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CLASSIFICATION OF ALGEBRAIC SOLUTIONS OF IRREGULAR GARNIER SYSTEMS.

KARAMOKO DIARRA AND FRANK LORAY

In the memory of Tan Lei

ABSTRACT. We prove that algebraic solutions of Garnier systems in the irregular case are of two types. The classical ones come from isomonodromic deformations of linear equations with diagonal or dihedral differential Galois group; we give a complete list in the rank $N = 2$ case (two indeterminates). The pull-back ones come from deformations of coverings over a fixed degenerate hypergeometric equation; we provide a complete list when the differential Galois group is $SL_2(\mathbb{C})$. By the way, we have a complete list of algebraic solutions for the rank $N = 2$ irregular Garnier systems.

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1. INTRODUCTION

During the last ten years, much have been done about classification of special solutions of isomonodromy equations. Recall how Painlevé and Garnier differential equations arise in the computation of monodromy-preserving deformations of linear equations over the Riemann sphere. Consider the general 2nd order linear differential equation

$$(1) \quad u'' + f(x)u' + g(x)u = 0, \quad (u' = \frac{du}{dx})$$

$$\begin{cases} f(x) = \frac{\theta_{N+1}}{x} + \frac{\theta_{N+2}}{x-1} + \sum_{i=1}^N \frac{\theta_i}{x-t_i} - \sum_{j=1}^N \frac{1}{x-q_j} \\ g(x) = \frac{c_0}{x} + \frac{c_1}{x-1} - \sum_{i=1}^N \frac{H_i}{x-t_i} + \sum_{j=1}^N \frac{p_j}{x-q_j} \end{cases}$$

with $2N + 3$ regular-singular points distributed as follows:

- $N + 3$ essential singular points $x = t_1, \dots, t_N, 0, 1, \infty$ with exponents θ_i , $i = 1, \dots, N + 3$, and
- N apparent singular points $x = q_1, \dots, q_N$ (with trivial local monodromy).

Coefficients c_0, c_1 and H_i can be explicitly determined as rational functions of all other parameters t_i 's, q_i 's, p_i 's, and θ_i 's after imposing the following constraints

- the singular point at $x = \infty$ is regular-singular with exponent θ_{N+3} ,
- the singular points $x = q_1, \dots, q_N$ are apparent.

Then, it follows from the works of Fuchs, Garnier, Okamoto, Kimura that an analytic deformation

$$t \rightarrow (p_1(t), \dots, p_N(t), q_1(t), \dots, q_N(t)) \quad \text{with} \quad t = (t_1, \dots, t_N),$$

of equation (1) is isomonodromic (i.e. with constant monodromy) if, and only if¹ all θ_i 's are fixed and other parameters satisfy the Hamiltonian system

$$(2) \quad \frac{dq_j}{dt_i} = \frac{\partial H_i}{\partial p_j} \quad \text{and} \quad \frac{dp_j}{dt_i} = -\frac{\partial H_i}{\partial q_j} \quad \forall i, j = 1, \dots, N.$$

The system (2) reduces to the Painlevé VI equation for $N = 1$, and to the Garnier system for $N > 1$. It is integrable, in the sense that it admits a local solution for each initial data. These local solutions are expected to be very transcendental in general, and this has been proved by Umemura in the Painlevé case $N = 1$: for any choice of θ_i 's, the general solution cannot be explicitly expressed in terms of solutions of linear differential equations (of any order), non linear

¹The “only if” needs that exponents θ_i 's are not integers.

differential equations of order 1, or algebraic functions. However, for special choices of θ_i 's, there are Riccati solutions or algebraic solutions. The first ones, called “classical”, have been classified by Watanabe; the later ones have been classified mainly by Boalch, and by Lisovsky and Tykhyi (see [7, 35]), after a long period of works by Hitchin, Dubrovin, Mazzocco, Andreev, Kitaev... [21, 16, 40, 15, 1, 2, 29, 30, 31, 32, 3, 4, 5, 6, 46, 47]. In the Garnier case $N > 1$, we expect a similar feature; classical solutions have been classified by Okamoto and Kimura in [43], and by Mazzocco in [41], but the classification of algebraic solutions is still open. Following Cousin, and Heu [11, 12], algebraic solutions are finite branch solutions and come from finite orbits of the Mapping-Class-Group on character varieties, or equivalently representations on the total space of an algebraic deformation of the punctured curve, extending the monodromy representation of (1). On the other hand, the result of Corlette and Simpson [10] shows that such representations are of three different origins:

- degenerate representations, i.e. taking values into a finite, dihedral or reducible group,
- factorization through a representation on a fixed curve,
- arithmetic quotient of a polydisc.

The reader will find a more precise statement in [10]. Let us just mention in the first case the works of Girand [19] and Komyo [33] for deformations of equation (1) with dihedral monodromy, and Cousin and Moussard [13] in the reducible case. Deformations of equation (1) having a finite group are algebraic and provide an algebraic Garnier solution in a systematic way, but computations can be very tedious as it has been in the works of Boalch for the Painlevé case $N = 1$.

In the second case, solutions are said of “pull-back type”: the deformation of equation (1) is given in this case by the pull-back of a fixed differential equation (or instance rigid, hypergeometric, i.e. $N = 0$) by a family of ramified covers $f_t : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ (see section 5.2). This method has been used by Doran, Andreev, Kitaev, Vidunas to construct Painlevé VI algebraic solutions. For Garnier systems $N > 1$, all pull-back solutions with non degenerate linear monodromy have been classified by the first author in [14] (see Proposition 27). However, in the last case, we do not know how to bound the arithmetic data in order to be able to classify. This is not using this trichotomy that algebraic solutions of Painlevé VI equation were found, but by brute force, which seems out of reach even in the case $N = 2$. Recently, Calligaris and Mazzocco [9] gave a partial classification by using confluence of poles in order to exploit the Painlevé classification [35].

So far, we have only considered linear differential equations with regular-singular points, leading to the Painlevé VI equation and Garnier systems. There is a similar approach for linear differential equations with irregular-singular points leading for instance, in the case $N = 1$, to the other Painlevé equations (see section 4). They can be deduced from the regular-singular case by confluence of poles, and whose solutions parametrize isomonodromic and iso-Stokes deformations of linear differential equations with 4 poles counted with multiplicity. In a similar way, we can define irregular Garnier systems and, in the case $N = 2$, they are listed in the papers of Kimura [28] and Kawamuko [26]. For general case $N > 1$, such integrable systems also exist, due to the work of Malgrange (see Heu [20] in the ramified case), and a general formula can be found in the work of Krichever [34]. Again we expect the general solution to be very transcendental, but there are classical and algebraic solutions. In the Painlevé case, a complete classification of these special solutions can be found in [42]; see section 5.5 for the list of algebraic solutions for Painlevé I to V equations. For Garnier systems, classical solutions have been investigated by Suzuki in [45]. For several formal types, Kawamuko and Suzuki listed rational/algebraic solutions in [25, 45, 27]. This is all what is known so far about algebraic solutions of irregular Garnier systems. Our main result is a complete classification, as well as a complete list in the case $N = 2$.

Recently, the second author, together with Pereira and Touzet, proved an irregular version of Corlette-Simpson Theorem in [36]. An immediate consequence is that an algebraic solution of an irregular Garnier system is of one of the two following types

- **classical**: comes from the deformation of a rank 2 differential system with diagonal or dihedral differential Galois group,
- **pull-back**: comes from the deformation obtained by pull-back of a fixed linear differential equation by a family ramified covers.

(see Corollary 15). The main result of the paper is the classification of solutions of pull-back type. Let us describe more precisely the construction. We consider a fixed meromorphic linear differential equation \mathcal{E}_0 on \mathbb{P}^1 , which can be a two-by-two system, a second-order scalar equation, or a rank two vector bundle with a connection. Then, we consider a family of ramified covers $(\phi_t : \mathbb{P}^1 \rightarrow \mathbb{P}^1)_t$ and the family of pull-back $\mathcal{E}_t := \phi_t^* \mathcal{E}_0$. Clearly, the deformation $t \mapsto \mathcal{E}_t$ is isomonodromic and isoStokes, and this gives rise to a partial solution of a (possibly irregular) Garnier system; moreover, if the family $(\phi_t)_t$ is algebraic, we get an algebraic partial solution. Here, partial means that the time variable is a function of t which may not be dominant, and it won't be for general \mathcal{E}_0 and $(\phi_t)_t$. When the dimension of deformation has the right dimension, namely $n - 3$ where n is the number of poles of \mathcal{E}_t counted with multiplicity, then we get a complete algebraic solution. Sections 6, 7, 8 and 9 are devoted to the classification of such solutions. Inspired by the similar classification in the logarithmic case established by the first author in [14], we define the irregular analogues of curve, Teichmüller and moduli spaces, Euler characteristic and Riemann-Hurwitz formula. Then we prove that, assuming \mathcal{E}_0 irregular with differential Galois group not reduced to the diagonal or dihedral group (to avoid classical solutions), \mathcal{E}_0 is of degenerate hypergeometric type (at most 3 poles counted with multiplicity) and the cover degree of ϕ_t is bounded by 6. Finally, the list of solutions is obtained by scattering poles to reduce to the list of [14]. In the pure Garnier case $N > 1$ (i.e. excluding Painlevé equations) we obtain 3 solutions, for Garnier systems of rank $N = 2$ or 3 (see Tables 4 and 5). They all come from pull-back of the degenerate (or ramified) Kummer equation $u'' + \frac{2}{3x}u' - \frac{1}{x}u = 0$ by coverings of degree 4 and 6. Consequently: *irregular Garnier systems of rank $N > 3$ admit only classical algebraic solutions.*

In order to describe our classification result, let us introduce for each singular point of the linear differential equation the following invariants:

- the Poincaré-Katz irregularity index $\kappa \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ which is such that, after putting the linear differential equation $u'' = g(x)u$ into Sturm-Liouville normal form, the coefficient $g(x)$ has a pole of order $2\kappa + 2$;
- the exponent $\theta \in \mathbb{C}$, defined up to a sign, which is the difference of eigenvalues of the residue for the differential equation in matrix form when $\kappa \in \mathbb{Z}_{\geq 0}$, and $\theta = 0$ in the ramified case $\kappa \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$.

Denote by $[\kappa]$ the smallest integer satisfying $\kappa \leq k$. This formal data can be algebraically computed from the differential equation, and is invariant under isomonodromic/isoStokes deformations. In fact, after normalizing the linear differential equation by birational gauge transformation, in order to minimize the number and order of poles, the differential equation is determined by

- its irregular curve, i.e. the base curve equipped with local coordinates up to order $[\kappa]$ at each pole (only the position for simple poles),
- the monodromy data including Stokes matrices.

(see [39, 34, 8, 44], and sections 2.4 and 2.5 for details). With this in hand, for each (global) formal data (and fixed genus g)

$$\begin{pmatrix} \kappa_1 & \cdots & \kappa_n \\ \theta_1 & \cdots & \theta_n \end{pmatrix} \quad \text{where} \quad \begin{cases} \kappa_i \in \mathbb{Z}_{\geq 0} & \Rightarrow \theta_i \in \mathbb{C} \\ \kappa_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0} & \Rightarrow \theta_i = 0 \end{cases}$$

we get a quasi-projective moduli space of linear differential equations (see [23, 24]), and on this moduli space we get a polynomial foliation whose leaves correspond to deformations of the differential equation (in fact of the spectral curve) with constant monodromy data: we call it *isomonodromic foliation*. In the genus $g = 0$ case, this is known as (degenerate) Garnier systems: we get a N dimensional foliation on a $3N$ -dimensional moduli space, where $N = \sum_{i=1}^n [\kappa_i] + 1$ is called the rank of the Garnier system. Leaves with algebraic closure correspond to algebraic solutions of the Garnier system. For $N = 1$, we find all Painlevé equations (see section 4) and algebraic solutions, in the irregular case ($\kappa_i > 0$ for one i at least), are listed in section 5.5.

There are biregular isomorphisms between these foliated moduli spaces, due to the fact that the normalization of a linear differential equation is not unique: after birational gauge transformation, one can shift θ_i 's by integers (and we can change their sign). We obviously classify algebraic solutions up to these isomorphisms.

Theorem 1. *Up to isomorphisms, there are exactly 3 non classical algebraic solutions, for irregular Garnier systems of rank $N > 1$. The list of corresponding formal data is as follows:*

$$\begin{pmatrix} 0 & 1 & 1 \\ \frac{1}{3} & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the rank $N = 2$ case, the list of explicit Garnier systems is provided in [28, 26]. The two first algebraic solutions are as follows under Kimura's notations [28]:

- $H(1, 2, 2; 2)$ with parameters $\varkappa_0 = \varkappa_1 = 0$ and $\varkappa = \frac{2}{9}$ (and $\eta_0 = \eta_1 = 1$): q_1 is implicitly defined by²

$$\left(\frac{q_1(q_1 + 1)}{(q_1 - 1)(q_1 - 2)} \right)^3 = \left(\frac{t_2}{t_1} \right)^2,$$

and other variables are given by

$$q_2 = \frac{q_1 + 1}{2q_1 - 1}, \quad p_1 = -\frac{1}{2} \frac{t_1}{(q_1 - 1)^2} - \frac{1}{2} \frac{t_2}{q_1^2} - \frac{1}{6} \frac{2q_1 - 1}{q_1(q_1 - 1)}$$

$$\text{and} \quad p_2 = -\frac{1}{2} \frac{(2q_1 - 1)^2 t_1}{(q_1 - 2)^2} - \frac{1}{2} \frac{(2q_1 - 1)^2 t_2}{(q_1 + 1)^2} + \frac{1}{2} \frac{2q_1 - 1}{(q_1 - 2)(q_1 + 1)}.$$

- $H(2, 3; 2)$ with parameters $\varkappa_0 = 0$ and $\varkappa_\infty = -\frac{1}{2}$ (and $\eta = 1$): q_1 is a solution of

$$\left(\frac{q_1(3q_1 + 2t_1)}{3} \right)^3 = 2t_2^2$$

and other variables are given by

$$q_2 = -q_1 - \frac{2}{3}t_1, \quad p_1 = \frac{q_1}{4} + \frac{t_1}{2} - \frac{1}{6q_1} - \frac{t_2}{2q_1^2}$$

$$\text{and} \quad p_2 = -\frac{q_1}{4} + \frac{t_1}{3} + \frac{1}{2(3q_1 + 2t_1)} - \frac{9t_2}{2(3q_1 + 2t_1)^2}.$$

²In Kimura's paper, canonical coordinates are denoted λ_i and μ_i instead of variables q_i and p_i respectively, and Hamiltonians K_i instead of H_i

In each case, the solutions $(t_1, t_2) \mapsto (p_1, p_2, q_1, q_2)$ satisfy the Hamiltonian system (2) for explicit Hamiltonians H_i given in section 11.

The second solution coincides with one of the two solutions found by Kawamuko in [27]. For the third solution, we are able to compute the algebraic isomonodromic deformation of the pull-back linear differential equation, but we don't know the explicit form of the Garnier system in that case.

Theorem 2. *Up to isomorphisms, classical algebraic solutions of irregular Garnier systems of rank $N = 2$ occur exactly for the following formal data*

- *infinite discrete family:*

$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

- *two-parameter families*

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \theta_1 & \theta_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \theta_1 & \theta_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \theta_1 & \theta_2 & -\theta_1 - \theta_2 \end{pmatrix}$$

- *one-parameter families*

$$\begin{pmatrix} 0 & 0 & 2 \\ \frac{1}{2} & \frac{1}{2} & \theta \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & \theta \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \theta \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{3}{2} & 0 \\ \frac{1}{2} & 0 & \theta \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 2 \\ 0 & \theta & -\theta \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & \theta & -\theta \end{pmatrix}$$

- *sporadic solutions*

$$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{5}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}.$$

In the first case, there are countably many distinct algebraic solutions of unbounded degree. In any other case, there is exactly one algebraic solution for each formal data.

Kawamuko already discovered the fourth one-parameter family of solutions in [27] and the third sporadic solution in [25]. We provide the first sporadic solution in section 11.

Corollary 3. *In the rank $N = 2$ case, irregular Garnier systems with the following formal data:*

$$\begin{pmatrix} \frac{1}{2} & 2 \\ 0 & \theta \end{pmatrix}, \quad \begin{pmatrix} 1 & \frac{3}{2} \\ \theta & 0 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ \theta \end{pmatrix}, \quad \begin{pmatrix} \frac{7}{2} \\ 0 \end{pmatrix},$$

have no algebraic solution.

Sections 2, 3 and 4 are folklore [38, 39, 34, 8, 44]. The Structure Theorem is presented in section 5. Sections 6, 7, 8 and 9 are devoted to the classification of pull-back type solutions; in these sections, the irregular Euler characteristic is introduced and the irregular Riemann-Hurwitz formula is established. Classical solutions are classified in section 10 for the case $N = 2$. Finally, explicit Hamiltonians are given in section 11 for the above explicit algebraic solutions.

2. LINEAR DIFFERENTIAL EQUATIONS

In this paper, we consider rank 2 **meromorphic connections** on curves. This consists in the data of a rank 2 holomorphic vector bundle $E \rightarrow C$ on a complete smooth curve C , together with a linear connection $\nabla : E \rightarrow E \otimes \Omega^1(D)$ where D is the effective divisor of poles. Precisely, ∇ is a \mathbb{C} -linear map satisfying the Leibniz rule $\nabla(f \cdot s) = df \otimes s + f \otimes \nabla(s)$ for any local function f on C and local section s of E . Locally on C , in trivializing coordinates for E , the connection writes

$$(3) \quad Y \mapsto \nabla(Y) = dY + A \cdot Y \quad \text{with} \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

where $\alpha, \beta, \gamma, \delta$ are local sections of $\Omega^1(D)$ (meromorphic 1-forms). In general, the vector bundle E is not trivial (globally) and we can only give such description in local charts on C .

2.1. Bundle transformations. There are two kinds of transformations we use to consider on connections. First of all, given a **birational bundle transformation** $\phi : E' \dashrightarrow E$ over C , we can define $\nabla' := \phi^*\nabla$ on E' as the unique connection whose horizontal sections are preimages by ϕ of ∇ -horizontal sections at a generic point of C . Locally, ϕ is defined by $Y = M \cdot Y'$ with M meromorphic, $\det(M) \neq 0$, and ∇' is defined by

$$(4) \quad A' = M^{-1}AM + M^{-1}dM.$$

Also, given a rank one meromorphic connection (L, ζ) over C , we can consider the **twist** $(E', \nabla') = (E \otimes L, \nabla \otimes \zeta)$ locally defined by $A' = A + \omega I$ where $\zeta = d + \omega$ and I is the identity matrix. All these transformations can be equivalently considered locally, in the holomorphic/meromorphic setting. Note that they can add, simplify or delete singular points of ∇ . We simply call **bundle transformation** the combination of these two kinds of transformations. They can be used to trivialize the vector bundle, or also to minimize the support and order of poles. In general, we cannot do this simultaneously, except on $C = \mathbb{P}^1$ (see Dekker's Theorem). We will also use local/global biholomorphic/bimeromorphic bundle transformations depending of the nature of ζ and M .

2.2. Local formal data. At the neighborhood of a singular point $x = 0$ on C , up to bimeromorphic bundle transformation and change of coordinate $x \rightarrow \varphi(x)$, we are in one of the following models:

$$(5) \quad \begin{array}{ll} \mathbf{Log} & A = \begin{pmatrix} \frac{\theta}{2} & 0 \\ 0 & -\frac{\theta}{2} \end{pmatrix} \frac{dx}{x}, & \theta \in \mathbb{C} \setminus \mathbb{Z} & \text{logarithmic non resonant} & \begin{pmatrix} 0 \\ \theta \end{pmatrix} \\ \mathbf{Log}^{\text{res}} & A = \begin{pmatrix} \frac{n}{2} & 0 \\ 0 & -\frac{n}{2} \end{pmatrix} \frac{dx}{x}, & n \in \mathbb{Z}_{\geq 0} & \text{logarithmic resonant} & \begin{pmatrix} 0 \\ n \end{pmatrix} \\ \mathbf{Irr}^{\text{un}} & A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \left(\frac{dx}{x^{k+1}} + \theta \frac{dx}{x} \right) + \tilde{A}, & \begin{cases} k \in \mathbb{Z}_{>0} \\ \theta \in \mathbb{C} \end{cases} & \text{irregular unramified case} & \begin{pmatrix} k \\ \theta \end{pmatrix} \\ \mathbf{Irr}^{\text{ram}} & A = \begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix} \frac{dx}{x^{k+1}} + \tilde{A}, & k \in \mathbb{Z}_{>0} & \text{irregular ramified case} & \begin{pmatrix} k - \frac{1}{2} \\ 0 \end{pmatrix} \end{array}$$

where \tilde{A} is holomorphic. The matrix column on the right will be explained later. The order of pole is minimal up to bimeromorphic bundle transformation in all these models, and it is therefore an invariant. We call θ the **exponent**; we set $\theta = n$ in the logarithmic non diagonal case $\mathbf{Log}^{\text{res}}$, and $\theta = 0$ in the irregular ramified case $\mathbf{Irr}^{\text{ram}}$. In fact, only $\cos(2\pi\theta)$ really makes sense up to bimeromorphic bundle transformations, since θ can be shifted by integers under birational bundle transformations; moreover, the variable permutation $y_1 \leftrightarrow y_2$ in Y changes the sign of θ .

We now define the Katz **irregularity index** $\kappa \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ by

- $\kappa = 0$ in the logarithmic case \mathbf{Log} and $\mathbf{Log}^{\text{res}}$,
- $\kappa = k$ in the irregular unramified case \mathbf{Irr}^{un} ,
- $\kappa = k - \frac{1}{2}$ in the irregular ramified case $\mathbf{Irr}^{\text{ram}}$.

The main property of θ and κ is that they are multiplicative under ramified cover:

Proposition 4. *If $\varphi(x) = x^n$, $n \in \mathbb{Z}_{>0}$, then $(\tilde{E}, \tilde{\nabla}) = \varphi^*(E, \nabla)$ has, up to bundle transformation, the following invariants*

$$\tilde{\kappa} = n\kappa \quad \text{and} \quad \tilde{\theta} = n\theta.$$

In particular, the class of irregular singular points is characterized by $\kappa \neq 0$ and is stable under ramified covers.

The proof is straightforward.

Remark 5. *In case of model \mathbf{Log} with exponent $\theta = \frac{p}{q}$ rational, the pole becomes apparent after a ramification $\varphi(x) = x^n$ with $n = mq$ a multiple of q : we get $\tilde{\theta} = mp \in \mathbb{Z}$ and the pole disappears after bundle transformation. In a similar way, in case of model $\mathbf{Irr}^{\mathbf{ram}}$, when $n = 2m$ is even, we get after bundle transformation an unramified pole $\mathbf{Irr}^{\mathbf{un}}$ with $\tilde{\theta} = m$, that can be normalized to $\tilde{\theta} = 0$ after an additional bundle transformation.*

In irregular models $\mathbf{Irr}^{\mathbf{un}}\text{-}\mathbf{Irr}^{\mathbf{ram}}$, we can further kill the holomorphic part \tilde{A} by formal bundle transformation; however it is divergent in general. This already shows that κ and θ are the only formal invariant, i.e. that can be algebraically computed. In the sequel, we denote by $\bar{\kappa} := k$ the smallest integer $\geq \kappa$, i.e. $\bar{\kappa} = \kappa$ or $\kappa + \frac{1}{2}$. There are other invariants, called Stokes matrices (see below), whose computation is very transcendental.

To resume, each singular point is characterized, up to base change and formal bundle transformation, by its irregularity $\kappa \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, and $\theta \in \mathbb{C}$, its exponent. We will call **local formal data** of a differential equation the matrix

$$(6) \quad \begin{pmatrix} \kappa_1 & \cdots & \kappa_n \\ \theta_1 & \cdots & \theta_n \end{pmatrix} \quad \text{where} \quad \begin{cases} \kappa_i \in \mathbb{Z}_{\geq 0} & \Rightarrow \theta_i \in \mathbb{C} \\ \kappa_i \in -\frac{1}{2} + \mathbb{Z}_{>0} & \Rightarrow \theta_i = 0 \end{cases}$$

specifying the formal type at each singular point. In (5), the formal type is indicated on the rightside.

2.3. Normalization.

Proposition 6. *Any (global) connection (E, ∇) is equivalent, up to birational bundle transformation, to a SL_2 -connection (E_0, ∇_0) which locally fits with one of the models \mathbf{Log} , $\mathbf{Log}^{\mathbf{res}}$, $\mathbf{Irr}^{\mathbf{un}}$ or $\mathbf{Irr}^{\mathbf{ram}}$ at any pole along the curve, for a convenient choice of coordinate x and trivialization of E . Such a reduction is not unique, and we can moreover assume, up to additional birational bundle transformation, that formal data satisfies*

$$0 \leq \Re(\theta_i) \leq \frac{1}{2} \quad \text{for } i = 1, \dots, n-1 \quad \text{and} \quad 0 \leq \theta_n < 1.$$

If $\kappa_i, \theta_i \notin \frac{1}{2}\mathbb{Z}$, then this latter reduction is unique (up to permutation of poles).

We call **normalized equation** a connection like (E_0, ∇_0) in the statement.

Proof. The algorithm is as follows. We alternate birational bundle transformations and twists in order to simplify the poles (minimize) and then apply a final twist to get the SL_2 -form. Let us firstly discuss the second step. Given a connection (E, ∇) , then its trace admits a square root

$$(\det(E), \mathrm{tr}(\nabla)) = (L, \zeta)^{\otimes 2}$$

if, and only if, $\deg(E)$ is even. In that case, $(E, \nabla) \otimes (L, \zeta)^{\otimes (-1)}$ is in SL_2 -form.

The first step is done by applying successive elementary transformations $E' \dashrightarrow E$ at each pole p , i.e. of the form

$$Y = M \cdot Y' \quad \text{with} \quad M = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$$

in convenient local coordinate x and trivializations Y, Y' . In a more intrinsic way, if $l \subset E|_p$ denotes the direction spanned by $Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then E' is defined as the locally free sheaf whose sections are those sections of E which, in restriction to $E|_p$, belong to the direction l . We note that E and E'

are canonically isomorphic over the complement $C \setminus p$; moreover, $\deg(E') = \deg(E) + 1$. Now, if ∇ is a connection on E with a pole at p , we say that the elementary transformation is ∇ -adapted if l is an eigendirection of the leading term of the matrix connection; it is equivalent to the fact that the induced connection ∇' on E' has a pole of order not greater than ∇ at p :

$$A = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \frac{dx}{x^{k+1}} \quad \Rightarrow \quad A' = M^{-1}AM + M^{-1}dM = \begin{pmatrix} a(x) & xb(x) \\ \frac{c(x)}{x} & d(x) + x^k \end{pmatrix} \frac{dx}{x^{k+1}}.$$

We now proceed to simplify poles by applying adapted elementary transformations.

- If $A(0)$ is scalar, i.e. of the form $I \frac{dx}{x^{k+1}}$, then it can be killed by a twist, and the order k decreases.
- If $A(0)$ is semi-simple but not scalar, then we can reduce to model **Log**, **Log^{res}** or **Irr^{un}** by biholomorphic bundle equivalence and change of coordinate (with possibly $\theta \in \mathbb{Z}$ in **Log**, in which case the pole can be deleted by birational bundle equivalence) and passing to the SL_2 -form.
- If $A(0)$ is not semi-simple, then we apply a ∇ -adapted elementary transformation; if ∇ and ∇' have same order at p , then one can check that we are in models **Log^{res}** or **Irr^{ram}** up to holomorphic bundle equivalence and change of coordinate.

Finally, after finitely many elementary steps, we arrive at one of the models, up to biholomorphic bundle transformation. At the end, if $\deg(E)$ is even, we obtain the SL_2 -form after a twist. If not, we can apply a ∇ -adapted elementary transformation at one of the poles to shift $\deg(E)$ by $+1$, so that it becomes even, and then normalize by a twist. The first part of the statement is proved.

The lack of unicity comes from

- the possibility of performing an even number of additional ∇ -adapted elementary transformation,
- the possibility of changing the SL_2 -normalization by twisting with a 2-torsion holomorphic connection (this freedom does not occur on $C = \mathbb{P}^1$).

The first operation has the effect to shift exponents $\theta_i \mapsto \theta_i + n_i$, $n_i \in \mathbb{Z}$, with $\sum \theta_i \in 2\mathbb{Z}$ (except in case **Irr^{ram}** where θ is always zero). This does not affect neither the type **Log**, **Log^{res}**, **Irr^{un}** or **Irr^{ram}** of the pole, nor irregularity κ , but only θ . Recall also that θ_i 's are defined up to a sign. We promptly deduce that all θ_i can be normalized with $0 \leq \Re(\theta_i) \leq \frac{1}{2}$ by birational bundle transformation: first put $-\frac{1}{2} \leq \Re(\theta_i) \leq \frac{1}{2}$ by shifting it by integers, and then switch to $-\theta_i$ if necessary. But since we need $\det(E)$ even for the final SL_2 -normalization, then we possibly need to apply an additional elementary transformation, and then shift one of the θ_i 's by one. \square

Remark 7. *If one of the poles has formal data of one of the following types*

$$\begin{pmatrix} \kappa \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

then one can furthermore assume that

$$0 \leq \Re(\theta_i) \leq \frac{1}{2} \quad \text{for all } i = 1, \dots, n.$$

Indeed, it suffices to note that these two types of poles are invariant by some ∇ -adapted elementary transformation. So the last one needed to get $\det(E)$ even can be performed on this pole.

Also note, in case $C = \mathbb{P}^1$ and no more than one pole is of the above type, that the normalization is unique in that case.

2.4. Irregular curve. To encode the global formal structure of the differential equation, we have to take into account the conformal type of the curve C (when genus $g > 0$), and the position of singular points. But it is important to recall here that, to reach the models $\mathbf{Irr}^{\text{un}}\text{-}\mathbf{Irr}^{\text{ram}}$, we really need a change of coordinate $x := \varphi(x)$ in general, in order to normalize the principal part of the differential equation. In fact, it is enough to consider φ polynomial of order $\bar{\kappa}$ (recall $\bar{\kappa} \in \mathbb{Z}_{\geq 0}$ is κ or $\kappa + \frac{1}{2}$); in the logarithmic case $\kappa = 0$, no change is needed. In other words, for irregular singular points, there are $\bar{\kappa}$ other formal invariants when considering only bundle transformations, and they can be killed by base change. For this reason, the conformal type of the base curve C should be enriched with the additional data, at each irregular point t_i , of a $\bar{\kappa}_i$ -jet of coordinate $x_i : (C, t_i) \rightarrow (\mathbb{C}, 0)$ in which the equation can be reduced to the models $\mathbf{Irr}^{\text{un}}\text{-}\mathbf{Irr}^{\text{ram}}$. We call **irregular curve** the data $X := (C, D, \{x_i\})$. The deformation space of the irregular curve locally identifies with $H^0((\Omega^1)^{\otimes 2}(D))$ and has dimension

$$(7) \quad T := 3g - 3 + \deg(D) = 3g - 3 + \sum_{i=1}^n (1 + \bar{\kappa}_i).$$

There is a global deformation space which is a principal bundle over the moduli space $M_{g,n}$ for the punctured curve $(C, |D|)$, whose fiber is the group product of diffeomorphism jets. Once we know the irregular curve and local formal data, it remains to add some extra analytic invariants given by the monodromy representation together with Stokes data. They can only be computed algebraically from the differential equation, i.e. by means of above invariants, when $T \leq 3$. For larger T , one can deform the equation with fixed irregular curve and formal data and these invariants we are going to describe are transcendental functions of the coefficients of the equation.

2.5. Monodromy and Stokes matrices. The **monodromy representation** of the equation is a group morphism

$$\rho_{\nabla} : \pi_1(C \setminus |D|, t_0) \rightarrow \text{GL}_2(\mathbb{C})$$

defined as the monodromy of a local basis of solutions B_0 . More precisely, we can cover $C \setminus |D|$ (where $|D|$ is the support of D , i.e. set of poles) by open sets U_i over each of which the differential equation admits a basis of solutions B_i ; on overlappings $U_i \cap U_j$, we get $B_i = M_{i,j} B_j$ for a transition matrix $M_{i,j} \in \text{GL}_2(\mathbb{C})$. The collection $(M_{i,j})$ defines an element ρ_{∇} of

$$H^1(C \setminus |D|, \text{GL}_2(\mathbb{C})) \simeq \text{Hom}(\pi_1(C \setminus |D|), \text{GL}_2(\mathbb{C}))$$

(point of view of **local systems**). In fact, any loop $\gamma \in \pi_1(C \setminus |D|, x_0)$ based at x_0 can be covered, by compactity, by a finite number of such open sets U_i , where γ crosses successively these open sets following the index order $i = 0, 1, \dots, m$. Then, the analytic continuation of the initial basis of solutions B_0 writes:

$$\underbrace{B_0}_{\text{on } U_0} = \underbrace{M_{0,1} B_1}_{\text{on } U_1} = \underbrace{M_{0,1} M_{1,2} B_2}_{\text{on } U_2} = \dots = \underbrace{M_{0,1} M_{1,2} \dots M_{m,0} B_0}_{(\text{back to } U_0)} =: B_0^\gamma,$$

$$\text{i.e. } B_0^\gamma = M^\gamma B_0 \text{ on } U_0, \quad \text{with } M^\gamma := M_{0,1} M_{1,2} \dots M_{m,0}.$$

We then define the monodromy morphism by setting $\rho_{\nabla}(\gamma) = M^\gamma$. It depends on the choice of the basis B_0 : it is well-defined up to conjugacy by an element $M \in \text{GL}_2(\mathbb{C})$:

$$B_0 = M \tilde{B}_0 \quad \Rightarrow \quad \tilde{M}^\gamma = M^{-1} M^\gamma M \quad \Rightarrow \quad \tilde{\rho}_{\nabla} = M^{-1} \rho_{\nabla} M.$$

For instance, in the first logarithmic model **Log**, one easily calculate

$$B_0(x) = \begin{pmatrix} x^{-\frac{\theta}{2}} & 0 \\ 0 & x^{\frac{\theta}{2}} \end{pmatrix} \quad \text{and} \quad M^\gamma = \begin{pmatrix} e^{i\pi\theta} & 0 \\ 0 & e^{-i\pi\theta} \end{pmatrix}$$

where $\gamma(t) = e^{2i\pi t}$, $t \in [0, 1]$.

For irregular singular points, it is more subtle since there are **Stokes matrices**, playing the role of infinitesimal monodromy. In the unramified case, the model **Irr^{un}** with $\tilde{A} \equiv 0$ has the fundamental basis and monodromy

$$B_0(x) = \begin{pmatrix} x^{-\frac{\theta}{2}} e^{\frac{1}{2kx^k}} & 0 \\ 0 & x^{\frac{\theta}{2}} e^{-\frac{1}{2kx^k}} \end{pmatrix} \quad \text{and} \quad M^\gamma = \begin{pmatrix} e^{i\pi\theta} & 0 \\ 0 & e^{-i\pi\theta} \end{pmatrix}$$

We call it the **formal monodromy**. In the general case $\tilde{A} \neq 0$, equation **Irr^{un}** cannot be reduced to the formal normal form $\tilde{A} \equiv 0$ by holomorphic bundle transformation, but only by (generically divergent) formal bundle transformation; we thus get a formal fundamental basis $\hat{B}(x)$ with monodromy M^γ . The same reduction can also be done holomorphically over each of the following $2\kappa = 2k$ sectors

$$V_0 := \left\{ x ; 0 < |x| < r, \quad \left| \arg(x) - \frac{\pi}{2k} \right| < \frac{\pi}{k} - \epsilon \right\}$$

(where $r, \epsilon > 0$ are small enough) and

$$V_l := \left\{ x ; e^{-i\pi \frac{l}{k}} x \in V_0 \right\}, \quad l = 0, \dots, 2k - 1.$$

Note that these sectors are covering the punctured disc $\{0 < |x| < r\}$. Going back to the initial model **Irr^{un}**, there exists a fundamental matrix B_l over each sector V_l , asymptotic to the formal one $\hat{B}(x)$. Then, we have $B_l = S_l B_{l+1}$ with $S_l \in \text{GL}_2(\mathbb{C})$ is unipotent, upper-triangular when l is even, and lower-triangular for l odd. Therefore, the monodromy around $x = 0$ canonically splits as

$$(8) \quad M^\gamma = \underbrace{\begin{pmatrix} e^{i\pi\theta} & 0 \\ 0 & e^{-i\pi\theta} \end{pmatrix}}_{\text{formal monodromy}} \underbrace{\begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & s_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t_k & 1 \end{pmatrix}}_{\text{Stokes matrices}}$$

and this **Stokes decomposition** is unique up to conjugacy by a diagonal matrix (the choice of $\hat{B}(x)$), so that equivalent decompositions write

$$\begin{pmatrix} e^{i\pi\theta} & 0 \\ 0 & e^{-i\pi\theta} \end{pmatrix} \begin{pmatrix} 1 & cs_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c^{-1}t_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & cs_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c^{-1}t_k & 1 \end{pmatrix}, \quad \text{for } c \in \mathbb{C}^*.$$

In fact, any two models **Irr^{un}** with the same k and λ are equivalent by holomorphic bundle transformation if, and only if, their Stokes decomposition coincide up to diagonal conjugacy. Moreover, any $2k$ -uple $(s_1, \dots, s_k, t_1, \dots, t_k)$ is realisable as Stokes decomposition of a model **Irr^{un}**. In particular, the differential equation is equivalent to the formal model by holomorphic bundle transformation if, and only if, $s_l = t_l = 0$ for $l = 1, \dots, k$. In the **ramified case**, we have a similar story and Stokes decomposition looks like

$$(9) \quad M^\gamma = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\text{formal monodromy}} \underbrace{\begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & s_k \\ 0 & 1 \end{pmatrix}}_{\text{Stokes matrices}}$$

where $k = \kappa + \frac{1}{2}$: there are 2κ matrices (like in the unramified case where irregularity is $\kappa = k$). This decomposition characterizes the analytic equivalence class of the differential equation, and any such Stokes data can be realized. We refer to [38, 39, 34, 8, 44] for more details.

2.6. Differential Galois group. The (differential) Galois group of a normalized differential equation (C, E, ∇) can be computed from the monodromy and Stokes data (see [39]). In the logarithmic case, the Galois group is just the Zariski closure of the monodromy group in $\mathrm{SL}_2(\mathbb{C})$. Recall that algebraic subgroups of $\mathrm{SL}_2(\mathbb{C})$ are in the following list (up conjugacy):

$$(10) \quad \begin{array}{l} \text{infinite} \\ \text{finite} \end{array} \left\{ \begin{array}{l} C_\infty = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} ; \lambda \in \mathbb{C}^* \right\}, \\ D_\infty = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}, ; \lambda \in \mathbb{C}^* \right\} = \langle C_\infty, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle, \\ P_\infty = \left\{ \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} ; \mu \in \mathbb{C} \right\}, \\ T_\infty = \left\{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix} ; \lambda \in \mathbb{C}^*, \mu \in \mathbb{C} \right\}, \\ T_n = \left\{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix} ; \lambda^n = 1, \mu \in \mathbb{C} \right\}, \\ \mathrm{SL}_2(\mathbb{C}) \\ C_n = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} ; \lambda^n = 1 \right\}, & \#n \\ D_n = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}, ; \lambda^n = 1 \right\}, & \#2n \\ \text{tetrahedral} \simeq A_4 \times \mathbb{Z}/2, & \#24 \\ \text{octahedral} \simeq S_4 \times \mathbb{Z}/2, & \#48 \\ \text{icosahedral} \simeq A_5 \times \mathbb{Z}/2, & \#120 \end{array} \right.$$

In the irregular case, we first define the local Galois group as

- at an unramified pole, it is generated in the same basis as (8) by the **exponential torus**

$$C_\infty = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} ; \lambda \in \mathbb{C}^* \right\}$$

together with the Stokes matrices: we obtain C_∞ , T_∞ or $\mathrm{SL}_2(\mathbb{C})$ depending if all Stokes matrices are trivial, if one over two is trivial (i.e. all s_i 's or all t_i 's), or else.

- at a ramified pole, it is generated in the same basis as (9) by the exponential torus, the permutation matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the Stokes matrices: we obtain D_∞ or $\mathrm{SL}_2(\mathbb{C})$ depending if all Stokes matrices are trivial, or not.

The global Galois group is the Zariski closure in $\mathrm{SL}_2(\mathbb{C})$ of all local Galois groups and the global monodromy group. In particular, irregular differential equations have always > 0 dimension due to the exponential torus: we can only have C_∞ , D_∞ , T_∞ or $\mathrm{SL}_2(\mathbb{C})$.

2.7. Link with scalar equations. On $C = \mathbb{P}^1$, meromorphic connections were historically defined by higher order **scalar differential equations**, and it is sometimes more convenient to work with them. Consider the differential equation

$$(11) \quad u'' + f(x)u' + g(x)u = 0,$$

where f, g rational/meromorphic, and $u' = \frac{du}{dx}$. Then, thinking of $(u(x_0), u'(x_0))$ as the space of initial conditions at a generic point x_0 , it is natural to associate the **companion system**

$$(12) \quad \nabla = d + A \quad \text{with} \quad A = \begin{pmatrix} 0 & -g \\ 1 & -f \end{pmatrix} dx.$$

Indeed, identifying the standard basis with $(u(x), u'(x))$, then the connection satisfies $\nabla \cdot u = u'$ and $\nabla \cdot u' = u'' = -fu' - gu$. In the projective coordinate $y = -y_1/y_2$, it induces the Riccati equation $y' + y^2 + fy + g = 0$, and we recover the initial scalar equation by setting $y = u'/u$. Conversely, given a more general system (3), then we can first apply a twist to set $\alpha = 0$ in the matrix A , and then use a gauge transformation of the form

$$(13) \quad M = \begin{pmatrix} 1 & G \\ 0 & F \end{pmatrix}, \quad F \neq 0,$$

(and a twist) to reduce the matrix A in the companion form (12): set $Fdx = \gamma$ and $G = 0$. One can further reduce (12), or accordingly (11), by gauge transformation (13) with $F \equiv 1$, or equivalently setting $u := u/\exp(\int G)$; by this way, we can arrive to the unique SL-form

$$(14) \quad u'' = \frac{s(x)}{2}u, \quad s = f' + \frac{f^2}{2} - g$$

also called ‘‘Sturm-Liouville operator’’. In fact, $s(x)$ is the **Schwarzian derivative** of the quotient of any two independant solutions $u_1(x), u_2(x)$ of the initial equation (11)

$$s(x) := \{\varphi, x\} = \left(\frac{\varphi''}{\varphi'} \right)' - \frac{1}{2} \left(\frac{\varphi''}{\varphi'} \right)^2, \quad \varphi = \frac{u_1}{u_2}.$$

The above reduction to scalar equation depends on the choice of coordinate Y , or more precisely on the choice of the so called ‘‘cyclic vector’’ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. For a general rank 2 meromorphic connection (E, ∇) , the reduction to 2nd order scalar equation depend on the choice of a line subbundle $L \subset E$. It is however important to notice that the irregularity index of an arbitrary connection ∇ (not necessarily of the form **Log-Log^{res}-Irr^{un}-Irr^{ram}**) is directly given by the order of poles of any reduction to scalar equation, namely

$$1 + \kappa = \max\{\text{ord}(f), \text{ord}(g)/2\}.$$

3. CONFLUENT HYPERGEOMETRIC EQUATIONS

On $C = \mathbb{P}^1$, when the polar locus D has degree 3, the connection can be determined, up to bundle equivalence, by its local formal data. After base change (i.e. applying a Moebius transformation in

x -variable), we can reduce to the following list of classical scalar equations.

$$\begin{array}{ccccc}
 \begin{pmatrix} 0 & 0 & 0 \\ \theta_0 & \theta_1 & \theta_\infty \end{pmatrix} & \longrightarrow & \begin{pmatrix} 0 & 1 \\ \theta_0 & \theta_\infty \end{pmatrix} & \longrightarrow & \begin{pmatrix} 2 \\ \theta_\infty \end{pmatrix} \\
 \text{Hypergeometric} & & \text{Kummer} & & \text{Weber} \\
 & & \searrow & & \searrow \\
 & & \begin{pmatrix} 0 & \frac{1}{2} \\ \theta_0 & 0 \end{pmatrix} & \longrightarrow & \begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix} \\
 & & & & \text{Airy}
 \end{array}$$

The **Gauss hypergeometric equation**

$$(15) \quad u'' + \left(\frac{(a+b+1)x-c}{x(x-1)} \right) u' + \frac{ab}{x(x-1)} u = 0$$

has 3 simple poles at $x = 0, 1, \infty$ with respective exponents $\theta_0 = c-1$, $\theta_1 = a+b-c$ and $\theta_\infty = a-b$.

The **Kummer equation** (also called ‘‘confluent hypergeometric’’)

$$(16) \quad u'' + \left(\frac{c}{x} - 1 \right) u' - \frac{a}{x} u = 0$$

has a logarithmic pole at $x = 0$ with exponent $\theta_0 = c$, and an irregular point at $x = \infty$ having irregularity index $\kappa = 1$ and exponent $\theta_\infty = 2a - c$. Its SL-form

$$u'' = \left(\frac{1}{4} + \frac{2a-c}{2x} + \frac{(c-1)^2 - 1}{4x^2} \right) u$$

is also known as Whittaker equation. A particular case is Bessel equation when $\lambda = 0$. The monodromy data can be described as follows:

$$M_0 \cdot M_\infty = I \quad \text{with} \quad M_\infty = \begin{pmatrix} e^{i\pi\theta_\infty} & 0 \\ 0 & e^{-i\pi\theta_\infty} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

$$\text{and} \quad \text{trace}(M_0) = 2 \cos(\pi\theta_0) = \text{trace}(M_\infty) = 2 \cos(\pi\theta_\infty) + e^{i\pi\theta_\infty} st.$$

In particular, one easily check that are equivalent:

$$\begin{aligned}
 \text{the Galois group reducible} &\Leftrightarrow st = 0 \Leftrightarrow \text{trace}(M_0) = \text{trace}(M_\infty) \\
 &\Leftrightarrow \theta_\infty = \pm\theta_0 \pmod{2\mathbb{Z}} \Leftrightarrow a \in \mathbb{Z} \text{ or } c-a \in \mathbb{Z}.
 \end{aligned}$$

To determine when the differential equation is totally reducible (i.e. with diagonal monodromy), we refer to [18]. When irreducible, the Galois group of the normalized equation is $\text{SL}_2(\mathbb{C})$.

The **Weber equation**

$$(17) \quad u'' = (x^2 - 2a)u$$

has a single irregular pole at $x = \infty$ with irregularity index $\kappa = 2$ and exponent $\theta_\infty = 2a - 1$. The monodromy data can be described as follows:

$$M_\infty = \begin{pmatrix} e^{i\pi\theta_\infty} & 0 \\ 0 & e^{-i\pi\theta_\infty} \end{pmatrix} \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t_2 & 1 \end{pmatrix} = I.$$

In particular, one easily check that are equivalent:

$$\text{the Galois group reducible} \Leftrightarrow \theta_\infty \in 2\mathbb{Z} \Leftrightarrow a \in \frac{1}{2} + \mathbb{Z}.$$

When irreducible, the Galois group of the normalized equation is $\mathrm{SL}_2(\mathbb{C})$.

The **degenerate confluent hypergeometric equation**

$$(18) \quad u'' + \frac{c}{x}u' - \frac{1}{x}u = 0$$

has one logarithmic pole at $x = 0$ with exponent $\theta_0 = c$ and a ramified irregular point at $x = \infty$ with irregularity index $\kappa = \frac{1}{2}$. Its SL-form is the “degenerate Whittaker” equation

$$(19) \quad u'' = \left(\frac{1}{x} + \frac{(c-1)^2 - 1}{4x^2} \right) u$$

that will be used in our computations. The monodromy data can be described as follows:

$$M_0 \cdot M_\infty = I \quad \text{with} \quad M_\infty = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -s \end{pmatrix}$$

and $s + 2 \cos(\pi\theta_0) = 0$.

The Galois group of the normalized equation is $\mathrm{SL}_2(\mathbb{C})$, except when $s = 0$ where the Galois group is dihedral. One easily check that the latter case holds if, and only if $c \in \frac{1}{2} + \mathbb{Z}$.

Finally, the most degenerate one, the **Airy equation**

$$(20) \quad u'' = xu$$

has a single irregular pole at $x = \infty$ with irregularity index $\kappa = \frac{3}{2}$. The monodromy data can be described as follows:

$$M_\infty = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = I$$

and the Galois group is $\mathrm{SL}_2(\mathbb{C})$.

4. ISOMONODROMIC DEFORMATIONS

When the degree of the polar divisor $N := \deg(D)$ satisfies $N > 3$, then local formal data fail to determine the differential equation and we have non trivial deformations, even with constant monodromy and Stokes data. We call them isomonodromic deformations.

To define monodromy data in family, we need to introduce the **irregular Teichmüller** space, which is the moduli space of $(C, D, \{x_i\}, \{\alpha_j, \beta_j, \gamma_i\})$ where we take into account a basis (α_j, β_j) for the fundamental group (with some base point t_0), as well as a loop γ_i from the base point to the singular point t_i , ending along $x_i \in \mathbb{R} > 0$, for each irregular singular point. This irregular Teichmüller space is described in [34, 20]. Then we can define monodromy representation in family, as well as Stokes matrices. This makes sense so talk about deformation with constant monodromy, that we call isomonodromic deformations.

Precisely, fix a genus g and a local formal data (6); denote by T the corresponding irregular Teichmüller dimension (7). Then, we can consider the moduli space of triples (X, E, ∇) where

- $X = (C, D, \{x_i\})$ is an irregular curve of genus g , i.e. C is a curve of genus g ,

$$D = \sum_{i=1}^n (1 + \bar{\kappa}_i)[t_i] \quad \text{and} \quad x_i \text{ is a local coordinate at } t_i;$$

- E is a rank 2 vector bundle with trivial determinant $\det(E) = \mathcal{O}_C$;
- $\nabla : E \rightarrow E \otimes \Omega_C^1(D)$ is a meromorphic trace-free connection with polar divisor D ;

- in local coordinate x_i , the connection is defined by **Log-Log^{res}-Irr^{un}-Irr^{ram}** with local formal data (6).

We can consider C belonging to the irregular Teichmüller space if we want to define the monodromy, or belonging to the irregular moduli space of curves (or a smooth finite cover) for explicit computations. The latter one is a quasi-projective variety of dimension $T = 3g - 3 + \deg(D)$, and the former one is its universal cover, with Mapping-Class-Group acting as covering transformations. Once the irregular curve X is fixed, the moduli space $\mathcal{M}(X)$ of (E, ∇) with above constraints is a quasi-projective variety of dimension $2T$, provided the local formal data (6) is generic enough; in special cases, we have to consider semi-stable connections for some choice of weights, or for instance irreducible connections, in order to get such a nice moduli space. The total space \mathcal{M} of connections with fixed genus g and local formal data (6) has therefore dimension $3T$. A point (X, E, ∇) on this moduli space is locally determined by an irregular curve and an irregular monodromy (representation + Stokes matrices). If we fix the curve X and deform the monodromy, we then get the moduli space $\mathcal{M}(X)$. If we now fix the monodromy and deform the curve, then we get the so-called (universal) **isomonodromic deformation**. There is a T -dimensional foliation on the moduli space \mathcal{M} whose leaves correspond to maximal isomonodromic/isoStokes deformations. We call it **isomonodromic foliation**. Isomonodromic leaves are locally parametrized by the irregular Teichmüller space. When considering X living in the moduli space of curves, the $3T$ -dimensional space is algebraic and the isomonodromic foliation is expected to be a polynomial foliation (i.e. defined by polynomial differential equations). This fact is well-known when $C = \mathbb{P}^1$ and D is reduced: isomonodromic differential equations are known as **Garnier systems** of rank T in that case; the particular case $T = 1$ leads to **Painlevé equations** (where \dot{q} denotes $\frac{dq}{dt}$):

$$\begin{aligned}
(21) \quad P_I & : \quad \ddot{q} = 6q^2 + t \\
P_{II}(\alpha) & : \quad \ddot{q} = 2q^3 + tq + \alpha \\
P_{III}(\alpha, \beta, \gamma, \delta) & : \quad \ddot{q} = \frac{(\dot{q})^2}{q} - \frac{\dot{q}}{t} + \frac{\alpha q^2 + \beta}{t} + \gamma q^3 + \frac{\delta}{q} \\
P_{IV}(\alpha, \beta) & : \quad \ddot{q} = \frac{(\dot{q})^2}{2q} + \frac{3}{2}q^3 + 4tq^2 + 2(t^2 - \alpha)q + \frac{\beta}{q} \\
P_V(\alpha, \beta, \gamma, \delta) & : \quad \ddot{q} = \left(\frac{1}{2q} + \frac{1}{q-1}\right) (\dot{q})^2 - \frac{\dot{q}}{t} + \frac{(q-1)^2}{t^2} \left(\alpha q + \frac{\beta}{q}\right) + \frac{\gamma q}{t} + \frac{\delta q(q+1)}{q-1} \\
P_{VI}(\alpha, \beta, \gamma, \delta) & : \quad \ddot{q} = \frac{1}{2} \left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right) (\dot{q})^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right) \dot{q} \\
& \quad + \frac{q(q-1)(q-t)}{t^2(t-1)^2} \left(\alpha + \frac{\beta t}{q^2} + \frac{\gamma(t-1)}{(q-1)^2} + \frac{\delta t(t-1)}{(q-t)^2}\right)
\end{aligned}$$

The corresponding isomonodromy equations are given by Table 1.

Let us recall the case of Painlevé II equation $P_{II}(\alpha)$. The general linear differential equation having a single pole at infinity with local formal data

$$\begin{pmatrix} 3 \\ 1 - 2\alpha \end{pmatrix}$$

can be normalized in scalar SL_2 -form (i.e. Sturm-Liouville operator)

$$(22) \quad u'' = \left(x^4 + tx^2 + 2\alpha x + 2H_{II} + \frac{3}{4(x-q)^2} - \frac{p}{x-q} \right) u$$

(coefficients of x^4 and x^3 have been normalized to 1 and 0 by an affine transformation in x -variable). Here, p, q are accessory parameters (i.e. initial conditions for the Painlevé equation) and

$$(23) \quad H_{II} = \frac{1}{2} (p^2 - q^4 - tq^2 - 2\alpha q)$$

TABLE 1. Painlevé equations as isomonodromy equations

Local formal data	Isomonodromy equation
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \theta_0 & \theta_1 & \theta_t & \theta_\infty \end{pmatrix}$	$P_{VI} \left(\frac{(\theta_\infty - 1)^2}{2}, -\frac{(\theta_0)^2}{2}, \frac{(\theta_1)^2}{2}, \frac{1 - (\theta_t)^2}{2} \right)$
$\begin{pmatrix} 0 & 1 & 0 \\ \theta_0 & \theta_1 & \theta_\infty \end{pmatrix}$	$P_V \left(\frac{(\theta_\infty)^2}{2}, -\frac{(\theta_0 + 1)^2}{2}, \theta_1, -\frac{1}{2} \right)$
$\begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \theta_0 & 0 & \theta_\infty \end{pmatrix}$	$P_V \left(\frac{(\theta_\infty)^2}{2}, -\frac{(\theta_0 + 1)^2}{2}, -2, 0 \right) \sim$ $P_{III} (-4(\theta_0 - \theta_\infty - 1), -4(\theta_0 + \theta_\infty), 4, -4)$
$\begin{pmatrix} 0 & 2 \\ \theta_0 & \theta_\infty \end{pmatrix}$	$P_{IV} (\theta_\infty, -2(\theta_0 + 1)^2)$
$\begin{pmatrix} 0 & \frac{3}{2} \\ \theta_0 & 0 \end{pmatrix}$	$P_{II} (\theta_0 - \frac{1}{2})$
$\begin{pmatrix} 1 & 1 \\ \theta_0 & \theta_\infty \end{pmatrix}$	$P_{III} (4\theta_\infty, -4\theta_0, 4, -4)$
$\begin{pmatrix} 1 & \frac{1}{2} \\ \theta_0 & 0 \end{pmatrix}$	$P_{III} (-8, -4\theta_0, 0, -4)$
$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$	$P_{III} (4, -4, 0, 0)$
$\begin{pmatrix} 3 \\ \theta_\infty \end{pmatrix}$	$P_{II} \left(\frac{1 - \theta_\infty}{2} \right)$
$\begin{pmatrix} \frac{5}{2} \\ 0 \end{pmatrix}$	P_I

is determined so that the pole $x = q$ is apparent. After getting rid of apparent singular point, we can transform equation (22) into the Riccati equation

$$(24) \quad y' + (x - q)y^2 + (2x^2 - 2q^2 + 2p)y + (2p - 2q^2 - t)x + (1 - 2\alpha + 2pq - 2q^3 - qt) = 0$$

or equivalently into the SL_2 -system (setting $y = y_1/y_2$)

$$(25) \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} -x^2 + q^2 - p & 2\alpha - 1 - 2pq + 2q^3 + qt \\ x - q & x^2 - q^2 + p \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

One can check, by direct computation, that the singularity at infinity (in variable $z = 1/x$) can be normalized by holomorphic gauge transformation to

$$(26) \quad dY + AY = 0 \quad \text{with} \quad A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \left(2\frac{dz}{z^4} + t\frac{dz}{z^2} + (1 - 2\alpha)\frac{dz}{z} + \text{holomorphic} \right).$$

We retrieve, in the principal part, the fact that the first two coefficients have been normalized by affine transformation in x , the third coefficient stands for the time variable, and the fourth one, for the local formal data θ_∞ . A deformation $t \mapsto (p(t), q(t))$ of equation (22) is isomonodromic (in fact, iso-Stokes in that case) if, and only if, it satisfies the Hamiltonian equations

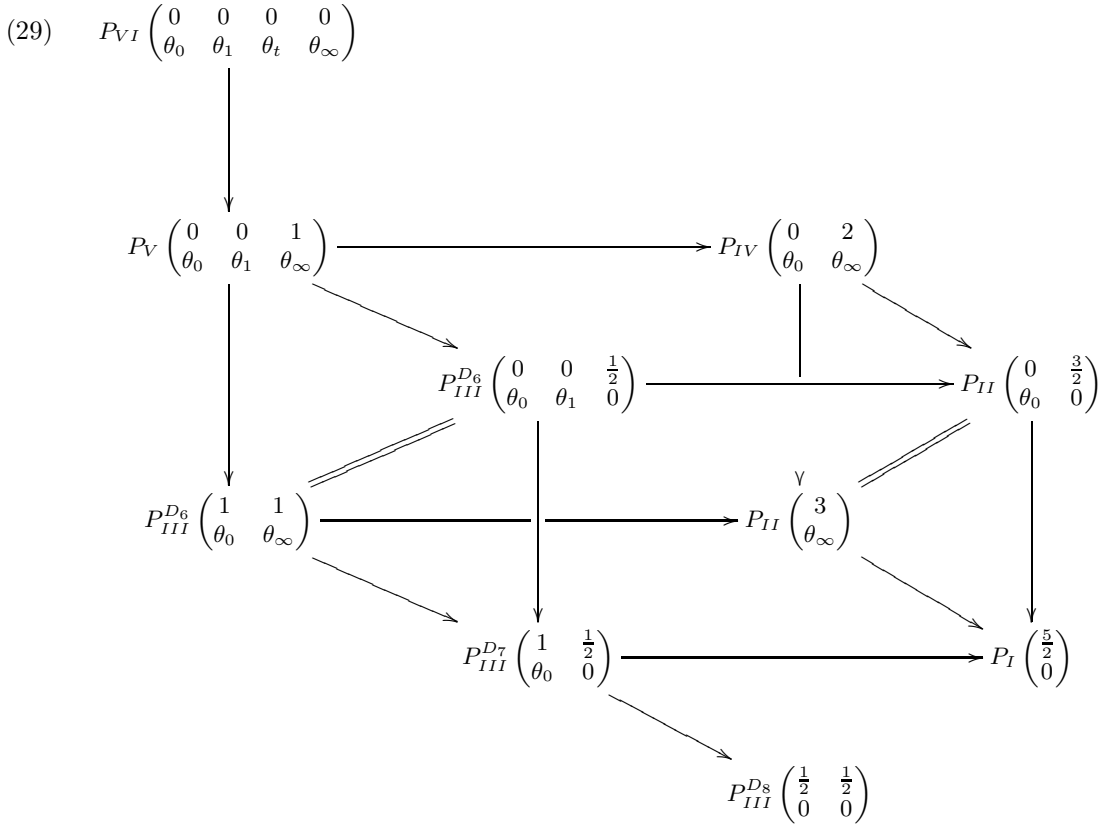
$$(27) \quad \frac{dp}{dt} = -\frac{\partial H_{II}}{\partial q} \quad \text{and} \quad \frac{dq}{dt} = \frac{\partial H_{II}}{\partial p}.$$

Equivalently, the corresponding curve in (t, p, q) -variables is in the kernel of the 2-form

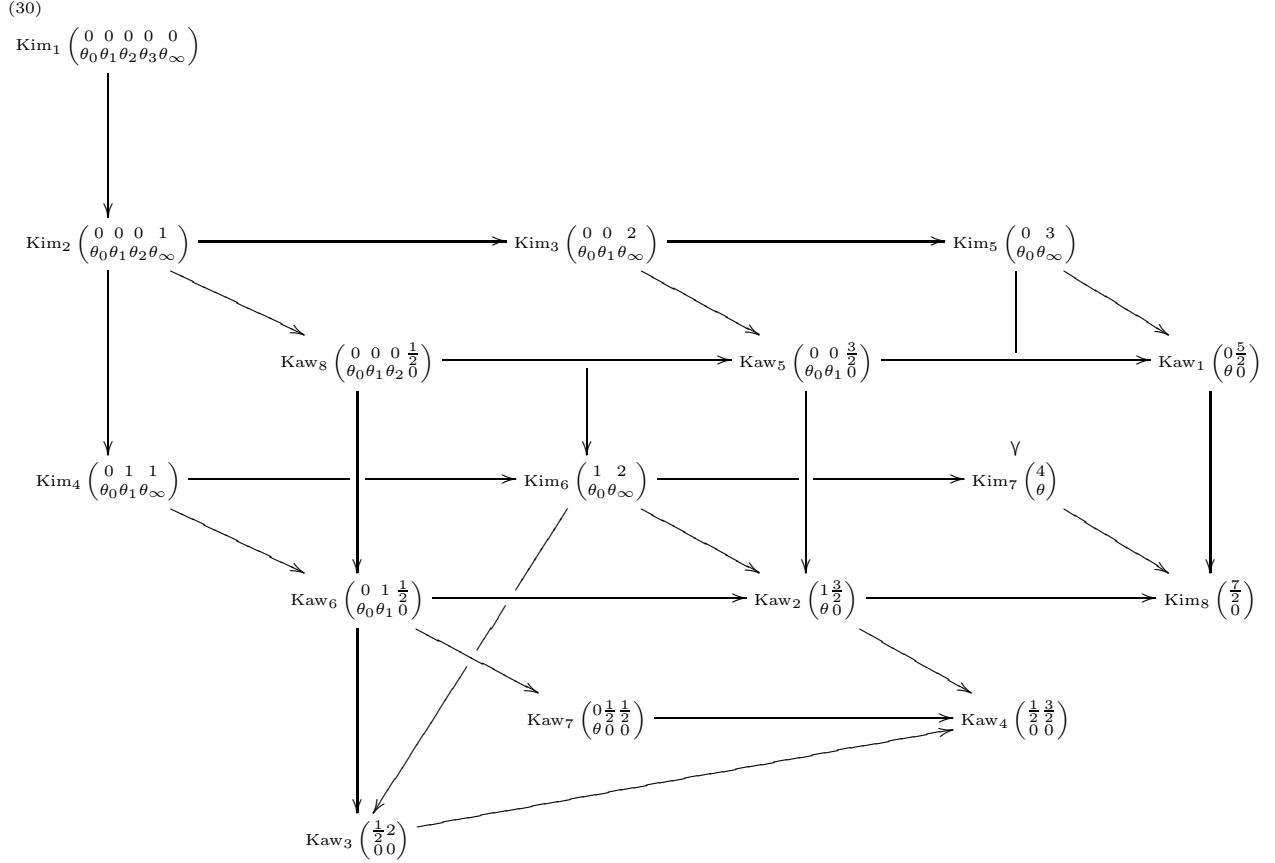
$$(28) \quad \omega = dp \wedge dq + dt \wedge dH_{II}.$$

Using the second equation (27), we can express p in terms of q and \dot{q} ; substituting in the first equation, we deduce that $q(t)$ is a solution of Painlevé II equation.

In (29) we see the list of formal data of Painlevé type, and how they confluence to each other.



In the case $T = 2$, the degeneration diagram is given in picture (30); the unramified part (and also the case $(7/2)$) of the list is treated by Kimura in [28], while the ramified part is studied by Kawamuko in [26]. We named $\text{Kim}_1, \dots, \text{Kim}_8, \text{Kaw}_1, \dots, \text{Kaw}_8$ these equations following the order of appearance in these two papers.



It is important to notice that isomonodromy condition is equivalent to the fact that the deformation of equation is induced by a flat meromorphic connection on the universal irregular curve, i.e. the total space of the family of irregular curve.

5. ALGEBRAIC SOLUTIONS OF IRREGULAR GARNIER SYSTEMS: EXAMPLES AND STRUCTURE

There are several methods to construct algebraic solutions of classical Garnier systems (logarithmic case), see [1, 7, 9, 14, 15, 16, 35] and references therein. In the irregular case, let us describe two methods to produce algebraic isomonodromic deformations.

5.1. Classical solutions: the Galois group is C_∞ or D_∞ . The rough idea is as follows. Since the differential Galois group of a linear differential equation can be determined from its coefficients by algebraic operations, it follows that iso-Galois deformations are of algebraic nature: we have an algebraic stratification of each moduli space

$$\mathcal{M}_g \begin{pmatrix} \kappa_1 & \cdots & \kappa_n \\ \theta_1 & \cdots & \theta_n \end{pmatrix}$$

as defined in section 4 where strata are defined in term of the Galois group. In fact, this is not exactly true since there might be infinitely many strata corresponding to finite groups in the

dihedral case. However, the locus of each finite group is algebraic and coincides with a finite number of isomonodromic leaves. It follows that the leaf associated to a finite linear group is algebraic. This has been extensively used in [21, 3, 4, 5, 6, 7] and references therein in the logarithmic case; this does not occur in the irregular case since the Galois group is never finite in that case. However, in a similar way, the locus of C_∞ or D_∞ is algebraic and is a finite union of isomonodromy leaves in many cases when $C = \mathbb{P}^1$.

Let us first explain the diagonal case C_∞ . The poles can only be of type **Log** and **Irr^{un}**, and in this latter case, the Galois group C_∞ coincides with all local exponential torii at irregular singular points and all Stokes matrices are trivial. The two eigendirections of the Galois group correspond to two ∇ -invariant line bundles $L, L^{-1} \subset E$ of the vector bundle for the normalized equation, and we have $E = L \oplus L^{-1}$. The connection ∇ restricts as meromorphic connections on $(L, \nabla|_L)$ and $(L, \nabla|_L)^{\otimes(-1)}$ and, at each pole t_i , the corresponding residues are $\pm \frac{\theta_i}{2}$: they are opposite for L and L^{-1} . Fuchs relation yields:

$$\sum_{i=1} n \epsilon_k \theta_k \in \mathbb{Z}, \quad \epsilon_i = \pm 1.$$

There are finitely many such relations for each formal data, and given one relation, the connection $(L, \nabla|_L)$ (and therefore (E, ∇)) can be uniquely determined by Mittag-Leffler's Theorem from the data of the irregular curve ($C = \mathbb{P}^1 +$ principal parts).

Something similar occur when the Galois group is D_∞ . The poles must be of type **Log**, **Irr^{un}** or **Irr^{ram}** and the normal subgroup $C_\infty \subset D_\infty$ must coincide with all local exponential torii at irregular singular points. In that case, Stokes matrices are trivial and, if $C = \mathbb{P}^1$, the global monodromy group is generated by matrices $M_i \in D_\infty$ (local monodromy at t_i) satisfying

$$M_1 \cdots M_n = I.$$

Precisely, for a normalized equation, we have:

$$\begin{aligned} M_i \text{ diagonal} &\Rightarrow \kappa_i \in \mathbb{Z}_{\geq 0} \\ M_i \text{ anti-diagonal} &\Rightarrow \begin{pmatrix} \kappa_i \\ \theta_i \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \text{ or } \begin{pmatrix} \frac{1}{2} + n \\ 0 \end{pmatrix}, \quad n \geq 0. \end{aligned}$$

The number of anti-diagonal matrices among M_1, \dots, M_n is even.

Remark 8. *One can check that, if $C = \mathbb{P}^1$ and there are only two anti-diagonal matrices among M_1, \dots, M_n , say M_{n-1} and M_n , then again the (algebraic) locus of D_∞ consists in a finite number of isomonodromic leaves. Indeed, the monodromy representation is determined by the M_i 's; for $i = 1, \dots, n-2$, we have*

$$M_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix} \quad \text{with } \lambda_i = e^{\pm \sqrt{-1} \pi \theta_i};$$

we can rescale $M_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ by diagonal conjugacy, and M_{n-1} is determined by the relation. The connection is also determined by the irregular curve in this case (principal parts).

5.2. Pull-back algebraic solutions. Another way to construct algebraic isomonodromic deformations (see [15, 1, 29, 2, 30, 31, 32, 46, 47, 14, 42]) is to fix a differential equation (C_0, E_0, ∇_0) and consider an algebraic family $\phi_t : C_t \rightarrow C_0$ of ramified covers, where $t \in P$ a projective variety. The pull-back $(C_t, \phi_t^* E_0, \phi_t^* \nabla_0)$ provides an algebraic isomonodromic deformation. Indeed, it is induced by the flat connection $(\Phi^* E_0, \Phi^* \nabla_0)$ over the total space $t : \mathcal{C} = \sqcup_t C_t \rightarrow P$ defined by pull-back via the total ramified cover $\Phi := (\phi_t, t) : \mathcal{C} \rightarrow C_0 \times P$.

5.3. Garnier algebraic solutions and apparent singular points. So far, we have neglected to consider apparent singular points in isomonodromic deformations since we were dealing with normalized equations. However, Garnier systems are derived from isomonodromic deformations of scalar differential equations (11) with N poles (counted with multiplicity) and $N - 3$ apparent singular points (see [28, 26]). This can be reinterpreted as the data of a connection ∇ on the bundle $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ with N poles, and we recover the scalar equation by taking $\mathcal{O}_{\mathbb{P}^1} \subset E$ as a cyclic vector. Equivalently, and closer to our point of view, one can consider a SL_2 -connection (E, ∇) with $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$, and the cyclic vector (with the right number of apparent singular points) is chosen to be the constant line bundle $L \subset E$ fitting with one of the eigendirection over one pole (typically at ∞ when normalizing the position of poles on \mathbb{P}^1). Here, the poles are of type **Log**, **Log^{res}**, **Irr^{un}** or **Irr^{ram}**, but in the case **Log**, we also allow $\theta \in \mathbb{Z}$ as a degenerate case of **Log^{res}** when the singular point becomes apparent. These considerations lead us to the following facts.

Proposition 9. *Let $(E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}, \nabla)$ be an irregular SL_2 -connection, and $L \subset E$ be a cyclic vector like above. The (local) isomonodromic deformation of (E, ∇) provides a (local) solution of the Garnier system if the line bundle L is not ∇ -invariant. If $\mathrm{Gal}(E, \nabla) = C_\infty$ and there is no apparent singular point, then L is ∇ -invariant and the deformation fails to provide a solution of the corresponding Garnier system.*

Proof. The condition that L is not ∇ -invariant is equivalent to the fact that it defines a cyclic vector, which allow to define the scalar system with N poles and $N - 3$ apparent singular points. This condition is preserved under isomonodromic deformations and the deformation of the scalar system leads to a Garnier solution. In the case (E, ∇) is normalized (i.e. without apparent singular points) and the Galois group is C_∞ , then $E = L_0 \oplus L_0^{-1}$ with L_0, L_0^{-1} two ∇ -invariant line bundles. Automatically, we have $L_0 \simeq \mathcal{O}_{\mathbb{P}^1}$ (a constant line bundle), and L_0 (resp. L_0^{-1}) coincide with an eigendirection over each pole. It follows that L_0 or L_0^{-1} coincides with L everywhere, and is L therefore ∇ -invariant. \square

In the presence of **apparent singular points**, the condition for a deformation to be isomonodromic and giving rise to a Garnier solution is more subtle. Let (E_0, ∇_0) be a meromorphic SL_2 -connection, which is normalized except at an apparent singular point $p_0 \in C$: it is of type **Log** with $\theta = n \in \mathbb{Z}_{>0}$. Then after a birational bundle transformation $\phi : E_0 \dashrightarrow E'_0$, we can erase the singular point; moreover, we can assume ϕ supported by p_0 , i.e. inducing a biholomorphic bundle transformation outside of p_0 . Generic local sections of E_0 at p_0 are transformed into sections of E'_0 , all tangent at the order $n - 1$ to a given ∇ -invariant analytic subbundle $L_0 \subset E'_0$. Then we have

Proposition 10. *Under notations above, given a a deformation of (E_0, ∇_0) induced by a flat connection (E, ∇) , are equivalent*

- (E, ∇) is logarithmic near p ,
- the deformation p of the singular point p_0 remains apparent and the corresponding deformation of line bundle $L_0 \subset E'_0$ is induced by a local analytic ∇ -invariant line bundle $L \subset E'$.

In this case we say that the deformation is isomonodromic.

Consequently, we will encode the data of such an equation (E_0, ∇_0) by the data of the normalized equation (E'_0, ∇'_0) together with the data of p_0 and the local ∇'_0 -invariant analytic line bundle L_0 , or better by its fiber $l_0 := L_0|_{p_0} \subset E'_0|_{p_0}$ (initial condition) that is usually called **parabolic data**. This method to deal with apparent singular points is used in [22].

Remark 11. *The case $n = 0$ also occur in solutions of Garnier systems as degenerate case of singular points of type $\mathbf{Log}^{\text{res}}$. Garnier solutions can also be interpreted as before as deformation of a normalized equation (E_0, ∇_0) together with the parabolic data $(p_0 \in C, l_0 \in E_0|_{p_0})$ and there are as many Garnier solutions as choices of initial condition l_0 .*

If we have several apparent singular points, the condition above must be imposed for each of them to get an isomonodromic solution, and therefore giving rise to Garnier solutions. By the way, we can have Garnier solutions corresponding to equation with Galois group C_∞ in the presence of apparent singular points as we will see for Painlevé IV equation (second line of Table 2).

Proposition 12. *The isomonodromic deformation of an irregular normalized connection (E_0, ∇_0) with an apparent singular point (p_0, l_0) (notations above) is algebraic if and only if*

- *the deformation of (E_0, ∇_0) is algebraic, with Galois group C_∞ or D_∞ ,*
- *the local ∇_0 -invariant analytic line bundle L_0 defined by l_0 has algebraic closure, i.e. corresponds to one of the two ∇_0 invariant line bundles in diagonal case C_∞ , or of the 2-multivalued line bundle in dihedral case D_∞ .*

Proof. The global deformation space of the irregular curve including the apparent singular point may be viewed as a fiber bundle over the deformation space of the strict irregular curve, with fiber C corresponding to the position of the apparent singular point for a fixed normalized equation. Clearly, the deformation of (E_0, ∇_0) is algebraic if and only if it is algebraic in each of these two directions. But to be algebraic along C -fibers implies, due to Proposition 10, that the local analytic line bundle L_0 have algebraic closure. And this implies that the Galois group must be C_∞ or D_∞ (the third possibility SL_2 in the irregular case is excluded here). \square

Proposition 12 can immediately be generalized to the case of several apparent singular points.

Remark 13. *In the dihedral case D_∞ , an algebraic solution of a Garnier system with apparent singular point, like in Proposition 12, always arises in family, as limit of algebraic solutions with linear monodromy D_∞ and no apparent singular points like in section 5.1. Indeed, from the monodromy side, this can be written as*

$$M_1 \cdots M_n = I \text{ with } M_{n-2} = \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & 0 \end{pmatrix}, M_{n-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \underbrace{M_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{apparent}}$$

$$\rightsquigarrow \tilde{M}_{n-2} = \begin{pmatrix} 0 & -\lambda_1 \\ \lambda_1^{-1} & 0 \end{pmatrix}, \tilde{M}_{n-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \underbrace{\tilde{M}_n = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}}_{\text{non apparent}}$$

where $\lambda_1 \lambda_2 = \lambda$. For the differential equation, the apparent point is replaced by a non apparent logarithmic singular point.

5.4. Structure theorem. The key result for our classification is the:

Theorem 14 ([36]). *Let (E, ∇) be a flat meromorphic \mathfrak{sl}_2 -connection on a projective manifold X . Then at least one of the following assertions holds true.*

- (1) *Maybe after passing to a (possibly ramified) two-fold cover $f : X' \rightarrow X$, the connection (E, ∇) is equivalent to a diagonal connection on the trivial bundle:*

$$\nabla = d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$$

with ω a rational closed 1-form on X .

- (2) There exists a rational map $\Phi : X \dashrightarrow C$ to a curve and a meromorphic connection (E_0, ∇_0) on C such that (E, ∇) is equivalent to $\Phi^*(E_0, \nabla_0)$ by bundle transformation.

A direct consequence is the

Corollary 15. *Any algebraic solution of an irregular Garnier system comes from*

- (1) either the deformation pull-back from a fixed connection (C_0, E_0, ∇_0) with $\mathrm{SL}_2(\mathbb{C})$ Galois group via an algebraic family of ramified covers,
- (2) or the deformation of a connection having Galois group D_∞ ,
- (3) or the deformation of a connection having Galois group C_∞ with apparent singular point(s) like in Proposition 12.

This result has been recently proved in the Painlevé case in [42] by checking *a posteriori* the known list of algebraic solutions. We expect to have a similar result for isomonodromy equations for curves of genus $g > 0$. However, it is still not known if these equations are polynomial in such a generality, so that the question of algebraicity of solutions does not make sense so far.

To prove the Corollary, we just notice that any solution (algebraic or not) comes from an isomonodromic deformation, or equivalently a flat connection over the total space \mathcal{C} of the deformation curve. If the solution is algebraic, then \mathcal{C} is algebraic, as well as the flat connection. We can therefore apply the above Theorem, and deduce the Corollary.

5.5. Algebraic solutions of Painlevé I-V equations. Table 2 provides the list of algebraic solutions for irregular Painlevé equations up to symmetries.

The name of solutions follows from [42]. Here “rat” and “alg” stand for “rational” and “algebraic” while “Lag” and “Her” stand for “Laguerre” and “Hermite” polynomials; also notation P_{34} refers to Gambier’s list. The P_V equation, when $\delta = 0$, is equivalent to P_{III} and solutions P_V -alg and $P_{III}^{D_6}$ -alg are therefore equivalent, but with different kind of linear local data, and this is why we keep the two solutions in the list. For the same reason, we might consider P_{34} -rat and P_{III} -rat as different solutions.

Remark 16. *In table 2, apart pull-back solutions listed in tables 4 and 5, there are two more pull-back solutions that correspond to dihedral Galois groups.*

6. RAMIFIED COVERS AND DIFFERENTIAL EQUATIONS

Given a ramified cover $\phi : C \rightarrow C_0$ of degree d , and given a point $c_0 \in C_0$, we can associate the pull-back divisor $\phi^*[c_0] = m_1[t_1] + \cdots + m_s[t_s]$ with $t_i \in C$ pair-wise distinct for $i = 1, \dots, s$. This means that $\{t_1, \dots, t_s\}$ is the fiber of ϕ , and through convenient local coordinates x_i near t_i , $\phi(x_i) = (x_i)^{m_i}$; we have $m_1 + \cdots + m_s = d$, and for generic t_0 , we have all $m_i = 1$. We can therefore associate to ϕ its **passport**

$$(31) \quad \left(\begin{array}{c} [m_{1,1}] \\ \vdots \\ [m_{1,s_1}] \end{array} \quad \cdots \quad \begin{array}{c} [m_{\nu,1}] \\ \vdots \\ [m_{\nu,s_\nu}] \end{array} \right), \quad \sum_{l=1}^{s_k} m_{k,l} = d \text{ for } k = 1, \dots, \nu$$

It is the data, for each critical value $c_1, \dots, c_\nu \in C_0$ of ϕ , of the corresponding partition $m_{k,1} + \cdots + m_{k,s_k} = d$: there are s_k points in the fiber $\phi^{-1}(c_k)$ with multiplicities $m_{k,l}$, $l = 1, \dots, s_k$. The

TABLE 2. Algebraic solutions of irregular Painlevé equations

Isomonodromy equation	solution	name	Local formal data	Galois group	pull-back ?	apparent pole ?
$P_V \left(\frac{\theta^2}{2}, -\frac{\theta^2}{2}, 0, -\frac{1}{2} \right)$	$q = -1$	P_V -rat	$\begin{pmatrix} 0 & 1 & 0 \\ \theta - 1 & 0 & \theta \end{pmatrix}$	SL_2	yes	no
$P_V \left(\frac{\theta^2}{2}, -\frac{1}{2}, \theta, -\frac{1}{2} \right)$	$q = \frac{t}{\theta} + 1$	P_V -Lag	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & \theta & \theta \end{pmatrix}$	C_∞	no	yes
$P_V \left(\frac{\theta^2}{2}, -\frac{1}{8}, -2, 0 \right)$	$q = 2\frac{\sqrt{t}}{\theta} + 1$	P_V -alg	$\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \theta \end{pmatrix}$	D_∞	no	no
$P_{IV} \left(0, -\frac{2}{9} \right)$	$q = -2t/3$	P_{IV} -rat	$\begin{pmatrix} 0 & 2 \\ -\frac{2}{3} & 0 \end{pmatrix}$	SL_2	yes	no
$P_{IV} \left(0, -2 \right)$	$q = -2t$	P_{IV} -Her	$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$	C_∞	yes	yes
$P_{III} \left(4\theta, -4\theta, 4, -4 \right)$	$q = \sqrt{t}$	$P_{III}^{D_6}$ -alg	$\begin{pmatrix} 1 & 1 \\ \theta & \theta \end{pmatrix}$	SL_2	yes	no
$P_{III} \left(4, -4, 0, 0 \right)$	$q = \sqrt{t}$	$P_{III}^{D_8}$ -alg	$\begin{pmatrix} \frac{1}{2} & 1 \\ 0 & 0 \end{pmatrix}$	D_∞	yes	no
$P_{III} \left(-8, 0, 0, -4 \right)$	$q = \left(-\frac{t}{2}\right)^{1/3}$	$P_{III}^{D_7}$ -alg	$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$	SL_2	yes	no
$P_{II} \left(0 \right)$	$q = 0$	P_{34} -rat	$\begin{pmatrix} 0 & \frac{3}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$	D_∞	yes	no
		P_{II} -rat	$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$	SL_2	yes	no

Riemann-Hurwitz formula writes

$$(32) \quad \underbrace{2 - 2g(C)}_{\chi(C)} = d \underbrace{(2 - 2g(C_0))}_{\chi(C_0)} - R \quad \text{where} \quad R = \text{total ramification} := \sum_{k=1}^n (d - s_k).$$

Given a differential equation (E_0, ∇_0) with polar divisor D on C_0 , we can consider the pull-back $(E, \nabla) := \phi^*(E_0, \nabla_0)$. It is easy to deduce the local formal data of (E, ∇) from that one of (E_0, ∇_0) and the passport. Precisely, let $p \in C$ be a point of multiplicity m for ϕ (ramification $r = m - 1$); then the local formal datas of (E_0, ∇_0) at $\phi(p)$ and (E, ∇) at p are related by:

$$(\kappa, \theta) = (m\kappa_0, m\theta_0)$$

except when $\kappa = 0$, $\theta_0 \in \mathbb{Q} \setminus \mathbb{Z}$ and $m\theta_0 \in \mathbb{Z}$ where the singular point becomes apparent, i.e. can be deleted by bundle transformation; in that latter case, we indeed delete the pole. We deduce, for the respective order of poles, that

$$\text{ord}_p(\nabla) \leq m \cdot \text{ord}_{\phi(p)}(\nabla_0) + 1 - m$$

with strict inequality in the special case above, and ramified case $\kappa_0 \notin \mathbb{Z}$ and $m > 1$. On the other hand, when $\phi(p)$ is not a pole for ∇_0 , then p is also non singular for ∇ .

In the sequel, we still fix the differential equation (C_0, E_0, ∇_0) and we want to deform the ramified cover and pull-back equation

$$\phi_t : C_t \rightarrow C_0, \quad (E_t, \nabla_t) := \phi_t^*(E_0, \nabla_0).$$

We consider an irreducible algebraic family, parametrized by say $P \ni t$ (irreducible and projective), and there is a Zariski open subset $U \subset P$ where the passport of ϕ_t is locally constant, as well as the number B of its critical values outside of the (fixed) polar locus of ∇_0 . Note that B bounds the dimension of P , and maybe switching to a larger family, i.e. with a larger parameter space P , we can assume $B = \dim(P)$. The deformation $t \mapsto (C_t, E_t, \nabla_t)$, being automatically isomonodromic, locally factors through the universal isomonodromic deformation (see [20]). In general, it defines a projective subvariety contained in the transcendental isomonodromy leaf \mathcal{L} of the corresponding isomonodromy foliation \mathcal{F} . But if $P \rightarrow \mathcal{L}$ is locally dominant, then it is globally dominant and the entire leaf \mathcal{L} itself is algebraic, giving rise to an algebraic solution of the corresponding isomonodromy equation. A necessary condition for this is that the dimension T of \mathcal{L} is bounded by the dimension B of P . We will call **admissible** the data of a differential equation (C_0, E_0, ∇_0) and a passport (31) such that $B \geq T$. For the sequel, it is convenient to add in the passport all trivial fibers

$$\underbrace{1 + \cdots + 1}_{d \text{ times}}$$

appearing over poles so that we may assume that $\nu = n + B$ in (31) with entries $k = 1, \dots, n$ corresponding to fibers over the poles of ∇_0 , and entries $k = n + 1, \dots, \nu$ corresponding to free critical points that are deformed along the family. Our aim now is to classify those admissible data such that the pull-back differential equation has irregular Teichmüller dimension $T \leq B$. We will see in the next section that this inequality gives very strong constraints.

7. SCATTERING RAMIFICATIONS

Suppose we are given a normalized differential equation (C_0, E_0, ∇_0) with poles $p_1, \dots, p_n \in C_0$, and local formal data $(\kappa_i, \theta_i)_{i=1, \dots, n}$ like (6). Suppose we are given a ramified covering $\phi : C \rightarrow C_0$ with passport $(d = m_{k,1} + \cdots + m_{k,s_k})_{k=1, \dots, \nu}$ over $c_1, \dots, c_\nu \in C_0$ like (31) where

- $(m_{k,l})_l$ for $k = 1, \dots, n$ correspond to the multiplicities of ϕ along fibers over the poles of ∇_0 ($c_k = p_k$), some of them being possibly unbranched, i.e. $m_{k,l} = 1$ for all l ;
- $(m_{k,l})_l$ for $k = n + 1, \dots, n + b$ correspond to fibers of ϕ over non singular points of ∇_0 , all of which are branching, i.e. $m_{k,l} > 1$ for at least one l .

Denote by $R_k := d - s_k$ the total ramification number of the fiber over c_k . We denote by (C, E, ∇) the normalized equation that can be deduced from the pull-back $\phi^*(E_0, \nabla_0)$ by bundle transformation. Let N_k denote the number of poles counted with multiplicity in the fiber over c_k . The irregular Teichmüller dimension of (C, E, ∇) is given by

$$T = 3g - 3 + N = 3g - 3 + N_1 + \cdots + N_n$$

where g is the genus of C , given by Riemann-Hurwitz Formula (32) with $R := \sum_{k=1}^\nu R_k$. We now explain how to compute N_k by means of (κ_k, θ_k) and $(m_{k,l})_l$.

- If $\kappa_k = 0$ and $\theta_k \notin \mathbb{Q} \setminus \mathbb{Z}$, then

$$N_k = s_k = d - R_k.$$

- If $\kappa_k = 0$ and θ_k has order $m > 1$ modulo \mathbb{Z} , then

$$N_k = d - R_k - \#\{l = 1, \dots, s_k \mid m \text{ divides } m_{k,l}\}.$$

- If $\kappa_k \in \mathbb{Z}_{>0}$, then

$$N_k = \sum_{l=1}^{s_k} (m_{k,l} + 1) = d(\kappa_k + 1) - R_k.$$

- If $\kappa_k \in \mathbb{Z}_{>0} - \frac{1}{2}$, then

$$N_k = d(\kappa_k + 1) - R_k + \#\{l = 1, \dots, s_k \mid 2 \text{ does not divide } m_{k,l}\}.$$

We assume ϕ admissible, i.e. that $T \leq B$.

In this section, we show that we can replace ϕ by another ramified cover ϕ' with more critical points but less multiplicity in fibers in such a way that $T - B$ can only decrease. The total ramification will be unchanged, but will be scattered outside of the polar locus. This will allow us to replace the deformation of ϕ by the wider deformation of ϕ' , so that we will be able to recover ϕ (and its deformation) by confluence of critical values. By the way, the passport of ϕ' will be as simple as possible and it will be easy to classify such covers.

The first step consists in scattering the branching points over critical values outside of the polar locus. We call **simple branching** a fiber of the form

$$\begin{bmatrix} m_{k,1} \\ m_{k,2} \\ \vdots \\ m_{k,s_k} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

i.e. with $R_k = 1$.

Lemma 17. *Let $n < k \leq \nu$, i.e. c_k not a pole of ∇ . We can deform $\phi \rightsquigarrow \phi'$ over a neighborhood of c_k so that the single fiber of ϕ with total ramification R_k is replaced by R_k simple branching fibers for ϕ' . We have increased B without changing R or T .*

Proof. Fix a disc $\Delta \subset C_0$ in which c_k is the unique critical value of ϕ . The monodromy of the ramified covering is given by the permutation that decomposes into the product of s_k cyclic permutations of orders $m_{k,1}, \dots, m_{k,s_k}$ with disjoint support. In order to construct ϕ' , it is enough to define its monodromy in Δ , namely the data of R_k transpositions whose product is the monodromy of ϕ . But it suffices to decompose each cyclic permutation of length $m_{k,l}$ above as the product of $m_{k,l} - 1$ transpositions, which indeed gives

$$\sum_{l=1}^{s_k} (m_{k,l} - 1) = d - s_k = R_k$$

transpositions making the job. □

After applying Lemma (17) to each critical fiber of ϕ outside the polar locus, we can now assume that all fibers $\phi^{-1}(c_k)$ are simple branching for $k > n$. By this way, we have maximized B without touching at fibers over poles of ∇_0 so far, so N has not changed. We now discuss how to simplify fibres over poles of ∇ by putting some of their branch points out, in additional simple branching fibers, without increasing $N - B$; this will however increase B . We do this in successive lemmatae discussing on the type of poles.

Lemma 18. *Let $0 \leq k \leq n$, i.e. $c_k = p_k$ is a pole of ∇ , and assume $\kappa_k = 0$ and $\theta_k \notin \mathbb{Q} \setminus \mathbb{Z}$. We can deform $\phi \rightsquigarrow \phi'$ over a neighborhood of c_k so that the single fiber of ϕ with total ramification R_k is replaced by a non branching fiber over p_k (i.e. $R'_k = 0$), and R_k simple branching fibers nearby. We have $N'_k = d = N_k + R_k$ and $B' = B + R_k$.*

Proof. We proceed like in the proof of Lemma 17 by replacing the monodromy of $\phi^{-1}(p_k)$ by the product of the identity and R_k transpositions. Here, the identity stands for the trivial monodromy of the non branching fiber over p_k . \square

Lemma 19. *Let $0 \leq k \leq n$, i.e. $c_k = p_k$ is a pole of ∇ , and assume $\kappa_k \in \mathbb{Z}_{>0}$. We can deform $\phi \rightsquigarrow \phi'$ over a neighborhood of c_k so that the single fiber of ϕ with total ramification R_k is replaced by a non branching fiber over p_k (i.e. $R'_k = 0$), and R_k simple branching fibers nearby. We have $N'_k = d(\kappa_k + 1) = N_k + R_k$ and $B' = B + R_k$.*

The proof is the same as before.

Lemma 20. *Let $0 \leq k \leq n$, i.e. $c_k = p_k$ is a pole of ∇ , and assume $\kappa_k = 0$ and θ_k has order $m > 1$ modulo \mathbb{Z} . We can deform $\phi \rightsquigarrow \phi'$ over a neighborhood of p_k so that the single fiber of ϕ with total ramification R_k is replaced by fiber with passport*

$$\begin{bmatrix} m'_{k,1} \\ \vdots \\ m'_{k,s'_k} \end{bmatrix} = \begin{bmatrix} m \\ \vdots \\ m \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

and only simple branching fibers nearby. We have $N'_k \geq N_k$, $B' \geq B$ and $N'_k - B' \leq N_k - B$.

Proof. Here, we cannot just proceed as before. Indeed, if m divides $m_{k,l}$, then there is not pole on the preimage (or an apparent one that disappears after normalization); however, replacing by $m_{k,l}$ non branching points would increase N_k by $m_{k,l}$, but increase B only by $m_{k,l} - 1$ so that $N - B$ increases by 1. We thus have to take care of those points with m dividing $m_{k,l}$.

Consider the euclidean division $m_{k,l} = s^0 \cdot m + s^1$. Then, we can replace the point with multiplicity $m_{k,l}$ by

- s^0 points of multiplicity m in the fiber $\phi^{-1}(p_k)$ (contributing to no pole),
- s^1 non branching points in the fiber $\phi^{-1}(p_k)$ (contributing to s^1 poles),
- and $s^0 + s^1 - 1$ additional simple branching fibers around.

To realize the deformation of ϕ , we have to realize the corresponding monodromy representation, which is easy in this case. Indeed, the concatenation of a cyclic permutation (or a cycle inside a permutation) runs as follows:

$$\underbrace{(1 \dots \mu)}_{\text{over } p_k} = \underbrace{(1 \dots \mu')(\mu' + 1 \dots \mu)}_{\text{over } p_k \text{ after concatenation}} \cdot \underbrace{(1, \mu' + 1)}_{\text{simple branching fiber}}$$

where $1 < \mu' < \mu$. We just have to repeat this procedure $s_0 + s_1 - 1$ times. By the way, we get $N'_k = N_k + s^1 - 1$ (or $N'_k = N_k = 0$ if m divides $m_{k,l}$) and $B' = B + s^0 + s^1 - 1$. We proceed similarly with all $m_{k,l} \neq m, 1$ in the fiber. \square

Lemma 21. *Let $0 \leq k \leq n$, i.e. $c_k = p_k$ is a pole of ∇ , and assume $\kappa_k \in \mathbb{Z}_{>0} - \frac{1}{2}$. We can deform $\phi \rightsquigarrow \phi'$ over a neighborhood of p_k so that the single fiber of ϕ with total ramification R_k is replaced*

by fiber with passport

$$\begin{bmatrix} m'_{k,1} \\ \vdots \\ m'_{k,s'_k} \end{bmatrix} = \begin{bmatrix} 2 \\ \vdots \\ 2 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

and only simple branching fibers nearby. We have $N'_k - B' = N_k - B$.

The proof is similar to that of Lemma 20. We call **scattered admissible covering** (with respect to (C_0, E_0, ∇_0)) an admissible covering ϕ satisfying the conclusion for ϕ' in Lemmae 17 - 21. We assume from now on that ϕ is scattered. We now deform the differential equation (E_0, ∇_0) on C_0 into a logarithmic one (E'_0, ∇'_0) without changing the ramified cover, in such a way that $N - B$ does not increase. This will allow us to conclude with the classification established by the first author [14] in the logarithmic case.

Lemma 22. *Let p_k be an irregular unramified pole of ∇ , i.e. $\kappa_k \in \mathbb{Z}_{>0}$. We can deform the differential equation $(E_0, \nabla_0) \rightsquigarrow (E'_0, \nabla'_0)$ over a neighborhood of p_k so that the deformed equation is also normalized with $\kappa_k + 1$ simple poles instead of a single pole of multiplicity $\kappa_k + 1$. If ϕ is a scattered covering, then $N' = N$ (and $B' = B$).*

Proof. It is similar to the previous proofs: instead of dealing with the monodromy of the covering, we use the monodromy of the differential equation. We can write the monodromy M of ∇_0 around p_k as the product of $\kappa_k + 1$ non trivial linear transformations:

$$M = M_0 \cdot M_1 \cdots M_{\kappa_k}, \quad M_i \in \mathrm{SL}_2(\mathbb{C}), \quad M_i \neq \pm I.$$

By standard arguments *à la Riemann-Hilbert*, we can first realize these matrices as local monodromy of a differential equation over a disc with $(\kappa_k + 1)$ simple poles in normal form, and total monodromy M . Next, by surgery over the disc, we replace the single pole p_k in (E_0, ∇_0) by this new differential equation, and get (E'_0, ∇'_0) with the desired properties. \square

Lemma 23. *Let p_k be an irregular and ramified pole of ∇ , i.e. $\kappa_k \in \mathbb{Z}_{>0} - \frac{1}{2}$. We can deform the differential equation $(E_0, \nabla_0) \rightsquigarrow (E'_0, \nabla'_0)$ over a neighborhood of p_k so that the deformed equation is also normalized with $\bar{\kappa}_k + 1 = \kappa_k + \frac{3}{2}$ simple poles instead of a single pole of multiplicity $\bar{\kappa}_k + 1$, one of which is at the critical point p_k for ϕ , with $\theta'_k = \frac{1}{2}$. If ϕ is a scattered covering, then $N' = N$ (and $B' = B$).*

Proof. The proof is similar, except that we need $M = M_0 \cdot M_1 \cdots M_{\bar{\kappa}_k}$ with $\mathrm{trace}(M_0) = 0$. Then we can realize M_0 as the monodromy of a logarithmic differential equation with exponent $\frac{1}{2}$. \square

8. IRREGULAR EULER CHARACTERISTIC

Following Poincaré, we can associate, to a fuchsian differential equation, an orbifold structure on the base curve C_0 (see also [14]) and get the notion of orbifold Euler characteristic. Here we define an irregular version of it. Given (C_0, E_0, ∇_0) in normal form, we define the **orbifold order** ν_k at a logarithmic singular point as the order of the local monodromy:

- $\nu_k \in \mathbb{Z}_{>1}$ is the order of $[\theta_k \pmod{\mathbb{Z}}]$ if $\theta_k \in \mathbb{Q} \setminus \mathbb{Z}$,
- $\nu_k = \infty$ if not.

We define the **irregular Euler characteristic** of the differential equation as

$$\chi^{\text{irr}}(C_0, E_0, \nabla_0) := 2 - 2g_0 - \sum_{k=1}^n (1 + \kappa_k) + \sum_{\kappa_k=0} \frac{1}{\nu_k}.$$

In the logarithmic case, this notion coincide to the orbifold Euler characteristic $\chi^{\text{irr}} = \chi^{\text{orb}}$. We note that the two operations of Lemmae 22 and 23, replacing (E_0, ∇_0) by a logarithmic equation (E'_0, ∇'_0) , does not change the irregular Euler characteristic. Moreover, likely as in [14, Prop. 2.5], we have the following characterization:

Proposition 24. *If $\chi^{\text{irr}}(C_0, E_0, \nabla_0) \geq 0$, then the Galois group $\text{Gal}(E_0, \nabla_0)$ is virtually abelian. In particular, in the irregular case, ∇_0 has only trivial Stokes and the Galois group is one-dimensional: it is diagonal, or dihedral.*

Proof. In the logarithmic case, the curve C_0 with its orbifold structure is a finite quotient of the sphere or the torus. The differential equation lifts as a differential equation with trivial or abelian monodromy group respectively. Since the Galois group is the Zariski closure of the monodromy group in the logarithmic case, we get that $\text{Gal}(E_0, \nabla_0)$ is virtually abelian. In the irregular case, $\chi^{\text{irr}} \geq 0$ gives us

$$-\chi^{\text{irr}} = 2g_0 - 2 + \underbrace{\sum_{\kappa_k=0} \left(1 - \frac{1}{\nu_k}\right)}_{\geq 0} + \underbrace{\sum_{\kappa_k>0} (1 + \kappa_k)}_{\geq 1 + \frac{1}{2}} \leq 0.$$

We promptly see that $g_0 = 0$, likely as in the logarithmic case, and we have the following possible formal local data up to bundle transformation

$$\begin{pmatrix} \kappa_1 \cdots \kappa_n \\ \theta_1 \cdots \theta_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

In the first case, we assume that we have an unramified irregular singular point, and the inequality for χ^{irr} gives no place for any other singular point; moreover, $\kappa = 1$. The monodromy around the unique singular point decomposes as

$$M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} \lambda(1+st) & \lambda s \\ \lambda^{-1}t & \lambda^{-1} \end{pmatrix}$$

which must be trivial, implying $\lambda = e^{i\pi\theta} = 1$ and $s = t = 0$. This means that $\theta \in \mathbb{Z}$, or equivalently $\theta = 0$ after bundle transformation, and we have trivial Stokes matrices. The local (and therefore global) Galois group is diagonal like the differential equation.

In the second and third cases, the irregular point is ramified and there is a place for a single logarithmic pole with orbifold order $\nu = 2$. The local monodromy decomposes as

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -s \end{pmatrix}$$

which is never the identity. The second case is therefore impossible. In the third case, M is also the local monodromy at the logarithmic pole: the trace must be zero, $s = 0$, implying trivial Stokes again. \square

Proposition 25. *Let (C_0, E_0, ∇_0) be a normalized differential equation with local formal data $(\kappa_k, \theta_k)_k$. Let $\phi : C \rightarrow C_0$ be a degree d ramified cover and (E, ∇) be the pull-back equation. Let T*

be the dimension of the irregular Teichmüller deformation space for (C, E, ∇) . Let B be the number of critical values of ϕ outside the poles of ∇_0 . Then we have

$$T - B \geq g - 1 - d \cdot \chi^{\text{irr}}(C_0, E_0, \nabla_0)$$

where g is the genus of C .

Proof. Let us decompose

$$\begin{aligned} N &= N_1 + \cdots + N_n \\ R &= R_1 + \cdots + R_n + B \end{aligned}$$

where N_k is the number of poles of ∇ (counted with multiplicity) along the fiber $\phi^{-1}(p_k)$, and R_k is the total ramification along $\phi^{-1}(p_k)$. Now we have

$$T = 3g - 3 + N = g - 1 + \underbrace{2g - 2}_{=d(2g_0-2)+R} + N$$

(by Riemann-Hurwitz) which gives

$$T - B = g - 1 + d(2g_0 - 2) + \sum_{k=1}^n (N_k + R_k).$$

Let us lower bound $N_k + R_k$ in function of the type of pole p_k for ∇_0 . We note that, along scattering in Lemmae 17 - 21, the value of $N_k + R_k$ can only decrease, so that it is enough to estimate a lower bound for a scattered covering ϕ .

If $\kappa_k = 0$ and $\nu_k \in \mathbb{Z}_{>0} \cup \{\infty\}$ is the orbifold order, then we have

$$d = N_k + m \cdot \nu_k \quad \text{and} \quad R_k = m \cdot (\nu_k - 1)$$

so that

$$N_k + R_k = N_k + \frac{d - N_k}{\nu_k} (\nu_k - 1) = d \left(1 - \frac{1}{\nu_k}\right) + \frac{N_k}{\nu_k} \geq d \left(1 - \frac{1}{\nu_k}\right).$$

If $\kappa_k \in \mathbb{Z}_{>0}$, then we find (after scattering)

$$N_k = d(1 + \kappa_k) \quad \text{and} \quad R_k = 0.$$

If $\kappa_k \in \mathbb{Z}_{>0} - \frac{1}{2}$, then we have

$$d = m_0 + 2m_1, \quad N_k = m_0(1 + \kappa_k + \frac{1}{2}) + m_1(1 + 2\kappa_k) \quad \text{and} \quad R_k = m_1$$

so that $m_1 = \frac{d - m_0}{2}$, and after substitution, we find

$$N_k + R_k = d(1 + \kappa_k) + \frac{m_0}{2} \geq d(1 + \kappa_k).$$

After summing for $k = 1, \dots, n$, we find the expected lower bound. \square

Corollary 26. *Under assumptions of Proposition 25, if ∇_0 is irregular with non trivial Stokes matrices and $T - B \leq 0$, then we have $\chi^{\text{irr}} < 0$, $g_0 = g = 0$, and*

$$d|\chi^{\text{irr}}| \leq 1.$$

Proof. The inequality $\chi^{\text{irr}} < 0$ directly follows from Proposition 24 and the fact we are assuming non trivial Stokes matrices. Then Proposition 25 gives

$$0 \geq T - B \geq g - 1 + \underbrace{d|\chi^{\text{irr}}|}_{>0}$$

which implies $g = 0$ (and therefore $g_0 = 0$). Then we deduce the expected inequality. \square

9. CLASSIFICATION OF COVERS

In this section, we classify pull-back algebraic solutions of irregular Garnier systems (see section 5). In other words, we list all differential equations (C_0, E_0, ∇_0) and ramified coverings $\phi : C \rightarrow C_0$ such that, by deforming $\phi \rightsquigarrow \phi_t$, we get a complete isomonodromic deformation $\phi_t^*(E_0, \nabla_0)$. In fact, we omit classical solutions which will be discussed in section 5.1 and therefore assume $\chi^{\text{irr}}(C_0, E_0, \nabla_0) < 0$ (see Proposition 24). Moreover, equations (E_0, ∇_0) are listed up to bundle transformation; in particular, we can assume without lack of generality that (E_0, ∇_0) is a normalized equation. In the sequel, we use notations of previous sections. In particular, T is the dimension of the irregular Teichmüller space of the irregular curve given by $\phi^*(C_0, E_0, \nabla_0)$, and B is the dimension of deformation of ϕ , obtained by moving the critical values of ϕ outside the poles of ∇_0 . To get a complete deformation, we need $T \leq B$, and we will assume $T > 0$ (otherwise there is no deformation).

Proposition 27 ([14]). *Assume (C_0, E_0, ∇_0) is a (normalized) logarithmic connection with at least one pole p_k having infinite orbifold order $\nu_k = \infty$, and assume $\phi : C \rightarrow C_0$ is a scattered ramified cover of degree $d \geq 2$. If $T \leq B$ and $\chi^{\text{irr}}(C_0, E_0, \nabla_0) < 0$, then (C_0, E_0, ∇_0) is hypergeometric, and up to bundle transformation, we are in the list of table 3.*

Proof. This proposition is proved in [14, second table, page 142]. It can be proved directly by using the inequality of Lemma 25. \square

TABLE 3. Logarithmic classification

Local formal data θ_k	Degree	Covering passport
$(\frac{1}{2}, \frac{1}{3}, \theta)$	6	$\left(\begin{array}{cc c c} [2] & & [1] & \\ [2] & [3] & \vdots & 3 \\ [2] & [3] & 1 & \end{array} \right)$
$(\frac{1}{2}, \frac{1}{3}, \theta)$	4	$\left(\begin{array}{cc c c} [2] & & [1] & \\ [2] & [3] & \vdots & 2 \\ & [1] & 1 & \end{array} \right)$
$(\frac{1}{2}, \frac{1}{3}, \theta)$	3	$\left(\begin{array}{cc c c} [2] & & [1] & \\ [1] & [3] & 1 & 1 \\ & & 1 & \end{array} \right)$
$(\frac{1}{2}, \frac{1}{4}, \theta)$	4	$\left(\begin{array}{cc c c} [2] & & [1] & \\ [2] & [4] & \vdots & 1 \\ & & 1 & \end{array} \right)$
$(\frac{1}{2}, \theta_1, \theta_\infty)$	2	$\left(\begin{array}{c c c c} [2] & [1] & [1] & \\ & 1 & 1 & 1 \end{array} \right)$

By confluence of the poles of the differential equation, we deduce the following:

Proposition 28. *Assume (C_0, E_0, ∇_0) is a (normalized) differential equation with at least one irregular pole p_k having non trivial Stokes matrices, and assume $\phi : C \rightarrow C_0$ is a scattered ramified cover of degree $d \geq 2$. If $T \leq B$, then (C_0, E_0, ∇_0) is a degenerate hypergeometric equation, and up to bundle transformation, we are in the list of table 4.*

TABLE 4. Irregular classification with scattered cover

Local formal data for (E_0, ∇_0)	Degree	Covering passport	Local formal data for (E, ∇)	Isomonodromy equation
$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{pmatrix}$	6	$\left(\begin{array}{c c} [3] & 2 \\ \hline [3] & 2 \end{array} \quad 3 \right)$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\text{Gar}^3(1, 1, 1)$
$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{pmatrix}$	4	$\left(\begin{array}{c c} [3] & 2 \\ \hline [1] & 2 \end{array} \quad 2 \right)$	$\begin{pmatrix} 0 & 1 & 1 \\ \frac{1}{3} & 0 & 1 \end{pmatrix}$	$\text{Gar}^2(0, 1, 1)$
$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{pmatrix}$	3	$\left([3] \quad \begin{array}{c c} 2 & \\ \hline 1 & \end{array} \quad 1 \right)$	$\begin{pmatrix} \frac{1}{2} & 1 \\ 0 & 0 \end{pmatrix}$	$P_{III}^{D_7}$
$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{4} & 0 \end{pmatrix}$	4	$\left([4] \quad \begin{array}{c c} 2 & \\ \hline 2 & \end{array} \quad 1 \right)$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$P_{III}^{D_6}$
$\begin{pmatrix} 0 & \frac{1}{2} \\ \theta & 0 \end{pmatrix}$	2	$\left(\begin{array}{c c} [1] & \\ \hline [1] & [2] \end{array} \quad 1 \right)$	$\begin{pmatrix} 0 & 1 & 0 \\ \theta - 1 & 0 & \theta \end{pmatrix}$	P_V
$\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \theta \end{pmatrix}$	2	$\left([2] \quad \begin{array}{c c} 1 & \\ \hline 1 & \end{array} \quad 1 \right)$	$\begin{pmatrix} 1 & 1 \\ \theta - 1 & \theta \end{pmatrix}$	$P_{III}^{D_6}$
$\begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}$	2	$\left([2] \quad 1 \right)$	$\begin{pmatrix} 3 \\ 0 \end{pmatrix}$	P_{II}

Proof. From Lemmae 22 and 23 the irregular poles can be scattered as several logarithmic poles with all of them having exponent θ with infinite orbifold order, except one in the ramified case, having exponent $\frac{1}{2}$. We are led to the list of table 3. We then deduce the list of table 4 by confluence of poles with infinite order θ , and possibly one of them $\frac{1}{2}$. In the first four entries of table 3, we have no other choice than make the two poles with exponent $\frac{1}{2}$ and θ confluing into a ramified irregular pole with $\kappa = \frac{1}{2}$. In the last entry however, we have several possible confluences, namely $\{\frac{1}{2}, \theta_1\} \rightsquigarrow \kappa = \frac{1}{2}$, $\{\theta_1, \theta_2\} \rightsquigarrow \kappa = 1$ and $\{\frac{1}{2}, \theta_1, \theta_2\} \rightsquigarrow \kappa = \frac{3}{2}$. \square

Finally, by confluence of ramification fibers of ϕ , we complete the list for pull-back solutions:

Proposition 29. *Assume (C_0, E_0, ∇_0) is a (normalized) differential equation with at least one irregular pole p_κ having non trivial Stokes matrices, and assume $\phi : C \rightarrow C_0$ is a non scattered ramified cover of degree $d \geq 2$. If $T \leq B$, then (C_0, E_0, ∇_0) is a degenerate hypergeometric equation, and up to bundle transformation, we are in the list of table 5.*

TABLE 5. Irregular classification with confluent cover

Local formal data for (E_0, ∇_0)	Degree	Covering passport	Local formal data for (E, ∇)	Isomonodromy equation
$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{pmatrix}$	6	$\left(\begin{array}{c c} [3] & 4 \\ \hline [3] & 2 \end{array} \quad 2 \right)$	$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	$\text{Gar}^2(1, 2)$
$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{pmatrix}$	6	$\left([3] \quad \begin{array}{c c} [6] & \\ \hline & \end{array} \quad 1 \right)$	$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$	P_{II}
$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{pmatrix}$	4	$\left([3] \quad \begin{array}{c c} [4] & \\ \hline [1] & \end{array} \quad 1 \right)$	$\begin{pmatrix} 0 & 2 \\ \frac{1}{3} & 1 \end{pmatrix}$	P_{IV}

Proof. We now inverse the scattering process of Lemmae 17 - 21. To do this, we replace simple branching fibers outside the poles of ∇_0 by additional ramifications over poles. Note that in table 4, each entry satisfies $T = B$ so that we cannot add ramifications over logarithmic poles with finite orbifold order, otherwise $T - B$ becomes > 0 (see Lemma 20). The only possibility is therefore to add ramifications over irregular poles of ∇_0 , or logarithmic poles with exponent θ having infinite orbifold order. Only the first two lines give examples with $T > 0$. \square

Remark 30. We observe that the algebraic solution of $P_{II}(0)$ can be constructed from two pull-back constructions (see table 4 last line and table 5 line 2). This comes from the fact that Airy equation $\begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}$ is itself pull-back from Kummer equation $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{pmatrix}$ by a 3-fold ramified cover. Similarly, the algebraic solution of $P_{III}^{D_6}$ appears twice in table 4 for $\theta = 0$ due to the fact that Kummer equation $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \theta \end{pmatrix}$ is the double cover of $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{4} & 0 \end{pmatrix}$ in that case.

10. CLASSIFICATION OF CLASSICAL SOLUTIONS IN THE CASE $N = 2$.

In section 9, we have given a complete classification of algebraic solutions of irregular Garnier systems whose linear Galois group is SL_2 . Indeed, we have classified those solutions of type (1) in Corollary 15. It does not make sense to do the same for solutions of type (2) or (3) in Corollary 15, since there are infinitely many, for arbitrary large rank N . However, for a given rank, it makes sense to classify, and we do this in this section for the case $N = 2$ which is the first open case after Painlevé equations. Recall that we have two solutions of type (1) in the case $N = 2$, one for each formal data:

$$\begin{pmatrix} 0 & 1 & 1 \\ \frac{1}{3} & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

10.1. Without apparent singular point. They correspond to the type (2) of Corollary 15 and have Galois group D_∞ .

$$(33) \quad \left\{ \begin{array}{ll} \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} & \text{infinite discrete family} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \theta_1 & \theta_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \theta_1 & \theta_2 \end{pmatrix} & \text{two-parameter families} \\ \begin{pmatrix} 0 & 0 & 2 \\ \frac{1}{2} & \frac{1}{2} & \theta \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & \theta \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \theta \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{3}{2} & 0 \\ \frac{1}{2} & 0 & \theta \end{pmatrix} & \text{one-parameter families} \\ \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \frac{5}{2} \\ \frac{1}{2} & 0 \end{pmatrix} & \text{sporadic solutions} \end{array} \right.$$

These formal data correspond to those for which the dihedral group D_∞ occur as a Galois group of the linear equation, up to bundle transformation. In order to find this list, we have to take into account the following constraints:

- there are 2 or 4 poles where the local monodromy (or Galois group) is anti-diagonal, and the local formal type must be

$$\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} k \\ \frac{k}{2} \\ 0 \end{pmatrix}, \quad \text{with } k \in \mathbb{Z}_{>0} \text{ odd.}$$

- other poles are of local formal type

$$\begin{pmatrix} k \\ \theta \end{pmatrix} \quad \text{with } k \in \mathbb{Z}_{\geq 0}, \theta \in \mathbb{C}.$$

We would like to insist that it is not necessary to consider particular values for θ as normalized equations with differential Galois group D_∞ having poles with diagonal local monodromy occur in family where each exponent θ can be deformed arbitrarily. This comes from the fact that the monodromy representation itself can be deformed as well.

The first entry corresponds to the unique case with 4 poles having local anti-diagonal monodromy. It is an irregular version of Picard-Painlevé equation (see [40, 37]): there are infinitely many algebraic solutions, in bijection with the orbits of $\mathbb{Q} \times \mathbb{Q}$ under the standard action of $\mathrm{SL}_2(\mathbb{Z})$. In fact, if $\tilde{C} \rightarrow C \simeq \mathbb{P}^1$ denotes the elliptic curve given by the 2-fold cover ramifying over the 4 poles of (E, ∇) , then Picard solutions are related with torsion points on \tilde{C} and how they vary when deforming the poles, and the curve \tilde{C} . Here, the story is the same. Indeed, the locus of D_∞ Galois group in the moduli space is closed algebraic, and a differential equation in this closed set has trivial Stokes matrices and is characterized by its monodromy representation.

For all other cases, recall that the differential equation with dihedral Galois group can be determined by means of exponents θ_i 's, as for its monodromy, once we know the irregular curve. Therefore, for each θ_i 's, we get exactly one algebraic solution.

10.2. With apparent singular point. They correspond to the type (3) of Corollary 15 and have Galois group C_∞ or D_∞ . However, as noticed in Remark 13, algebraic solutions with Galois group D_∞ and apparent singular points always arise as particular cases of more general algebraic solutions with Galois group D_∞ and arbitrary singular points as listed in section 10.1. It just remains to complete the list with those algebraic solutions with Galois group C_∞ and (at least one) apparent singular point.

$$(34) \quad \left\{ \begin{array}{ll} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \theta_1 & \theta_2 & \theta_3 \end{pmatrix}, & \theta_1 + \theta_2 + \theta_3 = 0 \quad \text{two-parameter family} \\ \begin{pmatrix} 0 & 0 & 2 \\ 0 & \theta & -\theta \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & \theta & -\theta \end{pmatrix} & \text{one-parameter families} \\ \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} & \text{sporadic solution} \end{array} \right.$$

For each value of θ_i 's, there is exactly one normalized equation once the irregular curve is fixed, and therefore exactly one algebraic solution. Solutions with 2 or more apparent singular points arise as particular cases of these ones by specifying θ_i 's.

11. EXPLICIT HAMILTONIANS FOR SOME IRREGULAR GARNIER SYSTEMS

Here we provide the linear differential equation and the Hamiltonians for some particular formal types (the complete list comes from [28, 26]). We translate our notations with Kimura and Kawamuko's.

11.1. $\text{Kim}_4(1, 2, 2)$. The general linear differential equation with (non apparent) poles $x = 0, 1, \infty$ and corresponding formal type $\begin{pmatrix} 2 & 2 & 1 \\ \theta_0 & \theta_1 & \theta_\infty \end{pmatrix}$ can be normalized into the form³

$$(35) \quad L(1, 2, 2) : \begin{cases} u'' + f(x)u' + g(x)u = 0 & \text{with} \\ f(x) = \frac{t_2}{x^2} + \frac{2-\theta_0}{x} + \frac{t_1}{(x-1)^2} + \frac{2-\theta_1}{x-1} - \sum_{k=1,2} \frac{1}{x-q_k} \\ g(x) = \frac{(\theta_0+\theta_1-1)^2-\theta_\infty^2}{4x(x-1)} - \frac{t_1 H_1}{x(x-1)^2} + \frac{t_2 H_2}{x^2(x-1)} + \sum_{k=1,2} \frac{q_k(q_k-1)p_k}{x(x-1)(x-q_k)} \end{cases}$$

Singular points $x = q_1, q_2$ are apparent if and only if coefficients H_1, H_2 are given by

$$(36) \quad H(1, 2, 2) : \begin{cases} H_1 = -\frac{q_1^2(q_1-1)^2(q_2-1)}{t_1(q_1-q_2)} \left(p_1^2 - \left(\frac{\theta_0}{q_1} - \frac{t_2}{q_1^2} + \frac{\theta_1-1}{q_1-1} - \frac{t_1}{(q_1-1)^2} \right) p_1 + \frac{(\theta_0+\theta_1-1)^2-\theta_\infty^2}{4q_1(q_1-1)} \right) \\ \quad + \frac{(q_1-1)q_2^2(q_2-1)^2}{t_1(q_1-q_2)} \left(p_2^2 - \left(\frac{\theta_0}{q_2} - \frac{t_2}{q_2^2} + \frac{\theta_1-1}{q_2-1} - \frac{t_1}{(q_2-1)^2} \right) p_2 + \frac{(\theta_0+\theta_1-1)^2-\theta_\infty^2}{4q_2(q_2-1)} \right) \\ H_2 = -\frac{q_1^2(q_1-1)^2q_2}{t_2(q_1-q_2)} \left(p_1^2 - \left(\frac{\theta_0-1}{q_1} - \frac{t_2}{q_1^2} + \frac{\theta_1}{q_1-1} - \frac{t_1}{(q_1-1)^2} \right) p_1 + \frac{(\theta_0+\theta_1-1)^2-\theta_\infty^2}{4q_1(q_1-1)} \right) \\ \quad + \frac{q_1q_2^2(q_2-1)^2}{t_2(q_1-q_2)} \left(p_2^2 - \left(\frac{\theta_0-1}{q_2} - \frac{t_2}{q_2^2} + \frac{\theta_1}{q_2-1} - \frac{t_1}{(q_2-1)^2} \right) p_2 + \frac{(\theta_0+\theta_1-1)^2-\theta_\infty^2}{4q_2(q_2-1)} \right) \end{cases}$$

A deformation of (35) is isomonodromic if, and only if, parameters satisfies Hamiltonian system (2).

$$(37) \quad \frac{dq_j}{dt_i} = \frac{\partial H_i}{\partial p_j} \quad \text{and} \quad \frac{dp_j}{dt_i} = -\frac{\partial H_i}{\partial q_j} \quad \forall i, j = 1, 2.$$

To construct the pull-back solution (second line of Table 4), we start with the differential equation $\frac{d^2 u}{dz^2} + \frac{2}{3z} \frac{du}{dz} - \frac{1}{z} u = 0$ and consider its pull-back by the branch cover

$$z = \phi(x) = \frac{t_2^2(2x - 4q_1x + q_1^2 + q_1)^3}{16q_1^3(q_1 + 1)^3x^2(x-1)^2}.$$

Comparing with (35), we get the first solution of Theorem 1.

11.2. $\text{Kim}_6(2, 3)$. The general linear differential equation with (non apparent) poles $x = 0, \infty$ and corresponding formal type $\begin{pmatrix} 2 & 3 \\ \theta_0 & \theta_\infty \end{pmatrix}$ can be normalized into the form⁴

$$(38) \quad L(2, 3) : \begin{cases} u'' + f(x)u' + g(x)u = 0 & \text{with} \\ f(x) = \frac{t_2}{x^2} + \frac{2-\theta_0}{x} - t_1 - \frac{x}{2} - \sum_{k=1,2} \frac{1}{x-q_k} \\ g(x) = \frac{\theta_0+\theta_\infty-1}{8} - \frac{H_1}{2x} + \frac{t_2 H_2}{x^2} + \sum_{k=1,2} \frac{q_k(q_k-1)p_k}{x(x-1)(x-q_k)} \end{cases}$$

Singular points $x = q_1, q_2$ are apparent if and only if coefficients H_1, H_2 are given by

$$(39) \quad H(2, 3) : \begin{cases} H_1 = \frac{2q_1^2}{q_1-q_2} \left(p_1^2 - \left(\frac{\theta_0}{q_1} - \frac{t_2}{q_1^2} + \frac{q_1}{2} + t_1 \right) p_1 + \frac{\theta_0+\theta_\infty-1}{8} \right) \\ \quad - \frac{2q_2^2}{q_1-q_2} \left(p_2^2 - \left(\frac{\theta_0}{q_2} - \frac{t_2}{q_2^2} + \frac{q_2}{2} + t_1 \right) p_2 + \frac{\theta_0+\theta_\infty-1}{8} \right) \\ H_2 = -\frac{q_1^2q_2}{t_2(q_1-q_2)} \left(p_1^2 - \left(\frac{\theta_0-1}{q_1} - \frac{t_2}{q_1^2} + \frac{q_1}{2} + t_1 \right) p_1 + \frac{\theta_0+\theta_\infty-1}{8} \right) \\ \quad + \frac{q_1q_2^2}{t_2(q_1-q_2)} \left(p_2^2 - \left(\frac{\theta_0-1}{q_2} - \frac{t_2}{q_2^2} + \frac{q_2}{2} + t_1 \right) p_2 + \frac{\theta_0+\theta_\infty-1}{8} \right) \end{cases}$$

³From Kimura's formulae [28], set $\eta_i = 1$, $\varkappa_i = \theta_i$, $\varkappa = \frac{1}{4}[(\theta_0 + \theta_1 - 1)^2 - \theta_\infty^2]$ and $K_i = H_i$.

⁴From Kimura's formulae [28], set $\eta_0 = 1$, $\varkappa_0 = \theta_0$, $\varkappa_\infty = \frac{\theta_0+\theta_\infty-1}{4}$ and $K_i = H_i$.

and isomonodromic deformations are defined by (37). To construct the pull-back solution (first line of Table 5), we start with the differential equation $\frac{d^2u}{dz^2} + \frac{2}{3z} \frac{du}{dz} - \frac{1}{z}u = 0$ and consider its pull-back by the branch cover

$$z = \phi(x) = \frac{(3x^2 + 8t_1x + 4q_1t_1 + 6q_1^2)^3}{6912x^2}.$$

Comparing with (38), we get the second solution of Theorem 1.

11.3. $\text{Kaw}_4(5/2, 3/2)$. The general linear differential equation with (non apparent) poles $x = 0, \infty$ and corresponding formal type $\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$ can be normalized into the form⁵ $u'' = g(x)u$ with

$$(40) \quad g(x) = \frac{t_2^2}{4x^3} + \frac{H_1}{x^2} + \frac{H_2}{x} + \frac{t_1}{2} + \frac{x}{4} + \sum_{k=1,2} \left(\frac{3}{4(x - q_k)^2} - \frac{p_k}{x - q_k} \right)$$

Then set

$$u_1 = q_1 + q_2, \quad u_2 = q_1q_2, \quad v_1 = \frac{q_1 + q_2}{2(q_1 - q_2)^2} + \frac{p_1q_1 - p_2q_2}{q_1 - q_2}$$

$$\text{and } v_2 = -\frac{1}{(q_1 - q_2)^2} - \frac{p_1 - p_2}{q_1 - q_2}$$

and

$$\begin{cases} K_1 = 2u_1v_1^2 + 4u_2v_1v_2 - 4v_1 - \frac{u_1^2}{2} - t_1u_1 + \frac{u_2}{2} + \frac{t_2^2}{2u_2} \\ K_2 = -2u_2v_1^2 + 2u_2^2v_2^2 - 2u_2v_2 + \frac{u_1u_2}{2} + t_1u_2 - \frac{t_2^2u_1}{2u_2} \end{cases}$$

Deformation of (40) is isomonodromic if, and only if

$$\frac{du_j}{dt_i} = \frac{\partial K_i}{\partial v_j} \quad \text{and} \quad \frac{dv_j}{dt_i} = -\frac{\partial K_i}{\partial u_j} \quad \forall i, j = 1, 2.$$

The first classical sporadic solution of Theorem 2, with dihedral linear Galois group, can be constructed by pulling back the differential equation $\frac{d^2u}{dz^2} = \left(\frac{1}{z} - \frac{3}{16x^2}\right)u$ by the ramified covering

$$z = \phi(x) = \frac{(x^2 + 3t_1x - 3t_2)^2}{36x};$$

after normalizing, and comparing with equation (40), we get the rational solution

$$(t_1, t_2) \mapsto (u_1, u_2, v_1, v_2) := \left(-t_1, t_2, 0, \frac{3}{4t_2}\right).$$

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⁵From Kawamuko's formulae [26], set $\lambda_i = q_i$, $\mu_i = p_i$ and $h_i^{04} = H_i$.

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