad (3.3)$$

where  $c_t \geq K_t$  is a stochastic process adapted to  $(\mathcal{F}_t)_{t \in [0, T]}$ . Notice that  $\tilde{A}_t = \tilde{E}_t + \tilde{D}_t$  is supposed to be positive. We interpret the difference  $c_t - K_t \geq 0$

as the amount of dividends delivered to equity holders at time  $t$  per unit of time. The financial interpretation of (3.3) is clear: there is no adding or withdrawn of wealth but the amount of debt which is reimbursed plus the amount of dividends paid on the period  $[t, t + dt]$  are withdrawn from the firm's asset.

In the following, for any  $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted stochastic process  $X$ , we use the notation  $\tilde{X}_t = X_t(S_t^0)^{-1}$  for the discounted value of  $X$ . As  $(S^0)^{-1}$  is of finite variations and continuous, precisely  $d(S_t^0)^{-1} = -r_t^0(S_t^0)^{-1}dt$ , we deduce by the integration by parts formula that

$$d\tilde{X}_t = -r_t^0\tilde{X}_tdt + (S_t^0)^{-1}dX_t, \quad t \in [0, T]. \quad (3.4)$$

For our model, we get that

$$d\tilde{A}_t = \theta_t d\tilde{S}_t - \tilde{c}_t dt, \quad t \in [0, T], \quad (3.5)$$

$$d\tilde{D}_t = (p_t - k_t)\tilde{D}_tdt, \quad t \in [0, T], \quad (3.6)$$

where  $p_t := r_t - r_t^0$  is the credit risk premium. When  $p = 0$ , the debt process  $D$  is denoted by  $D^0$ . In the following, we only consider *admissible* strategies  $\theta$  such that  $\tilde{A}_t \geq -\kappa^\theta$  for all  $t \in [t, T]$  where  $\kappa^\theta \geq 0$  is a constant. Observe that, in practice,  $\kappa^\theta$  is generally equal to 0. Moreover, contrarily to the Merton model where the asset  $A$  is defined ex ante, the dynamics above shows that  $A$  may depend on the risk premium through the process  $\tilde{c} \geq k\tilde{D}$ . This is very natural since we may think that the debt interest should adversely impact the firm's asset. Moreover, the firm is characterised by the financial position  $(-\tilde{D}, \tilde{A})$  whose liquidation value at time  $t$  is  $\tilde{L}_t := \tilde{A}_t - \tilde{D}_t$  so that  $\tilde{E} = \tilde{L}^+$ . At time 0, we suppose that  $\tilde{A}_0 - \tilde{D}_0 \geq 0$ . Observe that the dynamics of  $\tilde{L}$  is given by

$$d\tilde{L}_t = \theta_t d\tilde{S}_t - \tilde{d}_t dt - p_t \tilde{D}_t dt, \quad t \in [0, T], \quad (3.7)$$

where  $\tilde{d}_t = \tilde{c}_t - k_t \tilde{D}_t$  is the amount of dividends given to the share holders on the period  $[t, t + dt]$ . The dynamics above shows that the liquidation value of the firm's financial position is naturally controlled by the investment strategy  $\theta$  but it is adversely impacted by the dividends  $\tilde{d} \geq 0$  paid to the share holders and by the credit risk premium  $p$  as well. In particular, apart the risk provided by the risky asset  $S$ , there is a risk generated by the strategy  $\theta$



such that an increase of the credit risk premium may decrease the liquidation value  $\tilde{L}$ , which may leads to a bankruptcy when  $\tilde{L} = 0$ .

Observe that, by (3.5) and (3.6),  $\tilde{A}_T \geq \tilde{D}_T$  means that

$$\tilde{E}_0 + \int_0^T \theta_t d\tilde{S}_t - \int_0^T \tilde{d}_t dt \geq \int_0^T p_t \tilde{D}_t dt. \quad (3.8)$$

In the case where the inequality above holds, then it also holds for  $p = 0$ . Moreover, as  $D$  is an increasing function of  $p$ , the inequality is violated as soon as  $p$  is large enough. Therefore, it is possible for the debt holders to deliberately make the firm insolvent when increasing the credit risk premium. It is natural to use the convention that the credit risk premium is fixed to 0 as soon as the left hand side of the inequality above is non negative for  $p = 0$ . This convention is actually justified under the condition (MP) below, see Lemmas 4.4 and 4.9.

#### 4. Fair credit risk premium

We define the fair credit risk premium as the smallest process  $p = \hat{p}$  such that the inequality (3.8) holds, i.e.  $\tilde{A}_T \geq \tilde{D}_T$  where  $\tilde{D}$  satisfies the dynamics (3.6) with  $p = \hat{p}$ .

**Lemma 4.1.** *Let us consider a firm  $(D, A)$  as in Section 3. The fair credit risk premium is zero if and only if*

$$\tilde{E}_0 + \int_0^T \theta_t d\tilde{S}_t \geq \int_0^T \tilde{D}_t dt, \quad \text{a.s.} \quad (4.9)$$

*Proof.* If the credit risk premium is  $p = 0$ , then by (3.6),

$$\tilde{D}_T = \tilde{D}_0 - \int_0^T k_t \tilde{D}_t dt.$$

Moreover, by definition,  $p = 0$  implies that  $\tilde{A}_T \geq \tilde{D}_T$ . By (3.5), we then deduce that

$$\tilde{A}_0 + \int_0^T \theta_t d\tilde{S}_t - \int_0^T k_t \tilde{D}_t dt - \int_0^T \tilde{d}_t dt \geq \tilde{D}_T = \tilde{D}_0 - \int_0^T k_t \tilde{D}_t dt.$$

As  $\tilde{A}_0 = \tilde{E}_0 + \tilde{D}_0$ , the conclusion follows. Reciprocally, if (4.9) holds, then  $\tilde{A}_T \geq \tilde{D}_T$  when  $p = 0$ .  $\square$

#### 4.1. Application to a complete market model

Suppose that there exists  $Q \sim P$  such that  $\tilde{S}$  is a martingale under  $Q$ . We suppose that the market is complete, i.e. any integrable contingent claim may be replicated by a portfolio process which is a martingale under  $Q$ . Therefore, it is well known that  $Q$  is the unique risk-neutral probability measure for  $\tilde{S}$ . The dividend plans  $\tilde{d}$  are supposed to be admissible in the sense that the cumulated dividend plan  $\int_0^T \tilde{d}_t dt$  is supposed to be integrable under  $Q$ . Inspired by Lemma 4.1, since the market is complete, consider  $E_0^*$  and the admissible investment strategy  $\theta^*$  such that

$$E_0^* + \int_0^t \theta_u^* d\tilde{S}_u = \mathbb{E}_Q \left( \int_0^T \tilde{d}_u du | \mathcal{F}_t \right), \quad t \in [0, T], \text{ a.s.} \quad (4.10)$$

Observe that, if  $E_0 = E_0^*$ , then  $\theta^*$  is a strategy which allows a zero credit risk premium  $p = 0$  by Lemma 4.1. The following proposition shows that  $\theta^*$  is the unique strategy that yields the zero credit premium  $p = 0$  when  $E_0 = E_0^*$ .

**Proposition 4.2.** *Suppose that  $E_0 = E_0^*$ . Then,  $\tilde{A}_T \geq \tilde{D}_T$  if and only if  $\theta = \theta^* d[\tilde{S}, \tilde{S}]$  a.s. In particular, the investment strategies  $\theta$  and  $\theta^*$  generate the same asset  $\tilde{A}$ .*

*Proof.* Suppose that  $\tilde{A}_T \geq \tilde{D}_T$  and  $E_0 = E_0^*$ . Using (3.8) and (4.10), we deduce that

$$\int_0^T (\theta_u - \theta_u^*) d\tilde{S}_u \geq 0, \quad \text{a.s.}$$

Since  $\theta$  is admissible, and  $\tilde{c} \geq 0$ , we deduce the existence of a constant  $\kappa \geq 0$  such that

$$M_t := \int_0^t (\theta_u - \theta_u^*) d\tilde{S}_u \geq -\kappa - \int_0^t \theta_u^* d\tilde{S}_u, \quad t \in [0, T].$$

By (4.10), the process in the r.h.s. of the inequality above is a martingale. Therefore, using the Fatou's lemma, we deduce that the local martingale  $M$  is a supermartingale. Since  $M_T \geq 0$ , we deduce that  $M_t \geq 0$  for every  $t \in [0, T]$ . On the other hand,  $M_0 = 0$  and  $\mathbb{E}_Q(M_t | \mathcal{F}_0) \leq M_0$  by the supermartingale property. Therefore,  $M_t = 0$  a.s. for all  $t \leq T$  and, by [10, 4.54],

$$0 = \mathbb{E}_Q M_T^2 = \mathbb{E}_Q \int_0^T (\theta_t - \theta_t^*)^2 d[\tilde{S}, \tilde{S}]_t.$$

We deduce that  $\theta = \theta^* d[\tilde{S}, \tilde{S}]$  a.s. Reciprocally, if this property holds, then  $M = 0$  hence (3.8) holds with  $p = 0$ .  $\square$

Following the same arguments, we obtain the following general result:

**Proposition 4.3.**  $\tilde{A}_T \geq \tilde{D}_T$  if and only if  $E_0 \geq E_0^*$  and

$$E_0 + \int_0^t \theta_u d\tilde{S}_u \geq E_0^* + \int_0^t \theta_u^* d\tilde{S}_u, \quad \forall t \in [0, T].$$

*Proof.* As  $\theta$  is supposed to be an admissible strategy, we deduce that the local martingale  $t \mapsto \int_0^t \theta_u d\tilde{S}_u$  is a supermartingale. Therefore, with the inequality  $\tilde{A}_T \geq \tilde{D}_T$ , i.e. (3.8), we obtain that:

$$\begin{aligned} E_0 + \int_0^t \theta_u d\tilde{S}_u &\geq E_0 + \mathbb{E}_Q \left( \int_0^T \theta_u d\tilde{S}_u | \mathcal{F}_t \right), \\ &\geq \mathbb{E}_Q \left( \int_0^T \tilde{d}_u du | \mathcal{F}_t \right), \\ &\geq E_0^* + \int_0^t \theta_u^* d\tilde{S}_u. \end{aligned}$$

With  $t = 0$ , we deduce that  $E_0 \geq E_0^*$ . This proposition above means that the debt holder should require a non null credit risk premium as soon as the investment strategy is not efficient enough or when the initial equity capital is too small.

#### 4.2. Valuation under the risk-neutral probability measure

In the following, we suppose that  $A \geq 0$ . Let  $Q \sim P$  be the risk-neutral probability measure for  $\tilde{S}$  in the market model supposed to be complete. In the case of a possible default, e.g. when  $p \neq 0$ , the discounted payoff delivered to the debt holders is

$$\tilde{h}_T^D := \int_0^T k_t \tilde{D}_t dt + \tilde{A}_T \wedge \tilde{D}_T, \quad (4.11)$$

i.e. the sum of the cumulated amount of money reimbursed before  $T$  by the firm and the residual debt  $\tilde{D}_T$  when it is smaller than  $\tilde{A}_T$ . Otherwise, in the

case of bankruptcy, the debt holders receive the asset value  $\tilde{A}_T$ . The payoff delivered to the equity holders is

$$\tilde{h}_T^E := \int_0^T \tilde{d}_u du + (\tilde{A}_T - \tilde{D}_T)^+, \quad (4.12)$$

i.e. the sum of the cumulated dividends and the asset value minus the debt to be reimbursed. Observe that:

$$\tilde{h}_T^D + \tilde{h}_T^E = E_0 + D_0 + \int_0^T \hat{\theta}_u d\tilde{S}_u, \quad (4.13)$$

i.e. the terminal gain generated by the investment strategy  $\theta$  when starting from the initial endowment  $E_0 + D_0$  (i.e. the r.h.s. of (4.13)) is shared between the debt holders and the equity holders with a priority to the debt holders, see (4.11). Since the strategy  $\theta$  is supposed to be admissible, the  $Q$ -local martingale  $t \mapsto \int_0^t \theta_u d\tilde{S}_u$  is a  $Q$ -supermartingale hence  $\mathbb{E}_Q \int_0^t \theta_u d\tilde{S}_u \leq 0$  for every  $t \leq T$ . It is natural to require a stronger condition, i.e.  $\mathbb{E}_Q \int_0^T \theta_u d\tilde{S}_u = 0$  since the investors are willing to optimize their profits. Therefore, under this condition,  $N : t \mapsto \int_0^t \theta_u d\tilde{S}_u$  is a martingale under  $Q$ . Indeed, if  $\mathbb{E}_Q(N_T | \mathcal{F}_t) \leq N_t$  and if the inequality is strict on a non null set, we get that  $\mathbb{E}_Q(N_T) < \mathbb{E}_Q(N_t) \leq 0$ . So, the condition  $\mathbb{E}_Q(N_T) = 0$  yields a contradiction.

**Lemma 4.4.** *Suppose that  $N : t \mapsto \int_0^t \theta_u d\tilde{S}_u$  is a martingale under  $Q$ . Then,  $\mathbb{E}_Q(\tilde{h}_T^D(p)) = D_0$  and  $\mathbb{E}_Q(\tilde{h}_T^E(p)) = E_0$  if and only if  $\mathbb{E}_Q(\tilde{h}_T^D(p)) = D_0$ . This condition is called market pricing (MP).*

*Proof.* Using (4.13), we deduce that  $\mathbb{E}_Q(\tilde{h}_T^D) + \mathbb{E}_Q(\tilde{h}_T^E) = D_0 + E_0$ . This equality implies that  $\mathbb{E}_Q(\tilde{h}_T^D) = D_0$  and  $\mathbb{E}_Q(\tilde{h}_T^E) = E_0$  as soon as  $\mathbb{E}_Q(\tilde{h}_T^D) = D_0$ .  $\square$

Recall that

$$\begin{aligned} \tilde{A}_t &= A_0 + \int_0^t \theta_u d\tilde{S}_u - \int_0^t \tilde{d}_u du - \int_0^t k_u \tilde{D}_u du, \\ \tilde{D}_t &= D_0 + \int_0^t (p_u - k_u) \tilde{D}_u du. \end{aligned}$$

It follows that

$$\begin{aligned} \tilde{A}_T \geq \tilde{D}_T &\Leftrightarrow A_0 + \int_0^T \theta_u d\tilde{S}_u - \int_0^T p_u \tilde{D}_u du \geq D_0 + \int_0^T \tilde{d}_u du, \\ &\Leftrightarrow E_0 + \int_0^T \theta_u d\tilde{S}_u - \int_0^T \tilde{d}_u du \geq \int_0^T p_u \tilde{D}_u du. \end{aligned}$$

Therefore, we deduce that:

$$\begin{aligned}
\tilde{h}_T^D &= \tilde{h}_T^D(p) = \min \left( A_0 + \int_0^T \hat{\theta}_u d\tilde{S}_u - \int_0^T \tilde{d}_u du, D_0 + \int_0^T p_u \tilde{D}_u du \right), \\
&= \min \left( D_0 + \tilde{L}_T + \int_0^T p_u \tilde{D}_u du, D_0 + \int_0^T p_u \tilde{D}_u du \right), \\
&= D_0 + \int_0^T p_u \tilde{D}_u du - (\tilde{L}_T)^-, \tag{4.14}
\end{aligned}$$

$$\tilde{h}_T^E = \tilde{h}_T^E(p) = \max \left( \int_0^T \tilde{d}_u du, E_0 + \int_0^T \theta_u d\tilde{S}_u - \int_0^T p_u \tilde{D}_u du \right). \tag{4.15}$$

Recall that  $x^- = -\min(x, 0)$ . Notice that  $\tilde{h}_T^D(0) \leq D_0$  so that the debt holders should accept to lend money at the rate  $p = 0$  if and only if  $\tilde{h}_T^D(0) = D_0$ , i.e. if  $A_0 + \int_0^T \theta_u d\tilde{S}_u - \int_0^T \tilde{d}_u du \geq D_0$  a.s. Actually, we have the following

**Lemma 4.5.** *The condition  $\tilde{h}_T^D(p) \geq D_0$  holds a.s. if and only if  $A_0 + \int_0^T \theta_u d\tilde{S}_u - \int_0^T \tilde{d}_u du \geq D_0$  a.s. and, under this condition, (MP) holds if and only if  $p=0$ .*

*Proof.* Suppose that  $\tilde{h}_T^D \geq D_0$ . If  $\tilde{h}_T^D = A_0 + \int_0^T \theta_u d\tilde{S}_u - \int_0^T \tilde{d}_u du$ , then the inequality  $\tilde{h}_T^D \geq D_0$  implies that  $A_0 + \int_0^T \theta_u d\tilde{S}_u - \int_0^T \tilde{d}_u du \geq D_0$ . Otherwise,  $A_0 + \int_0^T \theta_u d\tilde{S}_u - \int_0^T \tilde{d}_u du \geq \tilde{h}_T^D = D_0 + \int_0^T p_u \tilde{D}_u du \geq D_0$ . The reverse implication is trivial. At last, under this last condition, note that  $\tilde{h}_T^D(p) \geq \tilde{h}_T^D(0) = D_0$ . Therefore, by (MP),  $\mathbb{E}_Q(\tilde{h}_T^D(p)) = D_0$  implies that  $\int_0^T p_u \tilde{D}_u du = 0$  a.s., i.e.  $p_u = 0$   $du$ -a.e.  $\square$

**Remark 4.6.** *Let us suppose that the inequality  $\tilde{h}_T^D(0) = \tilde{D}_0$  is not satisfied a.s. Therefore,  $P(\tilde{h}_T^D(0) < D_0) > 0$  and  $\tilde{h}_T^D(0) \leq D_0$  a.s. implies that  $\mathbb{E}_Q(\tilde{h}_T^D(0)) < D_0$ . Therefore, it is necessary to increase the credit risk premium  $p$  for Condition (MP) to be satisfied.*

### 4.3. Constant credit risk premium

In this section, we only consider constant credit risk premiums  $p_u = p_0 = p$  for every  $t \in [0, T]$ . The Merton model is a particular case where  $k = \tilde{d} = 0$  and the discounted face value  $\tilde{K}_T$  of the debt is a given deterministic constant. We then deduce the unique credit risk premium  $p$  by solving the equation  $\tilde{K}_T = D_0 e^{pT}$  where  $D_0$  is the market value of  $\tilde{h}_T^D = \tilde{A}_T \wedge \tilde{K}_T$ . In the following,

we show the uniqueness of such a constant credit risk premium for our model in the case where  $\mathbb{E}_Q \left( \int_0^T \tilde{d}_r dt \right) < E_0$ . Notice that, if this inequality does not hold, there is no economical reason for the debt holders to lend money as the firm is not solvent (see Lemma below) even if the condition (MP) actually holds whatever  $p$ . In that case, we take the convention that the credit risk prime is  $p^* = 0$  as  $\tilde{h}_T^D(p)$  does not depend on  $p$ .

**Proposition 4.7.** *Suppose that  $\mathbb{E}_Q \left( \int_0^T \tilde{d}_r dt \right) \geq E_0$ . Then,  $\mathbb{E}_Q(\tilde{h}_T^D(p)) \geq D_0$  if and only if  $\mathbb{E}_Q \left( \int_0^T \tilde{d}_r dt \right) = E_0$ , and  $\tilde{L}_T = \tilde{A}_T - \tilde{D}_T \leq 0$  a.s. i.e.*

$$E_0 + \int_0^T \theta_u d\tilde{S}_u - \int_0^T \tilde{d}_u du \leq \int_0^T p_u \tilde{D}_u du, \quad \text{a.s.}$$

Under these conditions,  $\tilde{h}_T^D(p)$  does not depend on  $p$  and  $\mathbb{E}_Q(\tilde{h}_T^D(p)) = D_0$ .

*Proof.* By (4.14),  $\mathbb{E}_Q(\tilde{h}_T^D(p)) \geq D_0$  if and only if

$$\mathbb{E}_Q((\tilde{L}_T)^-) \leq \mathbb{E}_Q \left( \int_0^T p_u \tilde{D}_u du \right)$$

or equivalently if

$$\mathbb{E}_Q \left( \left( \int_0^T p_u \tilde{D}_u du - \gamma \right)^+ \right) \leq \mathbb{E}_Q \left( \int_0^T p_u \tilde{D}_u du \right),$$

where

$$\gamma = L_0 + \int_0^T \theta_u d\tilde{S}_u - \int_0^T \tilde{d}_u du = \tilde{L}_T + \int_0^T p_u \tilde{D}_u du.$$

Since  $x^+ \geq x$ , we deduce that

$$\mathbb{E}_Q \left( \left( \int_0^T p_u \tilde{D}_u du - \gamma \right)^+ \right) \geq \mathbb{E}_Q \left( \int_0^T p_u \tilde{D}_u du \right) - \mathbb{E}_Q(\gamma).$$

Since  $L_0 = E_0$  and  $\mathbb{E}_Q \left( \int_0^T \tilde{d}_r dt \right) \geq E_0$ , then  $\mathbb{E}_Q(\gamma) \leq 0$ . It follows that  $\mathbb{E}_Q(\tilde{h}_T^D(p)) \geq D_0$  if and only if  $\mathbb{E}_Q(\gamma) = 0$ , i.e.  $\mathbb{E}_Q \left( \int_0^T \tilde{d}_r dt \right) = E_0$ , and

$$\mathbb{E}_Q \left( \left( \int_0^T p_u \tilde{D}_u du - \gamma \right)^+ \right) = \mathbb{E}_Q \left( \int_0^T p_u \tilde{D}_u du \right).$$

Therefore,  $\mathbb{E}_Q(\tilde{h}_T^D(p)) \geq D_0$  if and only if  $\mathbb{E}_Q(\gamma) = 0$  and

$$\left( \int_t^T p_u \tilde{D}_u du - \gamma \right)^+ = \int_t^T p_u \tilde{D}_u du - \gamma,$$

as the difference between the l.h.s. and the r.h.s. of the equality is non negative with a zero expectation. This implies that

$$\gamma \leq \int_0^T p_u \tilde{D}_u du, \quad \text{a.s.},$$

i.e.  $\tilde{L}_T = \tilde{A}_T - \tilde{D}_T \leq 0$  a.s.

Reciprocally, if  $\tilde{A}_T \leq \tilde{D}_T$ , by (4.11) we get that

$$\begin{aligned} \tilde{h}_T^D(p) &= \tilde{A}_T + \int_t^T k_u \tilde{D}_u du, \\ &= E_0 + D_0 + \int_0^T \theta_u d\tilde{S}_u - \int_0^T \tilde{d}_u du. \end{aligned}$$

Therefore,  $\tilde{h}_T^D(p)$  does not depend on  $p$  and satisfies  $\mathbb{E}_Q(\tilde{h}_T^D(p)) = D_0$ .  $\square$

**Proposition 4.8.** *Suppose that  $\mathbb{E}_Q\left(\int_0^T \tilde{d}_r dt\right) < E_0$ . Then, there exists a unique credit risk premium  $p^* \in \mathbf{R}_+$  such that (MP) holds.*

*Proof.* We use the same notations than in the proof of Proposition 4.7. Let us introduce the function  $\phi : p \in \mathbf{R}_+ \mapsto \mathbb{E}_Q(\tilde{h}_T^D(p)) - D_0$ . We have

$$\phi(p) = \mathbb{E}_Q \left( \int_0^T p_u \tilde{D}_u(p) du - \left( \int_0^T p_u \tilde{D}_u(p) du - \gamma \right)^+ \right).$$

Note that the function  $\delta(x) = x - (x - \gamma)^+$  is non decreasing and for  $x \geq 0$ ,  $|\delta(x)| \leq |\gamma|$ . Supposing that  $\gamma$  is integrable, we deduce by the dominated convergence theorem that  $\phi(\infty) = \mathbb{E}_Q(\gamma)$ . In particular, since  $\mathbb{E}_Q(\gamma) = E_0 - \mathbb{E}_Q\left(\int_0^T \tilde{d}_t dt\right)$ , we get that  $\phi(\infty) > 0$ . Moreover,  $\phi(0) = -\mathbb{E}_Q \gamma^- \leq 0$ . Therefore, as  $p \mapsto \mathbb{E}_Q(\tilde{h}_T^D(p))$  is continuous, there exists  $p^* \in \mathbf{R}^+$  such that  $\phi(p^*) = 0$ . Suppose that there are two real numbers  $p^1, p^2 \in \mathbf{R}^+$  such that

$\phi(p^1) = \phi(p^2) = 0$  and  $p^1 < p^2$ . Since  $\delta$  is strictly increasing on  $(-\infty, \gamma)$  and constant on  $[\gamma, \infty)$ , we obtain that

$$\delta \left( \int_0^T p^1 \tilde{D}_u(p^1) du \right) \leq \delta \left( \int_0^T p^2 \tilde{D}_u(p^2) du \right)$$

and, finally, the equality holds due to the assumption. Therefore, we necessarily have

$$\int_0^T p^2 \tilde{D}_u(p^2) du \geq \gamma$$

since  $\delta$  is strictly increasing on  $(-\infty, \gamma)$ . We deduce that  $\phi(p^2) \geq \mathbb{E}_Q(\gamma)$  where  $\mathbb{E}_Q(\gamma) > 0$  by assumption. This yields a contradiction.  $\square$

Note that, when  $\phi(0) = 0$ , then  $\gamma = \tilde{L}_T \geq 0$  a.s. hence  $\tilde{A}_T \geq \tilde{D}_T$  a.s. so that, by the proposition below,  $p^* = 0$  is the only risk premium satisfying  $\phi(p^*) = 0$  under (MP).

**Proposition 4.9.** *Suppose that (MP) holds and let  $p^* \in \mathbf{R}_+$  be such that  $\phi(p^*) = 0$ . Then,  $p^* = 0$  if and only if  $\tilde{A}_T \geq \tilde{D}_T$  a.s.*

In the following, we propose to implement an estimation of the credit risk premium in the case where  $k$  is a deterministic function. Recall that

$$\tilde{D}_t = D_0 e^{pt - \int_0^t k_u du} \quad t \in [0, T].$$

In particular,

$$\int_0^T p \tilde{D}_t dt = D_0 p \int_0^T e^{pt - \int_0^t k_u du} dt =: g(p).$$

Note that  $g$  is a strictly increasing function on  $[0, \infty)$  such that  $g(0) = 0$  and  $g(\infty) = \infty$ . As  $g$  is also continuous, the inverse  $g^{-1}$  exists and is  $[0, \infty)$ -valued.

We suppose that  $\mathbb{E}_Q \left( \int_0^T \tilde{d}_r dt \right) < E_0$  hence there is a unique credit risk prime  $p^* \in \mathbf{R}_+$  by Proposition 4.8 such that  $\phi(p^*) = 0$ . Note that  $p^*$  coincides with the unique solution to the equation  $g(p) = \mathbb{E}_Q(\gamma - g(p))^-$  where  $\gamma$  is defined by

$$\gamma := E_0 + \int_0^T \theta_t d\tilde{S}_t - \int_0^T \tilde{d}_t dt.$$



Here, we suppose that  $\gamma$  is square integrable. Let us consider the sequence  $p^n$  defined by  $p^0 = 0$  and

$$p^{n+1} = g^{-1} \left( \mathbb{E}_Q(\gamma - g(p^n))^- \right),$$

i.e.  $g(p^{n+1}) = \mathbb{E}_Q(\gamma - g(p^n))^-$ . As  $p^0 = 0$ , we have  $p^1 \geq p^0$ . Since  $g$  is increasing and  $x \mapsto x^-$  is non decreasing, we may show by induction that  $p^{n+1} \geq p^n$  for all  $n \geq 0$ . Similarly, as  $p^0 \leq p^*$ , we deduce that  $g(p^1) \leq \mathbb{E}_Q(\gamma - g(p^*))^- = g(p^*)$  hence  $p^1 \leq p^*$  and, by induction,  $p^n \leq p^*$  for all  $n \geq 0$ . We finally deduce that the increasing sequence  $(p^n)_n$  converges to  $p^*$ .

In practice, it seems to be difficult to implement the iteration scheme  $p^{n+1} = g^{-1}(\mathbb{E}_Q(\gamma - g(p^n))^-)$  as we need first to estimate the expectation of  $(\gamma - g(p^n))^-$ . Note that, since  $g$  is invertible, the problem is equivalent to solve the equation  $x = \mathbb{E}_Q(\gamma - x)^-$  whose solution is  $x^* = g(p^*)$  so that  $p^* = g^{-1}(x^*)$ . Clearly  $x^*$  does not depend on  $D_0$  while  $p^*$  does as it is the solution to the equation  $x^* = g(p^*)$ .

In order to estimate  $x^*$ , we propose a method inspired by the well known stochastic gradient descent. Precisely, we consider a sequence  $(\gamma_n)_{n \geq 1}$  of independent random variables distributed as  $\gamma$  and we introduce the following scheme:  $x_1 = 0$  and

$$x_{k+1} = x_k - \alpha_k H(\gamma_k, x_k), \quad k \geq 0,$$

where  $H(v, x) = x - (v - x)^-$ , and  $(\alpha_k)_{k \geq 1}$  is a positive sequence satisfying  $\sum_{k=1}^{\infty} \alpha_k = +\infty$  and  $\sum_{k=1}^{\infty} (\alpha_k)^2 < +\infty$ .

**Theorem 4.10.** *The sequence  $(x_k)_{k \geq 1}$  converges a.s. to  $x^*$ .*

*Proof.* Note that  $H(v, x) = x1_{x \leq v} + v1_{x > v}$ , in particular  $x \mapsto H(\gamma, x)$  is non decreasing. Let us define the non decreasing function  $C(x) = \mathbb{E}H(\gamma, x)$ . We have  $C(0) = -\mathbb{E}_Q\gamma^-$  and  $C(\infty) = \mathbb{E}_Q\gamma > 0$  by assumption. Recall also that  $C(x^*) = 0$  by assumption. Since  $\gamma$  is integrable, we also deduce that  $C$  is continuous. Therefore, as  $C$  is non decreasing and the equation  $C(x) = 0$  admits a unique solution, we deduce that, for every  $\epsilon > 0$ ,

$$\inf_{|x-x^*|>\epsilon} (x-x^*)C(x) > 0. \tag{4.16}$$

On the other hand, let us introduce  $D(x) = \mathbb{E}_Q H^2(\gamma, x)$ . We have

$$\begin{aligned} D(x) &= \mathbb{E}_Q (H(x, \gamma) - H(x^*, \gamma) + H(x^*, \gamma))^2, \\ &\leq 2\mathbb{E}_Q (H(x, \gamma) - H(x^*, \gamma))^2 + 2\mathbb{E}_Q H^2(x^*, \gamma), \\ &\leq 8(x - x^*)^2 + 2\mathbb{E}_Q \gamma^2. \end{aligned}$$

The last inequality is deduced from the fact that  $H(\cdot, \gamma)$  is 2-Lipschitz and that  $|H(x, \gamma)| \leq |\gamma|$  if  $x \geq 0$ .

Let us introduce the sequence  $d_k = (x_k - x^*)^2$ ,  $k \geq 1$  and the filtration  $\mathcal{F}_k = \sigma(\gamma_1, \dots, \gamma_{k-1})$ ,  $k \geq 2$ , and  $\mathcal{F}_1$  is the trivial  $\sigma$ -algebra induced by the negligible sets. Notice that  $x_k \in L^0(\mathbf{R}, \mathcal{F}_k)$  for all  $k$ . The goal is to show that  $d_k$  converges a.s. to zero. To do so, let us first compute

$$\begin{aligned} d_{k+1} - d_k &= (x_{k+1} - x^*)^2 - (x_k - x^*)^2 = (x_k - \alpha_k H(\gamma_k, x_k) - x^*)^2 - (x_k - x^*)^2, \\ d_{k+1} - d_k &= \alpha_k^2 H^2(\gamma_k, x_k) - 2\alpha_k H(\gamma_k, x_k)(x_k - x^*). \end{aligned}$$

We deduce that

$$\mathbb{E}(d_{k+1} - d_k | \mathcal{F}_k) = \alpha_k^2 \mathbb{E}_Q (H^2(\gamma_k, x_k) | \mathcal{F}_k) - 2\alpha_k (x_k - x^*) \mathbb{E}_Q (H(\gamma_k, x_k) | \mathcal{F}_k).$$

As  $\gamma_k$  is independent of  $\mathcal{F}_k$  and  $x_k$  is  $\mathcal{F}_k$ -measurable, we deduce that

$$\mathbb{E}_Q (H^2(\gamma_k, x_k) | \mathcal{F}_k) = \mathbb{E}_Q (H^2(\gamma, x_k)), \quad \mathbb{E}_Q (H(\gamma_k, x_k) | \mathcal{F}_k) = \mathbb{E}_Q (H(\gamma, x_k)).$$

Therefore, by the first step,

$$\mathbb{E}_Q (d_{k+1} - d_k | \mathcal{F}_k) \leq 8\alpha_k^2 (x_k - x^*)^2 + 2\alpha_k^2 \mathbb{E}_Q \gamma^2 - 2\alpha_k (x_k - x^*) \mathbb{E}_Q (H(\gamma, x_k)). \quad (4.17)$$

Therefore, as  $\alpha_k (x_k - x^*) \mathbb{E}_Q (H(\gamma, x_k)) \geq 0$ ,

$$\mathbb{E}_Q (d_{k+1} - d_k (1 + 8\alpha_k^2) | \mathcal{F}_k) \leq 2\alpha_k^2 \mathbb{E}_Q \gamma^2. \quad (4.18)$$

Let us introduce  $\delta_{k+1} = \mu_k d_{k+1}$ ,  $k \geq 0$ , where  $\mu_k := \prod_{j=1}^{k-1} (1 + 8\alpha_j^2)^{-1}$ ,  $k \geq 1$  and  $\mu_0 = 1$ . Multiplying (4.18) by  $\mu_k$ , we obtain that

$$\mathbb{E}_Q (\delta_{k+1} - \delta_k | \mathcal{F}_k) \leq 2\alpha_k^2 \mu_k \mathbb{E}_Q \gamma^2. \quad (4.19)$$

From above, we deduce that the process  $M_k := \delta_k - 2\mathbb{E}_Q \gamma^2 \sum_{j=1}^k \alpha_j^2 \mu_j$ ,  $k \geq 1$ , is a super martingale. As  $\delta_k \geq 0$  for all  $k \geq 1$ ,  $M$  is bounded from

above hence  $(M_k)_{k \geq 1}$  is a.s. convergent by the Doob's theorem. We deduce that  $(\delta_k)_{k \geq 1}$  is also convergent. As the sequence  $(\ln(\mu_k))_{k \geq 1}$  is convergent, we deduce that  $\mu_k \rightarrow \mu > 0$  hence the sequence  $d_k = (x_k - x^*)^2$  is convergent.

Taking the expectation in (4.17) and summing up, as  $d_k \geq 0$ , we get that

$$\mathbb{E}_Q \left( \sum_{k=1}^{\infty} \alpha_k (x_k - x^*) \mathbb{E}_Q (H(\gamma, x_k)) \right) < \infty,$$

and finally

$$0 \leq \sum_{k=1}^{\infty} \alpha_k (x_k - x^*) \mathbb{E}_Q (H(\gamma, x_k)) < \infty,$$

where we recall that  $\alpha_k (x_k - x^*) \mathbb{E}_Q (H(\gamma, x_k)) \geq 0$  a.s. for all  $k \geq 1$ .

Let us define  $r = \liminf_{k \rightarrow \infty} (x_k - x^*) \mathbb{E}_Q (H(\gamma, x_k))$ . Suppose that  $r > 0$ . Then,  $(x_k - x^*) \mathbb{E}_Q (H(\gamma, x_k)) > r/2$  for  $k \geq k_0$  large enough so that

$$\sum_{k=k_0}^{\infty} \alpha_k (x_k - x^*) \mathbb{E}_Q (H(\gamma, x_k)) > r/2 \sum_{k=k_0}^{\infty} \alpha_k = +\infty,$$

which yields a contradiction. We deduce that  $r = 0$  and, in the case where  $\lim_{k \rightarrow \infty} d_k = 2\epsilon^2 > 0$ , we get that  $(x_k - x^*)^2 \geq \epsilon^2$  for  $k$  large enough. By (4.16), we deduce a constant  $c$  such that  $(x_k - x^*)C(x_k) > c$  where we recall that  $C(x_k) = \mathbb{E}_Q (H(\gamma, x_k))$  hence a contradiction, i.e.  $\lim_{k \rightarrow \infty} d_k = 0$ .  $\square$

## 5. Conclusion

We consider the problem of characterizing and computing the fair credit risk premium that a non-financial firm should pay when borrowing money from a bank. In particular, we study the case where the risk premium depends explicitly on the firm's strategy such that the expected discounted value of the bank's payoff coincides with the loan's par value. The strategy is explicitly defined in terms of investment and consumption. We use a risk-neutral framework to show the existence of a unique credit risk premium for such a commercial loan. We then propose a numerical procedure to estimate the fair premium. A limit of this research is that it relies on the assumption of the absence of information asymmetry between the bank and the non-financial firm, which suggests a new direction for further research.

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