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REMAINDER PADÉ APPROXIMANTS FOR THE HURWITZ ZETA FUNCTION

MARC PRÉVOST

ABSTRACT. Following our earlier research, we use the method introduced by the author in [9] named Remainder Padé Approximant in [10], to construct approximations of the Hurwitz zeta function. We prove that these approximations are convergent on the positive real line. Applications to new rational approximations of $\zeta(2)$ and $\zeta(3)$ are provided.

1. INTRODUCTION

In [9], we gave a new proof of the irrationality of $\zeta(2) = \sum_{k=1}^{\infty} 1/k^2$ (and also of $\zeta(3) = \sum_{k=1}^{\infty} 1/k^3$) based on an explicit construction of some Padé approximants of the remainder term $R_2(1/n) = \sum_{k=n}^{\infty} 1/k^2$ with

$$R_2(z) = \sum_{k=0}^{\infty} \frac{z^2}{(zk+1)^2}.$$

More precisely, we have $\zeta(2) = \sum_{k=1}^{n-1} 1/k^2 + R_2(1/n)$. The function R_2 is meromorphic in $\mathbb{C} \setminus \{0, -1, -\frac{1}{2}, -\frac{1}{3}, \dots\}$ and so is not holomorphic at 0. However, R_2 is \mathcal{C}^{∞} at $z = 0$ and may be expressed as a Taylor series $\hat{R}_2(z) = \sum_{k=0}^{\infty} B_k z^{k+1}$ with a zero radius of convergence, B_k being the k th Bernoulli number.

The idea was to compute an explicit representation of diagonal Padé approximant $P_n(z)/Q_n(z) = [n/n]_{\hat{R}_2}(z) \in \mathbb{Q}(z)$ of the series $\hat{R}_2(z)$ with a good estimation of the error term $E_n(z) = R_2(z) - [n/n]_{\hat{R}_2}(z)$.

At last, we stated the relationship

$$Q_n(1/n)E_n(1/n) = Q_n(1/n)\zeta(2) - Q_n(1/n) \sum_{k=1}^{n-1} 1/k^2 - P_n(1/n) = q_n\zeta(2) - p_n$$

providing a sequence of rational approximation p_n/q_n and proving the irrationality of $\zeta(2)$. The surprise was to retrieve the sequences used by Apéry [1] for the same purpose.

The same method applied to the series $\zeta(3) = \sum_{k=1}^{\infty} 1/k^3$ provides also the Apéry numbers given in [1].

The term Remainder Padé Approximant (RPA) has been used later in a paper written with T. Rivoal [10] for the exponential function.

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In this paper, we apply the same method and we prove that some sequence of RPA for Hurwitz zeta function is convergent on the real line.

For $a \in \mathbb{C}$, $\Re(a) > 0$, the Hurwitz zeta function is defined as

$$\zeta(s, a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \quad (1.1)$$

where the Dirichlet series on the right hand side of (1.1) is convergent for $\Re(s) > 1$ and uniformly convergent in any finite region where $\Re(s) \geq 1 + \delta$, with $\delta > 0$. It defines an analytic function for $\Re(s) > 1$. The Riemann ζ function is $\zeta(s, 1)$.

An integral representation of $\zeta(s, a)$ is given by

$$\zeta(s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} e^{-ax}}{1 - e^{-x}} dx \quad \Re(s) > 1, \Re(a) > 0, \quad (1.2)$$

where $\Gamma(s) = \int_0^{\infty} y^{s-1} e^{-y} dy$ is the gamma function.

If $\Re(s) \leq 1$, the Hurwitz zeta function can be defined by the equation

$$\zeta(s, a) = \Gamma(1-s) \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{z^{s-1} e^{az}}{1 - e^z} dz,$$

where \mathcal{C} is a loop around the negative real axis; it starts at $-\infty$, encircles the origin once in the positive direction without enclosing any of the points $z = \pm 2i\pi, \pm 4i\pi, \dots$, and returns to $-\infty$.

Thus $\zeta(s, a)$ can be continued meromorphically to the whole complex s -plane except for a single pole at $s = 1$ with its residue 1.

If we write the formula (1.1) as :

$$\zeta(s, a) = \sum_{k=0}^{n-1} \frac{1}{(k+a)^s} + \sum_{k=0}^{\infty} \frac{1}{(n+a+k)^s} \quad (1.3)$$

and set

$$\Psi(s, x) := t^{1-s} \sum_{k=0}^{\infty} \left(\frac{t}{1+kt} \right)^s$$

then

$$\zeta(s, a) = \sum_{k=0}^{n-1} \frac{1}{(k+a)^s} + (n+a)^{1-s} \Psi \left(s, \frac{1}{n+a} \right). \quad (1.4)$$

Another expression of $\Psi(s, t)$ is given in [13, chapt. XIII] by

$$\Psi(s, t) = \frac{t^{1-s}}{\Gamma(s)} \int_0^{\infty} u^{s-1} \frac{e^{-u/t}}{1 - e^{-u}} du \quad (1.5)$$

whose Taylor series is

$$\Psi(s, t) = \sum_{k=0}^{\infty} \frac{B_k}{k!} (s)_{k-1} (-1)^k t^k, \quad (1.6)$$

which is convergent only at $t = 0$, where $(s)_k$ is the Pochhammer symbol ($(s)_p := s(s+1)(s+2)\cdots(s+p-1)$, $(s)_{-1} = 1/(s-1)$).

For a given pair (m_1, m_2) , let $[m_1/m_2]_{\Psi(s,t)} = \tilde{Q}_{m_1}(t)/\tilde{P}_{m_2}(t)$ be a Padé approximant to the function Ψ .

Then by replacing the remainder term $\Psi\left(s, \frac{1}{n+a}\right)$ in (1.4), by the Padé approximant $[m_1/m_2]_{\Psi(s,t)}$, computed at $t = 1/(n+a)$, we obtain the RPA approximant

$$RPA(n, m_1, m_2) := \sum_{k=0}^{n-1} \frac{1}{(k+a)^s} + (n+a)^{1-s} \frac{\tilde{Q}_{m_1}(1/(n+a))}{\tilde{P}_{m_2}(1/(n+a))}. \quad (1.7)$$

The crucial point is the convergence of this sequence $(RPA(n, m_1, m_2))$ when one or more of the parameters n, m_1, m_2 tends to infinity.

The convergence of Padé approximants is proved for many functions as meromorphic function, Stieltjes function, ... [3]. In this paper, when s is a real positive number, we prove that the function $\Psi(s, t)$ is a Stieltjes function with respect to the variable t . First we need to remind the definition and some properties of Padé approximants.

2. PADÉ APPROXIMANTS

Let f be a formal series whose Taylor expansion about $t = 0$ is

$$f(t) = \sum_{i=0}^{\infty} c_i t^i.$$

The Padé approximant $[m_1/m_2]_f$ to f is a rational fraction $\tilde{Q}_{m_1}/\tilde{P}_{m_2}$ whose denominator has degree m_2 and whose numerator has degree m_1 so that its expansion in ascending powers of t coincides with the expansion of f up to the degree $m_1 + m_2$, i.e.

$$\tilde{P}_{m_2}(t)f(t) - \tilde{Q}_{m_1}(t) = \mathcal{O}(t^{m_1+m_2+1}). \quad (2.1)$$

The theory of Padé approximation is linked to the theory of orthogonal polynomials as following [6].

Let us define the linear functionals c and $c^{(p)}$, $p \in \mathbb{Z}$ acting on \mathcal{P} , space of polynomials by

$$c : \mathcal{P} \rightarrow \mathbb{R} \text{ (or } \mathbb{C}) \quad (2.2)$$

$$x^i \rightarrow \langle c, x^i \rangle = c_i \quad i = 0, 1, 2, \dots \quad (2.3)$$

and

$$\langle c^{(p)}, x^i \rangle := \langle c, x^{i+p} \rangle = c_{i+p}, \quad i = 0, 1, 2, \dots (c_i = 0, i < 0). \quad (2.4)$$

Then the denominators of the Padé approximants $[m_1/m_2]_f$ satisfy the following orthogonality property

$$\langle c^{(m_1-m_2+1)}, x^i P_{m_2}(x) \rangle = 0 \quad i = 0, 1, 2, \dots, m_2 - 1$$

where $P_{m_2}(x) = x^{m_2} \tilde{P}_{m_2}(x^{-1})$ is the reversed polynomial.

Let us define the associated polynomials

$$R_{m_2-1}(t) := \left\langle c^{(m_1-m_2+1)}, \frac{P_{m_2}(x) - P_{m_2}(t)}{x - t} \right\rangle, R_{m_2-1} \in \mathcal{P}_{m_2-1}, \quad (2.5)$$

where $c^{(m_1-m_2+1)}$ acts on the variable x . Then

$$\tilde{Q}_{m_1}(t) = \left(\sum_{i=0}^{m_2-m_1} c_i t^i \right) \tilde{P}_{m_2}(t) + t^{m_1-m_2+1} \tilde{R}_{m_2-1}(t) \quad (2.6)$$

where $\tilde{R}_{m_2-1}(t) = t^{m_2-1} R_{m_2-1}(t^{-1})$.

If c admits an integral representation with a non decreasing function α , with bounded variation, such as

$$c_i = \int_L x^i d\alpha(x), \quad (2.7)$$

then f is a Stieltjes function and the theory of Gaussian quadrature shows that the polynomials P_n , orthogonal with respect to c , have all their roots in the support of the function α .

Moreover, if f satisfies the Carleman's criterion [3, p.240] then all paradiagonal sequences of $([m + J, m])_m$ Padé approximants with $J \geq -1$ converge to f .

The aim of the paper is to find, in the case of Hurwitz zeta function, the weight function $d\alpha$ depending on s and to prove that it is a positive function.

The error is defined by

$$f(t) - [m_1/m_2]_f(t) = \frac{t^{m_1+m_2+1}}{\tilde{P}_{m_2}^2(t)} c^{(m_1-m_2+1)} \left(\frac{P_{m_2}^2(x)}{1 - xt} \right). \quad (2.8)$$

The expression of the error above is understood as a formal one if c is only a formal linear functional [6, chapt. 3], but if c admits the integral representation (2.7) then the error rewrites as :

$$f(t) - [m_1/m_2]_f(t) = f(t) - \frac{\tilde{Q}_{m_1}(t)}{\tilde{P}_{m_2}(t)} = \frac{t^{m_1+m_2+1}}{\tilde{P}_{m_2}^2(t)} \int_L x^{m_1-m_2+1} \frac{P_{m_2}^2(x)}{1 - xt} d\alpha(x). \quad (2.9)$$

In the particular case $m_1 = m_2 - 1$,

$$f(t) - [m_2 - 1/m_2]_f(t) = \frac{t^{2m_2}}{\widetilde{P}_{m_2}^2(t)} \int_L \frac{P_{m_2}^2(x)}{1 - xt} d\alpha(x). \quad (2.10)$$

Note that

$$[m_2 + p/m_2]_f(t) = c_0 + c_1 t + \cdots + c_p t^p + t^{p+1} [m_2 - 1/m_2]_{f_p}(t)$$

where

$$f_p(t) = \sum_{i=0}^{\infty} c_{i+p+1} t^i.$$

3. STATEMENTS OF THE RESULTS

In this section, we will prove that it is possible to compute the weight function underlying the coefficients $\frac{B_k}{k!}(s)_{k-1}(-1)^k$ of the function $\Psi(s, t) = \sum_{k=0}^{\infty} \frac{B_k}{k!}(s)_{k-1}(-1)^k t^k$. The weight function w_s will depend on s . For particular values of s ($s > 0$), w_s is a positive weight function and thus $\Psi(s, t)$ is a Stieltjes function depending on the variable t and the convergence of RPA will be proved for $t = 1/(n + a)$.

Theorem 1. *For any complex numbers s, a , $\Re(s) > 0$, $\Re(a) > 0$ and any positive integer m , $m > \Re(s) - 1$, then*

$$\zeta(s, a) = \frac{1}{a^{s-1}} \left(\frac{1}{s-1} + \frac{1}{2a} + \int_0^{\infty} \frac{1}{a^2 + x^2} w_s(x) dx \right) \quad (3.1)$$

where the weight w_s is defined by

$$w_s(x) := \frac{2(-1)^m x^s}{\Gamma(s)\Gamma(m+1-s)} \int_x^{\infty} (t-x)^{m-s} \frac{d^m}{dt^m} \left(\frac{1}{e^{2\pi t} - 1} \right) dt.$$

We will prove in the section 4 that this formula (3.1) is a consequence of Hermite's formula for the function $\zeta(s, a)$.

As proved by Touchard [12], Bernoulli numbers satisfy

$$B_k = -i \frac{\pi}{2} \int_L x^k \frac{dx}{\sin^2(\pi x)},$$

where L is the line $L := -\frac{1}{2} + i\mathbb{R}$ and thus can be viewed as moments of order $k \geq 0$ of the positive weight function $1/\sin^2(\pi x)$ on L . Therefore, for $s = 2$, the coefficients $\frac{B_k}{k!}(s)_{k-1} = B_k$ of (1.6) are the moments of a positive weight. For $s = 3$, these coefficients appear as their derivative $(k+1)B_k$. The derivative of $1/\sin^2(\pi x)$ is a weight function, symmetric around $-1/2$ whose support is also the line L . But for integer value of s greater than 4, it is no more true for all $k \geq 0$. A much more difficult case is the case when s is real.

Nevertheless, from the previous Theorem, we obtain an integral representation of $\frac{B_k}{k!}(s)_{k-1}$ but for $k \geq 2$.

Theorem 2. *Integral representation of Bernoulli numbers.*

If s is a complex number such that $\Re(s) > 0$, then for $k \in \mathbb{N}$,

$$B_{k+2} \frac{(s)_{k+1}}{(k+2)!} = \begin{cases} (-1)^{k/2} \int_0^{+\infty} x^k w_s(x) dx & (\text{if } k \text{ even}) \\ 0 & (\text{if } k \text{ odd}) \end{cases} \quad (3.2)$$

with

$$w_s(x) := \frac{2(-1)^m x^s}{\Gamma(s)\Gamma(m+1-s)} \int_x^\infty (t-x)^{m-s} \frac{d^m}{dt^m} \left(\frac{1}{e^{2\pi t} - 1} \right) dt \quad (3.3)$$

where m is any integer satisfying $m > \Re(s) - 1$.

Proof We use the expansion of $\zeta(s, a)$ in terms of Bernoulli numbers [11, p.160]. We have

$$\zeta(s, a) = a^{-s} + \sum_{k=0}^n (s)_{k-1} \frac{B_k}{k!} a^{-k-s+1} + \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^t - 1} - \sum_{k=0}^n \frac{B_k}{k!} t^{k-1} \right) e^{-at} t^{s-1} dt$$

which is valid for $\Re(s) > -2n + 1$, $\Re(a) > 0$ and n a positive integer.

By comparing this formula with (3.1), we get the formula (3.2).

Remark. The positivity of the weight function is important because it guarantees the convergence of Padé approximants. In the case when s is a positive real number then w_s is positive on its support. This gives the following main theorem.

Theorem 3. *For every positive real number s , $s \neq 1$, for every complex number a , $\Re(a) > 0$, for all integers $n \geq 0, p \geq 1$, the following sequence*

$$RPA(n, m+p, m) := \sum_{k=0}^{n-1} \frac{1}{(k+a)^s} + (n+a)^{1-s} [m+p/m]_{\Psi(s,t)}|_{t=1/(n+a)}. \quad (3.4)$$

converges to $\zeta(s, a)$ when m tends to ∞ .

Theorems (1 and 3) will be proved in Section 4.

Remarks. For $n \in \mathbb{N}, p = -1$, Theorem 3 is proved only for $s = 2, 3$ in [9]. $RPA(n, n-1, n)$ (resp. $RPA(n, 2n-1, 2n)$) are the Apéry's numbers for $\zeta(2)$ (resp. $\zeta(3)$).

If $n = 0$, formula (3.4) yields

$$\forall s > 0, \forall a \in \mathbb{C}, \Re(a) > 0, \forall p \geq 1, \lim_{m \rightarrow \infty} a^{1-s} [m+p/m]_{\Psi(s,t)}|_{t=1/a} = \zeta(s, a).$$

Moreover, for the particular case $n = 0, a = 1$, $RPA(0, m+p, m)$ is a rational function depending on the variable s and which converges to $\zeta(s, 1) = \zeta(s)$ when m tends to ∞ (by Theorem 3).

For example, for $p = 1$,

$$RPA(0, 2, 1) = \frac{(s+2)(s+3)}{12(s-1)},$$

$$RPA(0, 3, 2) = \frac{(s+2)(s^2+12s+31)}{2(s-1)(s^2+3s+62)},$$

$$RPA(0, 4, 3) = \frac{(s+2)(s+4)(3s^2+88s+405)}{120(s-1)(s^2+7s+54)},$$

$$RPA(0, 5, 4) = \frac{(s+2)(s+4)(s^5+124s^4+2644s^3+23730s^2+92939s+122130)}{2(s-1)(s^6+101s^5+1755s^4+21415s^3+153884s^2+657564s+977040)}$$

$$RPA(0, 6, 5) = \frac{(s+2)(s+4)(s+6)(s^5+746s^4+28162s^3+498112s^2+3612925s+8457750)}{84(s-1)(s^6+429s^5+10387s^4+134511s^3+1044772s^2+4891020s+9666000)}$$

If $n \geq 1$ and $a = 1$, then $RPA(n, m+p, m)$ is no more a rational fraction since it contains powers of $(n+a)$ with s as exponent:

$$RPA(1, 2, 1) = \frac{2^{1-s}(s^2+11s+36)}{48(s-1)} + 1,$$

$$RPA(1, 3, 2) = \frac{2^{-s-1}(s^3+26s^2+231s+726)}{(s-1)(s^2+3s+242)} + 1,$$

$$RPA(1, 4, 3) = \frac{2^{-s-4}(3s^4+166s^3+2937s^2+22334s+64800)}{15(s-1)(s^2+7s+180)} + 1,$$

$$RPA(2, 2, 1) = \frac{3^{1-s}(s^2+17s+90)}{108(s-1)} + 2^{-s} + 1,$$

$$RPA(2, 3, 2) = \frac{3^{-s}(s^3+38s^2+527s+2710)}{2(s-1)(s^2+3s+542)} + 2^{-s} + 1.$$

Nevertheless, if s is an integer, $a = 1$ and $p \geq 1$, we obtain a sequence of rational approximations of $\zeta(s)$ by computing $RPA(n, m+p, m)$, depending on the two integers parameters n and m .

4. PROOF OF THEOREMS

4.1. Proof of Theorem 1. We start from Hermite's formula for $\zeta(s, a)$, $\Re(a) > 0$, which is a consequence of Plana's summation formula ([11], p.158):

$$\zeta(s, a) = \frac{1}{2} a^{-s} + \frac{a^{1-s}}{s-1} + 2 \int_0^\infty (a^2 + y^2)^{-s/2} \sin\left(s \arctan \frac{y}{a}\right) \frac{dy}{e^{2\pi y} - 1}. \quad (4.1)$$

Note that this integral converges for every complex number $s \neq 1$.

Let us define

$$I_s := \int_0^t \frac{x^s(t-x)^{-s}}{a^2 + x^2} dx \quad (t \geq 0, 0 < \Re(s) < 1).$$

The proof is based on the formula [8, p. 142]:

we have for $\Re(\alpha) > 0$ and $\Re(\beta) > 0$,

$$\int_0^1 \frac{u^{\alpha-1}(1-u)^{\beta-1}}{(u+z)^{\alpha+\beta}} du = B(\alpha, \beta) (1+z)^{-\alpha} z^{-\beta} \quad (z \in \mathbb{C}, |ph(z)| < \pi) \quad (4.2)$$

where $B(\alpha, \beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

With the change of variable $x = tu$, we have

$$\begin{aligned} I_s &= \int_0^1 \frac{tu^s(1-u)^{-s}}{a^2 + t^2u^2} du \\ &= \frac{i}{2} \int_0^1 u^{s-1}(1-u)^{-s} \left(\frac{1}{a+itu} - \frac{1}{a-itu} \right) du. \end{aligned}$$

By applying formula (4.2) with $\alpha = s, \beta = 1 - s$ and $z = \pm \frac{a}{it}$ ($|ph(z)| < \pi$ since $\Re(a) > 0$) we get

$$I_s = \Gamma(s)\Gamma(1-s) \frac{a^{s-1}}{2i} \left((a-it)^{-s} - (a+it)^{-s} \right).$$

Let us set $t = a \tan \theta$. We obtain

$$I_s = \Gamma(s)\Gamma(1-s)a^{-1} \cos^s(\theta) \sin(s\theta) = \Gamma(s)\Gamma(1-s)a^{s-1}(a^2 + t^2)^{-s/2} \sin \left(s \arctan \left(\frac{t}{a} \right) \right).$$

Thus, the following identity

$$\frac{1}{\Gamma(s)\Gamma(1-s)} \int_0^t \frac{x^s(t-x)^{-s}}{a^2 + x^2} dx = a^{s-1}(a^2 + t^2)^{-s/2} \sin \left(s \arctan \left(\frac{t}{a} \right) \right) \quad (4.3)$$

holds for all real $t \geq 0$ and complex $s, 0 < \Re(s) < 1$.

Now, for s complex satisfying $\Re(s) > 0$ and m an integer such that $m > \Re(s) - 1$, we claim that the following formula,

$$\frac{\sin(s \arctan t/a)}{(a^2 + t^2)^{s/2}} = \frac{a^{1-s}}{\Gamma(s)\Gamma(m+1-s)} \frac{d^m}{dt^m} \left(\int_0^t \frac{x^s(t-x)^{m-s}}{1+x^2} dx \right) \quad (4.4)$$

is satisfied.

First, it is true for $m = 0$ (4.3).

Let us assume that (4.4) is true for m then using

$$\begin{aligned} \frac{d^{m+1}}{dt^{m+1}} \left(\int_0^t \frac{x^s(t-x)^{m+1-s}}{1+x^2} dx \right) &= \frac{d^m}{dt^m} \frac{d}{dt} \left(\int_0^t \frac{x^s(t-x)^{m+1-s}}{1+x^2} dx \right) \\ &= \frac{d^m}{dt^m} \left(\int_0^t (m+1-s) \frac{x^s(t-x)^{m-s}}{1+x^2} dx \right), \end{aligned}$$

we obtain formula (4.4) by using the identity $\Gamma(m+2-s) = (m+1-s)\Gamma(m+1-s)$.

To prove Theorem 1, we replace $\frac{\sin(s \arctan(t/a))}{(a^2 + t^2)^{s/2}}$ (formula 4.4) in Hermite's formula (4.1), and we apply integration by parts m times:

$$\begin{aligned}
\zeta(s, a) &= \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + \frac{2 a^{1-s}}{\Gamma(s)\Gamma(m+1-s)} \int_0^\infty \frac{d^m}{dt^m} \left(\int_0^t \frac{x^s (t-x)^{m-s}}{a^2+x^2} dx \right) \frac{dt}{e^{2\pi t}-1} \\
&= \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + \frac{2(-1)^m a^{1-s}}{\Gamma(s)\Gamma(m+1-s)} \int_0^\infty \left(\int_0^t \frac{x^s (t-x)^{m-s}}{a^2+x^2} dx \right) \frac{d^m}{dt^m} \left(\frac{1}{e^{2\pi t}-1} \right) dt \\
&= \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + \frac{2(-1)^m a^{1-s}}{\Gamma(s)\Gamma(m+1-s)} \int_0^\infty \frac{x^s dx}{a^2+x^2} \int_x^\infty (t-x)^{m-s} \frac{d^m}{dt^m} \left(\frac{1}{e^{2\pi t}-1} \right) dt.
\end{aligned}$$

The permutation in the last integral is valid since the function

$\left| \frac{x^s}{a^2+x^2} (t-x)^{m-s} \frac{d^m}{dt^m} \left(\frac{1}{e^{2\pi t}-1} \right) \right|$ is less than $\frac{t^m}{a^2} \frac{d^m}{dt^m} \left(\frac{1}{e^{2\pi t}-1} \right)$ which is integrable in $[0, \infty[$.

Remark. If we consider $a = 1$, we get an integral representation of the Riemann zeta function:

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + \int_0^\infty \frac{1}{1+x^2} w_s(x) dx \quad (4.5)$$

with

$$w_s(x) := \frac{2(-1)^m x^s}{\Gamma(s)\Gamma(m+1-s)} \int_x^\infty (t-x)^{m-s} \frac{d^m}{dt^m} \left(\frac{1}{e^{2\pi t}-1} \right) dt \quad (4.6)$$

where m is an integer satisfying $m > \Re(s) - 1$.

So, if s is an integer, we can consider $m = s$ and

$$\zeta(s) = \frac{1}{2} + \frac{1}{(s-1)} + \frac{2(-1)^{s-1}}{\Gamma(s)} \int_0^\infty \frac{x^s}{1+x^2} \frac{d^{s-1}}{dx^{s-1}} \left(\frac{1}{e^{2\pi x}-1} \right) dx \quad (4.7)$$

$$= \frac{1}{2} + \frac{1}{(s-1)} + \frac{2}{\Gamma(s)} \int_0^\infty \frac{1}{e^{2\pi x}-1} \frac{d^{s-1}}{dx^{s-1}} \left(\frac{x^s}{1+x^2} \right) dx \quad (4.8)$$

which yields

$$\zeta(2) = \frac{3}{2} + \pi \int_0^\infty \frac{x^2}{1+x^2} \frac{1}{\sinh^2 \pi x} dx = \frac{3}{2} + \int_0^\infty \frac{1}{e^{2\pi x}-1} \frac{4x}{(1+x^2)^2} dx \quad (4.9)$$

$$\zeta(3) = 1 + \pi^2 \int_0^\infty \frac{x^3}{1+x^2} \frac{\cosh \pi x}{\sinh^3 \pi x} dx = 1 - \int_0^\infty \frac{1}{e^{2\pi y}-1} \frac{2x(x^2-3)}{(1+x^2)^3} dx \quad (4.10)$$

$$\zeta(4) = \frac{5}{6} + \frac{\pi^3}{3} \int_0^\infty \frac{x^4}{1+x^2} \frac{2 + \cosh(2\pi x)}{\sinh^4 \pi x} dx = \frac{5}{6} - \int_0^\infty \frac{1}{e^{2\pi x}-1} \frac{8x(x^2-1)}{(1+x^2)^4} dx. \quad (4.11)$$

4.2. Proof of Theorem 3. We consider the function $\Psi(s, t)$ given in (1.6) written as

$$\Psi(s, t) = \frac{B_0}{s-1} - B_1 t + t^2 \Psi_2(s, t) \quad (4.12)$$

where

$$\Psi_2(s, t) := \sum_{k=0}^{\infty} \frac{B_{k+2}}{(k+2)!} (s)_{k+1} (-1)^k t^k.$$

Thus the Padé approximant $[m + p/m]_{\Psi(s,t)}$ satisfies

$$[m + p/m]_{\Psi(s,t)} = \frac{B_0}{s-1} - B_1 t + t^2 [m + p - 2, m]_{\Psi_2(s,t)}.$$

For s a positive real number, the weight w_s is positive (Theorem 2) and so $\Psi_2(s, t)$ is a Stieltjes function

$$\Psi_2(s, t) = \int_0^{\infty} \frac{1}{1+x^2 t^2} w_s(x) dx.$$

Its coefficients $c_k := \frac{B_{k+2}}{(k+1)!} (s)_{k+1} (-1)^k$ are positive and satisfy the Carleman's criterion, i.e.

$$\sum_{k=0}^{\infty} c_k^{-1/2k} \text{ diverges.}$$

Actually, the Bernoulli numbers of even index satisfy

$$|B_{2k}| \sim 4\sqrt{\pi k} \left(\frac{k}{\pi e}\right)^{2k}$$

and so

$$c_{2k}^{-1/4k} \sim \left(\frac{k}{\pi e}\right)^{-1/2}$$

which is the general term of a divergent series. (Bernoulli of odd index greater than 2 are zero)

We now apply Theorem of [4, p.240] to conclude that for all integers $p, p \geq 1$,

$$\lim_{m \rightarrow \infty} t^{s-1} [m + p/m]_{\Psi(s,t)} = t^{s-1} \Psi(s, t) = \zeta(s, 1/t) \text{ for all complex numbers } t, \Re(t) > 0$$

5. PARTICULAR CASE: s IS AN INTEGER

For $p = 1, s = 2$ or $p = 2, s = 3$, we will give in this section the formal expression of $RPA(n, m + p, m)$ and the expression of the error. If $a = 1$, then the rational approximations $RPA(n, rn + 1, rn)$ to $\zeta(2)$ and $RPA(n, rn + 2, rn)$ to $\zeta(3)$ with some suitable value of r lead to another proofs of the Apéry's theorem.

The case for $s \geq 4$ remains an open problem.

5.1. **Case $s = 2$.** In this case, the weight in the expression (3.1) is defined as

$$w_2(x) := 2x^2 \int_x^\infty \frac{d^2}{dt^2} \left(\frac{1}{e^{2\pi t} - 1} \right) dt = \frac{\pi x^2}{\sinh^2(\pi x)}. \quad (5.1)$$

Another expression of $w_2(x)$ is

$$w_2(x) = \frac{1}{\pi} |\Gamma(1 + ix)\Gamma(1 - ix)|^2.$$

Generalizing a result by Carlitz [7], Askey and Wilson [2] gave an explicit expression for the orthogonal polynomial P_m with respect to the weight function w_2 :

$$P_m(x) = (m+1)(m+2) {}_3F_2 \left(\begin{matrix} -m, & m+3, & 1-x \\ & 2, & 2 \end{matrix} ; 1 \right) \quad (5.2)$$

$$= \sum_{k=0}^m \binom{m+1}{k+1} \binom{m+k+2}{k+1} \binom{x-1}{k}, \quad (5.3)$$

and

$$\int_{i\mathbb{R}} P_n(x)P_m(x) \frac{\pi x^2}{\sin^2 \pi x} dx = \begin{cases} 0 & (\text{if } n \neq m), \\ \frac{(-1)^n 2(n+1)(n+2)}{2n+3} & (\text{if } n = m). \end{cases}$$

The roots of the polynomials P_n are located on the imaginary axis since the weight $\frac{\pi x^2}{\sin^2 \pi x}$ is positive on this line (see 2.7).

The three-term recurrence relation is

$$P_{m+1}(x) = \frac{2(2m+3)}{(m+1)(m+2)} x P_m(x) + P_{m-1}(x) \quad (5.4)$$

with initial conditions: $P_{-1}(x) = 0, P_0(x) = 2$.

The Padé approximants are expressed with orthogonal polynomials and their associated polynomials (see 2.5).

Before computing these associated polynomials, we need some suitable modified moments, i.e., the moments of the binomial $\binom{x-1}{k}$.

Let us define the linear functional $c^{(s)}$ acting on the space of polynomials as

$$\langle c^{(s)}, x^j \rangle := \frac{B_j}{j!}(s)_{j-1}(-1)^j, \quad j \in \mathbb{N},$$

and $x^2 c^{(s)}$ by

$$\langle x^2 c^{(s)}, x^j \rangle := \langle c^{(s)}, x^{j+2} \rangle.$$

The following modified moments [9, p.228] satisfy

$$\left\langle c^{(2)}, \binom{x}{k} \right\rangle = \frac{(-1)^k}{k+1}. \quad (5.5)$$

The identity $x^2 \binom{x-1}{k} = (k+1)(k+2) \binom{x}{k+2} + (k+1)^2 \binom{x}{k+1}$ leads to the following formula for modified moments for the functional $x^2 c^{(2)}$:

$$\left\langle x^2 c^{(2)}, \binom{x-1}{k} \right\rangle = \frac{(-1)^k (k+1)}{(k+2)(k+3)}. \quad (5.6)$$

The associated polynomials R_{m-1} (2.5) are defined by

$$R_{m-1}(t) := \left\langle x^2 c^{(2)}, \frac{P_m(x) - P_m(t)}{x-t} \right\rangle$$

where the variable is x . From the expression (5.3) for P_m , we get the following formula for R_{m-1}

$$R_{m-1}(t) = \sum_{k=0}^m \binom{m+1}{k+1} \binom{m+k+2}{k+1} \left\langle x^2 c^{(2)}, \frac{\binom{x-1}{k} - \binom{t-1}{k}}{x-t} \right\rangle. \quad (5.7)$$

Using the expression of the polynomial $\frac{\binom{x-1}{k} - \binom{t-1}{k}}{x-t}$ on the Newton basis with abscissæ $\{0, 1, 2, \dots, k-1\}$

$$\frac{\binom{x-1}{k} - \binom{t-1}{k}}{x-t} = \binom{t-1}{k} \sum_{i=1}^k \frac{\binom{x-1}{i-1}}{i \binom{t-1}{i}},$$

we can write a compact formula for R_{m-1} :

$$R_{m-1}(t) = \sum_{k=0}^m \binom{m+1}{k+1} \binom{m+k+2}{k+1} \binom{t-1}{k} \sum_{i=1}^k \frac{(-1)^{i-1}}{\binom{t-1}{i} (i+1)(i+2)}.$$

So, we get the $[m+1/m]$ Padé approximant to the function $\Psi(2, t)$

$$[m+1/m]_{\psi(2,t)} = B_0 - B_1 t + t^2 \frac{\tilde{R}_{m-1}(t)}{\tilde{P}_m(t)} = B_0 - B_1 t + t \frac{R_{m-1}(1/t)}{P_m(1/t)}$$

and also an approximation of $\zeta(2, a)$:

$$\begin{aligned} \zeta(2, a) &\approx \sum_{k=0}^{n-1} \frac{1}{(k+a)^2} + \frac{1}{n+a} [m+1/m]_{\psi(2,t)} \Big|_{t=1/(n+a)} \\ &= \sum_{k=0}^{n-1} \frac{1}{(k+a)^2} + \frac{1}{n+a} + \frac{1}{2(n+a)^2} + \frac{1}{(n+a)^2} \varepsilon_m(a) =: \frac{v_{n,m}(a)}{u_{n,m}(a)} = RPA(n, m+1, m) \end{aligned}$$

where

$$\varepsilon_m(a) = \frac{\sum_{k=0}^m \binom{m+1}{k+1} \binom{m+k+2}{k+1} \sum_{i=1}^k \frac{\binom{n+a-i-1}{k-i} (-1)^{i-1}}{\binom{k}{i}^{(i+1)(i+2)}}}{\sum_{k=0}^m \binom{m+1}{k+1} \binom{m+k+2}{k+1} \binom{n+a-1}{k}}.$$

Irrationality of $\zeta(2)$

A consequence of the previous formula is another proof of the irrationality of $\zeta(2)$.

Indeed, if $d_k := LCM[1, 2, \dots, k]$, then [9, Lemma 3.1]

$$\frac{d_k}{i \binom{k}{i}} \in \mathbb{N}.$$

Consequently, the numerator of $\varepsilon_m(1)$ multiplied by d_{m+2}^2 is an integer and for all integers n, m , $d_{m+2}^2 v_{n,m}(1) \in \mathbb{N}$, $d_{m+2}^2 u_{n,m}(1) \in \mathbb{N}$.

The error (formula (2.10)) applied to the function $\Psi(2, t)$ rewrites as

$$\Psi(2, t) - [m + 1/m]_{\Psi(2,t)}(t) = \frac{t^{2m}}{\tilde{P}_m^2(t)} \int_{i\mathbb{R}} \frac{P_m^2(x)}{1 - xt \sin^2(\pi x)} \frac{x^2}{\sin^2(\pi x)} dx$$

and the error term satisfies

$$\left| \zeta(2) - \frac{v_{n,m}(1)}{u_{n,m}(1)} \right| \leq \frac{1}{P_m^2(n+1)} \int_{i\mathbb{R}} \left| \frac{P_m^2(x)}{1 - x/(n+1)} \right| \frac{x^2}{\sin^2(\pi x)} dx \quad (5.8)$$

$$\leq \frac{1}{P_m^2(n+1)} \int_{i\mathbb{R}} |P_m^2(x)| \frac{x^2}{\sin^2(\pi x)} dx \quad (5.9)$$

$$\leq \frac{1}{P_m^2(n+1)} \frac{2(m+1)(m+2)}{\pi(2m+3)}. \quad (5.10)$$

Note that $u_{n,m}(1) = P_m(n+1)$.

Now, we consider $r \in \mathbb{Q}$ such that $m = rn \in \mathbb{N}$. Applying the Stirling formula to the expression (5.3) for orthogonal polynomials P_n , we get

$$\begin{aligned} \lim_n (P_{rn}(n+1))^{1/n} &= \max_{t \in [0,1]} \frac{(r+t)^{r+t}}{t^{3t}(r-t)^{r-t}(1-t)^{1-t}} \\ &= \frac{(r+\sigma(r))^r}{(1-\sigma(r))(r-\sigma(r))^r} =: \rho(r) \end{aligned} \quad (5.11)$$

where $\sigma(r) = \frac{-r^2 + \sqrt{r^4 + 4r^2}}{2}$ is a zero of $t^2 + r^2 t - r^2 = 0$.

So, using the prime number theorem $d_n = e^{n+o(n)}$,

$$\begin{aligned} \limsup_n |\zeta(2)d_{rn+2}^2 u_{n,rn}(1) - d_{rn+2}^2 v_{n,rn}(1)|^{1/n} &\leq \lim_n (d_{rn+2}^2 u_{n,rn}(1))^{1/n} \lim_n \frac{1}{(P_{rn}(n+1))^{2/n}} \\ &= e^{2\max(r,1)}/\rho(r). \end{aligned}$$

The inequation $e^{2\max(r,1)}/\rho(r) < 1$ is satisfied for $r \in]0.74, 1.53[$.

The irrationality of $\zeta(2)$ follows from the following limit

$$\lim_n (\zeta(2)d_{rn+2}^2 u_{n,rn}(1) - d_{rn+2}^2 v_{n,rn}(1)) = 0.$$

5.2. Case $s = 3$. For $s = 3$, the weight in the expression (3.1) is

$$w_3(x) := -\frac{x^3}{3} \int_x^\infty \frac{d^3}{dt^3} \left(\frac{1}{e^{2\pi t} - 1} \right) dt = \frac{\pi^2 x^3 \cosh \pi x}{3 \sinh^3(\pi x)}. \quad (5.12)$$

Another expression of $x^2 w_3(x)$ is

$$x^2 w_3(x) = \frac{1}{12\pi} \frac{|\Gamma(1+ix)|^8}{|\Gamma(2ix)|^2}.$$

This weight has been investigated by Wilson [14]. The orthogonal polynomial P_n satisfies:

$$P_m(x^2) = (m+1)(m+2) {}_4F_3 \left(\begin{matrix} -m, & m+3, & 1-x, & 1+x \\ & 2, & 2, & 2 \end{matrix} ; 1 \right) \quad (5.13)$$

$$= \sum_{k=0}^m \binom{m+1}{k+1} \binom{m+k+2}{k+1} \binom{x-1}{k} \binom{x+k}{k} / (k+1) \quad (5.14)$$

and the orthogonality relation is

$$\int_{i\mathbb{R}} P_n(x^2) P_m(x^2) \frac{\pi^2 x^5 \cos \pi x}{3 \sin^3 \pi x} dx = \begin{cases} 0 & (\text{if } n \neq m), \\ \frac{1}{3} \frac{(-1)^n (n+1)(n+2)}{2n+3} & (\text{if } n = m). \end{cases}$$

The weight $\frac{x^5 \cos \pi x}{\sin^3 \pi x}$ is positive on the line $i\mathbb{R}$ thus, the roots of the polynomials P_n are located on the imaginary axis.

We set $\Pi_m(x) := P_m(x^2)$.

The three-term recurrence relation is

$$\Pi_{m+1}(x) = \left(\frac{2(2m+3)}{(m+1)(m+2)^2} x^2 + \frac{2m+3}{m+2} \right) \Pi_m(x) - \frac{m+1}{m+2} \Pi_{m-1}(x) \quad (5.15)$$

with initial conditions: $\Pi_{-1} = 0, \Pi_0 = 2$.

In the previous subsection, we have defined the linear functional $c^{(s)}$ acting on the space of polynomials as

$$\langle c^{(s)}, x^j \rangle := \frac{B_j}{j!} (s)_{j-1} (-1)^j, \quad j \in \mathbb{N},$$

and $x^4 c^{(s)}$ by

$$\langle x^4 c^{(s)}, x^j \rangle := \langle c^{(s)}, x^{j+4} \rangle.$$

Before computing the associated polynomials, we need the moments of $\binom{x-1}{k} \binom{x+k}{k}$ and $\binom{x-1}{k+1} \binom{x+k}{k}$.

Lemma 1. *Let us define the polynomials $\nu_{2j+1}(x)$ and $\nu_{2j}(x)$ by*

$$\begin{aligned} \nu_{2j+1}(x) &= \binom{x-1}{j+1} \binom{x+j}{j} \\ \nu_{2j}(x) &= \binom{x-1}{j} \binom{x+j}{j}. \end{aligned}$$

Then,

$$\langle x^4 c^{(3)}, \nu_{2k+1}(x) \rangle = (-1)^{k+1} \frac{(k+1)^2}{2(k+3)(k+2)} \quad (5.16)$$

$$\langle x^4 c^{(3)}, \nu_{2k}(x) \rangle = (-1)^k \frac{(k+1)^2}{2(k+3)(k+2)} \quad (5.17)$$

Proof. $c^{(2)}$ satisfies $\langle c^{(2)}, x^j \rangle = B_j(-1)^j$. We denote $c^{(2)'}$ by $\langle c^{(2)'}, x^j \rangle := \langle c^{(2)}, -jx^{j-1} \rangle$. From the definition of $c^{(3)}$, $\langle c^{(3)}, x^j \rangle = \frac{1}{2} B_j(j+1)(-1)^j$, we have

$$c^{(3)} = -\frac{1}{2} x c^{(2)'}$$

In [9] p. 230, it is proved that $\langle -c^{(2)'}, N_{2k}(x) \rangle = 0$, $\langle -c^{(2)'}, N_{2k+1}(x) \rangle = \frac{(-1)^k}{(k+1)^2}$ where

$$N_0(x) = 1, N_{2k}(x) = \binom{x}{k} \binom{x+k}{k}, N_{2k+1}(x) = \binom{x}{k+1} \binom{x+k}{k}.$$

Now, $\langle x^4 c^{(3)}, \nu_{2k+1}(x) \rangle = \langle -\frac{1}{2} x^5 c^{(2)'}, \nu_{2k+1}(x) \rangle = \frac{1}{2} \langle -c^{(2)'}, x^5 \nu_{2k+1}(x) \rangle$.

The proof of (5.16) follows from the expansion

$x^5 \nu_{2k+1}(x) = \alpha N_{2k+6}(x) + \beta N_{2k+5}(x) + \gamma N_{2k+4}(x) + \delta N_{2k+3}(x) + \mu N_{2k+2}(x) - 2\mu N_{2k+1}(x)$,
with

$$\begin{aligned} \alpha &= (k+1)(k+2)^2(k+3)^2, \\ \beta &= -2(k+1)(k+2)^3(k+3), \\ \gamma &= (k+1)(k+2)^2(2k^2+6k+5), \\ \delta &= -(k+1)(k+2)(2k+3)(2k^2+6k+5), \\ \mu &= (k+1)^5. \end{aligned}$$

For (5.17), we use the expansion

$$x^5 \nu_{2k}(x) = (k+1)^2(k+2)^2(k+3)N_{2k+5}(x) + (k+1)^2(k+2)(2k^2+6k+5)N_{2k+3}(x) + (k+1)^5 N_{2k+1}(x).$$

□

Now, the associated polynomials Θ_{m-1} (2.5) are defined as

$$\Theta_{m-1}(t) := \left\langle x^4 c^{(3)}, \frac{\Pi_m(x) - \Pi_m(t)}{x-t} \right\rangle$$

where the variable is x . From the expression (5.14) for Π_m , we get the following formula for $\Theta_{m-1} \in \mathcal{P}_{2m-2}$:

$$\Theta_{m-1}(t) = \sum_{k=0}^m \binom{m+1}{k+1} \binom{m+k+2}{k+1} \frac{1}{k+1} \left\langle x^4 c^{(3)}, \frac{\binom{x-1}{k} \binom{x+k}{k} - \binom{t-1}{k} \binom{t+k}{k}}{x-t} \right\rangle. \quad (5.18)$$

Using the expression of the polynomial $\frac{\binom{x-1}{k} \binom{x+k}{k} - \binom{t-1}{k} \binom{t+k}{k}}{x-t}$ on the Newton basis with abscissæ $\{0, 1, -1, 2, -2, \dots, k, -k\}$ [9, formula 55], i.e.

$$\frac{\binom{x-1}{k} \binom{x+k}{k} - \binom{t-1}{k} \binom{t+k}{k}}{x-t} = \sum_{j=0}^{2k-1} \frac{\nu_{2k}(t)}{\nu_{j+1}(t)} \frac{\nu_j(x)}{[(j+2)/2]}.$$

and Lemma 1,

$$\langle x^4 c^{(3)}, \nu_{2j+1}(x) \rangle = - \langle x^4 c^{(3)}, \nu_{2j}(x) \rangle = (-1)^j \frac{(j+1)^2}{2(j+3)(j+2)},$$

we can write a compact formula for Θ_{m-1} :

$$\begin{aligned} \Theta_{m-1}(t) &= \sum_{k=0}^m \binom{m+1}{k+1} \binom{m+k+2}{k+1} \frac{1}{k+1} \sum_{j=0}^{2k-1} \frac{\nu_{2k}(t)}{\nu_{j+1}(t)} \langle x^4 c^{(3)}, \nu_j(x) \rangle \\ &= \sum_{k=0}^m \binom{m+1}{k+1} \binom{m+k+2}{k+1} \frac{\nu_{2k}(t)}{k+1} \sum_{p=1}^k \frac{t}{\binom{t-1}{p} \binom{t+p}{p}} \frac{(-1)^p}{2(p+1)(p+2)}. \end{aligned}$$

So, for $m \geq 2$, we get the $[2m + 2/2m]$ Padé approximant to the function $\Psi(3, t)$

$$[2m + 2/2m]_{\Psi(3,t)} = \frac{B_0}{2} - B_1 t + \frac{3}{2} t^2 B_2 + t^5 \frac{\tilde{\Theta}_{m-1}(t)}{\tilde{\Pi}_m(t)} = \frac{B_0}{2} - B_1 t + \frac{3}{2} t^2 B_2 + t^4 \frac{\Theta_{m-1}(1/t)}{\Pi_m(1/t)},$$

and an approximation of $\zeta(3, a)$:

$$\zeta(3, a) \approx \sum_{k=0}^{n-1} \frac{1}{(k+a)^3} + \frac{1}{2(n+a)^2} + \frac{1}{2(n+a)^3} + \frac{1}{4(n+a)^4} + \frac{1}{(n+a)^5} \epsilon_m(a) =: \frac{f_{n,m}(a)}{g_{n,m}(a)}$$

where

$$\begin{aligned} \epsilon_m(a) &= \frac{\Theta_{m-1}(n+a)}{\Pi_m(n+a)} \\ &= \frac{\sum_{k=0}^m \binom{m+1}{k+1} \binom{m+k+2}{k+1} \frac{n+a}{k+1} \sum_{p=1}^k \frac{\binom{n+a-p-1}{k-p} \binom{n+a+k}{k-p} (-1)^p}{\binom{k}{p}^{2(p+1)(p+2)}}}{\sum_{k=0}^m \binom{m+1}{k+1} \binom{m+k+2}{k+1} \binom{n+a-1}{k} \binom{n+a+k}{k} / (k+1)}. \end{aligned}$$

Irrationality of $\zeta(3)$

The irrationality of $\zeta(3)$ has been proved by Apéry in a celebrated paper [1]. A little later, a particular straightforward and elegant proof of this irrationality was given by Beukers [5]. The author gave another proof in [9] using the RPA. Actually, $RPA(m, 2m-1, 2m)$ provides exactly the Apéry's numbers for $\zeta(3)$. In the following, we show that the previous approximation also proves irrationality of $\zeta(3)$ with RPA's depending on other parameters.

Actually,

$$\forall n, a \in \mathbb{N}, d_{m+1}^3 \Theta_{m-1}(n+a) \in \mathbb{Z}, d_{m+1}^3 f_{n,m}(n+a) \in \mathbb{Z} \text{ and } d_{m+1}^3 g_{n,m}(n+a) \in \mathbb{Z}.$$

Note that $g_{n,m}(n+a) = \Pi_m(n+a)$. The proof is similar than in the previous subsection. We consider the case $a = 1$.

The error (formula (2.10)) applied to the function $\Psi(3, t)$ becomes

$$\Psi(3, t) - [2m + 2/2m]_{\Psi(3,t)} = \frac{t^{2m}}{\widetilde{\Pi}_m^2(t)} \int_{i\mathbb{R}} \frac{\Pi_m^2(x) \pi^2 x^5 \cos \pi x}{1 - xt \ 3 \sin^3(\pi x)} dx$$

and the error term

$$\zeta(3) - \frac{f_{n,m}(1)}{g_{n,m}(1)}$$

satisfies

$$\begin{aligned} \left| \zeta(3) - \frac{f_{n,m}(1)}{g_{n,m}(1)} \right| &\leq \frac{1}{\Pi_m^2(n+1)} \int_{i\mathbb{R}} \frac{\Pi_m^2(x)}{|1 - x/(n+1)|} \frac{\pi^2 x^5 \cos \pi x}{3 \sin^3(\pi x)} dx \\ &\leq \frac{1}{\Pi_m^2(n+1)} \int_{i\mathbb{R}} \Pi_m^2(x) \frac{\pi^2 x^5 \cos \pi x}{3 \sin^3(\pi x)} dx \\ &\leq \frac{1}{\Pi_m^2(n+1)} \frac{1}{3} \frac{(m+1)(m+2)}{2m+3}. \end{aligned}$$

Now, we consider $r \in \mathbb{Q}$ such that $m = rn \in \mathbb{N}$. Using the Stirling formula in the expression of the orthogonal polynomials (5.14),

$$\begin{aligned} \lim_n (\Pi_{rn}(n+1))^{1/n} &= \max_{t \in [0,1]} \frac{(r+t)^{r+t}(1+t)^{1+t}}{t^{3t}(r-t)^{r-t}(1-t)^{1-t}} \\ &= \frac{(1+\mu(r))(r+\mu(r))^r}{(1-\mu(r))(r-\mu(r))^r} =: \eta(r) \end{aligned}$$

where $\mu(r) = \frac{r}{\sqrt{1+r^2}}$ is a zero of $(1-t^2)(r^2-t^2) = t^4$.

So

$$\begin{aligned} \limsup_n |\zeta(3)d_{rn+1}^3 g_{n,rn}(1) - d_{rn+1}^3 f_{n,rn}(1)|^{1/n} &\leq \lim_n (d_{rn+1}^3 g_{n,rn}(1))^{1/n} \lim_n \frac{1}{(g_{n,rn}(1))^{2/n}} \\ &= e^{3\max(r,1)}/\eta(r) \end{aligned}$$

The inequation $e^{3\max(r,1)}/\eta(r) < 1$ is satisfied for $r \in]0.74, 1.36[$.

The irrationality of $\zeta(3)$ then follows from the following limit

$$\lim_n (\zeta(3)d_{rn+1}^3 g_{n,rn}(1) - d_{rn+1}^3 f_{n,rn}(1)) = 0.$$

Conclusion

In section 3, for $a = 1$, we have proved that for all fixed integers $n, p \geq 1$, and for each $s > 0, s \neq 1$, then the sequence

$$(RPA(n, m+p, m))_m$$

converges to $\zeta(s)$.

If m and n are linked such that $m = rn$, then the convergence of

$$(RPA(n, rn+p, rn))_n$$

is proved for $s = 2$ and the rate of convergence is $\rho(r)$ (see (5.11)):

$$\limsup_n (|\zeta(s) - RPA(n, rn+p, rn)|^{1/n} \leq \rho(r).$$

Numerically, it seems that this property is valid for all $s > 0, s \neq 1$ and more generally for all complex numbers $s, s \neq 1$.

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