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Second Order Langevin Equation and Definition of Quantum Gravity By Stochastic Quantisation

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Abstract

Euclidean quantum gravity might be defined by stochastic quantisation that is governed by a higher order Langevin equation rather than a first order stochastic equation. In a transitory phase where the Lorentz time cannot be defined, the parameter that orders the evolution of quantum gravity phenomena is the stochastic time. This changes the definition of causality in the period of primordial cosmology. The prediction of stochastically quantised gravity is that there will a transition from an oscillating quantum phase to a semi-classical one, when the Lorentz time emerges. The end of the transition, as it can be observed from now and described by inflation models, is a diluted Universe, following the inflation phenomenological evolution. It is filled at the beginning with scattered classical primordial black holes. The smallest ones will quickly decay in matter, with a standard quantum field theory evolution till our period. The stable heavier black holes will remain, forming a good fraction of the dark matter and the large black holes observed in the galaxies. In a theoretically related way, this framework suggests the possibility of a gravitational parton content for “point-like” particles, in the same five dimensional quantum field theory context as in the primordial cosmology, with a $(+ - - - -)$ signature for the 5d metrics. The very precise and explicit result expressed in this paper is actually far more modest than its motivation. We compute explicitly the meaning of a second order Langevin equation in zero dimensions and define precisely what is second order stochastic quantisation in a soluble case.

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1 Introduction

It has been proposed in [1] that 4d Euclidean quantum gravity might obey the laws of a stochastic quantisation governed by a second order rather than a first order stochastic equation and, perhaps equivalently, by a possibly higher order Fokker-Planck equation and its 5d supersymmetric representation.

The idea of [1] is that in the case of gravity, this equilibrium can only be the classical gravity theory (coupled to ordinary standard particle QFT's), due to a given fluctuation that suddenly increases the volume of the space, with a brutal decrease of the cosmological constant, so that there is no more room for quantum gravity effects, and Lorentz time can be then defined. In this way one avoids the usual paradox that the Lorentz time cannot be defined when non-perturbative quantum gravity prevails. The system has two phases, one without Lorentz time and other one that has an “emerging” Lorentz time, but both phases rely on the same microscopic theory. It was observed that this may enlarge the scope of causality.

In this paper we explain in more details how stochastic quantisation can truly fit the proposition of [1] and find a modification of the theory that explains the exit from inflation. There is no contradiction between our proposition and the present understanding of various aspects of semi-classical gravity in the post-inflation regime.

Many aspects of the Langevin equation we propose here are in a separate and more mathematically oriented paper [2], published elsewhere. The purpose of the present paper is to introduce and justify physically a second order Langevin equation as a method of quantising gravity.

More importantly, we predict some physical consequences in cosmology and on the short distance behaviour of point-like particles. Some remarks will be also given to support the use of a second order Langevin equation for non-perturbative gravity. Stochastic quantisation in Euclidean quantum gravity is in fact an out-of-equilibrium system from the point of view of the Euclidean path integral. It is indeed well known that for $d > 2$ the Euclidean Einstein action S_E is not bounded from below and hence its exponential $\exp(-\frac{1}{\hbar}S_E)$ is not suitable for defining an equilibrium distribution.

In this paper we argue that the modification from the first order stochastic quantisation Langevin equation of Paris-Wu [3] to a second order one opens new possibilities during the relaxation phenomenon toward the quantum physics equilibrium. We expose the following claim: physically relevant fluctuations of the vacuum of the 5-dimensional supersymmetric quantum field theory that can be associated to any given Langevin equation from a 4d Euclidean quantum field theory may occur at finite stochastic time, in correlation with quantum effects consisting of an abundance of creations and annihilations of the quantised states of the systems. In the case of the stochastic quantisation of gravity, we call them 5-dimensional gravitational partons, in analogy with the physical scheme discussed in [1].

In the case of gravity, stochastic time fluctuations of the metric field change the structure of the space when the stochastic time evolves. The change in geometry can be so strong that it can be assimilated to a reset of the initial conditions, putting the system in a very different phase. This supports our view that they can trigger a transition from a regime where quantum gravity must be treated in a completely non-perturbative way, that is by the 5d stochastic quantisation method, to a semi-classical regime where it can be treated by the standard 4d perturbative methods.

Using a standard first order stochastic quantisation is actually a dead end in the case of gravity because it has no equilibrium distribution and its dynamics is ill-defined. But if one uses higher order stochastic quantisation, new phenomena may occur in finite stochastic time and they may help avoid the known difficulties encountered in the standard attempts of quantum gravity. The creation of pairs of microscopic 5d quantised states is perhaps the cause of a possible phase transition at any

given moment, explaining the exit from the primordial cosmology phase towards the phase where the universe is so diluted that it can be described by the standard quantum gauge field theories in a (semi)classical gravitational background. (At a much more modest but rather realistic level, some progress in the theory of earthquakes was achieved by using models with a second order Langevin equation rather than a first order one.)

Our picture for quantum gravity is therefore a very non-trivial generalised Brownian motion. Namely, it has a phase transition, marked by the exit from the inflation, the meaning of which will be made precise below. The occurrence of such oscillations is made rather explicitly (in the Appendices) in a zero-dimensional interacting model, treated perturbatively. In higher dimensions, counterbalancing between the oscillations of the vacuum and the productions and absorptions of partons may occur during the stochastic time evolution. The zero-dimensional case is of course simplistic in comparison to the one we propose for quantum gravity in four dimensions.

The second order Langevin equation that we propose here has a drift force for the metric that is the sum of the Einstein tensor (plus a possible contribution proportional to the cosmological constant) and the energy-momentum tensor of the matter. An additional gauge-fixing drift force takes into account the 4d reparametrisation invariance that must hold true at every given finite value of the stochastic time. The result is a very interesting and complicated differential equation involving a noise whose effect vanishes when $\hbar \rightarrow 0$. It generalises the flow equation of classical gravity within the scheme of stochastic quantisation by addition of a noise (multiplied by $\sqrt{\hbar}$) to the gravity drift forces. Some mathematical properties of this equation will be explored in [2].

Stochastic quantisation is intrinsically set up for defining the quantum Euclidean correlators of a theory classically defined by its equations of motion. It provides a wider perspective and deeper insights to the signification of the path integral method [3]. Given a drift force and certain initial conditions, it is logically possible not to be able to Wick rotate one of the Euclidean coordinates in the correlators that stochastic quantisation determines at a finite stochastic time. If the stochastic time is the true evolution parameter of physics, the Lorentz time simply doesn't exist during the phase of primordial cosmology. In this phase the Physics must be described in the 5-dimensional framework, just as one must use the microscopic time as the parameter of evolution when describing a Brownian motion far enough from its equilibrium. In fact, it would make no sense to have the coexistence of both the stochastic time and the Lorentz time for ordering phenomena from the knowledge of correlators. No theory can be defined with two independent parameters of evolution.

This point is actually an advantage in gravity theory, in which we know that the Lorentz time cannot be part of the hypothesis in the definition of quantum gravity beyond perturbation theory. This is for instance obvious because of the contradictions implied by the Wheeler-deWitt equation. By giving a physical meaning to the evolution in the stochastic time, our aim is in fact to obtain a substitute to order physical events before the transition identified by the exit of the inflation. So, given the impossibility of defining a Lorentz time when gravity has a non-perturbative quantum behaviour, we propose that the parameter that could truly describes the evolution of processes is the stochastic time. This results in a broader sense of causality in the primordial phase of the Universe, made explicit by the Langevin equation.

The usual description of causality in the evolution in Lorentz time is of course recovered after the exit from inflation when the Lorentz time can emerge and gravity can be described semi-classically. After this event, all the correlators of the fields can be computed directly by the standard path integral in the smooth and existing limit of infinite stochastic time. Their analytic continuation is possible, as a general result of perturbation theory, allowing the emerging Lorentz time to be used after Wick rotation as a phenomenological ordering parameter. It is only in this phase that quantum field theory predicts the existence of clocks built on the principle of using the computable decay rates of atoms or elementary particles. Notice that the determination of such rates only involves correlators that can

be directly computed in the purely Euclidean theory.

This fancy way of defining the Lorentz time (when it can possibly exist) can be also applied to classical physics. Indeed, one can replace the Hamilton equations of motion by flow equations. This may look as a formal remark, but the case of Ricci-flow equations in gravity is still an open question.

A conclusion of the heuristic scenario of [1] was that the interpretation of the Euclidean correlators at a finite stochastic time is what might define quantum gravity until one has a sudden relaxation to semi-classical gravity by a phase transition identified as the exit from inflation. The second order Langevin equation we present here make this scenario plausible in a more precise way. More mathematical details on the Langevin equation than what we will exhibit here will be given in [2].

We are aware that the widely discussed question of defining time in quantum gravity, the so-called “time problem” has been approached in many different ways. We will not refer to all these other approaches, because ours predicts that in the presence of non-perturbatively gravity there cannot be a possible definition of the Lorentz time. We believe therefore that there is not much sense to compare the content of this paper and that of other interesting articles about the question of the Lorentz time. Rather, we should only be concerned with the compatibility of our proposition and the already established facts in (semi-classical) quantum gravity, and we will discuss these topics in the paper.

Besides, the quantum gravity community is often unfamiliar with the possible improvements brought to the quantisation methods by the use stochastic quantisation. Part of our intuition relies on a comparison between the the spin-glass theory that cannot correspond to a regular Boltzmann thermalisation and quantum gravity that cannot undergo a smooth infinite stochastic limit in the framework of stochastic quantisation.

Let us explain why. Spin-glass are systems with an out of thermal equilibrium phase that can be reversibly melted into an ordinary phase with a regular thermal distribution. There is an analogy between the “phase transition” for the stochastic stochastic time evolution of gravity and what happens to spin glasses. In the spin glass phases, thermal equilibrium cannot be reached. Their distinct physics cannot be described by Gibbs equilibrium but depends on the initial conditions, and their causality properties are quite special, with with unusual evolutions and no stable equilibrium. However, at the microscopic level, spin glasses are made of standard matter. Thus they can undergo phase transitions, like melting or sublimation, in which the spin glass becomes a standard liquid or gas, whose properties can be described by a regular Boltzmann-Gibbs ensemble. In this way, the system can evolves toward one in an ordinary thermal equilibrium, under the influence of changes of external parameters. Technical advances have been done to compute the correlators of the out of thermal equilibrium phase of spin glass and describe their physics by using Langevin equations.

For gravity, the transition between its non-perturbative phase, without Lorentz time and standard path integral 4d formulation, and its semi-classical gravity phase might exist in a way similar to the possible transition for spin-glasses. In each case, both phases are described by the same Langevin equation of the same microscopic theory.

The non-perturbative phase of gravity is a non smoothly converging regime of pure fluctuations induced by the noise. There is no way out other than using a Langevin equation do compute the correlators, which give a different physics. For the semi-classical phase, the techniques of computations greatly simplify by using the path integral formula (as one uses the Boltzmann formula to describe a melted spin glass)*. Notice that in classical physics, the use of flow equations rather than equations of motion is the same as using a Langevin equation while neglecting the noise.

*Part of the comparison is that the stochastic time is like the physical molecule time for the collisions of particles in a Brownian motion, and the Lorentz time, when it can be defined, is like the adiabatic time of measurements when one measures the effects of the variation of external parameters for a system made of a very large number of the molecules in thermal equilibrium.

A tempting idea is therefore to compare of impossibility of non-perturbative quantum gravity to reach its infinite stochastic time limit “smoothly” i.e., without a sharp transition) with the impossibility of reaching a standard thermal equilibrium in the spin glass example. This is modulo the fact that the transition between the “phases” of quantum and classical gravity is “spontaneous”. Here, spontaneity means that the effect of a single fluctuation, without a change in external parameters, can trigger the change of phase, giving a possible very sharp transition. It can be so because the fluctuations of the gravitational field under the effect of the stochastic noise affect the structure of the space. In return, this effect can trigger a transition toward a phase where the system becomes basically semi-classical everywhere, as if the stochastic process had been brutally reset with boundary conditions compatible to a semi-classical description of the system.

In this phase, analogous to that of a melted spin-glass with a consistent Gibbs distribution, the Euclidean correlators admit a well defined and smooth large stochastic time limit, in which case their analytic properties are such that the Wick rotation is possible at very large stochastic time, a phenomenon that we called the “emergence” of Lorentz time. An example where ergodicity is spontaneously recovered by a phase transition with no need of changing the parameters is the decoherence of a microscopic quantum system, coupled to a larger classical system.

The comparison with spin glasses has some limitations. The exit from inflation in gravity is not a reversible phenomenon because it concerns with the whole system and the probability that it can spontaneously reverse its course is zero. The transition is spontaneous and induced by the fluctuations of the metric field, whose induced changes of correlators affect the boundaries of the system. In contrast, the transition in a spin glass is not spontaneous. Its melting must be imposed from the outside, for instance by external sources that change locally the temperature. In this way, it is reversible.

Let us recall that such a spontaneity scenario was already present of [1], and based on a Markov process, with the heuristic idea that in the primordial cosmology conditions, the time could flow in a discrete way. Here the interpretation of the oscillation of Euclidean correlators at a finite stochastic time defines precisely quantum gravity, and also predicts a possible sudden relaxation to classical gravity by a phase transition identified as the exit from inflation.

The paper is organised as follows. Section 2 justifies the use of stochastic time to order non-perturbative quantum gravity phenomenon. Section 3 explains stochastic quantisation with a first order Langevin equation. Sections 4 displays materials needed for second order Langevin equations. The use of such equations is illustrated in Appendix A by an interacting zero-dimensional example, making concrete the meaning of a second order Langevin equation (at the perturbative level). We compute the oscillation mechanism that is the basis of our description of gravity with a second order Langevin equation. Appendix B describes the supersymmetric representation for such second order Langevin equations. This gives some hint about the oscillation in stochastic time of a classical background field around which one defines a quantum theory. Section 5 gives some justifications for using stochastic quantisation for gravity. Section 6 presents the second order Langevin equation for the gravity. This equation will be studied in depth in [2] by checking how it can preserve the 4d reparametrisation invariance throughout the stochastic time evolution. Section 7 concludes with further speculations about gravitational phenomena suggested by the use of a second order Langevin equation for quantum gravity.

2 The stochastic time as the evolution parameter of primordial cosmology

One must be more precise about what we mean for the term “exit from inflation” through this article. The claim of this paper is that second order stochastic quantisation might be a key to define the non-perturbative quantum gravity fluctuations that must occur in primordial cosmology to reach a regime where gravity can be treated semi-classically everywhere in the Universe. In this framework, the exit from inflation period means the end of the relatively narrow cross-over domain between the stochastically disordered phase of quantum gravity and the phase of semi-classical gravity. At the very end of this cross-over, the Lorentz-time has “approximatively emerged,” but residuals of the non-perturbative gravity quantum regime persist, in a more and more negligible way, with a continuous evolution toward a situation where gravity can be treated semi-classically everywhere. For this end of the transition, one may use the 4d formalism by an ad hoc completion of the fundamental fields with a given set of inflaton fields (perhaps of no fundamental significance), allowing the use of an inflation model whose evolution is phenomenologically parametrised by a distorted Lorentz time. This provides a local quantum field theory that may reliably describes the physics of the end of the crossover.

The effect of inflaton fields fades away as they are no longer needed in the final phase where all possible effects of gravity can be treated semi-perturbatively. The current inflation theory scenario indicate that the phenomenological inflaton fields get stabilised to constant values after some delay. The cosmological constant gets equal the small constant we measure today and the semi-classical gravity regime stays irreversibly as it is now, with its own standard causality laws.

So, in our scheme, the “exit from inflation” is marked by the possibility of expressing the evolution of the physics either in Lorentz or in Euclidean formulation (equivalently) at infinite value of the stochastic time. Only afterward can one use the emerging Lorentz time as our privileged parameter to order the phenomena via a causal and local QFT, with a standard classical limit.

In the relatively very short duration of the exit of the inflation, there are some remnants of the non-perturbative effects of quantum gravity. The dynamics of both phases mixes during the crossover, but the virtue of the inflation theory is to phenomenologically describe these mixed effects. This happens till the non-perturbative quantum effect of gravity has disappeared from the dynamics of the Universe. Thus, for the special moment of the history of the Universe, using a Lorentz time is made possible at the cost of using ad hoc and non-fundamental inflaton fields.

Before the exit of the inflation, the stochastic quantum field theory framework is necessary to compute the equal finite stochastic time field correlators of the Euclidean correlators along the lines indicated in Sections 3 and 4, with some illustrations in Appendices A and B. The task is of course far of being understood for the case of gravity and we can only describe the method from a generic point of view.

Stochastic quantisation relates fields to their noises. The second order Langevin equation (10) that will be considered for gravity is a PDE that relates the metric field to its noise. When one solves it and computes field correlation functions by averaging over the noises at finite values of the stochastic time, the result depends on the initial field configuration at an earlier stochastic time. It follows that the early cosmology evolution will depend strongly on the new scales introduced by the Euclidean geometry one chooses as an initial condition for the stochastic process. The latter is an initial condition in the Cauchy problem of the Langevin equation. It is only under the condition that a stochastic equilibrium has been reached that the values of correlation functions become independent of the initial conditions. This property is nothing but the generalisation of what happens in the Brownian motion of a gas. Its distribution at finite stochastic time depends on the initial condition. It is only when the thermal equilibrium is reached is the dependence on the initial conditions is truly washed away, giving a Boltzmann partition function.

In the case of a second order Langevin equation, the rate of fluctuations influences the probability of a possible phase transition on the way to a thermal equilibrium. This rate is basically given by the dimensionful constant ΔT that must occur for dimensional reasons in the acceleration term in a second order Langevin equation as Eq. (10). The smaller ΔT is, the faster one has the possibility of a phase transition.

Consider now what may happen just after the phase transition we advocate, when gravity can be computed in its semi-classical approximation. The scale of variation of all observable physical phenomenon is much (almost infinitely) slower than the convergence of the stochastic process, as in a Brownian motion. One can take the limit of infinite stochastic time, and then the details of the stochastic time evolution become physically irrelevant for all realistic experiments. The dependence on the initial data of the correlation functions in the phase before the phase transition is in fact completely washed away after its relaxation to the phase where gravity has become effectively (semi)classical everywhere, as a typical thermalisation phenomenon, and standard clocks for the Lorentz time can exist and be located at any given point of the space.

For instance, such standard clocks, located at various places of our observable Universe, will never allow us to get information about the initial data that we might wish to know to solve (in theory) the Langevin equation in the primordial cosmology phase. Indeed, when one computes the rate of physical Lorentz time clocks in the semi-classical regime of gravity, there is no need to be concerned with the details of stochastic quantisation.

However, the understanding of the stochastic quantisation scheme can help figure out an appealing set of initial conditions right after the exit from inflation, giving a more theoretically sound framework to the physical description of [1]. It is interesting to compare the use of a Langevin equation with the discrete time scenario of primordial cosmology. As discussed in [1], there is a possible justification for having a sharp exit from the inflation by the hypothesis that the Lorentz time cannot be considered as a continuous parameter at very short distance and might run in a discrete way during the primordial cosmology. In the discrete time framework, ΔT is the interval between two successive steps of the discrete time, while in this paper, the scale ΔT is directly introduced as the coefficient of the second order term in the Langevin equation and the four Euclidian coordinates X^μ are continuous. This suggests the possibility of interpreting the space with coordinates $\{X^\mu, \tau\}$ as a 5d-space foliated by the stochastic time parameter τ . This will be done separately in the more mathematically motivated work [2].

A striking feature of the stochastic time picture is that it justifies quite logically the exit from inflation as a phase transition, allowing the Lorentz time to emerge at an age of around $10^{-30}s \gg \Delta T$ (or perhaps less, provided it is much bigger than ΔT), which is about the limit of how far we observe our past. In this way, when it can emerge, the Lorentz time is nothing but one of the Wick rotated coordinates X^μ , in complete agreement with the usual wisdom. This is quite a clarification compared to [1].

Only after this transition, when all correlators can be computed by the conventional 4d path integral methods, do the standard QFT properties imply analytic behaviours of the correlators, allowing their inverse Wick rotation. Non-perturbative gravitational effects can be neglected everywhere. At this stage of the Universe, the Lorentz time has basically emerged and can be used to compute the evolution of the physics in the framework of the current phenomenological theories. The recently created Lorentz time has then become a convenient parameter to order the relative successions of events, as they can be experimentally observed and precisely measured.

Therefore, an appealing novelty of the approach is that it doesn't postulate the existence of the Lorentz time. Rather, it considers the Lorentz time as a possibly measurable coordinate, which has been analytically continued. This can only be done after the exit of the inflation in the context of stochastically quantised Euclidean quantum field theory. After that one can show the consistency of

using the Lorentz as an evolution parameter both classically and quantumly, because gravity became (semi) classical, and clocks can exist in this phase.

The existence of the Lorentz time can (and must) be checked of the possibility of building and using clocks, e.g., made from the created particles. Let us stress that in our framework, the theory of the inflation becomes then a phenomenological model for the short period during which the existence of the Lorentz time is only “approximately” valid, in the sense that shortly after it the stochastic time dependence becomes undetectable by our observations, and one can use the 4d formalism for all physical fields.

After the transition, the Lorentz time can be used effectively sufficiently far away from the created primordial black holes singularities. In fact, even before the inflation period is completely over, the inflation phenomenological scheme is defined by a “time dependent cosmological constant” that quickly evolves towards a constant value in an irreversible process. It provides a local quantum field theory that is phenomenologically admissible for a “gracious exit” from inflation using appropriate effective (unphysical) inflaton fields, with an effective dependence on the Lorentz time and no contradiction with the CMB experimental observations. In [1] we estimated the frequency of the stochastic oscillations needed to reach a transition from primordial cosmology to our phase in which semi-classical gravity prevails to be of the order of $1/\Delta T \sim 10^{65} s^{-1}$ or more. This transition necessarily occurs with probability 1 after a large enough number of sequences.

The oscillatory regime of stochastic quantisation suggests that the phase transition that allows the emergence of Lorentz time is marked by a very large production of gravitational bound states enhanced by the vacuum fluctuations in the 5d theory[†]. Such predictions were heuristically justify in the framework of [1]. At most, [1], identified the exit of the inflation as the period after which one can approximate the Lorentz time as a continuous variable, so that all quantum correlators become independent of the scale variable ΔT . Its occurrence was justified by the energy fluctuations implied by the existence of a discrete time.

The scheme of stochastic quantisation scheme may indicate more precisely that, as soon as strong enough stochastic fluctuations occur, creation of an abundance of classical gravitational bound states at a given value of the stochastic time is what makes the space description semi-classical everywhere, except at the 4d singular points of the leftover primordial black holes, which are scattered everywhere after the exit from inflation. The discussion in the Appendix B is a modest step in this direction.

So we propose in this paper that, as a consequence of the description of primordial Cosmology by the 5d stochastically quantised gravity, the initial condition after the exit of inflation of the phenomenological inflation equations is a small universe, filled with a cloud of primordial black holes with a given mass distribution. The latters are remnants of the 5d-gravitational bound states on the 4d boundary at infinite stochastic time (our observable Universe).

This suggestive picture describes a bit more precisely than in [1] the phase transition from quantum to classical gravity and we claim that it is a genuine consequence of having a non-perturbative quantum gravity regime driven by an evolution in the stochastic time defined by a second order Langevin equation.

It predicts that whatever the exact shape of the end of inflation period is, it drives the Universe towards its semi-classical gravity phase where its 4d geometry is determined in an excellent approximation almost everywhere by solving the Einstein equation with a fixed cosmological constant with appropriate boundary conditions. The relatively short period in which an effective theory exists with a Lorentz time only approximatively true is a necessary feature. It makes a smooth extrapolation between the end of the purely quantum gravity regime and the beginning of the semi-classical regime of our phase. This semi-classical regime means the geometry can be treated classically at an excellent

[†]See Appendix B for some zero-dimensional example as a (partial) justification of this statement.

approximations modulo the possible emissions of perturbative gravitons, all other interactions being defined by their standard 4d quantum field description. No realistic experiments can detect in our phase the non-perturbative quantum gravity properties, although its Langevin equation is the same as in the primordial phase.

The detailed evolution at the very end of the inflation is just whatever it was, and is not so meaningful, nor is the exact shape of the curve describing the end of a cross-over region. It is in fact well known in other branches of physics that, for most phase transitions, the details of the precise cross-over curve are very non-trivial as they depend on initial conditions which are in general not under control. As a consequence, the period following the exit from inflation can occur with or without reheating, that is, with or without oscillations in the recently created Lorentz time. There is a wealth of inflation models that can describe all possibilities. Such details are phenomenologically important, although they occur in a relatively short duration of perhaps $\sim 10^{-33}$ s. They are of great help as their signals seem to be almost certainly observable from precise CMB measurements, which is a great success of the inflation model.

So, at this level, the use of stochastic quantisation as the quantum formalism for all interactions is consistent with the description of the last moment of the exit from inflation by a standard inflation theory. More importantly, our considerations predict a phase transition and the definition of initial conditions: our phase starts with a gas of primordial black holes, some of which would shortly decay towards standard matter, depending on their sizes, and possibly to more exotic particles, for instance the so-called WIMPS. The heavier black holes are stable at the time scale of the age of the Universe, and will remain as a substantial contribution to dark matter, in addition to what possibly comes from the many post-inflation scenarios.

Our scenario leaves undecided the mysterious question of whether or not a small black hole can really fully shrink to a state with no black hole at the end of the process.

The decay of primordial black holes can only occur when the expanded Universe is large enough compared with the dimension of particles for the latter to be possibly created. When this has happened a reliable evolution can be computed according to the laws of the numerous viable gauge models where gravity is treated semi-classically. In this phase the Lorentz time has emerged and it can be used to order observed physical events.

It is gratifying that this scheme provides a good explanation of the new ideas that galaxies might often be built around black holes. Indeed, as it has become obvious, many of these black holes cannot be obtained by the collapse of standard matter. We predict here that a large part of the standard matter was created by the decay of some of these early primordial black holes (e.g., by Hawking like effects) determined by the transition from quantum to classical gravity, while the standard black holes stemming from gravitational collapses of stars are perhaps in the minority. They are in fact of secondary importance. Later in the paper, we will also comment on a possible application of this 5d framework to giving a gravitational structure to the 4d point-like particles, providing meaning to their ultra short range properties.

To summarise, in the framework of stochastic quantisation of quantum gravity and cosmology, the presence of “primordial black holes” is a signature of an unavoidable stochastic time fluctuation that has triggered the existence of inflation. In the post-inflation regime, when the Universe is big enough for gravity to be classical and small enough for the standard matter not to exist yet, it is plausible that the early cosmology can be well approximated by a scattered cloud of primordial black holes, with a mass distribution that determines their further decay to standard matter within a background of dark matter that is the remnant of the heaviest primordial black holes. The advantage of using stochastic quantisation with a second order term is that the same equation gives a possibly adequate new physics for the primordial cosmology and reproduces all known results for post inflation physics.

3 Answers to possible counterarguments against the use of the stochastic quantisation scheme of gravity

More can be said on whether any quantum behaviour that cannot be described by a stochastic quantisation equilibrium has consequences consistent with the known results in semi-classical gravity.

A quite simple possible counterargument can be made against the non-existence of the Lorentz time in the primordial cosmology. More precisely, since time is emergent in our picture, it is unclear at the first sight if one can eventually end with a system which is locally Lorentz invariant. In fact, the Lorentz symmetry must be already rooted in the bulk of stochastic quantisation, although it has no room there, since the Wick rotation cannot be done at finite stochastic time. This is not a minor issue because there are very strict bounds on violations of Lorentz symmetry.

The answer is in [2], which shows that the gravity Langevin equation computes the evolution of leaves, with metrics $g_{\mu\nu}(\tau, X)$ and imposes reparametrisation invariance in the Euclidean coordinate x^μ in each leaf at constant τ . So Eq. (10) is not 5d invariant, but can be made X^μ covariant for each value of τ , in a typical foliation. Thus there is a local $SO(4)$ symmetry in each leaf at each value of τ . This symmetry is preserved in the τ evolution and survives the exit from inflation. So as soon as the stochastic process has converged effectively, one can do the Wick rotation and an exact emerging local $SO(1, 3)$ symmetry is warranted in our phase.

Notice that all measurements that relate our epoch to the moment of exit from inflation have no access to the primordial 5d physics prior to the exit. Thus our scheme of primordial cosmology is compatible with all realistically possible experiments about the strict bounds on violations of Lorentz symmetry.

Notice also that there is no way that we can observe the physics in the phase of quantum gravity. Indeed, we have no apparatus to detect the evolution in stochastic time, which runs at a scale given by the super high frequency $1/\Delta T$ of oscillations in the second order Langevin equation. All precise information about the early cosmology is washed away after the exit from inflation analogously as all global information about a spin glass is erased after it transforms into a standard liquid or solid.

Another subtle point that should be addressed is the following. According to our model, the exit from inflation terminates the regime of non-perturbative quantum gravity completely. The question is whether this is compatible with the arguments for constraints on the couplings of classical gravity to quantum field theory.

For instance, puzzling questions were presented by Page and Geilker [4] and in [5] and references therein, where gedanken (and possibly realistic) experiments point out the necessity of a mechanism for allowing the standard collapse of the wave function of quantum matter coupled in a non-negligible way to classical gravity. Page and Geilker [4] investigated the consequences of the equation $E_{\mu\nu} \sim \langle T_{\mu\nu} \rangle$ for the interactions between classical gravity and quantum matter. They investigated the necessity for having a wave function collapse of the matter of which $T_{\mu\nu}$ is the energy-momentum tensor. To solve the inconsistencies, it is necessary and probably sufficient that some perturbative quantum gravity holds after the exit from inflation. In this way, gaussian perturbative quantum excitation emitted by the classical gravity field defining the Einstein tensor $E_{\mu\nu}$ will forbid any possible paradoxes about a violation of the matter wave function collapse.

It is thus necessary that the scheme of stochastic quantisation allows the exchange of perturbative quantum gravitons in our phase, at least at a very microscopic level, in order to allow possible back reactions. The logic is that it must occur as in QED, where analogous wave function collapse paradoxes disappear when one admits, correctly, that a classical background can emit or absorb photons. So a good question is whether some kind of perturbative gravity truly survives the exit from inflation, even if its non-perturbative quantum gravity effects have disappeared effectively.

As a matter of fact, perturbative quantum gravity, although not a renormalisable theory, is perfectly compatible with stochastic quantisation at any given finite order of perturbation theory provided the perturbation UV quantities are suitably regularised[†]. In fact, what may follow the exit from inflation is the classical gravity completed by its standard perturbative theory, suitably regularised. This theory allows emission and absorption of perturbative on-shell gravitons, much like a classical electric field is able to emit and absorb perturbative photons. This possibility is enough to satisfy the first principles of quantum mechanics including the wave function collapse property in our phase, with experiments such as those suggested in [4] and [5].

Thus, although gravity is basically classical everywhere in our phase, any gravitational background can be considered as a coherent state rather than a classical field. This means that any kind of gravitational background can evolve towards a slightly different one, modulo the emission or absorption of spin 2 gravitons. As in the much better understood case of QED, this ensure the possibility of back-reactions of a gravitational background on a microscopic quantum system. This should be sufficient to avoid paradox when one confronts the topics of measuring the collapse of the matter wave function of quantum matter under the influence of a gravitational field[§].

Another general question is whether or not one can we fully trust the Hawking formula of black hole evaporation, which is at best a classical or semi-classical computation, while neglecting completely aspects of non-perturbative gravity. There is a huge literature about controversies if some remnants of the black holes should stay when the mass of the black hole reaches values as small as the Planck mass, or even less. A better understanding of possible emission of perturbative gravitons is probably insufficient for solving this question about the black holes loss of information.

Stochastic quantisation may or may not bring a modest contribution to these topics. On the one hand it provides a new physical cutoff because of the coefficient ΔT of the second order term of its Langevin equation, and the ΔT dependance might bring new interesting topics. On the other hand, when matter is sufficiently dilute, no process can carry enough energy to create conditions as those of the quantum gravity regime of early cosmology, which require the 5-dimensional framework. Thus, the use of stochastic quantisation will probably not help understanding the fate of a black hole when it shrinks to zero. We do not know if a combined knowledge of the very short distance of gravity on the one hand and a better handling of its semi-classical approximation far away from singularities on the other hand may clarify this issue.

A more positive note is however as follows. The Hawking evaporation has some analogy with the Schwinger effect of matter creation in a strong electric field. Thus it must involve some quantum back-reaction of the classical gravitation field. Since we explained that very weak effect involving emissions and absorptions of perturbative gravitons are consistent in the scheme of stochastic quantisation, this may warrant at least formally the possibly unitarity in the physics of the black hole evaporation.

4 Standard quantisation and first order stochastic equations

We start now to explain how to reach concretely a second order Langevin equation for quantum gravity. We first explain the basic features of the first order stochastic quantisation, the goal being the justification of formula (10).

[†]To get a formal proof of this fact, one must incorporate in its Langevin equation a gauge restoring force such as the one in the Langevin equation we will display and justify in a geometrical way in [2].

[§]Since we invoke that gravity may and must involves perturbative effects in our phase, we may push the question further. Can we regularise and compensate the ultra-violet divergencies of gravity that also occur in the stochastic quantisation of gravity by looking at the gravity theory in a stringy context? The answer seems positive, since one can for instance treat the string Polyakov action in a stochastic approach, and it looks as if one can follow diagrammatically the phenomenon of compensation of ultraviolet divergencies of gravity in the stochastic quantisation scheme.

Let us consider a scalar field $\Phi(X)$ in d -dimensional Euclidean space with coordinates X^μ . Stochastic quantisation involves an additional “stochastic time” variable τ , in the sense that one promotes the field $\Phi(X)$ depending on X to $\Phi(X, \tau)$ depending on both X and a stochastic time τ , and is aimed at defining the τ -dependent correlations functions $\langle\langle \Phi(X_1, \tau_1) \cdots \Phi(X_n, \tau_n) \rangle\rangle$. If the limit $\tau \rightarrow \infty$ exists, it must equate the Euclidean correlator of the d -dimensional Euclidean QFT obtained by the standard formalism, either canonical quantisation or the path integral, according to

$$\lim_{\tau \rightarrow \infty} \langle\langle \Phi(X_1, \tau_1) \cdots \Phi(X_n, \tau_n) \rangle\rangle \Big|_{\tau_1 = \dots = \tau_n = \tau} \equiv \langle \Phi(X_1) \cdots \Phi(X_n) \rangle. \quad (1)$$

In zero dimensions, $\Phi(X)$ reduces to a point x and $\Phi(X, \tau)$ is simply a real function $x(\tau)$. So, in zero dimensions, the concept of stochastic quantisation can be handled with standard knowledge in analysis and distribution theory, and one gets the Brownian motion justification of the Boltzmann distribution. The extension to QFT is a limit where the number of points becomes infinite, but the general features of the zero dimensional case must remain the same. Stochastic quantisation extends at the quantum level the classical notion of a flow equation.

The quantum field theory proposal of Parisi and Wu [3] is that the stochastic time dependence of fields is through a (first order) Langevin equation, where the drift force is the Euclidean equation of motion. Giving a physical interpretation to the stochastic time τ is a step forward. In fact, Langevin built its basic equation as a second order equation, for non relativistic systems made of a large numbers of particles, but he reduced it to a first order equation by neglecting the acceleration, which is often a negligible term when one approaches the thermal equilibrium.

Given a local action $S[\Phi]$ with the relevant properties, one may postulate the existence of a Fokker-Planck evolution kernel $P^{\text{FP}}(\Phi(X), \tau)$, such that

$$\langle\langle \Phi(X_1, \tau) \cdots \Phi(X_n, \tau) \rangle\rangle \equiv \int [d\Phi]_X \Phi(X_1) \cdots \Phi(X_n) P^{\text{FP}}(\Phi(X), \tau). \quad (2)$$

Having a smooth quantum field theory means that $P^{\text{FP}}(\Phi(X), \tau)$ satisfies the parabolic equation

$$\frac{\partial}{\partial \tau} P^{\text{FP}}[\Phi, \tau] = \int dX \frac{\delta}{\delta \Phi(X)} \left(\frac{\delta S}{\delta \Phi(X)} + \hbar \frac{\delta}{\delta \Phi(X)} \right) P^{\text{FP}}(\Phi(X), \tau), \quad (3)$$

One must choose any given initial condition $P_0(\Phi(X))$ at $\tau = \tau_0$ for P^{FP} , but one can prove that the limit, if it exists, is independent of the choice $P_0(\Phi, \tau_0)$, with an exponential damping in powers of $\exp(-\tau)$ when $\tau \rightarrow \infty$.

The connection with Euclidean Feynman standard path integral is by proving that the equilibrium distribution of the Fokker-Planck distribution is

$$\lim_{\tau \rightarrow \infty} P^{\text{FP}}[\Phi, \tau] = \exp -\frac{1}{\hbar} S[\Phi]. \quad (4)$$

This is a stationary solution of Eq. (3). Gauge invariances can be taken into account in the stochastic framework. However this necessitates a further degree of sophistication that involves the framework of gauge covariant TQFT's [7].

It turns out that all relevant QFTs for elementary particles that are not coupled to quantum gravity can be defined by such a first order Fokker-Planck equation. One gets therefore a probabilistic interpretation of all their Euclidean correlation functions.

However, some Lagrangians, and in particular the Einstein Lagrangian, are such that the limit (4) cannot be reached or does not make sense. Here we will suggest that this is not in contradiction with the possibility that its correlators are well defined at finite values of the stochastic time, with some possible oscillatory dependence when the stochastic evolution is generalised from first to second order. The point of this paper is that this property has a relevant physical interpretation.

A first order Langevin equation often implies a Fokker-Planck equation. The Langevin framework is more general because it allows one to define correlators at different values of the τ_i . It also has a supersymmetric formulation in $d + 1$ dimensions, whose Hamiltonian interpretation illuminates very interestingly the Fokker-Planck formulation of many well-behaved theories [9].

As a matter of fact, Parisi and Wu proposed in their seminal paper [3] that the stochastic time dependence of the fields in the correlation functions $\langle\langle\Phi(X_1, \tau_1) \cdots \Phi(X_n, \tau_n)\rangle\rangle$ is determined from the equation[¶]

$$\frac{\partial\Phi(X, \tau)}{\partial\tau} = \frac{\delta S[\Phi]}{\delta\Phi(X, \tau)} + \sqrt{\hbar}\eta(X, \tau), \quad (5)$$

where $\eta(X, \tau)$ is a noise whose correlation functions define the quantisation, together with some initial condition of Φ at $\tau = \tau_0$, say $\Phi(X, \tau_0) = \Phi_0(X)$. In this simplest formulation leading to the Fokker-Planck equation (4), the noise is Gaussian white, which means

$$\langle\langle\eta(X, \tau)\eta(X', \tau')\rangle\rangle = 2\delta(\tau - \tau')\delta(X - X'). \quad (6)$$

The Langevin equation is thus very much like a Brownian motion equation where $\frac{\delta S[\Phi]}{\delta\Phi(X, \tau)}$ is a drift force. Eq. (5) shows that rotational invariance is generally not ensured in $d + 1$ dimensions. Such an invariance can be however enforced in a few cases, e.g., for certain Chern-Simons actions [7]. The stochastic time dimension is the square of the dimension of space coordinates if the free part of the action is second order in the space derivatives.

The prescription to get rid of the τ dependence and define the Euclidean correlation functions $\langle\Phi(X_1) \cdots \Phi(X_n)\rangle^{\text{Euclidean}}$ of the d -dimensional smooth quantum field theory is by solving eq. (5) to express Φ as a function of τ . Then one computes the correlation functions of the $(d + 1)$ -dimensional theory at equal τ by averaging over the noises and one takes the limit $\tau \rightarrow \infty$, assuming for example that $\Phi = 0$ at $\tau = 0$. The result is, generically,

$$\lim_{\tau \rightarrow \infty} \langle\langle\Phi(X_1, \tau) \cdots \Phi(X_n, \tau)\rangle\rangle = \int [d\Phi]_X \Phi(X_1) \cdots \Phi(X_n) \exp\left(-\frac{1}{\hbar}S[\Phi]\right) \equiv \langle\Phi(X_1) \cdots \Phi(X_n)\rangle^{\text{Euclidean}}.$$

One can often prove the convergence of the stochastic process toward the same limit for arbitrary initial conditions at $\tau = \tau_0$, whatever the value of τ_0 is. The statistical Physics literature shows in many different ways that the Langevin equation formulation gives the same result as the Fokker-Planck formulation for $\lim_{\tau \rightarrow \infty} \langle\langle f(\Phi_\tau) \rangle\rangle$, but, as we already said, the Langevin equation is more fundamental. As will be explained in a separate publication, the Langevin equation formulation allows one to investigate the interesting question of the consequences of possible reparametrisations of the stochastic time.

In the Fokker-Planck formalism, it is rather easy to prove that the damping of the Langevin/Fokker-Planck process toward the usual path integral formula amounts to the possibility of normalising the Euclidean path integral, that is^{||}:

$$\int [d\Phi]_X \exp\left(-\frac{1}{2\hbar}S[\Phi]\right) < \infty. \quad (7)$$

This condition is fulfilled for all renormalisable theories that satisfies (7) and many quantum mechanical models with a discrete spectrum and a standard normalisable vacuum.

[¶]There is the possibility of inserting a kernel in factor of $\frac{\delta S[\Phi]}{\delta\Phi(X, \tau)}$ to improve the convergence, if any, of the stochastic process, without changing the conclusion of the foregoing discussion. A non-trivial kernel will appear in Eq. (10) and will be studied in great details in [2].

^{||}The necessity of this condition is a general result of statistical physics that can be proven in many ways. A simple proof is in [6]

But it is not fulfilled non-perturbatively in the case of 4d gravity, since in this case the field is $g_{\mu\nu}$ and $S[\Phi]$ amounts to the Euclidean Hilbert-Einstein action that is not positive definite. In this theory (7) cannot hold because metric fluctuations can change the sign of the scalar Euclidean space curvature. This is consistent with the fact that time cannot be globally well defined in quantum gravity.

We now discuss how stochastic quantisation may improve our understanding of quantum gravity.

5 A possible way out for the stochastic quantisation of gravity

For gravity, given some initial conditions to the Langevin equation, there is generally no smooth limit for the stochastic process at $\tau \rightarrow \infty$ beyond perturbation theory. We are lead to the following strategy.

We observe that the stochastic evolution is well defined at finite values of stochastic time. The metric correlators, generically denoted as $\langle\langle \mathcal{F}(g_{\mu\nu}(\tau, X)) \rangle\rangle$, describe some complicated 5d physics without possibility of defining a Lorentz time at finite values of τ . But if the 4d initial conditions corresponds to a semi-classical quantum field theory configuration, the stochastic evolution of $\langle\langle \mathcal{F}(g_{\mu\nu}(\tau, X)) \rangle\rangle$ will admit a well-defined, smooth $\tau \rightarrow \infty$ limit, which can be described by a 4d semiclassical path integral.

We thus have the following possibility. We can suppose that at some given finite value of τ there is a non-zero probability that a given fluctuation transforms an arbitrary configuration in a classical or semi-classical gravity configuration. This transition can be seen as a reset of the boundary condition of the stochastic process. After this reset, the stochastic process becomes smooth by definition, since the standard stochastic quantisation is compatible with the semi-classical quantum field theory.

In order to trigger such a fluctuation, we need to use the exact Langevin equation that contains a second order term, since it is only justified to neglect it when one approaches smoothly the equilibrium limit for $\tau \rightarrow \infty$. This is an advantage, since the energy carried by the stochastic acceleration can enhance the radical changes in the metric to get a larger probability for the wanted transition to a semi-classical configuration.

Another remark is that the difficulties brought by the Wheeler-DeWitt equation when one tries to quantise gravity in 4d are of no relevance in the 5d formulation, because the gravitational Hamiltonian for the stochastic evolution in 5d does not vanish. (As said earlier, there is no 5d general covariance, and the Fokker-Planck Hamiltonian is not vanishing, which implies that the τ -evolution cannot be frozen). The stochastic time τ has thus a status different from that of the 4 Euclidean coordinates X^μ . In [2], it will be shown in great details that the role of τ is to foliate the 5d space by Euclidean 4d leaves, each having its own 4d reparametrisation covariance.

When the convergence at $\tau \rightarrow \infty$ is ensured, one can in principle make a Wick rotation that is an analytic continuation of all correlators in one of the Euclidean coordinates X^μ , in the limit $\tau \rightarrow \infty$. Then all necessary Lorentzian correlators can be used to compute perturbatively S -matrix elements, giving a particle interpretation to the theory with well-defined scattering amplitudes and clocks. The standard perturbation theory allows one to prove this property.

The question of the possibility of a Wick rotation at finite stochastic time has not been extensively studied in the literature. A recent paper has studied the question of establishing Hilbert space positivity as a precise mathematical result at finite time [8]. A simple argument is that it usually makes no sense to have a theory with two distinct parameters of evolution to order phenomenon.

The equivalence proofs between standard path integral quantisation and stochastic quantisation with a first order Langevin/Fokker-Planck amount to show that there is a smooth damped relaxation of the stochastic process toward the Euclidean correlation functions defined by the path integral with

weight $\exp(-\frac{1}{\hbar}S)$ (the analogue of a Boltzmann weight in statistical physics). It can only be done if the equilibrium process is itself well defined.

About the classical limit, when the effect of the noise is less stringent, the solution is that the fields concentrate around the classical solutions, until they reach the classical limit. In fact, we call τ -dependent classical states solutions to the Langevin equations when the noise is neglected all along the τ evolution. This operation can be heuristically understood by taking the limit $\hbar \rightarrow 0$ in the Langevin equation for certain fields. In fact, stochastic coherent states can be also defined as solutions that depend on the noise and are as near as possible to stochastic classical fields when one computes correlators in a natural generalisation of the Schrödinger picture.

We pointed that, as in the theory of earthquakes, considering the effect of the stochastic acceleration can be a useful feature with relevant consequences. Thus, we now come to the generalisation of a first order stochastic evolution into a second order one.

6 Stochastic quantisation with second order Langevin equations

We now generalise the first order equation (5) into

$$\Delta T^2 \frac{\partial^2 \Phi(X, \tau)}{\partial^2 \tau} + 2b \frac{\partial \Phi(X, \tau)}{\partial \tau} = \frac{\delta S[\Phi]}{\delta \Phi(X, \tau)} + \sqrt{\hbar} \eta_{\Phi}(X, \tau). \quad (8)$$

In other words, we will analyse the new physical features of the dynamics of stochastic quantisation after $\partial_{\tau} \equiv \partial/\partial\tau$ is replaced by $\Delta T^2 \partial_{\tau}^2 + 2b \partial_{\tau}$. Here b is a dimensionless positive real number, a is a positive real number with the same dimension as $\sqrt{\tau}$ and can be perhaps used as a physical ultra-violet cutoff, $\sim 1/\Delta T$, for certain theories. For $\Delta T \rightarrow 0$, the standard stochastic quantisation occurs, and it is a consistent formulation provided we have a renormalisable QFT with an appropriate potential. If $b = 0$, we have oscillatory solutions that are unlikely to become stationary for $\tau \rightarrow \infty$. So we must always have a friction term, proportionally to b . A deeper understanding of the dissipation that occurs for $b \neq 0$ is of great interest.

For $\Delta T \neq 0$, as discussed in [1], the standard first order equation is effectively recovered when the action of the operator $\Delta T^2 \partial_{\tau}^2$ is much smaller than the action of $2b \partial_{\tau}$, at least perturbatively, if one has a renormalisable theory for S . The combination of the effect of the noise and of the acceleration term proportional to ΔT^2 can lead the system to nontrivial relaxations towards a possible equilibrium, like oscillating ones that cannot be predicted by a genuine first order equation. This justifies heuristically the intuition that $1/a$ is a physical UV cutoff, by comparing the time scale of stochastic time oscillations with those time scales that occur in standard quantisation when $\tau \rightarrow \infty$. In fact, beyond all heuristic justifications as those of [1], the original version of the Langevin equation was of second order. Langevin approximated it as a first order equation, only for the purpose of studying the thermal equilibrium approach.

In the example of [1], we considered a 6-dimensional scalar (one real and one complex field) renormalisable theory with interacting terms $g\phi^3 + e\phi\bar{\psi}\psi$. (This was to avoid the more confusing case of fields with genuine self-interactions.) In this theory the real field ϕ can be treated as a background coherent state ϕ_{cl} and ψ is a complex quantum field that couples with the background ϕ_{cl} by its current $e\bar{\psi}\psi$, similar to a background electromagnetic field coupling to an electron current, or a classical background gravitational field coupling to a microscopic quantum black hole pair.

Eq. (8) is translation invariant. Since it contains the friction term proportional to b , the stochastic process fundamentally involves a dissipation of the stochastic time energy. For the cases where there is dumping toward the equilibrium at infinite stochastic time, this stochastic energy will reach some constant. For $a \neq 0$, one has oscillations on the way, and the entropy increase follows a different

pattern. The physics at finite stochastic time is different for $\Delta T = 0$ and for $\Delta T \neq 0$. Before the possible equilibrium, there are stochastic time slices during which the term $\Delta T^2 d^2/dt^2$ dominates the term bd/dt .

Non-trivial physics occurs then with an approximate conservation of the energy, since one can neglect the friction term for a while. For these intervals of time, one can estimate which phenomena must balance the τ -energy variations in the τ oscillating background ψ_{cl} . The latter is computed by solving the Langevin equation for $\eta_\phi = 0$, and the quantum effects must be considered as those around this background, which one understands how to compute. Our scenario is that the quantum excitations for the field ψ during this τ -dependent process are analogous to pair creations of the Schwinger type.

The pair creations or absorptions that occur all along the τ -evolution are a back reaction that compensates the effects of the vacuum τ -oscillations. The whole process is driven by the effect of the noise and its relation to the field by a second order differential equation in τ . The stronger ϕ_{cl} is, the stronger is the counter effects of pair annihilation and creation for the field ψ . In principle, this can be perturbatively computed by solving the stochastic equation including the effect of the noise η_ψ in the background field ϕ_{cl} .

When one relaxes to an equilibrium, if any, the damping of the periodic growth and contractions of the energy of ϕ_{cl} stops, as well as the creations and absorptions of pairs as they given by the Langevin equation, whose consequences can be described by a supersymmetric theory in the bulk. We will be more explicit in what is the supersymmetric theory in the bulk in Appendix B using a concrete and very simple example. For gravity, we actually claim the equilibrium can be reached in a discontinuous way, under the form of a phase transition, because the field properties mix with the structure of the space. A comparison with what can happen with spin glasses was discussed in the Introduction.

In Appendix A, we shall justify some of these speculations by a careful analysis of a zero dimensional case, where the fields Φ and Ψ are replaced by two real numbers x and y , as an extreme dimensional reduction, to explain the methodology and the way to handle the initial data. It was argued in [1] that this early dynamics might even keep away the field excitations from dangerous part of the potential for a while.

There is a period where the damping is very slow as compared to the rapidity of the oscillations. During this period, one may approximate

$$\Delta T^2 \frac{\partial^2 \Phi(X, \tau)}{\partial^2 \tau} + 2b\partial\Phi(X, \tau)\partial\tau \sim \Delta T^2 \frac{\partial^2 \Phi(X, \tau)}{\partial^2 \tau} \quad (9)$$

The dashed line in the heuristic description by Figure 1 represents this regime.

7 Introducing the second order Langevin equation for the gravity quantisation

For gravity, one must promote the space metric $g_{\mu\nu}(X)$ to $g_{\mu\nu}(X, \tau)$. One needs to introduce of additional components $g_{\mu\tau}(X, \tau)$ and $g_{\tau\tau}(X, \tau)$ following the general method for defining the stochastic quantisation of a theory with a gauge invariance. Thanks to these extra fields, one can construct a covariant derivative $\nabla_\tau = \nabla/\partial\tau$, and the gravity stochastic equation can be written using the method of equivariant cohomology. (For a 2d example and the YM_4 case where such additional fields are introduced, see the last reference in [7].) The second order gravity Langevin equation must be

$$\begin{aligned} \Delta T \nabla_\tau^2 g_{\mu\nu}(X, \tau) + b \nabla_\tau g_{\mu\nu}(X, \tau) &= G_{\mu\nu\mu'\nu'}(R^{\mu'\nu'}(X, \tau) - \frac{1}{2}g^{\mu'\nu'}(X, \tau)R \\ &+ \kappa g^{\mu'\nu'}(X, \tau) - 8\pi G T^{\mu'\nu'}) + \sqrt{\hbar} \eta_{\mu\nu}(X, \tau), \end{aligned} \quad (10)$$

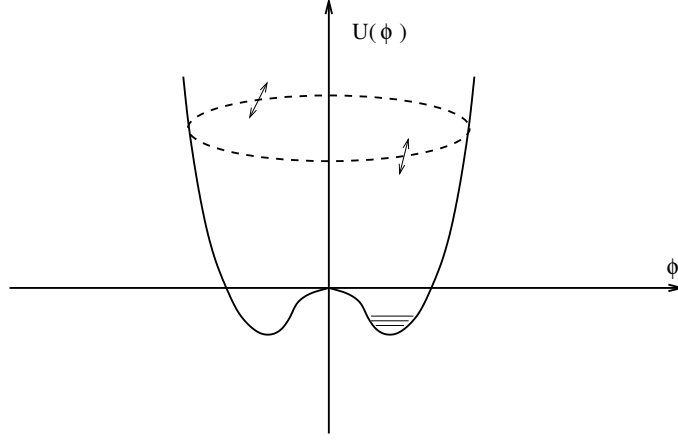


Figure 1: The dashed line represents one among the oscillating stochastic time dependent “vacua” (OSTDV) ϕ_{cl} , and the double arrows represent symbolically the pair creations and annihilations that occur to periodically compensate for the approximate conservation of the total energy when τ varies, as explained in the text. So the “inertial force” due to the second order terms keeps ϕ_{cl} away from the absolute minima of the potential for a while; the process is reminiscent of a turbulent phenomenon. Because of the friction terms in the second order Langevin equation, the angular velocity decreases as $\tau \rightarrow \infty$ and ϕ_{cl} will fall into one of the absolute minima of the potential. Then, either the ordinary quantisation or the classical behaviour will prevail with the damping of the τ dependence. For gravity, the equilibrium is a diluted Universe, filled with scattered classical primordial black holes, the smallest of which decaying then quickly to matter.

$\eta_{\mu\nu}$ is the noise for the Euclidean $g_{\mu\nu}$, and one must have a (non-white) Gaussian distribution

$$\langle\langle \eta_{\mu\nu}(x, \tau) \eta_{\mu'\nu'}(x', \tau') \rangle\rangle = G_{\mu\nu\mu'\nu'}^{-1} \delta^4(x - x') \delta(\tau - \tau'), \quad (11)$$

where the kernel $G_{\mu\nu\mu'\nu'}(x, \tau)$ is a metric on the space of metrics $g_{\mu\nu}(x, \tau)$ and is a quadratic functional of $g_{\mu\nu}$ for ensuring the reparametrisation invariance of (10) for each value of τ .

Eq. (10) is a complicated non-linear differential equation. It will be analysed in great details in [2], where most importantly, the role and the exact definition of $G_{\mu\nu\mu'\nu'}$ will be thoroughly examined. The construction and geometrical identification of the covariant stochastic speed and acceleration operators ∇_τ and ∇_τ^2 is a non-trivial task and will be also explained in [2].

If one solves Eq. (10) and eliminates the noise in the standard way as prescribed by stochastic quantisation method, it produces correlation functions at a finite stochastic time τ that depend on the initial data $g_{\mu\nu}(X, \tau = \tau_0)$. Therefore the dimensionful parameters that may occur in the solutions of the stochastically quantised quantum gravity are not only a , b and \hbar , κ and the Newton constant G , but also the initial condition that describes the initial Euclidean geometry of the universe at $\tau = \tau_0$.

The value of the stochastic time, measured in units of a , when the Universe exits from inflation depends on the initial data. After the exit from inflation, which was induced by a random fluctuation according to our scenario, gravity becomes by definition purely semi-classical. One has essentially a standard classical evolution of the gravity from a given geometry, modulo some negligible emissions of perturbative gravitons, which must fit the data of today’s observations.

The physical events, when gravity was behaving in quantum and non-perturbative way, were ordered by the evolution in stochastic time in a 5d framework. It is only when the stochastic time has reached the value when gravity can be treated effectively as a semi-classical theory everywhere can the correlators in the final 4d leave be computed by the standard path integrals with the semi-classical gravity weight $\exp(-\frac{1}{\hbar}S)$. All Physics can then computed by its value at the boundary $\tau \sim \infty$ and the

ΔT dependence is fully negligible. Modulo a very short period where one needs inflaton fields, gravity can then be treated for the whole future semi-classically while other interactions are still treated fully quantumly. It is only after this possible mathematical emergence of the Lorentz time, by the possibility of a Wick rotation of the Euclidean correlators in stochastic equilibrium, that causality can be introduced in the usual sense, giving the property that an observer can only detect phenomena having occurred in its future light cone.

Before the transition between primordial cosmology and the post-inflation semi-classical gravity regime, the impossibility of defining the Euclidean path integral of gravity is due to the fact that the Einstein action is not suited to define a normalised equilibrium description for $\hbar \neq 0$. In the language of statistical physics, it translates to saying that we cannot compute by a Boltzmann average the correlators of a system that has no thermalisation.

8 Conclusion and physical outputs

There is interesting physics for the correlators of a stochastically quantised Euclidean field theory at finite stochastic time when stochastic quantisation τ is of second order. It is ignored because we cannot build clocks that measure the evolution in function of the stochastic time.

We have done some elementary computations to illustrate the non-trivial oscillations undergone by correlators when the stochastic process converges smoothly.

In some cases, it may happen that no smooth limit $\tau \rightarrow \infty$ exists, independent of the initial conditions. This is the case for gravity. Our proposal, with non-trivial physical consequences, is that instabilities can occur on the way, especially when stochastic quantisation τ is of second order, so that the phase of the system can change abruptly.

The Introduction of this article has detailed qualitatively some physical predictions suggested by the definition of quantum gravity through stochastic quantisation. We will now make them a bit more detailed.

We suggest that stochastic quantisation opens a window for new speculations on primordial cosmology. It says that, given that the time scale ΔT of stochastic oscillations is extremely small (e.g., 10^{-65}s) compared to the scale of the length of the exit of inflation $\sim 10^{-32}\text{s}$, a fluctuation necessarily occur, producing a transition from the quantum gravity phase to the classical phase. The fundamental difference between the classical and the quantum phases is the existence or not of the Lorentz time.

One must check the possibility of having a Lorentz time that emerges by inspections of the analytical properties of the stochastic Euclidean correlators that stochastic quantisation defines and allows one to compute. This gives us two possible phases for gravity. The classical phase, after the exit from inflation, is the standard model coupled to classical gravity, where, effectively, all fields take their values at infinite stochastic time, and all phenomena can be consistently ordered by the Lorentz time, a property that can be proved theoretically (at least rigorously in perturbation) theory, but that one must check experimentally, instead of postulating.

The verification involves experiments done at effective infinite stochastic time. The accuracy of this verification is optimal in the big accelerators and in astrophysics, and it goes down to $\sim 10^{-27}\text{s}$ nowadays. Atomic clocks provide an accuracy of $\sim 10^{-18}\text{s}$. However, this accuracy is obviously totally out of scale as compared to the smallness of the time scale for detecting quantum gravity effects, as predicted by the Langevin equation. Any given experiment is helpless, since it has to be made with an instrumentation that is described by the physics of matter and fields that is only relevant in our (semi)classical gravity phase. This is unfortunate because the possible physical existence of the stochastic time seems theoretically favoured, giving a microscopic justification of the success of

standard QFTs at the scale of all known experiences.

The other phase, where quantum gravity is involved, predicts that there is no Lorentz time, which is in fact in agreement with the Wheeler-DeWitt constraint. Rather, there are oscillations in the stochastic time of Euclidean gravitational correlators with no smooth relaxation, which defines the physics of quantum gravity, and which we have pictured as exchanges of energy between the 5d classical solutions and the 5d quantum states (whose 4d wave functions could be related to that of a 4d instantons), as a generalisation of the Schwinger effect.

So, we have suggested that there are oscillations in the stochastic time of Euclidean correlators with no smooth relaxation, which define the physics of quantum gravity. The evolution in function of the stochastic time τ of this physics is governed by the 4d Hamiltonian \mathcal{H}_4 of the 5d (+ - - - -) supersymmetric theory corresponding to the stochastic Langevin equation defined by Eq. (10). Using Legendre transform, \mathcal{H}_4 can be computed from the second order Fokker-Planck supersymmetric action with metrics signature (+ - - - -) of the 4d Euclidean gravity. Fig. 1 sketches the oscillating exchanges of energy between the stochastic time dependent coherent state obtained by solving the Langevin equation at vanishing noise, which generalises into a 5d background in the case of gravity, with the annihilations or destructions of the 5-dimensional states around it, which are elements of the spectrum of \mathcal{H}_4 .

This regime of oscillations can be abruptly changed with a transition from the quantum to the classical phase provoked by a strong enough fluctuation, acting as a reset of the initial conditions of the stochastic process.

The way we visualise the mechanism of stochastic quantisation of gravity in a small enough volume implies that this transition results in the appearance of a diluted Universe, with a rather small dimension at the beginning, and where the Lorentz time has emerged (as a change in the analytical properties of Euclidean correlation functions). At the end of the inflation, it says that this small enough Universe is filled with faraway scattered primordial black holes before it expands. We may consider this as a physical prediction of the definition of Euclidean quantum gravity through a second order Langevin equation.

The spatial distribution and mass distribution of theses primordial black holes (meaning solutions of 4d classical Lorentz equations of motion) allow a classical description of gravity almost everywhere. It might be that this scheme gives a preferred initial condition for any model based on inflation theory. At this stage of the evolution of the Universe, it is by the definition of the change in phase that the recently “emerging” Lorentz time allows us to order phenomena by the standard QFT formalism with its own smooth causality. The inflation theory gives a phenomenological description of the last moments of the transition, as soon as the Lorentz time has emerged and can be used to order phenomena.

After the transition, the mass distribution of these 4d primordial black holes opens the possibility of having elementary particles in between, under the form of ordinary matter resulting from evaporation mechanisms of the lightest primordial black holes. In this 4d description, one assumes that one never encounters the unsolved question of 4d point singularities. This assumption is justified by the impossibility of concentrating enough energy at a given point to ignite the quantum gravity regime.

Giving such justified initial conditions for the post inflation Universe may provide a realistic description of the phase of the Universe as it is now, using the Lorentz time as an effective parameter for the standard causal evolution. The inflation theory gives a phenomenological description of the last moments of the transition, as soon as the Lorentz time has emerged and can be used to order phenomena.

In fact, for these last moments of the transition, one may say that the phenomenological smooth Lorentz time dependence is described by the choice of this inflaton field. The transition is definitely

over when the cosmological constant has become a fixed number and every manifestation of the former phase involving quantum gravity has faded away. Then, the inflaton field disappears from our description of physics, and all fundamental fields are enough to describe physics by the standard QFT formalism, where the stochastic time evolution has been thermalised, meaning that effectively physics can be always computed at $\tau = \infty$, and the Lorentz time can be used as an (almost) fundamental evolution parameter. One may presumably find realistic and simple distributions of these predicted primordial black holes with the present cosmological measurements as consistent initial conditions for the various cosmological inflation models.

It is worth trying to give a suggestive name to the 5d gravitational quantum states that are created and absorbed in the 5d space with signature $(+ - - -)$. It would be misleading to call them 5d black holes; rather they are more like particles in 5d, with a 4d particle shape. We suggest to call them gravitational partons. Before the phase transition, the metric has an oscillating evolution in stochastic time. It goes together with fast creations and annihilation of these 4-dimensional particles, which build the spectrum of \mathcal{H}_4 .

Now, in a sort of reverse way, let us consider a gedanken experiment in the 4d Lorentz space. Suppose we can build an accelerator to produce a collision at ultra short distances of two massive point-like particles existing in our phase, and observe their collision from afar. We can for example consider the ultra-deep inelastic collision $e + e \rightarrow$ inclusive set of observed particles. When both particles are at an ultra short distance from each other, each one “feels” the very strong gravitational field created by the other particle. This very short distance collision should be analysed in the context of quantum gravity. In the range of the interaction, the conditions are basically locally the same as they are everywhere in primordial cosmology, and the theory that one uses must be the one we just discussed, with its 5d metrics signature $(+ - - -)$.

It is suggestive that, for the very short range collision, each particle sees the other particle as a cloud of the above mentioned elements of the spectrum of \mathcal{H}_4 , and the interaction for the very short duration of the collision amounts to interactions between these microscopic constituents, governed by the 5d theory, which we just called 5d gravitational partons. In this description of the ultra high energy collision of 4d “point-like” particles, after the very short moment of the quantum gravity interactions, the energy that was exchanged between the 5d gravitational partons of both particles will be redistributed among standard 4d particle degrees of freedom, which will be detected sufficiently far away from the zone of interaction. This view is a wild generalisation of the parton model in QCD.

This illustration of what might be the ultra short collisions of “point-like” particles might perhaps justify a posteriori the transition between the primordial cosmology and the post-inflation period.

We can now rephrase the scenario. At the end of the phase change that terminates the epoch of primordial cosmology, namely the exit of the inflation, the states made of the 5d gravitational bound states are replaced by solutions of classical gravity with the Lorentz signature $(+ - - -)$. When one observes them at a far enough distance, some of them behave as ordinary 4d elementary particles and others are organised as classical 4d black holes. At the very beginning, it is most likely that we only have black holes, till the Universe is large enough. For all of them, effectively, the stochastic time can be estimated as blocked at its infinite value. Sufficiently far away from their singularities, we just have the semi-classical gravity coupled to the rest of the standard interactions. The evolution of the Universe can then be ordered in function of the Lorentz time, with an effective field theory which is that of the inflation. After the transition to its semi-classical phase, the microscopic quantum gravity 5d theory with the $(+ - - -)$ signature is effectively reduced to a 4d theory with a $(+ - - -)$ signature, modulo a Wick rotation from a $(- - - -)$, namely the standard semi-classical gravity.

After the exit from inflation, and for all realistic experiences, the standard semi-classical gravity is helpless to describe the “interior” of the elementary particles. This is compatible with the fact that the 4d description of quantum gravity cannot describe the physics at a very short distance such as the

scale ΔT . In fact, our model says that the internal QFT “inside” a so-called “point particles”, e.g., an electron or a quark, is the 5d quantum gravity theory.

The 5-dimensional properties imply that all particles have a gravitational structure and suppress the contradictions brought by the notion of being 4d point-like. This also eliminates the question of trying to give a physical sense to the gravitational singularities at a very short distance in 4d with a Lorentz metric: the gravity theory changes at extremely short distance in the 4d Lorentz space and becomes 5d. This theoretical extension might be useful to clarify aspects of the information paradox. Indeed in the 5d formulation, no particle from the outer space will ever penetrate in the usual sense the very short distance core of a black hole, at the scale where the gravity becomes quantum without leading to contradiction at long distances. It might be the case that all particles must have at least a very tiny mass in order to have a consistent QFT description in our 5d formulation.

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Appendix

A Second order Langevin equation in a perturbative zero-dimensional example

Consider the zero dimensional case of two real variables x and y with an action

$$S(x, y) = \frac{1}{2}M^2x^2 + \frac{1}{2}m^2y^2 + \frac{\lambda}{2}xy^2. \quad (12)$$

The stochastic quantisation of $S(x, y)$ introduces a bulk time τ and promotes x, y to $x(\tau), y(\tau)$, and one builds eventually a quantum mechanical path integral with a supersymmetric Lagrangian $\mathcal{L}_{\text{stoc}}(x, y, \psi_x, \psi_y)$, to be determined later on, such that for any observable $f(x, y)$, one has

$$\begin{aligned} & \int dx dy f(x, y) \exp\left(-\frac{1}{\hbar}S(x, y)\right) \\ &= \lim_{T \rightarrow \infty} \int [dx]_\tau [dy]_\tau [d\psi_x]_\tau [d\psi_y]_\tau [d\bar{\psi}_x]_\tau [d\bar{\psi}_y]_\tau f(x(\tau), y(\tau)) \exp\left(-\frac{1}{\hbar} \int_0^T d\tau \mathcal{L}_{\text{stoc}}(x, y, \psi_x, \bar{\psi}_x, \psi_y, \bar{\psi}_y)\right). \end{aligned} \quad (13)$$

Second order stochastic quantisation means in fact that, in correspondence with such a supersymmetric representation, there is an underlying Langevin equation

$$\begin{aligned} \left(a^2 \frac{d^2}{d\tau^2} + 2b \frac{d}{d\tau}\right) x(\tau) + M^2 x(\tau) + \frac{\lambda}{2} y^2(\tau) &= \sqrt{\hbar} \eta_x(\tau), \\ \left(a^2 \frac{d^2}{d\tau^2} + 2b \frac{d}{d\tau}\right) y(\tau) + m^2 y(\tau) + \lambda x(\tau) y(\tau) &= \sqrt{\hbar} \eta_y(\tau) \end{aligned} \quad (14)$$

that defines the τ evolution. Here η_x and η_y are taken as Gaussian noises, i.e.,

$$\langle\langle \eta_x(\tau) \rangle\rangle = \langle\langle \eta_y(\tau) \rangle\rangle = 0, \quad \langle\langle \eta_x(\tau) \eta_x(\tau') \rangle\rangle = \langle\langle \eta_y(\tau) \eta_y(\tau') \rangle\rangle = 2b\delta(\tau - \tau'). \quad (15)$$

All other Gaussian correlators for mean values of higher order products of the η 's follow by averaging with the Gaussian distribution $\int [d\eta]_\tau \exp(-\frac{1}{4b} \int d\tau \eta^2(\tau))$.

The case $a = 0$ and $b = \frac{1}{2}$ is for the standard first order stochastic quantisation of [3].

In what follows, both $a \neq 0$ and $b \neq 0$. For a more general potential $S = \frac{1}{2}M^2y^2 + \frac{1}{2}m^2x^2 + \lambda V(x, y)$, we must replace $\frac{\lambda}{2}y^2$ and λxy in Eq. (14) by λV_x and λV_y , respectively. Of course, not all choices of the “pre-potential” S produce proper convergence of solutions at infinite τ . For $a \neq 0$, we need to complete each Langevin equation with two boundary conditions, instead of one condition in the first order case. Here we assume that at two values of the time, $\tau = \tau_1$ and τ_2 , the coordinates x and y take the values $x(\tau_i) = x_i$ and $y(\tau_i) = y_i$ for $i = 1, 2$. For regular theories, the infinite time limit is expected to be independent of the choice of such conditions.

Our aim is to consider that one of the field, here $x(\tau)$, is a “coherent state”, which is, in the Schrödinger sense, a state that minimises maximally some quantum fluctuations. So $x(\tau)$ is a state that is as close as possible to a solution where one neglects everywhere η_x . This situation has been advocated to in [1], to define the primordial cosmology.

Moreover, since we understand intuitively that at finite stochastic time $y(\tau)$ undergoes quantum effects around the “strong” coherent state $x(\tau)$, we introduce an arbitrary given fixed position x_{cl} for x , which can be interpreted as the classical equilibrium value of $\langle\langle x(\tau) \rangle\rangle$ when $\tau \rightarrow \infty$. We thus reduce the above coupled Langevin equations, to the case where effectively $\eta_x = 0$, that is,

$$\begin{aligned} a^2\ddot{x} + 2b\dot{x} + M^2(x - x_{\text{cl}}) + \frac{\lambda}{2}y^2 &= 0, \\ a^2\ddot{y} + 2b\dot{y} + m^2y + \lambda xy &= \sqrt{\hbar}\eta_y. \end{aligned} \quad (16)$$

Solving perturbatively Eqs. (16) as a Taylor expansion on λ is of course possible. Using Green’s function techniques gives a hint of the physics at finite τ . In fact, some aspects of the loop expansion that will occur for $a \neq 0$ are better viewed by the redefinition of fields and coupling constant,

$$\hbar\lambda \rightarrow \lambda, \quad \frac{x}{\sqrt{\hbar}} \rightarrow x, \quad \frac{y}{\sqrt{\hbar}} \rightarrow y. \quad (17)$$

After such redefinitions, the Planck constant \hbar disappears from the Langevin equations, which become

$$D_\tau^M(x - x_{\text{cl}}) + \frac{\lambda}{2}y^2 = 0, \quad D_\tau^m y + \lambda xy = \eta_y, \quad (18)$$

where $D_\tau^M \equiv a^2\partial_\tau^2 + 2b\partial_\tau + M^2$, $D_\tau^m \equiv a^2\partial_\tau^2 + 2b\partial_\tau + m^2$, and the perturbative loop expansion of the correlators can be expanded in powers of the rescaled coupling constant $\lambda\hbar \rightarrow \lambda$.

A.1 Perturbative expansion and finite τ QFT behaviour

The perturbative expansion in λ for the τ evolution of x and y can be represented by Feynman diagrams with insertions of the noises analogous to those in [3], except that its propagators G_M and G_m have a double pole structure instead of being of parabolic type in [3]. There is a forward propagation of modes with positive and negative energy in the τ evolution, and one has insertions of the field $x(\tau)$ in addition to the insertions of η_x on the propagators of the $y(\tau)$. The Feynman diagrams that one can draw to describe perturbation theory involve closed loops.

In a Fourier transformation over τ , using the conjugate variable E , we must define particles of type y and antiparticles of type \bar{y} , with creation and annihilation operators acting on a Fock space, basically because we have solutions of positive and negative energy in a symmetric way.

Because $a \neq 0$, the τ dependence implies a relativistic quantum field theory framework rather than a non-relativistic framework as in the case $a = 0$. The closed loops can be interpreted as forward stochastic time propagations of particles of type y and antiparticles of type \bar{y} .

The closed loops are finite integrals over a one dimensional momentum space, with neither infra-red nor ultra-violet divergences, since $m \neq 0$ and $M \neq 0$. They occur in the perturbative expansion of $\langle\langle x^p(\tau)y^q(\tau)\rangle\rangle$ and can be interpreted as creations of a virtual pairs created by the vertex $\lambda xy\bar{y}$ of particle and antiparticle y and \bar{y} at a given value of the stochastic time, each one propagating forwardly in τ , until they annihilate at a further stochastic time, with possible interactions with the “classical field” $x(\tau)$.

This suggests that a double Fock space must be constructed. It is made of all possible states that can occur for the τ evolution, one for all possible “background vacua” of x , representing the stochastic oscillations of x and determined by solving the second order Langevin equation at $\eta_x = 0$, and the other one for the ordinary quanta of the field y emitted around this background. This description will become clearer by studying the Fokker-Planck Lagrangian $\mathcal{L}_{\text{stoc}}$ and Hamiltonian associated to the second order Langevin equation: $\mathcal{L}_{\text{stoc}}$ contains a higher order derivatives in the stochastic time, and implies a doubled phase space in the (stochastic time) Hamiltonian formalism.

Here we use the approximation that the field $x(\tau)$ is a coherent state, made up of its elementary quanta, in a way that minimises the uncertainty principle.

The elementary quantum processes that build the perturbation theory are the possible decay, annihilation and diffusion reactions

$$x_{\text{cl}} \rightarrow y + \bar{y}, \quad y + \bar{y} \rightarrow x_{\text{cl}}, \quad y + x_{\text{cl}} \rightarrow y + x_{\text{cl}}, \quad \bar{y} + x_{\text{cl}} \rightarrow \bar{y} + x_{\text{cl}} \quad (19)$$

whose strength is proportionally to λ . The τ translation symmetry of the Langevin equation implies that at each vertex there is a conservation of the τ energy, so that the τ evolution of the field x_{cl} can be accompanied by real decay and real annihilations of pairs of y and \bar{y} quanta, namely by a Schwinger type mechanism.

Because we have a friction term proportional to b , the phenomena that occur during the τ evolution will disappear in the limit $\tau \rightarrow \infty$, if it exists. This is the case in our example.

Once $x(t)$ and $y(t)$ have been diagrammatically expressed at a given order of perturbation, one can compute at the same order of perturbation theory $x^p(\tau)y^q(\tau)$, and then averaging, one can obtain $\langle\langle x^p(\tau)y^q(\tau)\rangle\rangle$ by using the fact that η has a Gaussian distribution.

Thus, there is a perturbation expansion involving interactions and propagators, with closed loops, which determines for every value of τ the expectation values $\langle\langle x^p(\tau)y^q(\tau)\rangle\rangle$ as a formal series in λ , and one can compute it at any given finite order in λ . The final result is expressed as

$$\begin{aligned} & \lim_{T \rightarrow \infty} \langle\langle x^p(\tau)y^q(\tau)\rangle\rangle \\ &= \int dx dy x^p y^q \delta(x - x_{\text{cl}} + o(\lambda)) \exp\left(-\frac{1}{2}(M^2(x - x_{\text{cl}})^2 + \frac{1}{2}m^2 y^2 + \frac{1}{2}\lambda xy^2)(1 + o(\lambda^2))\right) \\ &= Z_x^p x_{\text{cl}}^p \int dy y^q \exp\left(-\frac{1}{2}Z_m^2 m^2 y^2 + \frac{1}{2}Z_\lambda \lambda x_{\text{cl}} y^2\right). \end{aligned} \quad (20)$$

The Z factors are finite renormalisation factors with a Taylor expansion, $Z = 1 + \lambda Z^1 + \lambda^2 Z^2 + \dots$, where all the finite coefficients Z^n can be in principle computed in the perturbation theory in λ .

The last formula tells us that Langevin equations gives us a very complicated way to define a standard Gaussian integral in zero dimensions, with a well defined theory that computes the corrections in λ at any given finite order of perturbation theory! The way the limit is reached is an exponential damping, with a regime of very fast oscillations, as sketched in the figure (6).

We will check this explicitly for the one point and two point functions $\langle\langle x(\tau)\rangle\rangle$ and $\langle\langle y(\tau)y(\tau')\rangle\rangle$.

A.2 0-loop and 1-loop computations of two point functions

The Green's function $G^M(\tau)$ of the operator $D_\tau^M = a^2 \partial_\tau^2 + 2b \partial_\tau + M^2$ satisfying $D_\tau^M G^M = \delta(\tau)$ can be computed by a Laplace or Fourier transform. Suppose $aM > b > 0$. It is

$$G^M(\tau) = \theta(\tau) \frac{\exp(-E_+^M \tau) - \exp(-E_-^M \tau)}{a^2(E_-^M - E_+^M)} = \frac{i\theta(\tau)}{2\sqrt{a^2 M^2 - b^2}} (\exp(-E_+^M \tau) - \exp(-E_-^M \tau)), \quad (21)$$

where

$$E_\pm^M = \frac{1}{a^2} (b \pm i\sqrt{a^2 M^2 - b^2}) \quad (22)$$

satisfy $a^2 E^2 + 2bE + M^2 = (E + E_+^M)(E + E_-^M)$.

When $a \neq 0$, the free propagator for the τ evolution has still an exponential damping factor, with a characteristic time that is proportional to b^{-1} (when τ is counted in units of a^2), but there is a new phenomenon, which are τ -oscillations that can be of a very high frequency (in units of a) if the mass M is large enough.

This is the situation that was suggested generically in [1]. In the case of QFTs, one should replace M^2 by $M^2 + \vec{k}^2$, where \vec{k} stands for the momentum of the particle. Care must be given to the possible UV divergency when \vec{k}^2 becomes very large.

If $am > b$, we have $G^m(\tau)$ and E_\pm^m similarly for the operator $D_\tau^m = a^2 \partial_\tau^2 + 2b \partial_\tau + m^2$.

We consider the coupled Langevin equations (18), with only one noise for $y(\tau)$. We solve Eq. (18) perturbatively. Suppose

$$x(\tau) = x_0(\tau) + \lambda x_1(\tau) + o(\lambda^2), \quad y(\tau) = y_0(\tau) + \lambda y_1(\tau) + o(\lambda^2). \quad (23)$$

satisfy (18). Then the 0th order terms in λ satisfy

$$D_\tau^M (x_0 - x_{\text{cl}}) = 0, \quad D_\tau^m y_0 = \eta \quad (24)$$

whereas the first order terms in λ satisfy

$$D_\tau^M x_1 + \frac{1}{2} y_0^2 = 0, \quad D_\tau^m y_1 + x_0 y_0 = 0. \quad (25)$$

The solution to the first (homogeneous) equation for $x_0(\tau)$ in (24) is

$$x_0(\tau) = x_{\text{cl}} + c_+^M \exp(-E_+^M \tau) + c_-^M \exp(-E_-^M \tau) = x_{\text{cl}} + o(e^{-\tau}), \quad (26)$$

where c_\pm^M are constants that are determined by the chosen values of x_0 at some τ_1 and τ_2 , and $o(e^{-\tau})$ stands for any term that is dominated by $e^{-\epsilon\tau}$ for some $\epsilon > 0$ (including the oscillations) as $\tau \rightarrow \infty$. Thus

$$\langle\langle x_0(\tau) \rangle\rangle = x_{\text{cl}} + o(e^{-\tau}). \quad (27)$$

Indeed, when the stochastic process converges, the limit at infinite stochastic time of correlators of the fields are independent of the chosen values of the fields at τ_1 and τ_2 as well. Since E_\pm^M have a positive real part, the damping in the dependence of boundary conditions is exponentially fast in τ , times some oscillations.

On the other hand, the equation in (24) for $y_0(\tau)$ is inhomogeneous and we have

$$y_0(\tau) = (G^m * \eta)(\tau) + c_+^m \exp(-E_+^m \tau) + c_-^m \exp(-E_-^m \tau) = (G^m * \eta)(\tau) + o(e^{-\tau}) \quad (28)$$

for some constants c_{\pm}^m , where $*$ stands for the convolution. Similarly, c_{\pm}^m are related to the boundary values of $y_0(\tau)$ at some τ_1 and τ_2 , and their choice does not affect the correlators at infinite stochastic time. If $0 < \tau_1 \leq \tau_2$, we have

$$\begin{aligned}\langle\langle y_0(\tau_1)y_0(\tau_2)\rangle\rangle &= \int_0^{\tau_1} d\tau'_1 \int_0^{\tau_2} d\tau'_2 G^m(\tau_1 - \tau'_1) G^m(\tau_2 - \tau'_2) \langle\langle \eta(\tau'_1)\eta(\tau'_2)\rangle\rangle + o(e^{-\tau_1}) \\ &= 2b \int_0^{\tau_1} d\tau' G^m(\tau_1 - \tau') G^m(\tau_2 - \tau') + o(e^{-\tau_1}) \\ &= \frac{b}{a^4(E_+^m + E_-^m)(E_-^m - E_+^m)} \left(\frac{\exp(-E_+^m(\tau_2 - \tau_1))}{E_+^m} - \frac{\exp(-E_-^m(\tau_2 - \tau_1))}{E_-^m} \right) + o(e^{-\tau_1}).\end{aligned}\quad (29)$$

In particular, taking $\tau_2 = \tau_1 = \tau$, we have

$$\langle\langle y_0(\tau)^2 \rangle\rangle = \frac{b}{a^4(E_+^m + E_-^m)E_-^m E_+^m} + o(e^{-\tau}) = \frac{1}{2m^2} + o(e^{-\tau}).\quad (30)$$

For the next order, we solve $x_1(\tau)$ from Eq. (25) and obtain

$$x_1(\tau) = -\frac{1}{2} \int_0^{\tau} d\tau' G^M(\tau - \tau') y_0(\tau')^2 + o(e^{-\tau}).\quad (31)$$

Taking the expectation value, we have

$$\begin{aligned}\langle\langle x_1(\tau) \rangle\rangle &= -\frac{1}{4m^2} \int_0^{\tau} d\tau' G^M(\tau - \tau') + o(e^{-\tau}) \\ &= -\frac{1}{4m^2} \cdot \frac{1}{a^2(E_-^M - E_+^M)} \left(\frac{1}{E_+^M} - \frac{1}{E_-^M} \right) + o(e^{-\tau}) \\ &= -\frac{1}{4m^2 M^2} + o(e^{-\tau}).\end{aligned}\quad (32)$$

Similarly, solving $y_1(\tau)$ from Eq. (25), we obtain

$$\begin{aligned}y_1(\tau) &= -\int_0^{\tau} d\tau' G^m(\tau - \tau') x_0(\tau') y_0(\tau') + o(e^{-\tau}) \\ &= -x_{\text{cl}} \int_0^{\tau} d\tau' G^m(\tau - \tau') y_0(\tau') + o(e^{-\tau}).\end{aligned}\quad (33)$$

Therefore, taking the expectation value, we have

$$\begin{aligned}\langle\langle y_0(\tau)y_1(\tau) \rangle\rangle &= -x_{\text{cl}} \int_0^{\tau} d\tau' G^m(\tau - \tau') \langle\langle y_0(\tau)y_0(\tau') \rangle\rangle \\ &= \frac{bx_{\text{cl}}}{a^4(E_+^m + E_-^m)(E_-^m - E_+^m)} \int_0^{\tau} d\tau' G^m(\tau - \tau') \left(\frac{\exp(-E_+^m \tau')}{E_+^m} - \frac{\exp(-E_-^m \tau')}{E_-^m} \right) + o(e^{-\tau}) \\ &= -\frac{bx_{\text{cl}}}{2a^6(E_+^m + E_-^m)(E_+^m E_-^m)^2} + o(e^{-\tau}) \\ &= -\frac{x_{\text{cl}}}{4m^4} + o(e^{-\tau}).\end{aligned}\quad (34)$$

Combining the above 0th order and the 1st order contributions to the correlators, we have

$$\begin{aligned}\lim_{\tau \rightarrow \infty} \langle\langle x(\tau) \rangle\rangle &= \lim_{\tau \rightarrow \infty} \langle\langle x_0(\tau) \rangle\rangle + \lambda \lim_{\tau \rightarrow \infty} \langle\langle x_1(\tau) \rangle\rangle + o(\lambda^2) \\ &= x_{\text{cl}} - \frac{\lambda}{4m^2 M^2} + o(\lambda^2)\end{aligned}\quad (35)$$

from Eqs. (27) and (32), and

$$\begin{aligned}
\lim_{\tau \rightarrow \infty} \langle\langle y(\tau)^2 \rangle\rangle &= \lim_{\tau \rightarrow \infty} \langle\langle y_0(\tau)^2 \rangle\rangle + 2\lambda \lim_{\tau \rightarrow \infty} \langle\langle y_0(\tau) y_1(\tau) \rangle\rangle + o(\lambda^2) \\
&= \frac{1}{2m^2} + 2\lambda \cdot \left(-\frac{x_{\text{cl}}}{4m^4} \right) + o(\lambda^2) \\
&= \frac{1}{2(m^2 + \lambda x_{\text{cl}})} + o(\lambda^2)
\end{aligned} \tag{36}$$

from Eqs. (30) and (34). In the limit, the a and b dependence is washed away, as well as that on the initial conditions.

The shifts $x_{\text{cl}} \rightarrow x'_{\text{cl}} = x_{\text{cl}} - \lambda/4bm^2M^2$ and $m^2 \rightarrow m'^2 = m^2 + \lambda x_{\text{cl}}$ in Eqs. (35) and (36) have a simple explanation. In Eq. (18), substituting the expectation value $\langle\langle y_0^2 \rangle\rangle$ in (30) for y^2 in the equation for x , we have

$$D_\tau^M(x - x_{\text{cl}}) + \frac{\lambda}{2} \langle\langle y_0^2 \rangle\rangle = D_\tau^M(x - x'_{\text{cl}}) \tag{37}$$

while substituting the expectation value $\langle\langle x_0 \rangle\rangle$ in (27) for x in the equation for y , we have

$$D_\tau^m y + \lambda x \langle\langle y_0 \rangle\rangle = D_\tau^{m'} y. \tag{38}$$

Thus we have the same form of Eq. (18) but with the shifted constant parameters x'_{cl} and m'^2 . This is consistent with Eq. (20) with the finite renormalisation constants

$$Z_x = 1 - \frac{\lambda}{4bm^2M^2x_{\text{cl}}} + o(\lambda^2), \quad Z_m = 1 + \frac{\lambda x_{\text{cl}}}{2m^2} + o(\lambda^2), \quad Z_\lambda = 1 + o(\lambda). \tag{39}$$

A.3 Higher loops and diagrammatic expansions

Recall the system (18) of stochastic equations. Iteratively, we have

$$\begin{aligned}
x &= x_{\text{cl}} - \frac{\lambda}{2} G^M * y^2 = x_{\text{cl}} - \frac{\lambda}{2} G^M * (G^m * \eta - \lambda G^m * (xy))^2 \\
&= x_{\text{cl}} - \frac{\lambda}{2} G^M * (G^m * \eta)^2 + \lambda^2 x_{\text{cl}} G^M * ((G^m * \eta)(G^M * G^m * \eta)) + o(\lambda^3)
\end{aligned}$$

and

$$\begin{aligned}
y &= G^m * \eta - \lambda G^m * (xy) = G^m * \eta - \lambda G^m * \left(\left(x_{\text{cl}} - \frac{\lambda}{2} G^M * y^2 \right) (G^m * \eta - \lambda G^m * (xy)) \right) \\
&= G^m * \eta - \lambda x_{\text{cl}} G^m * G^m * \eta + \lambda^2 x_{\text{cl}}^2 G^m * G^m * \eta + \frac{\lambda^2}{2} G^m * ((G^M * (G^m * \eta)^2)(G^m * \eta)) + o(\lambda^3).
\end{aligned}$$

Diagrammatically, the above expansions of x and y can be represented by

$$x(\eta) = \begin{array}{c} \text{---} \circ \text{---} \\ \lambda^0 \end{array} + \begin{array}{c} \text{---} \text{wavy} \text{---} \text{---} \times \\ \lambda^1 \end{array} + \begin{array}{c} \text{---} \text{wavy} \text{---} \text{---} \text{---} \text{---} \times \\ \lambda^2 \end{array} + \dots$$

and

$$y(\eta) = \begin{array}{c} \text{---} \times \\ \lambda^0 \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \times \\ \lambda^1 \end{array} + \underbrace{\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \times \\ \lambda^2 \end{array}}_{\lambda^2} + \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \times \\ \lambda^3 \end{array} + \dots$$

Here a circle means an insertion of the constant term x_{cl} , a cross means attaching to the external noise η , a wiggly line means the propagator with the Green's function G^M of x , and a solid line means the propagator with the Green's function G^m of y . To each trivalent vertex with one wiggly line and two solid lines is assigned the coupling constant $-\lambda$ and to each circle on a solid line is assigned the factor $-\lambda x_{\text{cl}}$. As usual, each diagramme is divided by the order of its automorphism group. Note that the quantity represented by each diagramme, though τ -dependent, is determined by solely the equations in (18) and does not rely on the initial/boundary conditions of a particular solution.

We can express $\langle\langle x(\tau) \rangle\rangle$ and $\langle\langle y(\tau_1)y(\tau_2) \rangle\rangle$ by diagrammes. Since the same method of iteration is used both here and in Section A.2, the results to the order λ^1 must agree. To the order λ^2 , we have

$$\begin{aligned} \langle\langle x(\tau) \rangle\rangle &= \underset{(a)}{\tau \bigcirc} + \underset{(b)}{\tau \text{---} \tau' \bigcirc \times \tau''} + \underset{(c)}{\tau \text{---} \tau' \bigcirc \times \tau'' \bigcirc \times \tau'''} + \dots \\ &\quad \lambda^0 \qquad \qquad \lambda^1 \qquad \qquad \lambda^2 \end{aligned}$$

and

$$\begin{aligned} \langle\langle y(\tau_1)y(\tau_2) \rangle\rangle &= \underset{(d)}{\tau_2 \text{---} \tau' \times \tau_1} + \underbrace{\underset{(e)}{\tau_2 \text{---} \tau'_2 \times \tau' \bigcirc \tau_1} + \underset{(f)}{\tau_2 \text{---} \tau' \bigcirc \tau'_1 \times \tau_1}}_{\lambda^1} + \\ &+ \underbrace{\underset{(g)}{\tau_2 \text{---} \tau'_2 \bigcirc \tau' \times \tau'_1 \tau_1} + \underset{(h)}{\tau_2 \text{---} \tau'_2 \bigcirc \tau'_2 \times \tau' \tau_1} + \underset{(i)}{\tau_2 \text{---} \tau' \times \tau'_1 \bigcirc \tau'_1 \tau_1} + \underset{(j)}{\tau_2 \text{---} \tau'_2 \text{---} \tau'_1 \times \tau' \tau_1} + \underset{(k)}{\tau_2 \text{---} \tau' \text{---} \tau'_2 \times \tau'_1 \tau_1} + \underset{(l)}{\tau_2 \text{---} \tau'_2 \text{---} \tau'_2 \bigcirc \times \tau' \tau_1} + \underset{(m)}{\tau_2 \text{---} \tau' \text{---} \tau'_1 \bigcirc \times \tau'_1 \tau_1}}_{\lambda^2} + \dots, \end{aligned}$$

where a cross on a solid line means now contraction of two η 's in the diagrammatic expansions of $x(\eta)$ and $y(\eta)$ above using (15), giving rise to a factor of $2b$. Quantitatively, these diagrammes are

$$\begin{aligned} (a) &= x_{\text{cl}}, \\ (b) &= -b\lambda \int_0^\tau d\tau' G^M(\tau - \tau') \int_0^{\tau'} d\tau'' G^m(\tau' - \tau'')^2, \\ (c) &= 2b\lambda^2 x_{\text{cl}} \int_0^\tau d\tau' G^M(\tau - \tau') \int_0^{\tau'} d\tau'' G^m(\tau' - \tau'') \int_{\tau''}^{\tau'} d\tau''' G^m(\tau' - \tau''') G^m(\tau''' - \tau'') \end{aligned}$$

for $\langle\langle x(\tau) \rangle\rangle$. The diagrammes (a) and (b) agree with (35) when $\tau \rightarrow +\infty$. For $\langle\langle y(\tau_1)y(\tau_2) \rangle\rangle$, we have

$$\begin{aligned}
(d) &= 2b \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_1 - \tau') G^m(\tau_2 - \tau'), \\
(e) &= -2b\lambda x_{\text{cl}} \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_1 - \tau') \int_{\tau'}^{\tau_2} d\tau'_2 G^m(\tau_2 - \tau'_2) G^m(\tau'_2 - \tau'), \\
(f) &= -2b\lambda x_{\text{cl}} \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_2 - \tau') \int_{\tau'}^{\tau_1} d\tau'_2 G^m(\tau_1 - \tau'_2) G^m(\tau'_2 - \tau'), \\
(g) &= 2b\lambda^2 x_{\text{cl}}^2 \int_0^{\tau_1} d\tau'_1 G^m(\tau_1 - \tau'_1) \int_0^{\tau_2} d\tau'_2 G^m(\tau_2 - \tau'_2) \int_0^{\min(\tau'_1, \tau'_2)} d\tau' G^m(\tau'_1 - \tau') G^m(\tau'_2 - \tau'), \\
(h) &= 2b\lambda^2 x_{\text{cl}}^2 \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_1 - \tau') \iint_{\tau' \leq \tau'_2 \leq \tau'_2 \leq \tau_2} d\tau'_2 d\tau''_2 G^m(\tau_2 - \tau'_2) G^m(\tau'_2 - \tau''_2) G^m(\tau''_2 - \tau'), \\
(i) &= 2b\lambda^2 x_{\text{cl}}^2 \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_2 - \tau') \iint_{\tau' \leq \tau'_1 \leq \tau'_1 \leq \tau_1} d\tau'_1 d\tau''_1 G^m(\tau_1 - \tau'_1) G^m(\tau'_1 - \tau''_1) G^m(\tau''_1 - \tau'), \\
(j) &= 4b^2 \lambda^2 \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_1 - \tau') \iint_{\tau' \leq \tau'_1 \leq \tau'_2 \leq \tau_2} d\tau'_1 d\tau'_2 G^m(\tau'_1 - \tau') G^m(\tau_2 - \tau'_2) G^M(\tau'_1 - \tau'_2) \\
&\quad \int_{\tau'_1}^{\tau'_2} d\tau'' G^m(\tau'_2 - \tau'') G^m(\tau'' - \tau'_1) \\
(k) &= 4b^2 \lambda^2 \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_2 - \tau') \iint_{\tau' \leq \tau'_2 \leq \tau'_1 \leq \tau_1} d\tau'_1 d\tau'_2 G^m(\tau'_2 - \tau') G^m(\tau_1 - \tau'_1) G^M(\tau'_2 - \tau'_1) \\
&\quad \int_{\tau'_2}^{\tau'_1} d\tau'' G^m(\tau'_1 - \tau'') G^m(\tau'' - \tau'_2) \\
(l) &= 4b^2 \lambda^2 \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_1 - \tau') \int_{\tau'}^{\tau_2} d\tau'_2 G^m(\tau_2 - \tau'_2) G^m(\tau'_2 - \tau') \\
&\quad \int_0^{\tau'_2} d\tau''_2 G^M(\tau'_2 - \tau''_2) \int_0^{\tau''_2} d\tau'' G^m(\tau''_2 - \tau'')^2 \\
(m) &= 4b^2 \lambda^2 \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_2 - \tau') \int_{\tau'}^{\tau_1} d\tau'_1 G^m(\tau_1 - \tau'_1) G^m(\tau'_1 - \tau') \\
&\quad \int_0^{\tau'_1} d\tau''_1 G^M(\tau'_1 - \tau''_1) \int_0^{\tau''_1} d\tau'' G^m(\tau''_1 - \tau'')^2,
\end{aligned}$$

and diagrammes (d), (e), (f) agree with (36) when $\tau_1 = \tau_2 = \tau \rightarrow +\infty$.

B Supersymmetric representation and a possible Fokker-Planck Hamiltonian

Independent of the goals of obtaining a possible definition of quantum gravity and (ambitious) physical predictions as outlined just above, it was suggested in [1] that, by using the idea of stochastic time with a second order time evolution, a d -dimensional Euclidean QFT can be defined as the limit of a $(d+1)$ -dimensional QFT with a double Fock space structure. The latter contains the usual Fock space for the quantum particle degrees of freedom and another Fock space for the accessible levels of what we find suggestive to name the “stochastic time dependent oscillating vacuum (STDOV)”. This builds a good theoretical framework for the quantum gravity scenario in [1]. A higher order Langevin equation could also reveal some properties of simpler quantum mechanical models, such as the one with a conformal potential $\sim 1/x^2$, a system which never reaches an equilibrium but has interesting properties at a finite stochastic time.

This $(d+1)$ -dimensional QFT is Euclidean in all coordinates but the stochastic time, and it involves effectively a physical UV cutoff $1/\Delta T$ as a parameter that occurs naturally from dimensionality considerations in a second order Langevin equation.

The correlation functions stemming from a Langevin equation can be represented by a supersymmetric path integral, with the generating functional

$$Z[J_x, J_y] = \int [dx]_\tau [dy]_\tau [d\psi_x]_\tau [d\psi_y]_\tau [d\bar{\psi}_x]_\tau [d\bar{\psi}_y]_\tau [d\eta_x]_\tau [d\eta_y]_\tau \exp \left[-\frac{1}{\hbar} \int_0^T d\tau \left(\mathcal{L}_{\text{susy}}(x, y, \psi_x, \psi_y, \bar{\psi}_x, \bar{\psi}_y, \eta_x, \eta_y) + xJ_x + yJ_y \right) \right] \quad (40)$$

using the standard determinant manipulation [9]. For simplicity, we will write q for (x, y) , η for (η_x, η_y) , ψ for (ψ_x, ψ_y) , and $\bar{\psi}$ for $(\bar{\psi}_x, \bar{\psi}_y)$. The action is topological in the sense that $\mathcal{L}_{\text{susy}} = s_{\text{top}}(\dots)$, where the supersymmetry transformations

$$s_{\text{top}} q = \psi, \quad s_{\text{top}} \psi = 0, \quad s_{\text{top}} \bar{\psi} = \eta \quad (41)$$

satisfy $s_{\text{top}}^2 = 0$. In our case with a second order time evolution, one has

$$\begin{aligned} \mathcal{L}_{\text{susy}} &= s_{\text{top}} \left(\bar{\psi} \left(a^2 \ddot{q} + 2b \dot{q} + \frac{\partial S}{\partial q} - \frac{1}{2} \eta \right) \right) \\ &= -\frac{1}{2} \eta^2 + \eta \left(a^2 \ddot{q} + 2b \dot{q} + \frac{\partial S}{\partial q} \right) - \bar{\psi} \left(a^2 \ddot{\psi} + 2b \dot{\psi} + \frac{\partial^2 S}{\partial q^2} \psi \right) \\ &\sim \frac{1}{2} \left(a^2 \ddot{q} + 2b \dot{q} + \frac{\partial S}{\partial q} \right)^2 - \bar{\psi} \left(a^2 \ddot{\psi} + 2b \dot{\psi} + \frac{\partial^2 S}{\partial q^2} \psi \right), \end{aligned} \quad (42)$$

This Lagrangian $\mathcal{L}_{\text{susy}}$ was mentioned in Eq. (13) and can be called the Fokker-Planck supersymmetric Lagrangian.

The existence of a supersymmetric action that one can associate to an evolution equation involving a Gaussian noise, not necessarily a white one, adding to the drift force, is a general property [3]. This supersymmetry has been investigated with care in [9] in the context of stochastic quantisation. The relation between this fact and topological quantum field theory was noted in [7].

When one eliminates the fermions in the path integral by Berezin integration and computes correlation functions, one recovers the same results as from solving the Langevin equations and computing mean values of functions of q by elimination of the noise η .

It is actually useful to express by Legendre transform the Hamiltonian associated to the supersymmetric Lagrangian (42). Let us recall that, when $a = 0$, $b = \frac{1}{2}$, Ψ and $\bar{\Psi}$ are self-conjugate. In this case, the supersymmetric Hamiltonian can be represented as a 2×2 matrix whose diagonal terms are both advanced and retarded Fokker-Planck Hamiltonians for the first order stochastic time evolution, as they can be computed in statistical mechanics. The property of its spectrum determines the convergence of the stochastic process [9]. In this sense, the supersymmetry is truly a topological symmetry that determines the Fokker-Planck process from first principles. The word “topological” is justified by the fact that many interesting theories express in the simplest way their physics only in the slice $\tau = \infty$ [7].**

For $a \neq 0$, we have a more general supersymmetric Lagrangian that has higher order derivative terms, symbolically $L = L(q, \dot{q}, \ddot{q})$. This doesn't change our point of view that its Hamiltonian, once suitably defined, will generate the stochastic time evolution, and define the required generalisation of the Fokker-Planck evolution kernel.

To see how it works in an elementary way, we can use the simple η -dependence of the Lagrangian. We can write the bosonic part of the Lagrangian $\mathcal{L}_{\text{susy}}$ as

$$\mathcal{L}_{\text{susy}}^B = -a^2 \dot{\eta} \dot{q} + b(\eta \dot{q} - q \dot{\eta}) - \frac{1}{2} \eta^2 + \eta \frac{\partial S}{\partial q} \quad (43)$$

**In the general case of an action in d dimensions S_d , the $d+1$ dimensional supersymmetric stochastic Lagrangian that computes the correlators in the bulk is denoted by $\mathcal{L}_{\text{susy}}^{d+1}$. Its action is basically s_{top} -exact under a nilpotent operator, as in eq. (41). One can compute its supersymmetric Hamiltonian for the stochastic time evolution that we denote as \mathcal{H}_d for future use in the text. We express, without proof, the fact that the spectrum of \mathcal{H}_d is made of (bound) states evolving in $d+1$ dimensions, with d -dimensional wave functions related to the instantons of S_d .

up to a total derivative. Then the canonical momenta of q and η are

$$p_q \equiv \frac{\partial \mathcal{L}_{\text{susy}}^B}{\partial \dot{q}} = -a^2 \dot{\eta} + b \eta, \quad p_\eta \equiv \frac{\partial \mathcal{L}_{\text{susy}}^B}{\partial \dot{\eta}} = -a^2 \dot{q} - b q, \quad (44)$$

respectively. Taking a Legendre transform, the bosonic part of the Hamiltonian is

$$\begin{aligned} \mathcal{H}_{\text{susy}}^B &= p_q \dot{q} + p_\eta \dot{\eta} - \mathcal{L}_{\text{susy}}^B \\ &= -\frac{1}{a^2} (p_q - b \eta)(p_\eta + b q) + \frac{1}{2} \eta^2 - \eta \frac{\partial S}{\partial q}. \end{aligned} \quad (45)$$

It shows that we must consider a doubling of the phase space, $(q, p_q) \rightarrow (q, p_q, \eta, p_\eta)$, and that at finite values of τ , the limit $a \rightarrow 0$ is not smooth. The fermionic part contains the supersymmetric partners and has the same structure. This extension of the phase space is because the Langevin equation is of a higher order.^{††}

If we examine the Hamiltonian (45) from a perturbative point of view, its zeroth order is given by a quadratic approximation of S as a function of x . If we restrict to this approximation, one can diagonalise $\mathcal{H}_{\text{susy}}^B$ in the space $(x, p_x) \rightarrow (x, p_x, \eta, p_\eta)$ and get a double harmonic oscillator with a double harmonic spectrum, one for the particles and one for the noise through the quantisation of η . In our approximation where we neglect η_x , this extra quantisation is reported on the oscillations at finite time of x around x_{cl} , till it becomes stationary. These features survive perturbatively for a large class of potentials, and one may hope that they can be true non-perturbatively.

In a different approach, and basically with the same conclusion, we can analyse the Lagrangian $\mathcal{L}_{\text{susy}}$ directly after the elimination of η , which gives the last line in (42). The Lagrangian $\mathcal{L}_{\text{susy}}$ has higher order derivatives and can be studied by applying the standard Lagrangian/Hamiltonian formalism of Ostrogradsky (see for example [10] and references therein).^{‡‡} Given a Lagrangian depending on q, \dot{q}, \ddot{q} , where the dot means $\dot{} \equiv \frac{d}{d\tau}$, and an action $S = \int d\tau L(q, \dot{q}, \ddot{q})$, one finds that the extrema of S occur when the Euler-Lagrange equation of motion

$$\frac{\partial L}{\partial q} = \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}} - \frac{d^2}{d\tau^2} \frac{\partial L}{\partial \ddot{q}} \quad (46)$$

is satisfied. In fact, a general variation of L for arbitrary variations δq and $\delta \dot{q}$ is

$$\delta L = \left(\frac{\partial L}{\partial q} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{d\tau^2} \frac{\partial L}{\partial \ddot{q}} \right) \delta q + \frac{d}{d\tau} (p_0 \delta q_0 + p_1 \delta \dot{q}_1), \quad (47)$$

where

$$q_0 \equiv q, \quad q_1 \equiv \dot{q}, \quad p_0 \equiv \frac{\partial L}{\partial \ddot{q}} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}}, \quad p_1 \equiv \frac{\partial L}{\partial \dot{q}}. \quad (48)$$

This shows that the phase space is parametrised by the conjugate coordinates (q_0, q_1, p_0, p_1) . The canonical phase space is doubled, as can be simply understood because one needs twice as many initial conditions. The Ostrogradsky formalism is valid for higher order Lagrangians $L(q, \dot{q}, \ddot{q}, \dots, q^{(r)})$, giving an enlarged phase space of dimension $2r$.

This doubling of the phase space has non-trivial consequences. In particular, when $a \neq 0$, \dot{q} is not identified as the momentum of q in the Lagrangian (42), so q and \dot{q} can be measured simultaneously in the quantum mechanics defined by replacing the (anti)coordinates by operators and the Poisson brackets by (anti)commutators.

In fact, the uncertainty relation holds only between q_0 and p_0 (for $a \neq 0$), and between q_1 and p_1 (see the canonical commutation relations below).

^{††}An analogous phenomenon arises when one performs the BRST quantisation of supergravity, where the fermionic Lagrange multiplier for the gauge-fixing of the Rarita-Schwinger field becomes a propagating fermionic field.

^{‡‡}The same remark applies to the models presented in [7].

Eq. (47) shows that the conserved Hamiltonian associated to $L(q, \dot{q}, \ddot{q})$, which expresses the τ -evolution, is still given by a Legendre transformation

$$H \equiv p_0 \dot{q}_0 + p_1 \dot{q}_1 - L(q, \dot{q}, \ddot{q}), \quad (49)$$

where \dot{q}, \ddot{q} must be expressed as functions of q_0, p_0, q_1, p_1 using (48). One finds that the phase space equations of motion are

$$\dot{p}_0 = -\frac{\partial H}{\partial q_0}, \quad \dot{p}_1 = -\frac{\partial H}{\partial q_1}, \quad \dot{q}_0 = \frac{\partial H}{\partial p_1}, \quad \dot{q}_1 = \frac{\partial H}{\partial p_0}$$

They are first order equations, just as in the standard case, but a doubling. Quantisation in the operator formalism is then defined by regarding all coordinates and momenta as operators subject to the canonical relations

$$[\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0. \quad (50)$$

and for any operator \hat{A} , one has $\hat{\dot{A}} = [\hat{H}, \hat{A}]$. This more systematic construction explains in a different way the doubling of the Fock space that we directly derived, before the elimination of the noises. The positivity of the spectrum is not ensured, as it was obvious seen in (45), and its physical interpretation is different than of standard quantum mechanics in its Euclidean form. As explained in this paper, the later can be only obtained in limit $\tau \rightarrow \infty$, if the limit exists.

The construction of the phase space can be easily repeated for the entire supersymmetric $\mathcal{L}_{\text{susy}}$ in Eq. (42), giving a graded symplectic structure and a supersymmetric Hamiltonian, where both the Ψ and $\bar{\Psi}$ have their own independent momenta for $a \neq 0$ and there is also a doubling for the fermionic part of the phase space as compared to the case $a = 0$.

All this suggests that a well-defined Fokker-Planck Hamiltonian, which defines the stochastic time evolution by first order Hamiltonian equations, can be associated to the second order Langevin equation. This explains in a better way the expression (45), where the noise was kept in a rather artificial way. The supersymmetric representation (42) of the second order Langevin equation is expected to play a key role in the understanding of the τ -evolution as it did in the case of a first order Langevin equation. It actually used in a separate reference where we analyse the covariance questions of the gravity stochastic equation. Appendix B describes this for the example given in Appendix A.

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