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PITMAN TRANSFORMS AND BROWNIAN MOTION IN THE INTERVAL VIEWED AS AN AFFINE ALCOVE

PHILIPPE BOUGEROL AND MANON DEFOSSEUX

ABSTRACT. Pitman's theorem states that if $\{B_t, t \geq 0\}$ is a one dimensional Brownian motion, then $\{B_t - 2 \inf_{s \leq t} B_s, t \geq 0\}$ is a three dimensional Bessel process, i.e. a Brownian motion conditioned to remain forever positive. This paper gives a similar representation for the Brownian motion conditioned to remain in an interval. Due to the double barrier condition, this representation is more involved and only asymptotic. One uses the fact that the interval is an alcove of the Kac-Moody affine Lie algebra $A_1^{(1)}$, the Littelmann path approach of representation theory and a dihedral approximation.

ABSTRACT. Le théorème de Pitman affirme que si $\{B_t, t \geq 0\}$ est un mouvement brownien unidimensionnel, alors $\{B_t - 2 \inf_{s \leq t} B_s, t \geq 0\}$ est un processus de Bessel de dimension trois, c'est-à-dire un brownien conditionné à rester positif. Nous donnons dans cet article une représentation analogue pour le brownien conditionné à rester dans l'intervalle. En raison de la présence de deux extrémités, cette représentation est plus compliquée que celle du théorème original, et seulement asymptotique. Nous utilisons le fait que l'intervalle est une alcôve pour l'algèbre de Kac-Moody affine $A_1^{(1)}$, l'approche par les modèles de chemins de Littelmann de la théorie des représentations et une approximation diédrale.

couleurs de la corde
dépôt de cette image
cristaux du temps
traces d'espace

Raymond Queneau

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1. INTRODUCTION

1.1. The probability transition of the Brownian motion conditioned to stay positive forever is the difference of two heat kernels. This is a consequence of the reflection principle at 0. Pitman's theorem ([32]) of 1975 gives the path representation of this process as

$$\mathcal{P}B(t) = B_t - 2 \inf_{0 \leq s \leq t} B_s$$

where B is a standard Brownian motion with $B_0 = 0$. The transform $\mathcal{P}B$ is written with the reflection at 0. We consider Z_t , $t \geq 0$, a Brownian motion conditioned to stay in the interval $[0, 1]$ forever (see Definition 5.1). It can be seen as the Doob transform of the Brownian motion with Dirichlet conditions at 0 and 1. Its probability transition is an alternating infinite sum which can be obtained by applying successive reflection principles at 0 and 1 (method of images). It is therefore natural to ask if Pitman's theorem has an analogue for Z , written with an infinite number of transforms at 0 and 1. The main result of this article is to show that, to our surprise, this is not exactly the case. A small correction (a Lévy transform) has to be added. The same correction occurs in an asymptotic property of the highest weight representations of the affine Lie algebra $A_1^{(1)}$.

1.2. Let us state our main probabilistic result. We consider, for a continuous real path $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $\varphi(0) = 0$, for $t \geq 0$,

$$\begin{aligned} \mathcal{L}_1\varphi(t) &= \varphi(t) - \inf_{0 \leq s \leq t} \varphi(s), \\ \mathcal{P}_1\varphi(t) &= \varphi(t) - 2 \inf_{0 \leq s \leq t} \varphi(s), \end{aligned}$$

that we call the classical Lévy and Pitman transforms of φ . We introduce

$$\begin{aligned} \mathcal{L}_0\varphi(t) &= \varphi(t) + \inf_{0 \leq s \leq t} (s - \varphi(s)), \\ \mathcal{P}_0\varphi(t) &= \varphi(t) + 2 \inf_{0 \leq s \leq t} (s - \varphi(s)). \end{aligned}$$

The Brownian motion with drift μ is $B_t^\mu = B_t + t\mu, t \geq 0$. For $n \in \mathbb{N}$, we let $\mathcal{P}_{2n} = \mathcal{P}_0, \mathcal{L}_{2n} = \mathcal{L}_0$ and $\mathcal{P}_{2n+1} = \mathcal{P}_1, \mathcal{L}_{2n+1} = \mathcal{L}_1$. The aim of this paper is the following representation theorem (see Theorem 3.1).

Theorem. *Let $\mu \in [0, 1]$. For any $t > 0$, almost surely,*

$$\lim_{n \rightarrow \infty} t \mathcal{L}_{n+1} \mathcal{P}_n \cdots \mathcal{P}_1 \mathcal{P}_0 B^\mu(1/t) = \lim_{n \rightarrow \infty} t \mathcal{L}_{n+1} \mathcal{P}_n \cdots \mathcal{P}_2 \mathcal{P}_1 B^\mu(1/t) = Z_t,$$

where Z is a Brownian motion conditioned to stay in $[0, 1]$ forever, and $Z_0 = \mu$.

1.3. Briefly, the strategy of the proof is the following. We will use the fact that, when $Z_0 = \mu$, Z is the space component of the time inverted process of $A^{(\mu)}$, where $A^{(\mu)}$ is the space-time Brownian motion $B_t^{(\mu)} = (t, B_t^\mu), t \geq 0$, conditioned to stay in the affine cone

$$C_{\text{aff}} = \{(t, x) \in \mathbb{R}^2; 0 < x < t\},$$

(see Appendix 5.3). We define a sequence of non-negative random processes $\xi_n(t), t \geq 0, n \in \mathbb{N}$, by

$$\xi_n(t) = - \inf_{0 \leq s \leq t} \{s 1_{2\mathbb{N}}(n) + (-1)^{n-1} \mathcal{P}_{n-1} \cdots \mathcal{P}_0 B^\mu(s)\}.$$

Then

$$\mathcal{P}_n \cdots \mathcal{P}_0 B^\mu(t) = B_t^\mu + 2 \sum_{k=0}^n (-1)^{k+1} \xi_k(t),$$

and

$$\mathcal{L}_{n+1} \mathcal{P}_n \cdots \mathcal{P}_0 B^\mu(t) = \mathcal{P}_n \cdots \mathcal{P}_0 B^\mu(t) + (-1)^n \xi_{n+1}(t).$$

When $0 < \mu < 1$, the random variables $\xi_n(\infty) = \lim_{t \rightarrow +\infty} \xi_n(t), n \in \mathbb{N}$, are finite a.s. and we show that their laws have a simple explicit representation with independent exponential random variables. The law of $\xi_n(t), n \in \mathbb{N}$, can be deduced by a truncation argument. This allows us to show that for all $t \geq 0$ the limit of

$$(1.1) \quad \mathcal{P}_n \cdots \mathcal{P}_0 B^\mu(t) + (-1)^n \xi_{n+1}(t)$$

exists a.s. and has the law of the space component of $A^{(\mu)}$. We also prove that for $t > 0$, $\xi_n(t)$ tends to 2 when n tends to ∞ , which shows the necessity of the correction.

Actually, to prove these results, we approximate the space-time Brownian motion $B^{(\mu)}$ by planar Brownian motions with proper drifts and $A^{(\mu)}$ by these planar Brownian motions conditioned to remain in a wedge in \mathbb{R}^2 of dihedral angle π/m . We use the results of Biane et al. [3] in this situation. Due to the need of the correction term, the approximation is not immediate.

1.4. Before describing the plan of the paper, let us make an observation. At the heart of our approach is the fact that the interval $[0, 1]$ is an alcove for the Kac-Moody affine Lie algebra $A_1^{(1)}$ and that $A^{(\mu)}$ can be seen as a process conditioned to remain in a Weyl chamber of $A_1^{(1)}$. This is linked to highest weight representations of $A_1^{(1)}$ through Littelmann path approach. We have chosen to present the proof of our main probabilistic result without explicit reference to Kac-Moody algebras, so that it can be read by a probabilist. But let us now

explain the ideas from representation theory behind the scenes because this has been a source of inspiration. This may be helpful for some readers.

We first recall the link between Littelmann path theory for the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ (see Littelmann [28]) and the classical Pitman theorem as explained in Biane et al. [3]. Consider a real line $V = \mathbb{R}\alpha$ where α is a positive root. A path π in V is a continuous function $\pi : \mathbb{R}^+ \rightarrow V$ such that $\pi(0) = 0$. It can be written for $s \geq 0$ as $\pi(s) = f(s)\alpha$ with $f(s) \in \mathbb{R}$. A dominant path is a path with values in the Weyl chamber, which is here $\mathbb{R}_+\alpha$, so that π is dominant when $f(s) \geq 0$ for all $s \geq 0$. We fix a $t > 0$. An integral path on $[0, t]$ is a piecewise linear path such that $2f(t)$ and $2\min_{s \leq t} f(s)$ are in \mathbb{Z} . For an irreducible highest weight $\mathfrak{sl}_2(\mathbb{C})$ -module, one chooses a dominant integral path π on $[0, t]$, such that $\pi(t)$ is its highest weight. A path realization of the Kashiwara crystal associated to this module is the Littelmann module, which is the set of integral paths η on $[0, t]$ such that $\mathcal{P}_\alpha \eta(s) = \pi(s)$ for all $s \in [0, t]$, where \mathcal{P}_α is the path transform defined by

$$(1.2) \quad \mathcal{P}_\alpha \eta(s) = (\varphi(s) - 2 \inf_{0 \leq u \leq s} \varphi(u))\alpha,$$

when $\eta(s) = \varphi(s)\alpha$. One recognizes the Pitman transform.

Let us consider now the Littelmann path theory for the affine Lie algebra $A_1^{(1)}$. A realization of a real Cartan subalgebra $\mathfrak{h}_\mathbb{R}$ is given by \mathbb{R}^3 and we write the canonical basis as $\{c, \tilde{\alpha}_1, d\}$, and the dual canonical basis of $\mathfrak{h}_\mathbb{R}^*$ as $\{\Lambda_0, \alpha_1/2, \delta\}$. We choose $\tilde{\alpha}_0 = c - \tilde{\alpha}_1$ and $\alpha_0 = \delta - \alpha_1$. The Weyl group is generated by the reflections s_0, s_1 on $\mathfrak{h}_\mathbb{R}^*$ defined by, for $v \in \mathfrak{h}_\mathbb{R}^*$ and $i \in \{0, 1\}$,

$$s_i(v) = v - \tilde{\alpha}_i(v)\alpha_i,$$

which are reflections along the walls of the Weyl chamber

$$C_W = \{t\Lambda_0 + x\alpha_1/2 + y\delta, (t, x) \in C_{\text{aff}}, y \in \mathbb{R}\} = C_{\text{aff}} \times \mathbb{R}.$$

In Littelmann's theory, a path is now a continuous map $\eta : \mathbb{R}^+ \rightarrow \mathfrak{h}_\mathbb{R}^*$ such that $\eta(0) = 0$. One defines path transforms \mathcal{P}_{s_i} , $i \in \{0, 1\}$, by

$$\mathcal{P}_{s_i} \eta(t) = \eta(t) - \inf_{0 \leq s \leq t} \tilde{\alpha}_i(\eta(s))\alpha_i.$$

A dominant path is a path with values in the closure of C_W and one can define integral paths on $[0, t]$. For a fixed $t > 0$ and an integral dominant path π on $[0, t]$, the Littelmann module generated by π is the set of integral paths η on $[0, t]$ for which there exists $n \in \mathbb{N}$ such that

$$\mathcal{P}_{s_n} \mathcal{P}_{s_{n-1}} \cdots \mathcal{P}_{s_1} \mathcal{P}_{s_0} \eta(s) = \pi(s)$$

when $0 \leq s \leq t$, where $s_{2k} = s_0$ and $s_{2k+1} = s_1$ for $k \in \mathbb{N}$. This gives a description of the Kashiwara crystal of highest weight $\pi(t)$ (see Kashiwara [26], Littelmann [28]). For an integral path η on $[0, t]$ and more generally for a continuous piecewise C^1 path (see Proposition 5.8), there is a $k \in \mathbb{N}$ such that for all $n \geq k$,

$$\mathcal{P}_{s_n} \cdots \mathcal{P}_{s_0} \eta(s) = \mathcal{P}_{s_k} \cdots \mathcal{P}_{s_0} \eta(s),$$

for $0 \leq s \leq t$. This new path is dominant. One can ask if, similarly to Pitman's theorem, at least the limit when n tends to infinity of $\mathcal{P}_{s_n} \cdots \mathcal{P}_{s_0} \eta$ exists when η is replaced by a space-time Brownian motion.

We show in this paper that the answer is no. Nevertheless one proves that a slight modification converges in $\mathfrak{h}_{\mathbb{R}}^*/\mathbb{R}\delta$ to a space-time Brownian motion $B^{(\mu)}$ conditioned to remain in the affine cone C_{aff} . It will be enough for us to work in the space $V = \mathfrak{h}_{\mathbb{R}}^*/\mathbb{R}\delta$, so our paths and transforms will be defined on V .

We call the sequence $\{\xi_k(t), k \in \mathbb{N}\}$ defined above the affine string parameters, by analogy with the string parameters in the Littelmann model because, for integral paths, they are the string parameters of the element corresponding in the highest weight crystal (see Kashiwara [25], Littelmann [29]). Likewise, we call $\xi_k(\infty), k \in \mathbb{N}$, the Verma affine string parameters, by analogy with the string parameters of the crystal $B(\infty)$ of Kashiwara [25] associated with the Verma module of highest weight 0.

1.5. The paper is organized as follows. As above the process Z is the space component of the time inverted process of the space-time process $A^{(\mu)}$. Dealing with the two dimensional process $A^{(\mu)}$ linearizes the situation and so is easier. We first study it in Section 2, by using approximations by processes in dihedral cones.

In Section 2.1 one recalls the Pitman representation theorem for a planar Brownian motion in a dihedral cone and gives a precise description of the law of their string and Verma string parameters, due to Biane et al. [3],[4]. One studies their asymptotic behaviour in Section 2.2 and obtains a description of the Verma affine string parameters $\xi(\infty) = \{\xi_k(\infty), k \in \mathbb{N}\}$ of the space-time Brownian motion $B^{(\mu)}$. In Section 2.3 one proves that the series $\sum_{k=0}^n (-1)^k \xi_k(\infty)$ converges up to a correcting term. In Section 2.5 one shows the almost sure convergence of (1.1) and identifies the limit. One obtains in Section 2.6 a representation theorem for the conditioned space-time Brownian motion by applying at last a Lévy transform to the iterates of Pitman's ones. In Section 2.8 one makes some comments on the distributions of the first string parameter $\xi_1(\infty)$ of the space-time Brownian motion and of its affine Verma weight.

In Section 3 one proves the main result of the paper on the Brownian motion in the interval. As mentioned above we only use probabilistic notions and arguments up to this point.

In Section 4, we now introduce the affine Lie algebra $A_1^{(1)}$ and show how our results are related to its highest weight representations. We show that the conditional law of the Brownian motion is a Duistermaat Heckman measure for a circle action, describing the semiclassical behaviour of the weights of a representation when its highest weight is large. We also show that the Lévy correction term occurs in the behaviour of the elements of large weight of the Kashiwara crystal $B(\infty)$ (which is a non-probabilistic statement).

Section 5 is an appendix where we define rigorously the conditioned Brownian motion in the interval $[0, 1]$ and the conditioned space-time Brownian motion in the affine Weyl chamber C_{aff} . We prove in Theorem 5.7 that the first one is equal in law, up to a time inversion, to the space component of the second one.

1.6. In conclusion we see that the, a priori simple, Brownian motion in the interval can be studied thanks to its links with the affine Lie algebra $A_1^{(1)}$. It is an example of integrable probability in the sense of Borodin and Petrov ([5]).

It would be interesting to study the higher rank case $A_n^{(1)}$, $n \geq 2$, which occurs in the analysis of $n + 1$ non colliding Brownian motions on a circle (see Hobson and Werner [21]). This requires new ideas.

1.7. We thank Philippe Biane, Persi Diaconis and the referee for their advice.

2. REPRESENTATION OF THE CONDITIONED SPACE-TIME REAL BROWNIAN MOTION IN THE AFFINE WEYL CONE

In order to represent the Brownian motion Z conditioned to stay in $[0, 1]$ forever, we will linearize the problem and use the fact that Z is the space component of the time inverted process of $A^{(\mu)}$, where $A^{(\mu)}$ is a space-time Brownian motion conditioned to stay in the affine cone $C_{\text{aff}} = \{(t, x) \in \mathbb{R}^2; 0 < x < t\}$, rigorously defined in Definition 5.5. This process $A^{(\mu)}$ will be approached by planar Brownian motions conditioned to remain in dihedral cones.

2.1. Pitman representation for the planar Brownian motion in the dihedral cone C_m . In this section we describe the path representation of the planar Brownian motion conditioned to stay in a dihedral cone, using the results of Biane et al. [3],[4].

2.1.1. Dihedral Coxeter system. The dihedral group $I(m)$, where $m \in \mathbb{Z}_+$, is the finite group generated by two involutions s_0^m, s_1^m with the only relation $(s_0^m s_1^m)^m = 1$. In the two dimensional Euclidean vector space $V = \mathbb{R}^2$ identified with its dual \tilde{V} , one chooses two pairs $(\alpha_0^m, \tilde{\alpha}_0^m), (\alpha_1^m, \tilde{\alpha}_1^m)$ in $V \times \tilde{V}$, associated with the matrix $(\tilde{\alpha}_i^m(\alpha_j^m))_{0 \leq i, j \leq 1}$ given by

$$\begin{pmatrix} 2 & -2 \cos(\pi/m) \\ -2 \cos(\pi/m) & 2 \end{pmatrix}.$$

Namely, one takes

$$\begin{aligned} \alpha_0^m &= (2 \sin(\pi/m), -2 \cos(\pi/m)), \tilde{\alpha}_0^m = \alpha_0^m / 2 \\ \alpha_1^m &= (0, 2), \tilde{\alpha}_1^m = \alpha_1^m / 2. \end{aligned}$$

The following two linear reflections s_0^m, s_1^m of \mathbb{R}^2 ,

$$s_i^m(v) = v - \tilde{\alpha}_i^m(v) \alpha_i^m, \quad v \in \mathbb{R}^2,$$

generate the group $I(m)$. The usual scalar product on \mathbb{R}^2 is denoted by $\langle \cdot, \cdot \rangle$, and with our convention, for $v \in V$, $\tilde{\alpha}_i^m(v) = \langle \tilde{\alpha}_i^m, v \rangle$.

Definition 2.1. *The convex dihedral cone C_m , with closure \bar{C}_m , is*

$$\begin{aligned} C_m &= \{v \in \mathbb{R}^2; \tilde{\alpha}_i^m(v) > 0, i = 0, 1\} \\ &= \{(r \cos \theta, r \sin \theta) \in \mathbb{R}^2; r > 0, 0 < \theta < \pi/m\}. \end{aligned}$$

Let $\mathcal{C}_0(\mathbb{R}^2)$ be the set of continuous path $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^2$ such that $\eta(0) = 0$. The following path transforms are introduced in [3].

Definition 2.2. *The Pitman transforms $\mathcal{P}_{s_i^m}^m, i = 0, 1$, are defined on $\mathcal{C}_0(\mathbb{R}^2)$ by the formula, for $\eta \in \mathcal{C}_0(\mathbb{R}^2)$, $t \geq 0$,*

$$\mathcal{P}_{s_i^m}^m \eta(t) = \eta(t) - \inf_{0 \leq s \leq t} \tilde{\alpha}_i^m(\eta(s)) \alpha_i^m.$$

They depend only on s_i^m . For each $i \in \mathbb{N}$, when we write $\alpha_i^m, \tilde{\alpha}_i^m, s_i^m, \dots$, we take i modulo 2.

Theorem 2.3 (Biane et al. [3]). *Let $w = s_{i_r}^m \cdots s_{i_1}^m$ be a reduced decomposition of $w \in I(m)$ where $i_1, \dots, i_r \in \{0, 1\}$. Then*

$$\mathcal{P}_w^m := \mathcal{P}_{s_{i_r}^m}^m \cdots \mathcal{P}_{s_{i_1}^m}^m$$

depends only on w and not on its chosen decomposition.

In $I(m)$ there is a unique longest element, with two reduced decompositions, namely

$$(2.1) \quad w_0 = s_{m-1}^m \cdots s_1^m s_0^m = s_m^m \cdots s_2^m s_1^m.$$

Proposition 2.4 ([3]). *For any path $\eta \in \mathcal{C}_0(\mathbb{R}^2)$, the path $\mathcal{P}_{w_0}^m \eta$ takes values in the closed dihedral cone \bar{C}_m .*

Let $\gamma \in \bar{C}_m$ and let $W^{(\gamma)}$ be the standard planar Brownian motion in \mathbb{R}^2 with drift γ and identity covariance matrix, starting from the origin. The conditioned planar Brownian motion in the cone C_m with drift γ is intuitively given by $W^{(\gamma)}$ conditioned to stay in C_m forever. It is rigorously defined in Appendix 5.2. The following is proved in Biane et al. [3] when there is no drift, the case with drift follows easily and is proved below in Proposition 5.3.

Theorem 2.5 (Biane et al. [3]). *Let $W^{(\gamma)}$ be a planar Brownian motion with drift γ , where $\gamma \in \bar{C}_m$. Then $\mathcal{P}_{w_0}^m W^{(\gamma)}$ is a conditioned planar Brownian motion in the cone C_m with drift γ .*

2.1.2. String parameters in $\mathcal{C}_0(\mathbb{R}^2)$. We fix an integer $m \geq 1$. For simplicity of notations, and without loss of generality, one chooses one of the two decompositions of the longest element w_0 in $I(m)$, namely,

$$w_0 = s_{m-1}^m \cdots s_1^m s_0^m.$$

For $\eta \in \mathcal{C}_0(\mathbb{R}^2)$, $0 \leq k \leq m-1$ and $0 \leq t \leq +\infty$, let

$$x_k^m(t) = - \inf_{0 \leq s \leq t} \tilde{\alpha}_k^m(\mathcal{P}_{s_{k-1}^m}^m \cdots \mathcal{P}_{s_1^m}^m \mathcal{P}_{s_0^m}^m \eta(s)).$$

Definition 2.6. *We call*

$$x^m(t) = (x_0^m(t), x_1^m(t), \dots, x_{m-1}^m(t))$$

the string parameters of the path η on $[0, t]$ and we call $x^m(\infty)$ its Verma string parameters.

Notice that

$$(2.2) \quad \mathcal{P}_{w_0}^m \eta(t) = \eta(t) + \sum_{k=0}^{m-1} x_k^m(t) \alpha_k^m,$$

and that, for $t \geq 0$, for all $k = 0, \dots, m-1$,

$$(2.3) \quad 2x_k^m(t) = \tilde{\alpha}_k^m(\mathcal{P}_{s_k^m}^m \cdots \mathcal{P}_{s_0^m}^m \eta(t)) - \mathcal{P}_{s_{k-1}^m}^m \cdots \mathcal{P}_{s_0^m}^m \eta(t).$$

When

$$(2.4) \quad \lim_{s \rightarrow +\infty} \tilde{\alpha}_0^m(\eta(s)) = \lim_{s \rightarrow +\infty} \tilde{\alpha}_1^m(\eta(s)) = +\infty$$

holds, then the Pitman transforms $\mathcal{P}_w^m \eta$, $w \in I(m)$, have the same property and for all $t \geq 0$ and $0 \leq k \leq m-1$,

$$(2.5) \quad 0 \leq x_k^m(t) \leq x_k^m(\infty) < +\infty,$$

and $x_k^m(t) = x_k^m(\infty)$ for t large enough. Let, for $1 \leq k < m$,

$$a_k^m = \sin(k\pi/m).$$

Definition 2.7. *The cone Γ_m in \mathbb{R}^m is defined as*

$$\Gamma_m = \{(x_0, \dots, x_{m-1}) \in \mathbb{R}^m; \frac{x_1}{a_1^m} \geq \frac{x_2}{a_2^m} \geq \dots \geq \frac{x_{m-1}}{a_{m-1}^m} \geq 0, x_0 \geq 0\}.$$

For $\lambda \in \bar{C}_m$, the polytope $\Gamma_m(\lambda)$ is

$$\Gamma_m(\lambda) = \{(x_0, \dots, x_{m-1}) \in \Gamma_m; 0 \leq x_r \leq \tilde{\alpha}_r^m(\lambda - \sum_{n=r+1}^{m-1} x_n \alpha_n^m), 0 \leq r \leq m-1\}.$$

Remark the particular role of x_0 . The following proposition is proved in [4], Propositions 4.4 and 4.7.

Proposition 2.8. *For $t > 0$, the set of string parameters of paths on $[0, t]$ is Γ_m .*

2.1.3. String parameters of the planar Brownian motion. When $\gamma \in C_m$, the planar Brownian motion $W^{(\gamma)}$ with drift γ satisfies (2.4) by the law of large numbers, so its Verma string parameters are finite.

We consider a random vector $\mathcal{E}^m = (\mathcal{E}_0^m, \dots, \mathcal{E}_{m-1}^m)$ in \mathbb{R}^m whose components \mathcal{E}_k^m are independent exponentials with respective parameters $\gamma_k^m = \langle \gamma, \alpha_k^m \rangle$.

Theorem 2.9. *Let $\xi^m(t)$ be the string parameters of $W^{(\gamma)}$ on $[0, t]$, and $\xi^m(\infty)$ be its Verma string parameters.*

(i) *When $\gamma \in \bar{C}_m$, for any $t > 0$, conditionally on the σ -algebra*

$$\sigma(\mathcal{P}_{w_0}^m W^{(\gamma)}(s), s \leq t),$$

the random vector $(\xi_0^m(t), \dots, \xi_{m-1}^m(t))$ is distributed as the vector \mathcal{E}^m conditioned on the event $\{\mathcal{E}^m \in \Gamma_m(\mathcal{P}_{w_0}^m W^{(\gamma)}(t))\}$.

(ii) *When $\gamma \in C_m$, the random vector $(\xi_0^m(\infty), \dots, \xi_{m-1}^m(\infty))$ is distributed as the vector \mathcal{E}^m conditioned on the event $\{\mathcal{E}^m \in \Gamma_m\}$.*

Proof. Let $\mathcal{F}_t^\gamma = \sigma(\mathcal{P}_{w_0}^m W^{(\gamma)}(s), s \leq t)$. It is shown in Biane et al. [4], Theorem 5.2, that when $\gamma = 0$, the law of $\xi^m(t)$ conditionally on \mathcal{F}_t^0 is the normalized Lebesgue measure on $\Gamma_m(\mathcal{P}_{w_0}^m W^{(0)}(t))$. Let $\psi : \mathcal{C}_0(\mathbb{R}^2) \rightarrow \mathbb{R}$ be a bounded measurable function. It is easy to see by applying the definition of the conditional expectation and the Cameron Martin formula twice that we have the following Bayes formula,

$$(2.6) \quad \mathbb{E}(\psi(W_s^{(\gamma)}, 0 \leq s \leq t) | \mathcal{F}_t^\gamma) = \varphi(\mathcal{P}_{w_0}^m W^{(\gamma)}(t))$$

where

$$\varphi(\mathcal{P}_{w_0}^m W^{(0)}(t)) = \frac{\mathbb{E}(\psi(W_s^{(0)}, 0 \leq s \leq t) e^{\langle \gamma, W_t^{(0)} \rangle} | \mathcal{F}_t^0)}{\mathbb{E}(e^{\langle \gamma, W_t^{(0)} \rangle} | \mathcal{F}_t^0)}.$$

It follows from (2.2) that $W_t^{(0)} = \mathcal{P}_{w_0}^m W^{(0)}(t) - \sum_{k=0}^{m-1} \xi_k^m(t) \alpha_k^m$, where ξ^m are the string parameters of $W^{(0)}$, so that

$$\varphi(\mathcal{P}_{w_0}^m W^{(0)}(t)) = \frac{\mathbb{E}(\psi(W_s^{(0)}, 0 \leq s \leq t) e^{-\sum_{k=0}^{m-1} \xi_k^m(t) \langle \gamma, \alpha_k^m \rangle} | \mathcal{F}_t^0)}{\mathbb{E}(e^{-\sum_{k=0}^{m-1} \xi_k^m(t) \langle \gamma, \alpha_k^m \rangle} | \mathcal{F}_t^0)}.$$

If we take $\psi(W_s^{(\gamma)}, 0 \leq s \leq t) = F(\xi^m(t))$ where $F : \mathbb{R}^m \rightarrow \mathbb{R}$ is bounded and continuous, and now ξ^m are the string parameters of $W^{(\gamma)}$, and if dx is the Lebesgue measure on \mathbb{R}^m we obtain that

$$\mathbb{E}(F(\xi^m(t)) | \mathcal{F}_t^\gamma) = \frac{\int 1_{\Gamma_m(\mathcal{P}_{w_0}^m W^{(\gamma)}(t))}(x) F(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx}{\int 1_{\Gamma_m(\mathcal{P}_{w_0}^m W^{(\gamma)}(t))}(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx},$$

which proves (i). We now suppose that $\gamma \in C_m$, so

$$\begin{aligned} \mathbb{E}(F(\xi^m(\infty))) &= \lim_{t \rightarrow +\infty} \mathbb{E}(F(\xi^m(t))) \\ &= \lim_{t \rightarrow +\infty} \mathbb{E}(\mathbb{E}(F(\xi^m(t)) | \mathcal{F}_t^\gamma)) \\ &= \lim_{t \rightarrow +\infty} \mathbb{E}\left(\frac{\int 1_{\Gamma_m(\mathcal{P}_{w_0}^m W^{(\gamma)}(t))}(x) F(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx}{\int 1_{\Gamma_m(\mathcal{P}_{w_0}^m W^{(\gamma)}(t))}(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx}\right) \\ &= \frac{\int 1_{\Gamma_m}(x) F(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx}{\int 1_{\Gamma_m}(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx}, \end{aligned}$$

since, a.s., $\mathcal{P}_{w_0}^m W^{(\gamma)}(t)/t$ converges to γ as t tends to $+\infty$ and thus $\Gamma_m(\mathcal{P}_{w_0}^m W^{(\gamma)}(t))$ tends to Γ_m for $\gamma \in C_m$. This proves (ii). \square

Proposition 2.10. *We suppose that $\gamma \in C_m$. Let $\xi^m(\infty)$ be the Verma string parameters of $W^{(\gamma)}$. If we let*

$$\left(\xi_0^m(\infty), \frac{\xi_1^m(\infty)}{a_1^m} - \frac{\xi_2^m(\infty)}{a_2^m}, \frac{\xi_2^m(\infty)}{a_2^m} - \frac{\xi_3^m(\infty)}{a_3^m}, \dots, \frac{\xi_{m-1}^m(\infty)}{a_{m-1}^m}\right)$$

equal to

$$\left(\frac{\varepsilon_0^m}{\gamma_0^m}, \frac{\varepsilon_1^m}{\gamma_1^m a_1^m}, \frac{\varepsilon_2^m}{\gamma_1^m a_1^m + \gamma_2^m a_2^m}, \dots, \frac{\varepsilon_{m-1}^m}{\gamma_1^m a_1^m + \dots + \gamma_{m-1}^m a_{m-1}^m}\right),$$

then the ε_n^m are independent exponential random variables with parameter 1.

Proof. This follows from Theorem 2.9 since, on \mathbb{R}_+^m ,

$$\begin{aligned} \mathbb{P}(\xi_0^m(\infty) \in dx_0, \dots, \xi_{m-1}^m(\infty) \in dx_{m-1}) \\ &= C e^{-\gamma_0^m x_0} dx_0 \prod_{k=2}^m e^{-\gamma_{k-1}^m x_{k-1}} 1_{\left\{\frac{x_{k-1}}{a_{k-1}^m} \geq \frac{x_k}{a_k^m}\right\}} dx_{k-1} \\ &= C e^{-\gamma_0^m x_0} dx_0 \prod_{k=2}^m e^{-(\gamma_1^m a_1^m + \dots + \gamma_{k-1}^m a_{k-1}^m) \left(\frac{x_{k-1}}{a_{k-1}^m} - \frac{x_k}{a_k^m}\right)} 1_{\left\{\frac{x_{k-1}}{a_{k-1}^m} \geq \frac{x_k}{a_k^m}\right\}} dx_{k-1} \end{aligned}$$

where by convention $x_m/a_m^m = 0$ and C is a normalizing constant. \square

Notice that this is similar to Renyi's representation of order statistics ([34]).

2.2. Affine string parameters of the space-time Brownian motion. We will use a terminology inspired by the Kac-Moody affine algebra $A_1^{(1)}$ (see in particular Kac [24], Kashiwara [25], and Section 4). The infinite dihedral group $I(\infty)$ is the infinite group generated by two involutions s_0, s_1 with no relation. In $V = \mathbb{R}^2$ identified with its dual \tilde{V} , let $(\alpha_0, \tilde{\alpha}_0), (\alpha_1, \tilde{\alpha}_1)$ in $V \times \tilde{V}$ be given by

$$\begin{cases} \alpha_0 = (0, -2), \\ \alpha_1 = (0, 2), \\ \tilde{\alpha}_0 = (1, -1), \\ \tilde{\alpha}_1 = (0, 1). \end{cases}$$

The matrix $(\tilde{\alpha}_i(\alpha_j))_{0 \leq i, j \leq 1}$ is the Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The two linear reflections s_0, s_1 of \mathbb{R}^2 ,

$$s_i(v) = v - \tilde{\alpha}_i(v)\alpha_i, \quad v \in \mathbb{R}^2,$$

generate the group $I(\infty)$. Notice that s_0 is a non orthogonal reflection. For each $k \in \mathbb{N}$, when we write $\alpha_k, \tilde{\alpha}_k, s_k, \dots$, we take k modulo 2, as above. Thus $\alpha_k = (-1)^k \alpha_0$ (but $\tilde{\alpha}_k$ is not $(-1)^k \tilde{\alpha}_0$).

Definition 2.11. *The affine Weyl cone (or chamber) is*

$$C_{\text{aff}} = \{v \in \mathbb{R}^2; \tilde{\alpha}_i(v) > 0, i = 0, 1\} = \{(t, x) \in \mathbb{R}^2; 0 < x < t\},$$

and \bar{C}_{aff} is its closure.

We define the Pitman transform $\mathcal{P}_{s_i}, i = 0, 1$, on $\mathcal{C}_0(\mathbb{R}^2)$ by the formula:

$$\mathcal{P}_{s_i}\eta(t) = \eta(t) - \inf_{0 \leq s \leq t} \tilde{\alpha}_i(\eta(s))\alpha_i,$$

where $\eta \in \mathcal{C}_0(\mathbb{R}^2)$. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the involutive transformation $T(t, x) = (t, t - x)$. Then

$$(2.7) \quad \mathcal{P}_{s_1}T = T\mathcal{P}_{s_0}.$$

We will use mainly space-time paths, i.e. paths which can be written as $\eta(t) = (t, \varphi(t))$, $t \geq 0, \varphi(t) \in \mathbb{R}$. In this case,

$$\begin{aligned} \mathcal{P}_{s_0}\eta(t) &= (t, \varphi(t) + 2 \inf_{s \leq t} (s - \varphi(s))) \\ \mathcal{P}_{s_1}\eta(t) &= (t, \varphi(t) - 2 \inf_{s \leq t} \varphi(s)). \end{aligned}$$

One recognizes in the second component the transforms $\mathcal{P}_0, \mathcal{P}_1$ defined in Section 1.2. Let η be a path in $\mathcal{C}_0(\mathbb{R}^2)$. One defines $\eta_0 = \eta$, and, for $k \geq 1$,

$$\eta_k = \mathcal{P}_{s_{k-1}} \dots \mathcal{P}_{s_1} \mathcal{P}_{s_0} \eta,$$

and, for $t \leq \infty$, $x(t) = \{x_k(t), k \geq 0\}$ given by

$$x_k(t) = - \inf_{0 \leq s \leq t} \tilde{\alpha}_k(\eta_k(s)).$$

We call $x_k(t), k \geq 0$, the affine string parameters of the path η on $[0, t]$ and we call $x_k(\infty), k \geq 0$, its Verma affine string parameters.

We fix a real μ such that $0 \leq \mu \leq 1$ and we consider the space-time process

$$B_t^{(\mu)} = (t, B_t + t\mu)$$

where B is a standard real Brownian motion starting from 0.

Definition 2.12. We let $\xi(t) = \{\xi_k(t), k \geq 0\}$ be the affine string parameters of $B^{(\mu)}$ on $[0, t]$, and $\xi(\infty) = \{\xi_k(\infty), k \geq 0\}$ be its Verma affine string parameters.

One has, for any $n \geq 0$,

$$(2.8) \quad \mathcal{P}_{s_n} \cdots \mathcal{P}_{s_0} B^{(\mu)}(t) = B_t^{(\mu)} + \sum_{k=0}^n \xi_k(t) \alpha_k.$$

When $0 < \mu < 1$, $\xi_k(\infty) < +\infty$ for each $k \geq 0$ by the law of large numbers.

Let $\tau_m : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$\tau_m(t, x) = \left(\frac{\pi t}{m}, x \right),$$

for $(t, x) \in \mathbb{R}^2$. We will frequently use that for $v \in \mathbb{R}^2$,

$$(2.9) \quad \tau_m v = \left(\frac{\pi}{m \sin \frac{\pi}{m}} (\tilde{\alpha}_0^m(v) + \tilde{\alpha}_1^m(v) \cos \frac{\pi}{m}), \tilde{\alpha}_1^m(v) \right),$$

so the asymptotics of $\tau_m v$ and $(\tilde{\alpha}_0^m(v) + \tilde{\alpha}_1^m(v), \tilde{\alpha}_1^m(v))$ are the same as m tends to $+\infty$. It is easy to see that,

Lemma 2.13. For $i = 0, 1$,

$$\begin{aligned} \lim_{m \rightarrow +\infty} \tau_m \alpha_i^m &= \alpha_i, \\ \lim_{m \rightarrow +\infty} \tau_m^{-1} \tilde{\alpha}_i^m &= \tilde{\alpha}_i, \\ \lim_{m \rightarrow +\infty} \tau_m \circ s_i^m \circ \tau_m^{-1} &= s_i. \end{aligned}$$

We will approach $B^{(\mu)}$ by $\tau_m W^{(\frac{m}{\pi}, \mu)}$ where

$$W^{(\frac{m}{\pi}, \mu)}(t) = \left(\beta_t + t \frac{m}{\pi}, B_t + t\mu \right),$$

β and B being two independent real Brownian motions starting from the origin.

Definition 2.14. We define $\xi^m(t)$ as the string parameters of $W^{(\frac{m}{\pi}, \mu)}$ on $[0, t]$ and $\xi^m(\infty)$ as its Verma string parameters. The dihedral highest weight process is

$$\Lambda_m(t) = \mathcal{P}_{w_0}^m W^{(\frac{m}{\pi}, \mu)}(t), \quad t \geq 0.$$

When $0 < \mu < 1$, we will always suppose that m is large enough to have $(\frac{m}{\pi}, \mu) \in C_m$. This ensures that $\xi_m(\infty)$ is finite.

Proposition 2.15. Almost surely, for all $t > 0$, $k \in \mathbb{N}$,

$$\lim_{m \rightarrow +\infty} \xi_k^m(t) = \xi_k(t),$$

and when $0 < \mu < 1$,

$$\lim_{m \rightarrow +\infty} \xi_k^m(\infty) = \xi_k(\infty).$$

Proof. Let $\eta_0^m = W^{(\frac{m}{\pi}, \mu)}$, $\eta_0 = B^{(\mu)}$ and, for $k \geq 1$,

$$\eta_k^m = \mathcal{P}_{s_{k-1}^m}^m \dots \mathcal{P}_{s_0^m}^m \eta_0^m, \quad \eta_k = \mathcal{P}_{s_{k-1}} \dots \mathcal{P}_{s_0} \eta_0.$$

We prove by induction on $k \in \mathbb{N}$, that for any $T > 0$, a.s., uniformly on $[0, T]$,

$$\begin{aligned} \lim_{m \rightarrow +\infty} \tau_m \eta_k^m(t) &= \eta_k(t) \\ \lim_{m \rightarrow +\infty} \xi_k^m(t) &= \xi_k(t). \end{aligned}$$

This is clear for $k = 0$. If it holds for $k - 1$, then, by Lemma 2.13,

$$\tau_m \eta_k^m(t) = \tau_m \mathcal{P}_{s_{k-1}^m}^m \eta_{k-1}^m(t) = \tau_m \eta_{k-1}^m(t) + \xi_{k-1}^m(t) \tau_m \alpha_{k-1}^m$$

converges uniformly to $\eta_{k+1}(t)$. Since one can write

$$\xi_k^m(t) = - \inf_{s \leq t} \tilde{\alpha}_k^m(\eta_k^m(s)) = - \inf_{0 \leq s \leq t} \tau_m^{-1}(\tilde{\alpha}_k^m)(\tau_m \eta_k^m(s)),$$

$\xi_k^m(t)$ converges to $\xi_{k+1}(t)$.

We now suppose that $0 < \mu < 1$. First we notice that, a.s., for k fixed, for any $C > 0$, there is a time $t_C > 0$ such that, for any $m \in \mathbb{N}$,

$$(2.10) \quad \tilde{\alpha}_i^m(\eta_0^m(t)) \geq C,$$

when $t \geq t_C$, $i = 0, 1$. Let us show that a.s., for t large enough, $\xi_k^m(\infty) = \xi_k^m(t)$ for all $m \in \mathbb{N}$, large enough. This will imply that $\xi_k^m(\infty)$ tends to $\xi_k(\infty)$. Since $\tilde{\alpha}_k(\eta_k^m(0)) = 0$, it is enough to show that almost surely, there exists $T_k > 0$ such that for $t \geq T_k$, for all $m \in \mathbb{N}$ $\tilde{\alpha}_k^m(\eta_k^m(t)) \geq 0$. This follows from (2.10) and from the fact, easily proved by induction on k , that almost surely, there is a $C_k > 0$ such that, for all $t \geq 0$, $m \in \mathbb{N}$,

$$\|\eta_k^m(t) - \eta_0^m(t)\| \leq C_k.$$

□

Theorem 2.16. *For $0 < \mu < 1$, one can write, $\xi_0(\infty) = \varepsilon_0/2(1 - \mu)$, and for all $k \geq 1$,*

$$\frac{\xi_k(\infty)}{k} = \sum_{n=k}^{+\infty} \frac{2\varepsilon_n}{n(n+1) + (1-2\mu)\nu_n},$$

where $\varepsilon_n, n \geq 0$, are independent exponential random variables with parameter 1, and where $\nu_n = n$ when n is even and $\nu_n = -(n+1)$ when n is odd.

Proof. We consider the independent exponentially distributed random variables ε_n^m of Proposition 2.10 for the drift $\gamma = (m/\pi, \mu)$. One has

$$\begin{aligned} \gamma_{2k}^m &= \langle (\frac{m}{\pi}, \mu), \alpha_0^m \rangle = 2(\frac{m}{\pi} \sin(\frac{\pi}{m}) - \mu \cos(\frac{\pi}{m})), \\ \gamma_{2k+1}^m &= \langle (\frac{m}{\pi}, \mu), \alpha_1^m \rangle = 2\mu. \end{aligned}$$

Proposition 2.15 implies that the random variables $\xi_0^m(\infty)$, and

$$m \left(\frac{\xi_k^m(\infty)}{a_k^m} - \frac{\xi_{k+1}^m(\infty)}{a_{k+1}^m} \right)$$

converges almost surely when m goes to infinity for $k \geq 1$. Therefore the ε_n^m converge a.s. to independent exponentially distributed random variables ε_n with parameter 1. Moreover $\xi_0 = \frac{\varepsilon_0}{2(1-\mu)}$ and for every $n \geq 1$,

$$\frac{\xi_n(\infty)}{n} - \frac{\xi_{n+1}(\infty)}{n+1} = \frac{2\varepsilon_n}{n(n+1) + (1-2\mu)\nu_n}.$$

Proposition 2.15 and the description of the cone Γ_m imply that $\xi_k(\infty)/k, k \geq 1$, is a positive decreasing sequence. We denote by S its limit, which is a non negative random variable. Thus one has

$$\frac{\xi_k(\infty)}{k} = \sum_{n=k}^{+\infty} \frac{2\varepsilon_n}{n(n+1) + (1-2\mu)\nu_n} + S.$$

Moreover Fatou's lemma implies that

$$(2.11) \quad \mathbb{E}\left(\frac{\xi_k(\infty)}{k}\right) \leq \liminf_m \mathbb{E}\left(\frac{\xi_k^m(\infty)}{k}\right) = \liminf_m \frac{\pi}{m} \sum_{n=k}^{m-1} \frac{1}{a_1^m \gamma_1^m + \dots + a_n^m \gamma_n^m}$$

Let $c = \max\{1/\gamma_i^m, m \in \mathbb{N}, i = 0, 1\}$. Since $0 \leq 2t \leq \sin \pi t$ when $0 \leq t \leq 1/2$, one has, for $k \leq [m/2]$,

$$\sum_{n=1}^k a_n^m \gamma_n^m \geq \frac{1}{c} \sum_{n=1}^k \sin\left(\frac{n\pi}{m}\right) \geq \sum_{n=1}^k \frac{2n}{cm} \geq \frac{k^2}{cm},$$

and, for $k > [m/2]$,

$$\sum_{n=1}^k a_n^m \gamma_n^m \geq \sum_{n=1}^{[m/2]+1} a_n^m \gamma_n^m \geq \frac{m}{4c}.$$

Therefore, for $N \leq m-1$,

$$\frac{\pi}{m} \sum_{n=N}^{m-1} \frac{1}{a_1^m \gamma_1^m + \dots + a_n^m \gamma_n^m} \leq c\pi \sum_{n=N}^{[m/2]} \frac{1}{n^2} + 4c\pi \sum_{n=[m/2]+1}^m \frac{1}{m^2}$$

is as small as we want for N large enough. As one has for every $N > 0$,

$$\lim_m \frac{\pi}{m} \sum_{n=k}^N \frac{1}{a_1^m \gamma_1^m + \dots + a_n^m \gamma_n^m} = \sum_{n=k}^N \frac{2}{n(n+1) + (1-2\mu)\nu_n},$$

the lim inf in (2.11) is equal to $\sum_{n=k}^{+\infty} \frac{2}{n(n+1) + (1-2\mu)\nu_n}$ and therefore

$$\mathbb{E}\left(\frac{\xi_k(\infty)}{k}\right) \leq \sum_{n=k}^{+\infty} \frac{2}{n(n+1) + (1-2\mu)\nu_n}.$$

This proves that $\mathbb{E}(S) = 0$, which implies that $S = 0$, and finishes the proof. \square

An important for us and maybe unexpected result is:

Theorem 2.17. *For $0 < \mu < 1$, almost surely, $\lim_{k \rightarrow +\infty} \xi_k(\infty) = 2$.*

Proof. It is easy to see that $\mathbb{E}(\xi_k(\infty))$ tends to 2 and that $\sum_{k=1}^{\infty} \mathbb{E}((\xi_k(\infty) - \mathbb{E}(\xi_k(\infty)))^4)$ is finite, which implies the almost sure convergence. \square

2.3. Affine Verma weight $L^{(\mu)}(\infty)$. In this subsection we introduce the affine Verma weights. We consider, for $0 < \mu < 1$ and $k \geq 1$,

$$M_k = \frac{1}{2}\xi_k(\infty)\alpha_k + \sum_{n=0}^{k-1} \xi_n(\infty)\alpha_n,$$

where $\xi(\infty) = \{\xi_k(\infty), k \geq 0\}$ are the affine Verma string parameters of $B^{(\mu)}$. Notice that, more simply, since $\alpha_n = (0, (-1)^{n+1}2)$,

$$M_k = (0, (-1)^{k+1}\xi_k(\infty) + 2 \sum_{n=0}^{k-1} (-1)^{n+1}\xi_n(\infty)),$$

the notation with the α_n 's may be strange for the reader. It is explained by its natural interpretation in $A_1^{(1)}$ (see 4.1).

Proposition 2.18. *Let $\mu \in (0, 1)$. When k goes to infinity, M_k converges almost surely and in L^2 to*

$$L^{(\mu)}(\infty) = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{\varepsilon_{2n}}{n+1-\mu} \alpha_0 + \frac{\varepsilon_{2n+1}}{n+\mu} \alpha_1 \right).$$

Proof. One has the key relation

$$\begin{aligned} M_{2p+2} - M_{2p} &= (0, -\xi_{2p}(\infty) + 2\xi_{2p+1}(\infty) - \xi_{2p+2}(\infty)) \\ &= (0, \frac{\varepsilon_{2p+1}}{p+\mu} - \frac{\varepsilon_{2p}}{p+1-\mu}), \end{aligned}$$

hence $\{M_{2p} - \mathbb{E}(M_{2p}), p \geq 0\}$ is a martingale bounded in L_2 . Since $\mathbb{E}(M_{2p})$ converges, this shows that M_{2p} converges a.s. and its limit is $L^{(\mu)}(\infty)$. As

$$M_{2p+1} = M_{2p} + \frac{1}{2}(\xi_{2p}(\infty) - \xi_{2p+1}(\infty))\alpha_0$$

and $\xi_k(\infty)$ tends to 2 by Theorem 2.17, M_{2p+1} has the same limit $L^{(\mu)}(\infty)$. \square

Notice that $L^{(\mu)}(\infty)$ is not the limit of the series $\sum_{k=0}^n \xi_k(\infty)\alpha_k$ since $\xi_k(\infty)$ tends to 2 (Theorem 2.17).

Definition 2.19. *We call $L^{(\mu)}(\infty)$ the affine Verma weight of $B^{(\mu)}$.*

The terminology is due to the fact, that we will see in Section 4, that $L^{(\mu)}(\infty)$ plays the role of an asymptotic weight in the Verma module of highest weight 0 in the case of the affine Lie algebra $A_1^{(1)}$ (more correctly of minus this weight modulo δ). In coordinates, $L^{(\mu)}(\infty) = (0, D^\mu(\infty))$ where

$$(2.12) \quad D^\mu(\infty) = \sum_{n=0}^{\infty} \left(\frac{\varepsilon_{2n+1}}{n+\mu} - \frac{\varepsilon_{2n}}{n+1-\mu} \right).$$

For a sequence $x = (x_k) \in \mathbb{R}_+^{\mathbb{N}}$ we let

$$(2.13) \quad \sigma(x) = \lim_{n \rightarrow +\infty} \sum_{k=0}^{n-1} x_k \alpha_k + \frac{1}{2} x_n \alpha_n,$$

and write $\sigma(x) \in \mathbb{R}^2$ when this limit exists in \mathbb{R}^2 . When $0 < \mu < 1$,

$$\sigma(\xi(\infty)) = L^{(\mu)}(\infty).$$

Definition 2.20. One defines, for $\lambda \in \bar{C}_{\text{aff}}$,

$$\Gamma = \{x = (x_k) \in \mathbb{R}^{\mathbb{N}} : \frac{x_k}{k} \geq \frac{x_{k+1}}{k+1} \geq 0, \text{ for all } k \geq 1, x_0 \geq 0, \sigma(x) \in \mathbb{R}^2\},$$

$$\Gamma(\lambda) = \{x \in \Gamma : x_k \leq \tilde{\alpha}_k(\lambda - \sigma(x) + \sum_{i=0}^k x_i \alpha_i), \text{ for every } k \geq 0\}.$$

We remark that $x \in \Gamma$ is in $\Gamma(\lambda)$ if and only if for all $k \geq 0$,

$$(2.14) \quad \tilde{\alpha}_k(\sigma(x) - \sum_{i=0}^{k-1} x_i \alpha_i - \frac{1}{2} x_k \alpha_k) \leq \tilde{\alpha}_k(\lambda).$$

Notice the occurrence of the coefficient $1/2$ here which will explain the correction term in the representation theorem. The same coefficient already occurs in the description of the crystal $B(\lambda)$ of $A_1^{(1)}$ given by Nakashima [31] which is the discrete analogue of $\Gamma(\lambda)$. In a sense Γ will have the role of a continuous Verma crystal and $\sigma(x)$ has a role of the weight of $x \in \Gamma$. In the same way $\Gamma(\lambda)$ will be a kind of continuous crystal of highest weight λ .

Proposition 2.21. When $0 < \mu < 1$, the Verma string parameters $\xi(\infty)$ of $B^{(\mu)}$ are a.s. in $\cup\{\Gamma(\lambda), \lambda \in C_{\text{aff}}\}$ and for each $\lambda \in C_{\text{aff}}$, $\mathbb{P}(\xi(\infty) \in \Gamma(\lambda)) > 0$.

Proof. We have seen that, almost surely the sequence $\xi_k(\infty), k \geq 0$, is bounded and that $\sigma(\xi(\infty)) = L^{(\mu)}(\infty)$. So this sequence is in some $\Gamma(\lambda)$ for $\lambda \in C_{\text{aff}}$ (and actually in all λ large enough). Now

$$L^{(\mu)}(\infty) - M_{2k} = \sum_{p=k}^{\infty} (M_{2p+2} - M_{2p}) = (0, R_k)$$

where $R_k = \sum_{n=k}^{\infty} (\frac{\varepsilon_{2n+1}}{n+\mu} - \frac{\varepsilon_{2n}}{n+1-\mu})$, $k \geq 0$. Let

$$S_k = R_k + \sum_{n=2k}^{+\infty} \frac{2\varepsilon_n}{n(n+1) + (1-2\mu)\nu_n},$$

$$T_k = \frac{\varepsilon_{2k+1}}{(k+1)(2k+2\mu)} - \frac{\varepsilon_{2k}}{k+1-\mu}.$$

Then

$$\begin{aligned} L^{(\mu)}(\infty) - M_{2k+1} &= L^{(\mu)}(\infty) - M_{2k} - (M_{2k+1} - M_{2k}) \\ &= (0, R_k + \xi_{2k+1}(\infty) - \xi_{2k}(\infty)) \\ &= (0, S_k + T_k) \end{aligned}$$

since

$$\xi_{k+1}(\infty) - \xi_k(\infty) = \frac{\xi_{k+1}(\infty)}{k+1} - \frac{2k\varepsilon_k}{k(k+1) + (1-2\mu)\nu_k}.$$

Let $a < 0 < b$. As the sequences R_n, S_n, T_n converge almost surely to 0, we can choose $n_0 \geq 1$ such that

$$\mathbb{P}(R_n, S_n, T_n \in [a/2, b/2] \text{ for all } n \geq n_0) > 0.$$

One has

$$\begin{aligned} & \mathbb{P}(R_n, S_n + T_n \in [a, b], \text{ for all } n \geq 0) \geq \\ & \mathbb{P}(R_k - R_{n_0}, T_k + S_k - S_{n_0} \in [a/2, b/2], \text{ for } 0 \leq k < n_0, \\ & \quad R_n, S_n, T_n \in [a/2, b/2], \text{ for } n \geq n_0) \geq \\ & \mathbb{P}(R_k - R_{n_0}, T_k + S_k - S_{n_0} \in [a/2, b/2], \text{ for } 0 \leq k < n_0) \\ & \quad \times \mathbb{P}(R_n, S_n, T_n \in [a/2, b/2], \text{ for } n \geq n_0) > 0. \end{aligned}$$

This allows to finish the proof, since, by (2.14), $\xi(\infty) \in \Gamma(\lambda)$ if and only if, for all $k \geq 0$,

$$\tilde{\alpha}_k(L^{(\mu)}(\infty) - M_k) \leq \tilde{\alpha}_k(\lambda)$$

where for $\lambda \in C_{\text{aff}}$, $\tilde{\alpha}_k(\lambda)$ takes only two positive values. \square

2.4. Some technical results. In this subsection we prove some properties which will be used to study the string parameters $\xi_k(t)$, $k \in \mathbb{N}$, and conclude the proof of our main theorem. We take here $\mu \in (0, 1)$ and m large enough to ensure that $(\frac{m}{\pi}, \mu) \in C_m$. Let, for $0 \leq k \leq m$, with the convention that $\xi_m^m(\infty) = 0$,

$$M_k^m = \frac{1}{2} \xi_k^m(\infty) \alpha_k^m + \sum_{n=0}^{k-1} \xi_n^m(\infty) \alpha_n^m.$$

Proposition 2.22. *For any $k \geq 0$, $\tau_m M_k^m$ converges to M_k a.s. when m goes to infinity.*

Proof. This follows from Lemma 2.13 and Proposition 2.15. \square

Proposition 2.23. *One has, in probability,*

$$(2.15) \quad \lim_{m \rightarrow +\infty} \sup_{1 \leq k \leq m} \|\tau_m M_k^m - M_k\| = 0.$$

Proof. We use Proposition 2.10 and its notations. For $1 \leq 2p+1 \leq m$,

$$M_{2p+2}^m - M_{2p}^m = \frac{1}{2} \xi_{2p+2}^m(\infty) \alpha_{2p+2}^m + \xi_{2p+1}^m(\infty) \alpha_{2p+1}^m + \frac{1}{2} \xi_{2p}^m(\infty) \alpha_{2p}^m.$$

Therefore,

$$\begin{aligned} \tilde{\alpha}_0^m(M_{2p+2}^m - M_{2p}^m) &= \xi_{2p+2}^m(\infty) - 2 \cos\left(\frac{\pi}{m}\right) \xi_{2p+1}^m(\infty) + \xi_{2p}^m(\infty) \\ &= -2 \cos\left(\frac{\pi}{m}\right) a_{2p+1}^m \frac{\varepsilon_{2p+1}^m}{\gamma_1^m a_1^m + \gamma_0^m a_2^m + \cdots + \gamma_1^m a_{2p+1}^m} \\ & \quad + a_{2p}^m \left(\frac{\varepsilon_{2p+1}^m}{\gamma_1^m a_1^m + \cdots + \gamma_1^m a_{2p+1}^m} + \frac{\varepsilon_{2p}^m}{\gamma_1^m a_1^m + \cdots + \gamma_0^m a_{2p}^m} \right), \end{aligned}$$

since

$$a_{2p+2}^m - 2 \cos\left(\frac{\pi}{m}\right) a_{2p+1}^m + a_{2p}^m = 0.$$

One deduces that

$$\tilde{\alpha}_0^m(M_{2p}^m) = 2 \sin \frac{\pi}{m} \sum_{l=0}^{p-1} \left(\frac{\varepsilon_{2l}^m \cos \frac{l\pi}{m}}{\gamma_1^m \sin \frac{l}{m} \pi + \gamma_0^m \sin \frac{l+1}{m} \pi} - \frac{\varepsilon_{2l+1}^m \cos \frac{(l+1)\pi}{m}}{\gamma_1^m \sin \frac{l+1}{m} \pi + \gamma_0^m \sin \frac{l}{m} \pi} \right),$$

by using the relations

$$\sum_{k=1}^n a_{2k}^m = \frac{\sin \frac{n\pi}{m} \sin \frac{(n+1)\pi}{m}}{\sin \frac{\pi}{m}}, \quad \sum_{k=1}^n a_{2k-1}^m = \frac{\sin^2 \frac{n\pi}{m}}{\sin \frac{\pi}{m}}, \quad \sum_{k=1}^n a_k^m = \frac{\sin \frac{n\pi}{2m} \sin \frac{(n+1)\pi}{2m}}{\sin \frac{\pi}{2m}}.$$

Similarly, one finds

$$\begin{aligned} \tilde{\alpha}_1^m(M_{2p}^m) &= -2 \cos \frac{\pi}{m} \xi_0^m(\infty) + \sin \frac{\pi}{m} \tan \frac{p\pi}{m} \xi_{2p}^m(\infty) \\ &+ 2 \sin \frac{\pi}{m} \sum_{l=1}^{p-1} \left(\frac{\varepsilon_{2l+1}^m \cos \frac{l\pi}{m}}{\gamma_1^m \sin \frac{l+1}{m}\pi + \gamma_0^m \sin \frac{l}{m}\pi} - \frac{\varepsilon_{2l}^m \cos \frac{(l+1)\pi}{m}}{\gamma_1^m \sin \frac{l}{m}\pi + \gamma_0^m \sin \frac{l+1}{m}\pi} \right). \end{aligned}$$

On the other hand, one has

$$M_{2p+1}^m - M_{2p}^m = \frac{1}{2} (\xi_{2p}^m(\infty) \alpha_0^m + \xi_{2p+1}^m(\infty) \alpha_1^m).$$

Thus the proposition follows from the next lemma and from Lemma 2.13. \square

Lemma 2.24. *Let $\gamma_0^m, \gamma_1^m, m \in \mathbb{N}$, be real numbers such that γ_0^m tends to $2(1-\mu)$ and γ_1^m tends to 2μ . Let for $p \in \{0, \dots, [m/2]\}$,*

$$\begin{aligned} S_p^m &= \sin \frac{\pi}{m} \sum_{l=0}^{p-1} \left(\frac{\varepsilon_{2l}^m \cos \frac{l\pi}{m}}{\gamma_1^m \sin \frac{l}{m}\pi + \gamma_0^m \sin \frac{l+1}{m}\pi} - \frac{\varepsilon_{2l+1}^m \cos \frac{(l+1)\pi}{m}}{\gamma_1^m \sin \frac{l+1}{m}\pi + \gamma_0^m \sin \frac{l}{m}\pi} \right), \\ S_p &= \sum_{l=0}^{p-1} \left(\frac{\varepsilon_{2l}}{2l + 2(1-\mu)} - \frac{\varepsilon_{2l+1}}{2l + 2\mu} \right), \end{aligned}$$

where $(\varepsilon_n^m)_{n \geq 0}$ is a sequence of independent exponentially distributed random variables with parameter 1 that converges almost surely to $(\varepsilon_n)_{n \geq 0}$ when m goes to infinity. Then in probability

$$\lim_{m \rightarrow \infty} \sup_{1 \leq p \leq m/2} |S_p^m - S_p| = 0.$$

Proof. We have

$$(2.16) \quad \sup_{1 \leq p \leq m/2} |S_p^m - S_p| \leq \sup_{1 \leq p \leq m/2} |S_p^m - S_p - \mathbb{E}(S_p^m - S_p)| + \sup_{1 \leq p \leq m/2} |\mathbb{E}(S_p^m - S_p)|.$$

Let us show that the first term of the right hand-side converges to 0 in probability. One has for $N \leq [m/2]$,

$$(2.17) \quad \begin{aligned} \sup_{1 \leq p \leq m/2} |S_p^m - S_p - \mathbb{E}(S_p^m - S_p)| &\leq 2 \sup_{1 \leq p \leq N} |S_p^m - S_p - \mathbb{E}(S_p^m - S_p)| \\ &+ \sup_{N \leq p \leq m/2} |\tilde{S}_p^m - \tilde{S}_p - \mathbb{E}(\tilde{S}_p^m - \tilde{S}_p)|, \end{aligned}$$

where $\tilde{S}_p^m = S_p^m - S_N^m$ and $\tilde{S}_p = S_p - S_N$, for $p \geq N$. The first term of the right hand-side of (2.17) converges to 0, for any N , when m goes to infinity. As for each $m \in \mathbb{N}$, $\{\tilde{S}_p^m - \tilde{S}_p - \mathbb{E}(\tilde{S}_p^m - \tilde{S}_p), p = N, \dots, m/2\}$ is a martingale, Doob's martingale inequality gives, for $a > 0$,

$$(2.18) \quad \mathbb{P}\left(\sup_{N \leq p \leq m/2} |\tilde{S}_p^m - \tilde{S}_p - \mathbb{E}(\tilde{S}_p^m - \tilde{S}_p)| \geq a \right) \leq \frac{1}{a^2} \text{Var}(\tilde{S}_{[m/2]}^m - \tilde{S}_{[m/2]}).$$

Besides, as $\text{Var}(X + Y) \leq 2\text{Var}(X) + 2\text{Var}(Y)$, one has

$$\begin{aligned} \text{Var}(\tilde{S}_{[m/2]}^m - \tilde{S}_{[m/2]}) &\leq 2 \sum_{l=N}^{[m/2]-1} \left[\frac{\sin^2 \pi/m}{(\gamma_1^m \sin \frac{l}{m} \pi + \gamma_0^m \sin \frac{l+1}{m} \pi)^2} + \frac{1}{(2l + 2(1 - \mu))^2} \right. \\ &\quad \left. + \frac{\sin^2 \pi/m \cos(l+1)\pi/m}{(\gamma_1^m \sin \frac{l+1}{m} \pi + \gamma_0^m \sin \frac{l}{m} \pi)^2} + \frac{1}{(2l + 2\mu)^2} \right]. \end{aligned}$$

As $\frac{2x}{\pi} \leq \sin x \leq x$ for $0 \leq x \leq \frac{\pi}{2}$, one has the majorations

$$\frac{\sin \frac{\pi}{m}}{\gamma_1^m \sin \frac{l}{m} \pi + \gamma_0^m \sin \frac{l+1}{m} \pi} \leq \frac{\pi/2}{\gamma_1^m l + \gamma_0^m (l+1)},$$

and

$$\frac{\sin \frac{\pi}{m}}{\gamma_1^m \sin \frac{l+1}{m} \pi + \gamma_0^m \sin \frac{l}{m} \pi} \leq \frac{\pi/2}{\gamma_1^m (l+1) + \gamma_0^m l},$$

for $N \leq l \leq [m/2] - 1$. Thus for any $\varepsilon > 0$, there is a $N_0 \in \mathbb{N}$ such that for all $m \geq 2(N_0 + 1)$, the right hand-side term of (2.18) is smaller than ε . Hence $\sup_{1 \leq p \leq m/2} |S_p^m - S_p - \mathbb{E}(S_p^m - S_p)|$ converges in probability to 0. Let us deal with the second term of the right hand-side of (2.16). One has for $p, q \in \{0, \dots, [m/2]\}$, $p \geq q$,

$$|\mathbb{E}(S_p^m - S_q^m)| \leq \sum_{l=q}^{p-1} \frac{\pi^2(\gamma_0^m + \gamma_1^m)}{4(\gamma_1^m l + \gamma_0^m (l+1))(\gamma_1^m (l+1) + \gamma_0^m l)}$$

and

$$\mathbb{E}(S_p - S_q) = \sum_{l=q}^{p-1} \frac{2(2\mu - 1)}{(2l + 2(1 - \mu))(2l + 2\mu)}.$$

As for a fixed q , $\mathbb{E}(S_q^m)$ converges to $\mathbb{E}(S_q)$ when m goes to infinity, one obtains that $\sup_{0 \leq p \leq m/2} |\mathbb{E}(S_p^m - S_p)|$ converges to 0, which finishes the proof of the lemma. \square

Recall that $\xi^m(\infty)$, resp. $\xi(\infty)$, are the Verma string parameters of $W^{(\frac{m}{\pi}, \mu)}$, resp. $B^{(\mu)}$. The following proposition will allow us to pass from Verma affine string parameters to affine string parameters in the next subsection.

Proposition 2.25. *Let (λ_m) be a sequence of \mathbb{R}^2 such that $\lambda_m \in C_m$ and such that $\tau_m \lambda_m$ tends to λ when m tends to ∞ where $\lambda \in C_{\text{aff}}$. The random sets $\{\xi^m(\infty) \in \Gamma_m(\lambda_m)\}$ converge in probability to $\{\xi(\infty) \in \Gamma(\lambda)\}$.*

Proof. To prove convergence in probability of a sequence, it is enough to show that each subsequence has a subsequence which converges almost surely. Therefore, by Propositions 2.23 and 2.18, working with a subsequence, we can suppose that the set of $\omega \in \Omega$ for which

$$\sup_{1 \leq k \leq m} \|\tau_m M_k^m(\omega) - M_k(\omega)\| \rightarrow 0,$$

and $M_k(\omega)$ tends to $L^{(\mu)}(\infty)(\omega)$ is of probability one. If

$$X_k^m = M_m^m - M_k^m, \quad X_k = L^{(\mu)}(\infty) - M_k,$$

then (see (2.9))

$$(2.19) \quad \sup_{0 \leq k \leq m} |\tilde{\alpha}_k^m(X_k^m(\omega)) - \tilde{\alpha}_k(X_k(\omega))| \rightarrow 0, \tilde{\alpha}_k(X_k(\omega)) \rightarrow 0.$$

For these ω we will show that

$$\limsup_{m \rightarrow \infty} \{\xi^m(\infty) \in \Gamma_m(\lambda_m)\} \subset \{\xi(\infty) \in \Gamma(\lambda)\} \subset \liminf_{m \rightarrow \infty} \{\xi^m(\infty) \in \Gamma_m(\lambda_m)\}.$$

Notice that $\tilde{\alpha}_k^m(\lambda_m)$ tends to $\tilde{\alpha}_k(\lambda)$ for $k = 0, 1$ by Lemma 2.13. One has $\xi^m(\infty) \in \Gamma_m(\lambda_m)$ if and only if $\tilde{\alpha}_k^m(\lambda_m - X_k^m) \geq 0$ for $0 \leq k < m$, and $\xi(\infty) \in \Gamma(\lambda)$ if and only if $\tilde{\alpha}_k(\lambda - X_k) \geq 0$ for every $k \in \mathbb{N}$. The left inclusion above follows from (2.19). Now, suppose that $\xi(\infty)(\omega) \in \Gamma(\lambda)$. Since λ is fixed and since the distribution of X_k is continuous, for any $k \geq 0$,

$$\mathbb{P}(\tilde{\alpha}_k(\lambda - X_k) = 0) = 0,$$

and one can suppose that for all $k \geq 0$,

$$\tilde{\alpha}_k(\lambda - X_k(\omega)) > 0.$$

We choose $\varepsilon > 0$ such that $\tilde{\alpha}_i(\lambda) > \varepsilon, i = 0, 1$. Using (2.19), one can choose m_0 such that $m \geq m_0$ implies that

$$\sup_{0 \leq k \leq m} \tilde{\alpha}_k^m(X_k^m(\omega)) - \tilde{\alpha}_k(X_k(\omega)) < \varepsilon,$$

and then we choose k_0 such that $k \geq k_0$ implies that

$$\tilde{\alpha}_k(X_k(\omega)) < \tilde{\alpha}_k(\lambda) - \varepsilon.$$

As for each k , $\tilde{\alpha}_k^m(X_k^m)(\omega)$ converges to $\tilde{\alpha}_k(X_k)(\omega)$ when m goes to infinity, one takes m_1 such that when $m \geq m_1$,

$$\tilde{\alpha}_k^m(X_k^m(\omega)) < \tilde{\alpha}_k(\lambda),$$

for $k = 1, \dots, k_0$. Then for $m \geq m_0, m_1$ one has for $k \geq k_0$

$$\tilde{\alpha}_k^m(X_k^m(\omega)) = \tilde{\alpha}_k^m(X_k^m(\omega)) - \tilde{\alpha}_k(X_k(\omega)) + \tilde{\alpha}_k(X_k(\omega)) < \tilde{\alpha}_k(\lambda),$$

and for $k \leq k_0$,

$$\tilde{\alpha}_k^m(X_k^m(\omega)) < \tilde{\alpha}_k(\lambda),$$

which proves the right inclusion above, and shows the proposition. \square

2.5. The highest weight process $\Lambda^{(\mu)}$. In this subsection we introduce the highest weight process $\Lambda^{(\mu)}$ associated with the space-time Brownian motion $B^{(\mu)}$ which will appear as the limit of its Pitman's transforms (with a correction) and show that it coincides in law with $B^{(\mu)}$ conditioned to remain in C_{aff} forever. Let $0 \leq \mu \leq 1$, we define for $k \geq 0$ and $t \in \mathbb{R}^+$,

$$M_k(t) = \frac{1}{2} \xi_k(t) \alpha_k + \sum_{n=0}^{k-1} \xi_n(t) \alpha_n,$$

where $\xi(t) = \{\xi_n(t), n \geq 0\}$ are the affine string parameters of $B^{(\mu)}$ on $[0, t]$ and

$$M_k^m(t) = \frac{1}{2} \xi_k^m(t) \alpha_k^m + \sum_{n=0}^{k-1} \xi_n^m(t) \alpha_n^m,$$

where $\xi^m(t) = \{\xi_n^m(t), 0 \leq n < m\}$ are the string parameters of $W^{(\frac{m}{\pi}, \mu)}$ on $[0, t]$ (by convention $\xi_n^m(t) = 0$ when $n \geq m$). In particular, $M_m^m(t) = \sum_{n=0}^{m-1} \xi_n^m(t) \alpha_n^m$.

Theorem 2.26. *When $\mu \in [0, 1]$, for each $t \geq 0$, $M_k(t)$ converges almost surely when k goes to infinity. We denote by $L^{(\mu)}(t)$ the limit.*

Proof. By the Cameron-Martin theorem, one can suppose that $\mu \in (0, 1)$. For every $k \geq 1$ and $t \geq 0$, $\tau_m M_k^m(t)$ converges to $M_k(t)$ almost surely when m goes to infinity by Proposition 2.15 and Lemma 2.13. Let $\varepsilon > 0$. For $p, q \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}(\|M_{p+q}(t) - M_p(t)\| \geq \varepsilon) &= \lim_{m \rightarrow \infty} \mathbb{P}(\|\tau_m M_{p+q}^m(t) - \tau_m M_p^m(t)\| \geq \varepsilon) \\ &= \lim_{m \rightarrow \infty} \mathbb{E}(f_m(\Lambda_m(t))) \end{aligned}$$

where $\Lambda_m(t)$ is given in Definition 2.14 and

$$f_m(\Lambda_m(t)) = \mathbb{E}(1_{\|\tau_m M_{p+q}^m(t) - \tau_m M_p^m(t)\| \geq \varepsilon} | \sigma(\Lambda_m(t))).$$

One has, by Theorem 2.9,

$$f_m(\lambda) = \mathbb{P}(\|\tau_m M_{p+q}^m - \tau_m M_p^m\| \geq \varepsilon | \xi^m(\infty) \in \Gamma_m(\lambda)).$$

If $\lambda \in C_{\text{aff}}$ and $\tau_m \lambda_m$ tends to λ , then by Propositions 2.25 and 2.22, $f_m(\lambda_m)$ tends to $f(\lambda)$ where

$$f(\lambda) = \mathbb{P}(\|M_{p+q} - M_p\| \geq \varepsilon | \xi(\infty) \in \Gamma(\lambda)).$$

Notice that this is well defined by Proposition 2.21. On the other hand we will show in Theorem 2.45, independently of this proof, that when m tends to infinity, $\tau_m \Lambda_m(t)$ converges in law to $A^{(\mu)}(t)$, where $A^{(\mu)}$ is the Brownian motion conditioned to remain in C_{aff} defined in section 5.3. Therefore $\mathbb{E}(f_m(\Lambda_m(t)))$ tends to $\mathbb{E}(f(A^{(\mu)}(t)))$. This shows that

$$\mathbb{P}(\|M_{p+q}(t) - M_p(t)\| \geq \varepsilon) = \int_{\Omega} \mathbb{P}(\|M_{p+q} - M_p\| \geq \varepsilon | \xi(\infty) \in \Gamma(A^{(\mu)}(t)(\omega))) d\mathbb{P}(\omega).$$

Since M_k converges a.s. to $L^{(\mu)}(\infty)$ (Proposition 2.18) we see that $M_p(t)$, $p \in \mathbb{N}$, is a Cauchy sequence for the convergence in probability, and thus converges in probability. We will prove the almost sure convergence at the end of this subsection. \square

Notice that the first component of $L^{(\mu)}(t)$ is 0, so one can write

$$L^{(\mu)}(t) = (0, D^\mu(t)),$$

where $D^\mu(t) \in \mathbb{R}$.

Proposition 2.27. *When $0 < \mu < 1$ there is an almost surely finite random time $\sigma \geq 0$ such that $L^{(\mu)}(t) = L^{(\mu)}(\infty)$ for $t \geq \sigma$.*

Proof. Let $\sigma_0 = \max\{t \geq 0, B_t^{(\mu)} \notin C_{\text{aff}}\}$, and

$$\sigma_{n+1} = \max\{t \geq \sigma_n, B_t^{(\mu)} + \sum_{k=0}^n \xi_k(t) \alpha_k \notin C_{\text{aff}}\}$$

for $n \geq 0$. Then, a.s., $\sigma_0 < +\infty$ and for $t > \sigma_0$, $\xi_0(t) = \xi_0(\infty)$, hence

$$\sigma_1 = \max\{t \geq \sigma_0, B_t^{(\mu)} + \xi_0(\infty) \alpha_0 \notin C_{\text{aff}}\}.$$

Recursively, one has $\sigma_n < +\infty$ and

$$\sigma_{n+1} = \max\{t \geq \sigma_n, B_t^{(\mu)} + \sum_{k=0}^n \xi_k(\infty) \alpha_k \notin C_{\text{aff}}\}.$$

We know that $\sum_{k=0}^n \xi_k(\infty)\alpha_k$ is bounded, a.s., therefore $\sigma = \sup \sigma_n < +\infty$ and for $t \geq \sigma$, $\xi_k(t) = \xi_k(\infty)$ for all $k \geq 0$. So, for $t \geq \sigma$, $L^{(\mu)}(t) = L^{(\mu)}(\infty)$. \square

Proposition 2.28. *In probability, as m tends to ∞ , $\tau_m M_m^m(t)$ converges to $L^{(\mu)}(t)$.*

Proof. By Cameron-Martin's theorem, it is enough to prove the proposition for $\mu \in (0, 1)$. In that case, one has for $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}(\|\tau_m M_m^m(t) - L^{(\mu)}(t)\| > \varepsilon) &\leq \mathbb{P}\left(\sup_{1 \leq k \leq m} \|\tau_m M_k^m(t) - M_k(t)\| > \frac{\varepsilon}{2}\right) \\ &\quad + \mathbb{P}(\|M_m(t) - L^{(\mu)}(t)\| > \frac{\varepsilon}{2}). \end{aligned}$$

For the first term, we condition by $\Lambda_m(t)$ as in the proof of Theorem 2.26 and use Proposition 2.23. Convergence in probability of Theorem 2.26 gives the convergence of the second one. \square

Notice that

$$(2.20) \quad \Lambda_m(t) = \mathcal{P}_{w_0}^m W_t^{(\frac{m}{\pi}, \mu)}(t) = W_t^{(\frac{m}{\pi}, \mu)} + M_m^m(t).$$

We introduce the following definition,

Definition 2.29. *For $0 \leq \mu \leq 1$, we define the (affine) highest weight process of $\{B_t^{(\mu)}, t \geq 0\}$ by, for $t \geq 0$,*

$$\Lambda^{(\mu)}(t) = B_t^{(\mu)} + L^{(\mu)}(t).$$

In the analogy with the Littelmann model, for $t > 0$ fixed, $\{B_s^{(\mu)}, 0 \leq s \leq t\}$ is a path with weight $B_t^{(\mu)}$, with highest weight $\Lambda^{(\mu)}(t)$, and $L^{(\mu)}(t)$ is the weight seen from the highest weight.

Proposition 2.30. *In probability,*

$$\begin{aligned} \lim_{m \rightarrow +\infty} \tau_m W_t^{(\frac{m}{\pi}, \mu)} &= B_t^{(\mu)}, \\ \lim_{m \rightarrow +\infty} \tau_m \Lambda_m(t) &= \Lambda^{(\mu)}(t). \end{aligned}$$

Proof. The first statement is obvious. The second one is then a consequence of Proposition 2.28 and (2.20). \square

Let $\{A^{(\mu)}(t), t \geq 0\}$ be the conditioned space-time Brownian motion in the affine Weyl chamber C_{aff} with drift μ starting from the origin, defined in 5.3.

Theorem 2.31. *For any $\mu \in [0, 1]$, in law,*

$$\{\Lambda^{(\mu)}(t), t \geq 0\} = \{A^{(\mu)}(t), t \geq 0\}.$$

Proof. This follows from Proposition 2.30, and Theorem 2.45. \square

This shows in particular that $\Lambda^{(\mu)}$ has a continuous version. In order to prove the almost sure convergence in Theorem 2.26, let us show:

Proposition 2.32. *For $0 < \mu < 1$ and $f : \mathbb{R}^2 \times \Gamma \times \mathbb{R}^2 \rightarrow \mathbb{R}$, bounded and measurable,*

$$\mathbb{E}(f(B_t^{(\mu)}, \xi(t), \Lambda^{(\mu)}(t)) | \sigma(\Lambda^{(\mu)}(s), s \leq t)) = g(\Lambda^{(\mu)}(t))$$

for each $t \geq 0$, where

$$g(\lambda) = \mathbb{E}(f(\lambda - L^{(\mu)}(\infty), \xi(\infty), \lambda) | \xi(\infty) \in \Gamma(\lambda)).$$

Proof. It is enough to prove the theorem when f is a bounded continuous function depending on a finite number of variables. By Theorem 2.9, the conditional distribution of $\xi^m(t)$ knowing the sigma-algebra $\sigma(\Lambda_m(s), s \leq t)$ is the one of $\xi^m(\infty)$ knowing $\{\xi^m(\infty) \in \Gamma_m(\Lambda_m(t))\}$. Hence

$$\mathbb{E}(f(\tau_m \Lambda_m(t) - \tau_m M_m^m(t), \xi^m(t), \Lambda_m(t) | \sigma(\Lambda_m(s), s \leq t))) = g_m(\Lambda_m(t))$$

where

$$g_m(\lambda) = \mathbb{E}(f(\tau_m \lambda - \tau_m M_m^m, \xi^m(\infty), \lambda) | \xi^m(\infty) \in \Gamma_m(\lambda)).$$

Using the convergences in probability of the propositions 2.28 and 2.30, if h is bounded, continuous, and depends only on a finite number of variables,

$$\begin{aligned} & \mathbb{E}(f(\Lambda^{(\mu)}(t) - L^{(\mu)}(t), \xi(t), \Lambda^{(\mu)}(t))h(\Lambda^{(\mu)}(s), s \leq t)) \\ &= \lim_{m \rightarrow +\infty} \mathbb{E}(f(\tau_m \Lambda_m(t) - \tau_m M_m^m(t), \xi^m(t), \tau_m \Lambda_m(t))h(\tau_m \Lambda_m(s), s \leq t)) \\ &= \lim_{m \rightarrow +\infty} \mathbb{E}(g_m(\Lambda_m(t))h(\tau_m \Lambda_m(s), s \leq t)). \end{aligned}$$

Since $\tau_m \Lambda_m(t)$ tends to $\Lambda^{(\mu)}(t)$, we know from Proposition 2.25 that, in probability, $\{\xi^m(\infty) \in \Gamma_m(\Lambda_m(t))\}$ tends to $\{\xi(\infty) \in \Gamma(\Lambda^{(\mu)}(t))\}$. Therefore the limit above is equal to

$$\mathbb{E}(g(\Lambda^{(\mu)}(t))h(\Lambda^{(\mu)}(s), s \leq t))$$

which proves the proposition. \square

Applying the proposition with $f(b, \xi, \lambda) = 1_{\Gamma(\lambda)}(\xi)$ we obtain that

Corollary 2.33. *Almost surely, $\xi(t)$ is in $\Gamma(\Lambda^{(\mu)}(t))$.*

End of the proof of Theorem 2.26. By the Cameron-Martin theorem it is enough to prove the theorem for $\mu \in (0, 1)$. In that case, by Proposition 2.32, one has

$$\mathbb{E}(1_{\{\lim M_k(t) = L^{(\mu)}(t)\}} | \sigma(\Lambda^{(\mu)}(t))) = \mathbb{E}(1_{\{\lim M_k = L^{(\mu)}(\infty)\}} | \xi(\infty) \in \Gamma(\Lambda^{(\mu)}(t))).$$

As the right hand-side term is equal to 1 by Proposition 2.18, one obtains that

$$\mathbb{P}(\lim M_k(t) = L^{(\mu)}(t)) = \mathbb{E}(\mathbb{E}(1_{\{\lim M_k(t) = L^{(\mu)}(t)\}} | \sigma(\Lambda^{(\mu)}(t)))) = 1,$$

which finishes the proof of the theorem. \square

2.6. Representation using Pitman and Lévy transforms. Let us remind where we stand. For $B_t^{(\mu)} = (t, B_t + t\mu)$ we have written, see (2.8),

$$\mathcal{P}_{s_n} \cdots \mathcal{P}_{s_1} \mathcal{P}_{s_0} B^{(\mu)}(t) = B_t^{(\mu)} + \sum_{i=0}^n \xi_i(t) \alpha_i,$$

and

$$M_{n+1}(t) = \frac{1}{2} \xi_{n+1}(t) \alpha_{n+1} + \sum_{i=0}^n \xi_i(t) \alpha_i.$$

We have seen that when $\mu \in [0, 1]$, for $t > 0$, $\lim_{k \rightarrow +\infty} M_k(t) = L^{(\mu)}(t)$ a.s. and that the process $\Lambda^{(\mu)}(t) = B_t^{(\mu)} + L^{(\mu)}(t)$, $t \geq 0$, has the same distribution as the process $A^{(\mu)}(t)$, $t \geq 0$. Hence,

Theorem 2.34. *When $0 \leq \mu \leq 1$, for $t \geq 0$, almost surely,*

$$\lim_{n \rightarrow +\infty} \mathcal{P}_{s_n} \cdots \mathcal{P}_{s_1} \mathcal{P}_{s_0} B^{(\mu)}(t) + \frac{1}{2} \xi_{n+1}(t) \alpha_{n+1}$$

exists, and the limiting process has the same distribution as $\{A^{(\mu)}(t), t \geq 0\}$.

To interpret the correction term, it is worthwhile to introduce the Lévy transform (sometimes called Skorokhod transform). A theorem of Lévy (see Revuz and Yor [35], VI.2) states that if β is the standard Brownian motion, then

$$\mathcal{L}\beta(t) = \beta_t - \inf_{0 \leq s \leq t} \beta_s, t \geq 0,$$

has the same law as $|\beta_t|, t \geq 0$, and that $-\inf_{0 \leq s \leq t} \beta_s$ is the local time of $\mathcal{L}\beta$ at 0. We define here the following Lévy transform (sometimes the Lévy transform of β is defined as $\int_0^t \text{sign}(\beta_s) d\beta_s$, this is related to the transform here, but different).

Definition 2.35. *For $\eta \in \mathcal{C}_0(\mathbb{R}^2)$ and $i = 0, 1$, the Lévy transform $\mathcal{L}_{s_i}\eta$ of η is*

$$\mathcal{L}_{s_i}\eta(t) = \eta(t) - \frac{1}{2} \inf_{0 \leq s \leq t} \tilde{\alpha}_i(\eta(s)) \alpha_i.$$

Another way to state the former theorem is

Theorem 2.34 bis. *For $t \geq 0$,*

$$\lim_{n \rightarrow +\infty} \mathcal{L}_{s_{n+1}} \mathcal{P}_{s_n} \cdots \mathcal{P}_{s_1} \mathcal{P}_{s_0} B^{(\mu)}(t)$$

exists a.s. and the limiting process has the same distribution as $A^{(\mu)}$.

The following proposition indicates that the presence of the Lévy transform is due to the bad behavior of $A^{(\mu)}(t)$ for t near 0.

Proposition 2.36. *For all $t > 0$, a.s. $\lim_{k \rightarrow +\infty} \xi_k(t) = 2$.*

Proof. When $0 < \mu < 1$, $\xi_k(\infty)$ tends to 2 almost surely (Theorem 2.17), and the proposition follows from Proposition 2.32 by conditioning by $\sigma(\Lambda^{(\mu)}(t))$. The case when $\mu \in \{0, 1\}$ follows from the Cameron-Martin theorem. \square

It implies that, for $t > 0$,

$$\lim_{n \rightarrow +\infty} \mathcal{P}_{s_n} \cdots \mathcal{P}_{s_1} \mathcal{P}_{s_0} B^{(\mu)}(t) + (-1)^n \alpha_1 = A^{(\mu)}(t).$$

So without correction, the iterates of Pitman's transform do not converge.

In the whole paper we have chosen, in the iterations of Pitman's transforms, to begin by first applying \mathcal{P}_{s_0} to $B^{(\mu)}$. Let us show the non trivial fact that we obtain the same limits if we begin with \mathcal{P}_{s_1} . More precisely, if we denote with a tilde the quantities previously defined when we begin by \mathcal{P}_{s_1} rather than by \mathcal{P}_{s_0} , one has, for each $\mu \in [0, 1]$,

Theorem 2.37. *(i) Almost surely,*

$$\lim_{n \rightarrow +\infty} \mathcal{L}_{s_{n+1}} \mathcal{P}_{s_n} \cdots \mathcal{P}_{s_1} \mathcal{P}_{s_0} B^{(\mu)}(t) = \lim_{n \rightarrow +\infty} \mathcal{L}_{s_{n+1}} \tilde{\mathcal{P}}_{s_n} \cdots \tilde{\mathcal{P}}_{s_2} \tilde{\mathcal{P}}_{s_1} B^{(\mu)}(t).$$

(ii) $\tilde{\xi}(t)$ defined for $B^{(\mu)}$ has the same law as $\xi(t)$ defined for $B^{(1-\mu)}$.

Proof. For (i) we have to prove that $L^{(\mu)}(t) = \tilde{L}^{(\mu)}(t)$. We have seen in Proposition 2.28 that $\tau_m M_m^m(t)$ converges in probability to $L^{(\mu)}(t)$. By the same proof $\tau_m \tilde{M}_m^m(t)$ converges in probability to $\tilde{L}^{(\mu)}(t)$. Now we use the equality $M_m^m(t) = \tilde{M}_m^m(t)$ which follows from Theorem 2.3 and (2.1). Thus $L^{(\mu)}(t) = \tilde{L}^{(\mu)}(t)$ a.s.. (ii) follows from the relation (2.7) and the fact that $TB^{(\mu)}$ has the same law as $B^{(1-\mu)}$. \square

2.7. The law of $B_t^{(\mu)}$ conditionally to $\{\Lambda^{(\mu)}(s), s \leq t\}$. We will compute the conditional law of $B_t^{(\mu)}$ by approaching it by the dihedral case. The alternating polynomial associated to the dihedral group $I(m)$ is given by (see Dunkl and Xu [14], 6.2.3), for $v = (a, b) \in \mathbb{R}^2$,

$$h_m(v) = h_m(a, b) = \Im((a + ib)^m).$$

It is equal to the product of the roots for the root system of $I(m)$. For $w \in I(m)$ let $l(w)$ be the length of its shortest expression with s_0^m and s_1^m . Let, for $v \in \mathbb{R}^2, \gamma \in \bar{C}_m$,

$$(2.21) \quad \psi_v^m(\gamma) = \sum_{w \in I(m)} (-1)^{l(w)} e^{\langle w(\gamma) - \gamma, v \rangle},$$

and let $W^{(\gamma)}$ be the standard planar Brownian motion in \mathbb{R}^2 with drift $\gamma \in \bar{C}_m$.

Lemma 2.38. For $\zeta \in \mathbb{R}^2$ and $v = \mathcal{P}_{w_0}^m W^{(\gamma)}(t)$,

$$\mathbb{E}(e^{\langle \zeta, W_t^{(\gamma)} \rangle} | \sigma(\mathcal{P}_{w_0}^m W^{(\gamma)}(s), 0 \leq s \leq t)) = \frac{\psi_v^m(\zeta + \gamma)}{\psi_v^m(\gamma)} \frac{h_m(\gamma)}{h_m(\zeta + \gamma)} e^{\langle \zeta, v \rangle}.$$

Proof. Theorem 5.5 in [4] gives, for $\mathcal{F}_t^{(0)} = \sigma(\mathcal{P}_{w_0}^m W^{(0)}(s), 0 \leq s \leq t)$,

$$\mathbb{E}(e^{\langle \zeta, W_t^{(0)} \rangle} | \mathcal{F}_t^{(0)}) = C \frac{\psi_\alpha^m(\zeta)}{h_m(v) h_m(\zeta)} e^{\langle \zeta, v \rangle},$$

for $v = \mathcal{P}_{w_0}^m W^{(0)}(t)$ and a constant $C > 0$ independent of v and ζ . We conclude with the Bayes formula (2.6). \square

For $\alpha \in \mathbb{R}$, we define a theta function φ_α on $\mathbb{R}_+^* \times \mathbb{R}$ first when $\alpha \notin \mathbb{Z}$ by

$$(2.22) \quad \varphi_\alpha(t, x) = \frac{e^{-\alpha x}}{\sin(\alpha\pi)} \sum_{k \in \mathbb{Z}} \sinh(\alpha(2kt + x)) e^{-2(kx + k^2 t)},$$

for $t > 0, x \in \mathbb{R}$, and then by continuity for all $\alpha \in \mathbb{R}$ using the next lemma (which follows from the Poisson summation formula (see Bellman [1]) for the $2t$ -periodic function $x \mapsto e^{-\frac{x^2}{2t}} \varphi_\alpha(t, x)$), which implies that $\varphi_\alpha(t, x) = \varphi_{1-\alpha}(t, t-x)$.

Lemma 2.39. For $\alpha \notin \mathbb{Z}, t > 0, x \in \mathbb{R}$,

$$\varphi_\alpha(t, x) = \frac{\sqrt{\pi} e^{\frac{(x-\alpha t)^2}{2t}}}{\sqrt{2t} \sin(\alpha\pi)} \sum_{k \in \mathbb{Z}} \sin(k\pi\alpha) \sin(k\pi x/t) e^{-\frac{k^2 \pi^2}{2t}}.$$

The importance for us of φ_α is in particular its harmonicity (Proposition 5.4).

Lemma 2.40. *Let $\alpha_m \in \mathbb{R}^2, \gamma_m \in \bar{C}_m$ be such that, as m tends to $+\infty$, $\tau_m \gamma_m$ tends to $(1, \alpha)$ and $\tau_m \alpha_m$ tends to (t, x) , then*

$$(2.23) \quad \lim_{m \rightarrow \infty} \psi_{\alpha_m}^m(\gamma_m) = \frac{\sin(\alpha\pi)}{2} \varphi_\alpha(t, x),$$

$$(2.24) \quad \lim_{m \rightarrow \infty} \left(\frac{\pi}{m}\right)^m h_m(\gamma_m) = \sin(\alpha\pi).$$

Proof. Let r be the rotation of \mathbb{R}^2 of angle $2\pi/m$ and s be the symmetry $s(a, b) = (a, -b)$, $(a, b) \in \mathbb{R}^2$. Let $J(m) = \{-[m/2], \dots, [m/2]\}$ when m is odd and $J(m) = \{-[m/2] + 1, \dots, [m/2] - 1\} \cup \{m/2\}$ when m is even. The dihedral group $I(m)$ is $I(m) = \{r^k, k \in J(m)\} \rtimes \{Id, s\}$, and $l(r) = 2$, $l(s) = 1$. Therefore

$$\begin{aligned} \psi_{\alpha_m}^m(\gamma_m) &= e^{-\langle \gamma_m, \alpha_m \rangle} \sum_{k \in J(m)} (e^{\langle r^k(\gamma_m), \alpha_m \rangle} - e^{\langle r^k s(\gamma_m), \alpha_m \rangle}) \\ &= 2e^{-\gamma_m^2 \alpha_m^2} \sum_{k \in J(m)} I(m, k) \end{aligned}$$

where

$$I(m, k) = e^{\gamma_m^1 \alpha_m^1 (\cos \frac{2k\pi}{m} - 1) + \gamma_m^1 \alpha_m^2 \sin \frac{2k\pi}{m}} \sinh\left(\gamma_m^2 \left(-\alpha_m^1 \sin \frac{2k\pi}{m} + \alpha_m^2 \cos \frac{2k\pi}{m}\right)\right)$$

and where we write $\alpha_m = (\alpha_m^1, \alpha_m^2)$, $\gamma_m = (\gamma_m^1, \gamma_m^2)$ with α_m equivalent to $(mt/\pi, x)$ and γ_m equivalent to $(m/\pi, \alpha)$. For $\varepsilon > 0$, we choose $m_0 \geq 0$ such that, for $m \geq m_0$,

$$\sum_{m/4 \leq |k| \leq m/2} |I(m, k)| \leq \varepsilon,$$

by using the inequality $\cos(\frac{2k}{m}\pi) \leq 0$ when $\frac{m}{4} \leq |k| \leq \frac{m}{2}$. Besides, when $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then $\cos t \leq 1 - \frac{t^2}{\pi}$ and $|\sin(t)| \leq |t|$, hence, for $|k| \leq \frac{m}{4}$,

$$|I(m, k)| \leq e^{-c_1 k^2 + c_2 k}$$

where $c_1, c_2 > 0$ do not depend on m . So, one can choose N such that for $m \geq 1$

$$\sum_{m/4 \geq |k| \geq N} |I(m, k)| \leq \varepsilon.$$

Since, for N fixed,

$$\lim_{m \rightarrow +\infty} \sum_{k=-N}^N I(m, k) = e^{-\alpha x} \sum_{k=-N}^N \sinh(\alpha(2kt + x)) e^{-2(kx + k^2 t)},$$

we obtain (2.23). The relation (2.24) is immediate. \square

For $0 \leq \mu \leq 1$ and $\lambda = (t, x) \in \bar{C}_{\text{aff}}$, we consider the probability measure $\nu_\lambda^{(\mu)}$ on \mathbb{R}^2 carried by the line $\{(t, y), y \in \mathbb{R}\}$ which has the Laplace transform

$$(2.25) \quad \int e^{\langle \zeta, v \rangle} d\nu_\lambda^{(\mu)}(v) = \frac{e^{\langle \zeta, \lambda \rangle} \varphi_{\zeta_2 + \mu}(\lambda)}{\varphi_\mu(\lambda)},$$

for $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2$, introduced by Frenkel ([16]) in his study of orbital measures (see below Section 4.2).

Theorem 2.41. *The conditional law of $B_t^{(\mu)}$ on $\sigma(\Lambda^{(\mu)}(s), s \leq t)$ is the probability measure $\nu_{\Lambda^{(\mu)}(t)}^{(\mu)}$.*

Proof. The formula

$$\mathbb{E}(e^{\langle \zeta, B_t^{(\mu)} \rangle} | \sigma(\Lambda^{(\mu)}(s), s \leq t)) = \frac{e^{\langle \zeta, \Lambda^{(\mu)}(t) \rangle} \varphi_{\zeta_2 + \mu}(\Lambda^{(\mu)}(t))}{\varphi_{\mu}(\Lambda^{(\mu)}(t))}$$

is obtained by letting m goes to infinity in the Lemma 2.38 and using Proposition 2.30 and Lemma 2.40. \square

2.8. Remarks on the laws of $L^{(\mu)}(\infty)$ and $\xi_1(\infty)$.

2.8.1. Law of $L^{(\mu)}(\infty)$. We suppose that $0 < \mu < 1$ and write $L^{(\mu)}(\infty) = (0, D^\mu(\infty))$. The Laplace transform of $D^\mu(\infty)$ is, by (2.12), for $\tau > 0$,

$$\mathbb{E}(e^{-\tau D^\mu(\infty)}) = \prod_{n=0}^{\infty} \left(\left(1 + \frac{\tau}{n + \mu}\right) \left(1 - \frac{\tau}{n + 1 - \mu}\right) \right)^{-1}.$$

Using

$$(2.26) \quad \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \gamma)\Gamma(\beta - \gamma)} = \prod_{n=0}^{+\infty} \left(1 + \frac{\gamma}{n + \alpha}\right) \left(1 - \frac{\gamma}{n + \beta}\right),$$

(Formula 8.325.1 of [19]) and $\Gamma(z)\Gamma(1 - z) = \pi/\sin(\pi z)$, we obtain that

$$(2.27) \quad \mathbb{E}(e^{-\tau D^\mu(\infty)}) = \frac{\sin(\pi\mu)}{\sin(\pi(\mu + \tau))}.$$

In particular,

$$(2.28) \quad \mathbb{E}(D^\mu(\infty)) = \pi \cot(\pi\mu).$$

Corollary 2.42. *The density of $D^{1/2}(\infty)$ is $1/(\pi \cosh x)$.*

Proof. One uses that the Fourier transform of $1/(\pi \cosh x)$ is $1/\cosh(\lambda\pi/2)$. \square

Notice that the law of $\sum_{n=1}^{\infty} \left(\frac{\varepsilon_{2n+1}}{n} - \frac{\varepsilon_{2n}}{n+1}\right)$ appears in Diaconis et al. [9].

2.8.2. Law of $\xi_1(\infty)$. The Bessel process $\rho^{(\nu)}$ of dimension 3 with drift $\nu \geq 0$ is the norm of a Brownian motion in \mathbb{R}^3 with a drift of length ν .

Corollary 2.43. *For $0 \leq \mu \leq 1$, $\xi_1(\infty)$ has the same law as $\sup_{t \geq 0} (\varrho_t^{(1-\mu)} - t)$.*

Proof. By Pitman and Rogers [33], $\varrho_t^{(1-\mu)}$ has the same law as $\mathcal{P}_{s_1} B_t^{(1-\mu)}$, so the claim follows from (2) of Theorem 2.37 when $0 < \mu < 1$, and by continuity when $\mu = 1$. For $\mu = 0$, $\xi_1(\infty) = +\infty$. \square

When $\mu = 1$, $\xi_0(\infty) = +\infty$ and

$$\xi_1(\infty) = \frac{1}{2} \sum_{n=1}^{+\infty} \frac{\varepsilon_{2n-1} + \varepsilon_{2n}}{n^2}.$$

Its distribution is studied in Biane et al. [2] where it is symbolized $\pi^2 S_2/4$. Its Laplace transform is given by, for $\tau \geq 0$,

$$\mathbb{E}(e^{-2\tau\xi_1(\infty)}) = \frac{\pi^2\tau}{\sinh^2(\pi\sqrt{\tau})},$$

and its distribution function is (cf. Table 1 in [2])

$$F(x) = 1 + 2 \sum_{k=1}^{+\infty} (1 - 4k^2 x) e^{-2k^2 x},$$

in this case the corollary is also given in Example 20 of Salminen and Yor [37].

When $\mu = 1/2$,

$$\xi_1(\infty) = \sum_{n=1}^{+\infty} \frac{2\varepsilon_n}{n(n+1)}.$$

Proposition 2.44. *When $\mu = 1/2$, the Laplace transform of $\xi_1(\infty)$ is, for $\tau \geq 0$,*

$$\mathbb{E}(e^{-\tau \xi_1(\infty)}) = \frac{2\pi\tau}{\cosh(\pi\sqrt{2\tau - 1/4})},$$

its density is

$$\sum_{n=0}^{+\infty} (-1)^{n+1} n(n+1)(2n+1) e^{-n(n+1)x/2},$$

and, in law,

$$\xi_1(\infty) = \sup_{n>0, i=1,2,3} \frac{\varepsilon_n^{(i)}}{n},$$

where the $\varepsilon_n^{(i)}$ are exponential independent random variables with parameter 1.

Proof. Using the formula

$$\cosh \pi z = (1 + 4z^2) \prod_{n=1}^{+\infty} \left(1 + \frac{z^2}{(n + 1/2)^2}\right),$$

it is easy to see that

$$\mathbb{E}(e^{-\tau \xi_1(\infty)}) = \prod_{n=1}^{+\infty} \left(1 + \frac{2\tau}{n(n+1)}\right)^{-1} = \frac{2\pi\tau}{\cosh(\pi\sqrt{2\tau - 1/4})}.$$

Let

$$g(x) = 2 \sum_{n=0}^{+\infty} (-1)^n (2n+1) e^{-\frac{\pi}{2}(n+\frac{1}{2})^2 x}.$$

Since $1/\cosh \sqrt{2\tau}$ is the Laplace transform of $\pi g(\pi^2 x)/2$, (e.g. [2]),

$$\frac{2\pi\tau}{\cosh \pi\sqrt{2\tau - 1/4}} = \int_0^\infty \tau e^{-\tau x} e^{x/8} g(x) dx = \int_0^\infty e^{-\tau x} (e^{x/8} g(x))' dx,$$

by an integration by parts. Computing the derivative $(e^{x/8} g(x))'$ gives the density. By integration, the distribution function is

$$\sum_{n=0}^{+\infty} (-1)^n (2n+1) e^{-n(n+1)x/2}$$

which is equal to $\prod_{n=1}^\infty (1 - e^{-nx})^3$ by a formula of Jacobi ([18], Theorem 357). \square

2.9. Convergence of the dihedral highest weight to the affine one. Let $0 \leq \mu \leq 1$ and $m \geq 1$. We consider the planar Brownian motion $W^{(\frac{m}{\pi}, \mu)}$ and the dihedral highest weight

$$\Lambda_m(t) = \mathcal{P}_{w_0}^m W^{(\frac{m}{\pi}, \mu)}(t).$$

Theorem 2.45. *As a process, $\{\tau_m \Lambda_m(t), t \geq 0\}$ converges in law to $\{A^{(\mu)}(t), t \geq 0\}$ when m tends to $+\infty$.*

We will prove this theorem after the following proposition. Let $\{Z_t, t \geq 0\}$ be the conditioned Brownian motion in $[0, 1]$ starting from μ (cf. Definition 5.1).

Proposition 2.46. *For $m \in \mathbb{N}^*$, let $X_t^m, t \geq 0$, be the \mathbb{R}^2 -valued continuous process such that $X_{m^2 t}^m$ is the conditioned planar Brownian motion in the cone C_m , with drift $\gamma_m = m^2 e^{i\pi\mu/m}$ where $0 \leq \mu \leq 1$. One writes in polar coordinates*

$$X_t^m = R_t^m \exp i\pi\theta_t^m,$$

$R_t^m > 0, \theta_t^m \in [0, 1/m]$. Then, when m tends to $+\infty$, as processes, $m\theta_t^m$ tends to $Z_{1/\pi^2 t}$ and R_t^m tends to t .

Proof. As shown in Appendix 5.2, $X_{m^2 t}^m, t \geq 0$, is a radial multidimensional Dunkl process with drift and $R_{m^2 t}^m$ is a Bessel process of dimension $2(m+1)$ with drift m^2 , starting from 0. In other words, one can write

$$(R_{m^2 t}^m)^2 = (m^2 t + B_t(1))^2 + \sum_{k=2}^{2(m+1)} (B_t(k))^2$$

where $B_t(1), \dots, B_t(2(m+1))$ are independent standard real Brownian motions. Since

$$\mathbb{E}\left(\sum_{k=1}^{2(m+1)} B_{t/m^2}(k)^2\right) = 2t(m+1)/m^2$$

tends to 0 as m tends to $+\infty$, R_t^m converges to t in L^1 . It is shown in Gallardo and Yor [17] that the process $Y_t = X_{m^2 t}^m$ has the following time inversion property: $tY_{1/t}$ is the conditioned planar Brownian motion in \bar{C}_m without drift but starting from γ_m . Using its skew product decomposition, one can write

$$\pi\theta_{m^2 t}^m = \sigma_{a_t^m}^m, \quad \text{with } a_t^m = \int_t^{+\infty} \frac{1}{(R_{m^2 s}^m)^2} ds,$$

where the process σ_t^m is a solution of the following stochastic differential equation

$$d\sigma_t^m = dB_t + m \cot(m\sigma_t^m) dt$$

where B is a Brownian motion independent of R^m and $\sigma_0^m = \mu\pi/m$ (see Demni [13]). One remarks that $Z_t = \frac{m}{\pi} \sigma_{\pi^2 t/m^2}^m$ satisfies to

$$(2.29) \quad dZ_t = d\beta_t + \pi \cot(\pi Z_t) dt$$

for another Brownian motion β , and is therefore the conditioned Brownian motion in $[0, 1]$ starting from μ (see Appendix 5.1). As m tends to $+\infty$, a_{t/m^2}^m is equivalent to $1/tm^2$. Therefore $m\theta_t^m = \frac{m}{\pi} \sigma_{a_{t/m^2}^m}^m$ converges to $Z_{1/t\pi^2}$. \square

Proof of Theorem 2.45. Let $X_t^m = \frac{1}{m\pi}\Lambda_m(\pi^2 t)$. Since

$$\frac{1}{m\pi}W_{\frac{m^2\pi^2 t}{m^2\pi^2}}^{(\frac{m}{\pi}, \mu)} = (\beta_t + tm^2, B_t + tm\pi\mu),$$

one sees that $X_{m^2 t}^m$ is the conditioned planar Brownian motion in the cone C_m with a drift equivalent to $m^2 e^{i\pi\mu/m}$ (see Theorem 2.5). One writes its polar decomposition as $X_t^m = R_t^m \exp i\pi\theta_t^m$. Using the continuity of the solution of (2.29) with respect to the initial condition, we see that Proposition 2.46 also holds when the drift γ_m is only equivalent to $m^2 e^{i\pi\mu/m}$. Therefore $m\theta_t^m$ tends to $Z_{1/t\pi^2}$, R_t^m tends to t , and the process

$$\tau_m \Lambda_m(t) = (\pi^2 R_{t/\pi^2}^m \cos(\pi\theta_{t/\pi^2}^m), m\pi R_{t/\pi^2}^m \sin(\pi\theta_{t/\pi^2}^m)), t \geq 0,$$

converges in law to $(t, tZ_{1/t}), t \geq 0$, which is equal in law to $A^{(\mu)}$ by Theorem 5.7. \square

3. REPRESENTATION OF THE CONDITIONED BROWNIAN MOTION IN $[0, 1]$

For a continuous real path $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$, such that $\varphi(0) = 0$, we have defined

$$\begin{aligned} \mathcal{L}_0\varphi(t) &= \varphi(t) + \inf_{0 \leq s \leq t} (s - \varphi(s)), \mathcal{P}_0\varphi(t) = \varphi(t) + 2 \inf_{0 \leq s \leq t} (s - \varphi(s)), \\ \mathcal{L}_1\varphi(t) &= \varphi(t) - \inf_{0 \leq s \leq t} \varphi(s), \mathcal{P}_1\varphi(t) = \varphi(t) - 2 \inf_{0 \leq s \leq t} \varphi(s). \end{aligned}$$

For $B_t^\mu = B_t + t\mu, t \geq 0$, the theorem stated in the introduction is

Theorem 3.1. *Let $\mu \in [0, 1]$. For any $t > 0$, almost surely,*

$$\lim_{n \rightarrow \infty} t \mathcal{L}_{n+1} \mathcal{P}_n \cdots \mathcal{P}_1 \mathcal{P}_0 B^\mu(1/t) = \lim_{n \rightarrow \infty} t \mathcal{L}_{n+1} \mathcal{P}_n \cdots \mathcal{P}_2 \mathcal{P}_1 B^\mu(1/t) = Z_t,$$

where $Z_t, t \geq 0$, is a Brownian motion conditioned to stay in the interval $[0, 1]$ forever, starting from $Z_0 = \mu$.

Proof. The proof is just the juxtaposition of the theorems 2.34, 2.37 and 5.7. \square

Remark that one also has that, for $t > 0$, a.s.,

$$\lim_{n \rightarrow \infty} t(\mathcal{P}_n \cdots \mathcal{P}_1 \mathcal{P}_0 B^\mu(1/t) + (-1)^n 2) = Z_t,$$

which clearly shows the need of a correction for the Pitman transforms. As an illustration let us show that:

Proposition 3.2. *When $0 < \mu < 1$ and $Z_0 = \mu$, there is a standard Brownian motion β and a β -stopping time $\tau > 0$ a.s. such that for $0 \leq t \leq \tau$,*

$$Z_t = \beta_t + \mu + tD^\mu(\infty).$$

Proof. Let $\sigma \geq 0$ be the random time given by Proposition 2.27. For $t > \sigma$, $L^{(\mu)}(t) = L^{(\mu)}(\infty)$ hence $A^{(\mu)}(t) = B_t^{(\mu)} + L^{(\mu)}(\infty)$, which implies the relation for $\tau = 1/\sigma$ and $\beta_t = tB_{1/t}$, since $(t, tZ_{1/t}) = A^{(\mu)}(t)$. \square

Notice that, by (2.28), $\mathbb{E}(D^\mu(\infty)) = \pi \cot \pi\mu$ as expected from the drift of the generator $\frac{1}{2}d^2/dx^2 + (\pi \cot \pi x)d/dx$ of Z_t .

4. SOME ASYMPTOTICS FOR REPRESENTATIONS OF THE AFFINE ALGEBRA $A_1^{(1)}$

As we said in Section 1.4, there are strong links between what we have done and the affine Kac-Moody Lie algebra $A_1^{(1)}$. In this section we will see that quantities we have met also occur in the description of some semi-classical limits of its highest weight representations. In particular a Duistermaat Heckman probability measure is given by the conditional law of the space-time Brownian motion knowing the highest weight. The asymptotic behaviour of the large weights of the infinity crystal $B(\infty)$ of Kashiwara is given by the Verma affine string coordinates $\xi(\infty)$.

4.1. The Kac-Moody algebra $A_1^{(1)}$. We consider the affine Lie algebra $A_1^{(1)}$. For our purpose, we only need to define and consider a realization of a real Cartan subalgebra. We introduce, as in the introduction, two copies of \mathbb{R}^3 ,

$$\mathfrak{h}_{\mathbb{R}} = \text{Span}_{\mathbb{R}}\{c, \tilde{\alpha}_1, d\}, \quad \mathfrak{h}_{\mathbb{R}}^* = \text{Span}_{\mathbb{R}}\{\Lambda_0, \alpha_1, \delta\},$$

where $c = (1, 0, 0)$, $\tilde{\alpha}_1 = (0, 1, 0)$, $d = (0, 0, 1)$, and $\Lambda_0 = (1, 0, 0)$, $\alpha_1 = (0, 2, 0)$, $\delta = (0, 0, 1)$ in \mathbb{R}^3 . We let $\tilde{\alpha}_0 = (1, -1, 0)$ and $\alpha_0 = (0, -2, 1)$, so that $c = \tilde{\alpha}_0 + \tilde{\alpha}_1$ and $\delta = \alpha_0 + \alpha_1$. Notice that these α_0 and α_1 project on the ones given in 2.2 by the projection on $\mathfrak{h}_{\mathbb{R}}^*/\mathbb{R}\delta$, identified with $\mathbb{R}\Lambda_0 \oplus \mathbb{R}\alpha_1$, and thus also with the space $V = \mathbb{R}^2$ of Section 2. These notations are frequently used in this context. For instance α_0, α_1 are called the two positive simple roots of $A_1^{(1)}$ and $\tilde{\alpha}_0, \tilde{\alpha}_1$ their coroots. One considers the set of integral weights

$$P = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : \lambda(\tilde{\alpha}_i) \in \mathbb{Z}, i = 0, 1\},$$

and the set of dominant integral weights

$$P_+ = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : \lambda(\tilde{\alpha}_i) \in \mathbb{N}, i = 0, 1\}.$$

For a dominant integral weight λ one defines the character of the irreducible representation $V(\lambda)$ of $A_1^{(1)}$ with highest weight λ as the formal series

$$(4.1) \quad \text{char}_{\lambda} = \sum_{\beta \in P} \dim(V_{\beta}(\lambda)) e^{\beta},$$

where $V_{\beta}(\lambda)$ is the weight space of $V(\lambda)$ corresponding to the weight β . If we let $e^{\beta}(h) = e^{\beta(h)}$ for $h \in \mathfrak{h}_{\mathbb{R}}$, and evaluate this formal series at h , the series converges absolutely or diverges, and it converges when $\delta(h) > 0$. For more details about affine Lie algebras and their representations, we refer to Kac [24].

4.2. A Duistermaat Heckman measure. A way to define the Duistermaat Heckman measure for a semi-simple complex Lie algebra, is as an approximation of the distribution of the weights of an irreducible finite dimensional representation when its highest weight is large (see Heckman [20]). Let us explain how the same approach is possible in $A_1^{(1)}$. Let $r > 0$, and λ_r be a highest weight. Since the associated irreducible representation $V(\lambda_r)$ is infinite dimensional, we put a Boltzman factor on each weight. More precisely let $h_r \in \mathfrak{h}_{\mathbb{R}}$ be such that $\delta(h_r) > 0$, we define a measure γ_r on $\mathfrak{h}_{\mathbb{R}}^*$, letting

$$(4.2) \quad \gamma_r = \sum_{\beta \in P} \dim(V_{\beta}(\lambda_r)) e^{\beta(h_r)} \delta_{r^{-1}\beta},$$

where $\delta_{r^{-1}\beta}$ is the Dirac measure at $r^{-1}\beta$.

Proposition 4.1. *Let $\mu \in [0, 1]$, $\lambda = t\Lambda_0 + x\alpha_1/2 \in C_{\text{aff}}$, $\{\lambda_r, r \in \mathbb{N}^*\}$ be a sequence of dominant integral weights such that $\lim_{r \rightarrow +\infty} \lambda_r/r = \lambda$, and let $h_r = \frac{1}{r}(\mu\tilde{\alpha}_1 + 2d)$. As $r \rightarrow \infty$, for $\tau \in \mathbb{R}$, $u \geq 0$,*

$$\int_{\mathfrak{h}_{\mathbb{R}}^*} e^{\beta(\tau\tilde{\alpha}_1 + ud)} \gamma_r(d\beta) \sim \sqrt{\frac{4+2u}{\pi r}} e^{\frac{r\pi^2}{2(2+u)} + (\tau+\mu)x} \varphi_{\frac{2(\tau+\mu)}{2+u}}\left(\frac{(2+u)t}{2}, \frac{(2+u)x}{2}\right).$$

Proof. One has for any $v \in \mathfrak{h}$,

$$\int_{\mathfrak{h}_{\mathbb{R}}^*} e^{\beta(v)} \gamma_r(d\beta) = \text{char}_{\lambda_r}(r^{-1}v + h_r).$$

When $\lambda = n\Lambda_0 + m\frac{\alpha_1}{2}$, with $(m, n) \in \mathbb{N}^2$ such that $0 \leq m \leq n$, $a \in \mathbb{R}$, and $b \in \mathbb{R}_+^*$, the Weyl-Kac character formula implies that

$$\text{char}_{\lambda}(a\tilde{\alpha}_1 + bd) = \frac{\sum_{k \in \mathbb{Z}} \sinh(a(m+1) + 2ak(n+2)) e^{-b(k(m+1) + k^2(n+2))}}{\sum_{k \in \mathbb{Z}} \sinh(a + 4ak) e^{-b(k+2k^2)}},$$

(see Kac [24]). So for $v = \tau\tilde{\alpha}_1 + ud$, one has immediately that the numerator of the character $\text{char}_{\lambda_r}(\frac{v}{r} + h_r)$ converges, when r goes to infinity, to

$$e^{(\tau+\mu)x} \sin\left(\frac{2\pi(\tau+\mu)}{2+u}\right) \varphi_{\frac{2(\tau+\mu)}{2+u}}\left(\frac{(2+u)t}{2}, \frac{(2+u)x}{2}\right).$$

Besides Lemma 2.39 implies that the denominator of this character is equivalent to

$$\frac{\sqrt{\pi r}}{\sqrt{4+2u}} e^{-\frac{1}{2(u+2)}r\pi^2} \sin\left(\frac{2\pi(\tau+\mu)}{2+u}\right),$$

which finishes the proof. \square

By taking $u = 0$ we obtain,

Corollary 4.2. *For $\tau \in \mathbb{R}$,*

$$\lim_{r \rightarrow +\infty} \frac{1}{\text{char}_{\lambda_r}(h_r)} \int_{\mathfrak{h}_{\mathbb{R}}^*} e^{\tau\beta(\tilde{\alpha}_1)} \gamma_r(d\beta) = e^{\tau x} \frac{\varphi_{\tau+\mu}(t, x)}{\varphi_{\mu}(t, x)}.$$

We denote by ν_r the probability measure on $\mathfrak{h}_{\mathbb{R}}^*$ given by

$$\nu_r = \frac{1}{\text{char}_{\lambda_r}(h_r)} \gamma_r.$$

Recall that the probability measure $\nu_{\lambda}^{(\mu)}$ defined in (2.25) is, for any $t > 0$, the conditional law of $B_t^{(\mu)}$ when $\Lambda^{(\mu)}(t) = \lambda$.

Theorem 4.3. *Under the hypotheses of Proposition 4.1, the sequence of the push forward probabilities of $\{\nu_r, r \geq 1\}$ by the quotient map from $\mathfrak{h}_{\mathbb{R}}^*$ to $\mathfrak{h}_{\mathbb{R}}^*/\mathbb{R}\delta$ converges to $\nu_{\lambda}^{(\mu)}$, when $r \rightarrow +\infty$.*

Proof. This follows from Corollary 4.2 and Theorem 2.41. \square

Let $F_r\Lambda_0 + G_r\alpha_1/2 + H_r\delta$ be a random variable in $\mathfrak{h}_{\mathbb{R}}^*$ with law ν_r . From the theorem, as r tends to infinity, the law of (F_r, G_r) tends to $\nu_{\lambda}^{(\mu)}$, F_r tends to t . But H_r tends to $-\infty$, as follows from Proposition 4.1 with $s = 0$ and $u \geq 0$.

For a compact connected Lie group, the Duistermaat Heckman measure also appears as the pushforward measure of the Liouville measure on a coadjoint orbit

by the projection on a Cartan subalgebra. Frenkel has shown in [16] that the law of a Brownian motion on $\mathfrak{su}(2)$ indexed by the time in $[0, 1]$, given the conjugacy class of the endpoint of its wrapping on $SU(2)$, plays the role of a normalized Liouville measure on a coadjoint orbit of the loop group $L(SU(2))$. Identifying $\mathfrak{su}(2)$ and its dual, $\nu_\lambda^{(\mu)}$ is the projection of this law on the diagonal torus of $\mathfrak{su}(2)$ (see Frenkel [16] and also Defosseux [12]) for $\mu = 0$. This projection is a moment map. In particular $\nu_\lambda^{(\mu)}$ can be seen as a normalized Duistermaat-Heckman measure corresponding to the action of the torus of $SU(2)$ on the coadjoint orbit through λ .

4.3. Asymptotics for the crystal $B(\infty)$ of $A_1^{(1)}$. The infinity crystal $B(\infty)$ of Kashiwara ([25]) is the crystal of the Verma module with highest weight 0 of $A_1^{(1)}$. This crystal is important since any irreducible highest weight crystal may be obtained from $B(\infty)$. It is shown in Nakashima and Zelevinski ([30]) that using string parametrizations, a realization of $B(\infty)$ is given by

$$B(\infty) = \{x \in \mathbb{N}^{\mathbb{N}}; \text{ for some } n \in \mathbb{N}^*, \frac{x_1}{1} \geq \frac{x_2}{2} \geq \dots \geq \frac{x_n}{n} > 0, x_k = 0 \text{ for } k > n\}.$$

Notice that the only condition on x_0 is $x_0 \in \mathbb{N}$. For $x \in B(\infty)$, we let

$$\sigma(x) = \sum_{k=0}^{+\infty} x_k \alpha_k \in \mathfrak{h}_{\mathbb{R}}^*,$$

when $x_k = 0$ for $k > n$. By [30], $-\sigma(x)$ is the weight of x in the crystal $B(\infty)$. For $\tilde{\rho} = \tilde{\alpha}_1/2 + 2d$, we let

$$s(x) = \sum_{k=0}^{+\infty} x_k = \sigma(x)(\tilde{\rho}).$$

The character char_∞ of the Verma module of highest weight 0 is defined as in (4.1), and is given (see Kac [24], (9.7.2)) by

$$(4.3) \quad \text{char}_\infty = \prod_{\beta \in R_+} (1 - e^{-\beta})^{-1},$$

where

$$(4.4) \quad R_+ = \{\alpha_0 + n\delta, \alpha_1 + n\delta, (n+1)\delta, n \in \mathbb{N}\}$$

are the positive roots of $A_1^{(1)}$. As previously, if we let $e^\beta(h) = e^{\beta(h)}$ for $h \in \mathfrak{h}_{\mathbb{R}}$, and evaluate the formal character at h , it converges if and only if $\delta(h) > 0$. Let $r \in \mathbb{R}_+^*$. On each element x of the crystal $B(\infty)$ we put the Boltzman weight $e^{-s(x)/r}$. We introduce the discrete probability distribution β_r on $B(\infty)$ by

$$\beta_r(\{x\}) = \frac{e^{-s(x)/r}}{Z_r},$$

when $x \in B(\infty)$, where

$$Z_r = \sum_{x \in B(\infty)} e^{-s(x)/r} = \text{char}_\infty(\tilde{\rho}/r).$$

Since $B(\infty)$ is a Borel subset of Γ we may consider β_r as a probability on Γ . The following theorem indicates that the affine Brownian model describes a kind of continuous version of the infinity crystal $B(\infty)$ for the affine Lie algebra $A_1^{(1)}$.

Theorem 4.4. *Let $X^{(r)}, r \in \mathbb{N}$, be random variables in Γ with distribution β_r . Then $X^{(r)}/r$ converges in distribution to the Verma parameter $\xi(\infty)$ of $B^{(1/2)}$.*

Proof. We use the results on anti-lecture hall compositions recalled in Section 5.5. First it is clear that $X_0^{(r)}/r$ converges in law to $\xi_0(\infty)$. Let us now prove that $X_1^{(r)}/r$ converges in law to $\xi_1(\infty)$. It follows from (5.9), (5.10) that, for $a \geq 0$ and $q = e^{-1/r}$,

$$\begin{aligned} \mathbb{P}(X_1^{(r)} \leq ar) &= \frac{\sum_{\{x \in B(\infty); x_1 \leq [ar]\}} q^{s(x)}}{\sum_{\{x \in B(\infty)\}} q^{s(x)}} = \frac{\sum_{\{\lambda \in A_\infty; \lambda_1 \leq [ar]\}} q^{|\lambda|}}{\sum_{\{\lambda \in A_\infty\}} q^{|\lambda|}} \\ &= \frac{1}{1-q} (q; e^{-[ar]/r} q^2)_\infty (e^{-[ar]/r} q; e^{-[ar]/r} q^2)_\infty (e^{-[ar]/r} q^2; e^{-[ar]/r} q^2)_\infty, \end{aligned}$$

where $(p; q)_\infty = \prod_{n=0}^{\infty} (1 - pq^n)$. Since $(q; e^{-[ar]/r} q^2)_\infty$ is equivalent to $(1 - q)(e^{-a}; e^{-a})_\infty$ when q tends to 1, one has, by Proposition 2.44,

$$\lim_{r \rightarrow \infty} \mathbb{P}(X_1^{(r)} \leq ra) = (e^{-a}; e^{-a})_\infty^3 = \prod_{n=1}^{\infty} (1 - e^{-na})^3 = \mathbb{P}(\xi_1(\infty) \leq a).$$

So $X_1^{(r)}/r$ converges to $\xi_1(\infty)$. We now consider the full sequence $X_k^{(r)}/r, k \in \mathbb{N}$. For any r and $n \geq 1$, one has

$$\frac{X_1^{(r)}}{1} \geq \frac{X_2^{(r)}}{2} \geq \dots \geq \frac{X_n^{(r)}}{n} \geq 0,$$

which implies that for any $n \in \mathbb{N}$, the collection of laws of $(\frac{1}{r} X_1^{(r)}, \dots, \frac{1}{r} X_n^{(r)})_{r > 0}$ is tight since $X_1^{(r)}/r$ converges in law. By Cantor's diagonal argument, we construct an increasing sequence $\varphi(r) \in \mathbb{N}, r \in \mathbb{N}$, such that the random variables $\frac{1}{\varphi(r)} X_k^{(\varphi(r))}, k \geq 0$, converge in finite dimensional distribution when r goes to infinity. Let us denote by $(R_k, k \geq 0)$ the limit, and let us first prove that $\{\frac{R_k}{k} - \frac{R_{k+1}}{k+1}, k \geq 1\}$ has the same distribution as $\{2\varepsilon_k/k(k+1), k \geq 1\}$ where the ε_k 's are independent exponential random variables with parameter 1. For $x \in B(\infty)$, one has, since $s(x) = x_0 + \frac{1}{2} \sum_{k=2}^{+\infty} (kx_{k-1} - (k-1)x_k)$,

$$\begin{aligned} \mathbb{P}(X^{(r)} = x) &= \frac{e^{-\frac{1}{r}x_0}}{\text{char}_\infty(r^{-1}\tilde{\rho})} \prod_{k=2}^{\infty} e^{-\frac{1}{2r}(kx_{k-1} - (k-1)x_k)} \mathbf{1}_{\{\frac{x_{k-1}}{k-1} \geq \frac{x_k}{k}\}} \\ &= \frac{e^{-\frac{1}{r}x_0}}{\text{char}_\infty(r^{-1}\tilde{\rho})} \prod_{k=1}^{\infty} e^{-\frac{k+1}{2r}(x_k - \lceil \frac{k}{k+1} x_{k+1} \rceil)} e^{-\frac{1}{2r}(k \lceil \frac{k-1}{k} x_k \rceil - (k-1)x_k)} \mathbf{1}_{\{x_k \geq \lceil k \frac{x_{k+1}}{k+1} \rceil\}}, \end{aligned}$$

where $\lceil \cdot \rceil$ is the ceiling function. Let $Y_k^{(r)}, k = 0, \dots, n$, be independent geometric random variables with values in \mathbb{N} , where $Y_0^{(r)}$ has the parameter 1 and $Y_k^{(r)}$ has the parameter $e^{-\frac{k+1}{2r}}$ when $k \geq 1$. As,

$$e^{-\frac{k}{2r}} \leq e^{-\frac{1}{2r}(k \lceil \frac{k-1}{k} x_k \rceil - (k-1)x_k)} \leq 1,$$

one obtains that for $t_0, \dots, t_n \geq 0$,

$$C(r, n)(1 + O(\frac{1}{r})) \leq \frac{\mathbb{P}(X_0^{(r)} \leq t_0, \dots, X_n^{(r)} - \lceil \frac{n}{n+1} X_{n+1}^{(r)} \rceil \leq t_n)}{\mathbb{P}(Y_0^{(r)} \leq t_0, \dots, Y_n^{(r)} \leq t_n)} \leq C(r, n),$$

where $C(r, n)$ is independent of t_0, \dots, t_n , and tends to 1 when r tends to $+\infty$. This proves that for any n ,

$$\frac{1}{r} X_0^{(r)}, \frac{1}{r} (X_k^{(r)} - \lceil \frac{k}{k+1} X_{k+1}^{(r)} \rceil), \quad 1 \leq k \leq n,$$

converge jointly to $\varepsilon_0, \frac{2}{k+1} \varepsilon_k, 1 \leq k \leq n$, when r goes to infinity. Besides

$$\lim_{r \rightarrow \infty} \frac{1}{\varphi(r)} (X_k^{(\varphi(r))} - \lceil \frac{k}{k+1} X_{k+1}^{(\varphi(r))} \rceil) = R_k - \frac{k}{k+1} R_{k+1}.$$

Thus, for $k \geq 1$, $R_k - \frac{k}{k+1} R_{k+1}$ are independent random variables with the same law as $\frac{2}{k+1} \varepsilon_k$. The positive sequence R_k/k is decreasing. Let S be its limit. We have the identity in law, for all $k \geq 1$,

$$\frac{R_k}{k} = \sum_{n=k}^{+\infty} \frac{2\varepsilon_n}{n(n+1)} + S.$$

We have proved that in law $R_1 = \xi_1(\infty)$, so $S = 0$ which finishes the proof. \square

Recall (Corollary 2.42) that $L^{(1/2)}(\infty) = D^{1/2}(\infty)\alpha_1/2$ where the density of $D^{1/2}(\infty)$ is $1/(\pi \cosh x)$, and that $-\sigma(x)$ is the weight of $x \in B(\infty)$.

Proposition 4.5. *When r goes to infinity, in law,*

- (1) *In the quotient space $\mathfrak{h}_{\mathbb{R}}^*/\mathbb{R}\delta$, the normalized weights $\sigma(X^{(r)})/r$ converge to $L^{(1/2)}(\infty)$.*
- (2) *The coordinate of $\sigma(X^{(r)})/r$ along δ goes to $+\infty$.*

Proof. One has for any $u \in \mathfrak{h}_{\mathbb{R}}$,

$$\mathbb{E}(e^{-\sigma(X^{(r)})(u)}) = \frac{\text{char}_{\infty}(u + r^{-1}\tilde{\rho})}{\text{char}_{\infty}(r^{-1}\tilde{\rho})}.$$

In view of (4.4), the expression (4.3) of the character gives the Laplace transform of $\sigma(X^{(r)})$ and shows that, in distribution,

$$\sigma(X^{(r)}) = \sum_{n \geq 0} (G_0(n)(\alpha_0 + n\delta) + G_1(n)(\alpha_1 + n\delta) + G_2(n)(n+1)\delta),$$

where $G_i(n)$, $i = 0, 1, 2$, $n \in \mathbb{N}$, are independent random variables such that $G_0(n)$, $G_1(n)$ and $G_2(n)$, are geometrically distributed with respective parameter $e^{-(\alpha_0+n\delta)(r^{-1}\tilde{\rho})}$, $e^{-(\alpha_1+n\delta)(r^{-1}\tilde{\rho})}$, and $e^{-(n+1)\delta(r^{-1}\tilde{\rho})}$, i.e. $e^{-2(n+1/2)/r}$, $e^{-2(n+1/2)/r}$ and $e^{-2(n+1)/r}$. The proposition follows easily. \square

Remark 4.6. *Due to the Lévy correction, the formal expression $\sigma(\lim_{r \rightarrow +\infty} \frac{X^{(r)}}{r})$ is not equal to $\lim_{r \rightarrow +\infty} \sigma(\frac{X^{(r)}}{r})$.*

5. APPENDIX

5.1. The conditioned Brownian motion in $[0, 1]$. We recall some well known facts about the Brownian motion conditioned to stay forever in the interval $[0, 1]$. It first appeared in Knight ([27]), called there the taboo process, as the limit when t tends to ∞ of a standard Brownian motion starting in the interval, conditioned to reach the boundary after time t . To define it rigorously, first consider the Brownian motion in $[0, 1]$ killed at the boundary. Its generator is $\frac{1}{2} \frac{d^2}{dx^2}$ with Dirichlet boundary conditions. Its maximal eigenvalue is $-\pi^2/2$ with positive eigenvector

$$h(x) = \sin(\pi x),$$

called the ground state. We consider the associated h -Doob process $\{Z_t, t \geq 0\}$. It is the Markov process with transition probability density $q_t(x, y)$ given when $x, y \in (0, 1)$ by

$$(5.1) \quad q_t(x, y) = \frac{\sin(\pi y)}{\sin(\pi x)} e^{\pi^2 t/2} u_t(x, y),$$

where $u_t(x, y)$ is the transition probability density of the killed Brownian motion. It can also be viewed as the diffusion in $[0, 1]$ with generator

$$\frac{1}{2} \frac{d^2}{dx^2} + \pi \cot(\pi x) \frac{d}{dx}.$$

It is well known (and follows from the reflection principle), that, for $x, y \in (0, 1)$,

$$(5.2) \quad u_t(x, y) = \sum_{k \in \mathbb{Z}} (p_t(x, y + 2k) - p_t(x, -y + 2k)),$$

where p_t is the standard heat kernel (see Ito and McKean [23]). Using the Poisson formula, or the spectral decomposition of the generator, one has,

$$(5.3) \quad u_t(x, y) = \sum_{n \in \mathbb{Z}} \sin(n\pi x) \sin(n\pi y) e^{-\pi^2 n^2 t/2}.$$

By scale function techniques, one sees that 0 and 1 are entrance non-exit boundaries. In other words, (Z_t) can be started from the boundaries and does not touch them at positive time. Let us remark that Z can also be defined by the latitude of the Brownian motion on the 3-dimensional sphere (see Ito and McKean [23], Section 7.15) or by the argument of an eigenvalue of the Brownian motion in $SU(2)$. The boundaries behaviour is also clear from these descriptions.

The entrance density measure starting from 0 is the limit of $q_t(x, y)$ when x tends to 0, which is for $y \in (0, 1)$, by (5.3),

$$(5.4) \quad q_t(0, y) = \sin(\pi y) \sum_{n \in \mathbb{Z}} n \sin(n\pi y) e^{-\frac{t}{2} \pi^2 (n^2 - 1)}.$$

Definition 5.1. We call (Z_t) the conditioned Brownian motion process in $[0, 1]$.

Another interpretation of Z is given by the distance between two non colliding Brownian motions on a circle (see Hobson and Werner [21]).

5.2. The conditioned planar Brownian motion in a dihedral cone. It is intuitively given by the two-dimensional Brownian motion with drift starting from 0 conditioned to stay in the cone C_m forever. It is rigorously defined in 5.1 of [4] for a planar Brownian motion without drift. The definition when there is a drift is the following. We recall (see Section 2.7) that for $v = (a, b) \in \mathbb{R}^2$,

$$h_m(v) = h_m(a, b) = \Im((a + ib)^m),$$

and, for $\gamma \in \bar{C}_m$

$$\psi_\gamma^m(v) = \sum_{w \in I(m)} (-1)^{l(w)} e^{\langle w(\gamma) - \gamma, v \rangle}.$$

For $x, y \in C_m$, let

$$r_t(x, y) = \frac{\psi_\gamma^m(y)}{\psi_\gamma^m(x)} v_t^\gamma(x, y),$$

where v_t^γ is the semi group density of the standard planar Brownian motion with drift γ killed at the boundary of C_m .

Definition 5.2. *The conditioned planar Brownian motion $A_m^{(\gamma)}$ in the dihedral cone C_m with drift $\gamma \in \bar{C}_m$, is the continuous Markov process with values in $C_m \cup \{0\}$ with transition probability density $r_t(x, y)$ for $x, y \in C_m$, such that $A_m^{(\gamma)}(0) = 0$, and with entrance probability density at time t proportional to, for $x \in C_m$,*

$$h_m(x) \frac{\psi_\gamma^m(x)}{h_m(\gamma)} e^{-\frac{1}{2t} \langle x, x \rangle}.$$

In the definition above, $\psi_\gamma^m(x)/h_m(\gamma)$ is obtained by analytical continuation when $h_m(\gamma) = 0$.

Proposition 5.3. *Let $W^{(\gamma)}$ be the planar Brownian motion with drift $\gamma \in \bar{C}_m$ starting from the origin. The process $\mathcal{P}_{w_0}^m W^{(\gamma)}$ has the same law as $A_m^{(\gamma)}$.*

Proof. For $0 < t_1 < \dots < t_n$, and a bounded measurable function $F : \mathbb{R}^n \rightarrow \mathbb{R}$, the Cameron–Martin formula gives

$$\begin{aligned} \mathbb{E}(F(\mathcal{P}_{w_0}^m W^{(\gamma)}(t_1), \dots, \mathcal{P}_{w_0}^m W^{(\gamma)}(t_n))) &= \\ \mathbb{E}(F(\mathcal{P}_{w_0}^m W^{(0)}(t_1), \dots, \mathcal{P}_{w_0}^m W^{(0)}(t_n)) e^{-\frac{1}{2} \langle \gamma, \gamma \rangle t_n + \langle \gamma, W_{t_n}^{(0)} \rangle}). \end{aligned}$$

By Theorems 5.1 and 5.5 of [4], which deals with the case $\gamma = 0$, this is equal to

$$C \mathbb{E}(F(A_m^{(0)}(t_1), \dots, A_m^{(0)}(t_n)) e^{-\frac{1}{2} \langle \gamma, \gamma \rangle t_n} \frac{\psi_\gamma^m(A_m^{(0)}(t_n))}{h_m(\gamma) h_m(A_m^{(0)}(t_n))} e^{\langle \gamma, A_m^{(0)}(t_n) \rangle}),$$

where C is a constant. This implies easily the proposition. \square

Another quick manner to define rigorously $A_m^{(\gamma)}$ is to use Dunkl processes and the approach given by Gallardo and Yor in [17]. One considers the Dunkl Laplacian with multiplicity one associated with $I(m)$ given, for $f \in C^2(\mathbb{R}^2)$, by

$$\tilde{\Delta} f = \Delta f + 2 \sum_{\alpha \in R^+} \frac{\langle \alpha, \nabla f(x) \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle^2}$$

where σ_α is the orthogonal reflection with respect to $\{x \in \mathbb{R}^2, \langle x, \alpha \rangle = 0\}$, Δ is the usual Laplacian and ∇ the gradient. Roesler and Voit ([36], Section 3) show that for any $\gamma \in \mathbb{R}^2$, there exists a càd-làg process $X_\gamma(t), t \geq 0$, with generator $\tilde{\Delta}/2$ such that $X_\gamma(0) = \gamma$. Let $p : \mathbb{R}^2 \rightarrow \bar{C}_m$ be the projection where \bar{C}_m is identified with $\mathbb{R}^2/I(m)$. Then ([36], Theorems 4.10, 4.11) $\bar{X}_\gamma(t) = p(X_\gamma(t))$ is a continuous process and its norm is a Bessel process of dimension $2(m+1)$. According to Gallardo and Yor ([17], Section 3.3), the conditioned Brownian motion in C_m with drift $\gamma \in \bar{C}_m$, starting from 0 is defined as $t\bar{X}_\gamma(1/t)$. Its probability transitions are given by (25) of [17], which shows that it coincides in law with $A_m^{(\gamma)}$.

5.3. The conditioned space-time Brownian motion in C_{aff} . Let (B_t) be the standard real Brownian motion and $B_t^{(\mu)} = (t, B_t + t\mu)$ be the space-time Brownian motion with drift μ . We define rigorously the process $A^{(\mu)}(t), t \geq 0$, which is the process $B^{(\mu)}$ conditioned to stay forever in the affine cone $C_{\text{aff}} = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 < x < t\}$ starting from $(0, 0)$. It has been introduced and studied in Defosseux [10, 11, 12], where more motivations can be found.

We suppose that $\mu \in [0, 1]$. Let $\{K_t^{(\mu)}, t \geq 0\}$ be the space-time process $B_t^{(\mu)}$ killed at the boundary of \bar{C}_{aff} . This is the process in the cone with generator $\frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x}$ and Dirichlet boundary conditions. We have defined in (2.22) the function φ_μ , given in particular when $\mu \notin \mathbb{Z}$, by

$$(5.5) \quad \varphi_\mu(t, x) = \frac{e^{-\mu x}}{\sin(\mu\pi)} \sum_{k \in \mathbb{Z}} \sinh(\mu(2kt + x)) e^{-2(kx + k^2 t)}.$$

Proposition 5.4. *The function φ_μ is a space-time non negative harmonic function for the killed process $K^{(\mu)}$ on C_{aff} , vanishing on its boundary.*

Proof. The harmonicity is clear by computation. The boundary condition $\varphi_\mu(t, 0) = 0$, resp. $\varphi_\mu(t, t) = 0$, follows from the change of variable from k to $-k$, resp. k to $-1 - k$. Positivity follows for instance from the lemmas 2.38 and 2.40. \square

Using the Cameron-Martin formula and Defosseux ([11], Proposition 2.2), one obtains that, if we write $K_t^{(\mu)} = (t, K_t^\mu)$, the density $w_t^\mu((r, x), (t+r, \cdot))$ of K_{t+r}^μ given $K_r^\mu = x$ is

$$(5.6) \quad \begin{aligned} w_t^\mu((r, x), (t+r, y)) &= e^{\mu(y-x) - \frac{t}{2}\mu^2} w_t^0((r, x), (t+r, y)) \\ &= \sum_{k \in \mathbb{Z}} e^{-2(kx + k^2 r)} (e^{\mu(2kr+x)} p_t^\mu(x + 2kr, y) \\ &\quad - e^{-\mu(2kr+x)} p_t^\mu(-x - 2kr, y)), \end{aligned}$$

where p_t^μ is the standard heat kernel with a drift μ . For $0 \leq \mu \leq 1$, we define a Markov process $\{A^{(\mu)}(t), t \geq 0\}$ in $C_{\text{aff}} \cup \{(0, 0)\}$ in the following way: $A^{(\mu)}(t) = (t, A_t^\mu)$ where the law of A_{t+r}^μ given $A_r^\mu = x$ has the density

$$(5.7) \quad s_t^\mu((r, x), (r+t, y)) = \frac{\varphi_\mu(r+t, y)}{\varphi_\mu(r, x)} w_t^\mu((r, x), (r+t, y)),$$

for $(r, x), (r + t, y) \in C_{\text{aff}}$, $A^{(\mu)}(0) = (0, 0)$ and the entrance density is given by, for $(t, y) \in C_{\text{aff}}$, by

$$(5.8) \quad s_t^\mu((0, 0), (t, y)) = \varphi_\mu(t, y) \sin\left(\frac{y}{t}\pi\right) e^{-\frac{1}{2t}(y-\mu t)^2}.$$

Definition 5.5. For $0 \leq \mu \leq 1$, the process $\{A^{(\mu)}(t), t \geq 0\}$ is the space-time Brownian motion conditioned to stay in C_{aff} .

Recall that u_t , resp. q_t , is the transition probability density of the killed Brownian motion in $[0, 1]$, resp. of Z (see (5.1)).

Lemma 5.6. For $0 < r \leq t, 0 \leq x \leq r$, and $0 \leq y \leq t$,

$$u_{1/r-1/t}(y/t, x/r) e^{-\frac{1}{2t}y^2} = w_{t-r}^0((r, x), (t, y)) e^{-\frac{1}{2r}x^2}$$

and for $0 \leq x, y \leq 1$,

$$q_t(x, y) = c_t \sin(\pi y) e^{-\frac{1}{2t}(y-x)^2} \varphi_x(1/t, y/t).$$

Proof.

$$\begin{aligned} u_{1/r-1/t}(y/t, x/r) &= \sum_{k \in \mathbb{Z}} (p_{1/r-1/t}(y/t + 2k, x/r) - p_{1/r-1/t}(-y/t - 2k, x/r)) \\ &= \sum_{k \in \mathbb{Z}} \frac{p_r(0, x)}{p_t(0, y + 2kt)} (p_{t-r}(x, y + 2kt) - p_{t-r}(x, -y - 2kt)) \\ &= \frac{p_r(0, x)}{p_t(0, y)} \sum_{k \in \mathbb{Z}} e^{-2kx - 2k^2r} (p_{t-r}(x + 2kt, y) - p_{t-r}(-x - 2kr, y)), \end{aligned}$$

which gives the first equality, by (5.6). The second one follows from Lemma 2.39 and (5.3). \square

Theorem 5.7. For $\mu \in [0, 1]$, the processes $\{A^{(\mu)}(t), t \geq 0\}$ and $\{(t, tZ_{1/t}), t \geq 0\}$, where Z starts from μ , are equal in law.

Proof. For $0 < t_1 < \dots < t_n$, the quantity $\mathbb{P}(t_1 Z_{1/t_1} \in dx_1, \dots, t_n Z_{1/t_n} \in dx_n)$ is equal to

$$q_{\frac{1}{t_n}}(\mu, \frac{x_n}{t_n}) q_{\frac{1}{t_{n-1}} - \frac{1}{t_n}}(\frac{x_n}{t_n}, \frac{x_{n-1}}{t_{n-1}}) \dots q_{\frac{1}{t_1} - \frac{1}{t_2}}(\frac{x_2}{t_2}, \frac{x_1}{t_1}) dx_1 \dots dx_n.$$

Identity (5.1) and the first identity of Lemma 5.6 imply that

$$\prod_{k=2}^n q_{\frac{1}{t_{k-1}} - \frac{1}{t_k}}(\frac{x_k}{t_k}, \frac{x_{k-1}}{t_{k-1}})$$

is equal to

$$c_{t_1, \dots, t_n} e^{\frac{1}{2t_n}x_n^2 - \frac{1}{2t_1}x_1^2} \frac{\sin(\pi x_1/t_1)}{\sin(\pi x_n/t_n)} \prod_{k=2}^n w_{t_k - t_{k-1}}^0((t_{k-1}, x_{k-1}), (t_k, x_k)),$$

where c_{t_1, \dots, t_n} is a positive constant which doesn't depend on x_1, \dots, x_n . We conclude by using the second identity of Lemma 5.6 for $q_{\frac{1}{t_n}}(\mu, \frac{x_n}{t_n})$, the fact that

$$w_{t-s}^\mu((s, x), (t, y)) = e^{\mu(y-x) - \frac{t-s}{2}\mu^2} w_{t-s}^0((s, x), (t, y))$$

and identities (5.7), and (5.8). \square

This implies that $A^{(\mu)}(t)$ is really in the interior of the cone for $t > 0$.

5.4. A property of Pitman transform for piecewise C^1 paths. As mentioned in the introduction, the need of an infinite number of Pitman transforms to represent the space-time Brownian motion in C_{aff} is due to its wild behaviour. The situation is much simpler for regular space-time curves, as shown by the following proposition (recall that paths in \bar{C}_{aff} are fixed under the Pitman transforms).

Proposition 5.8. *Let $\varphi : [0, T] \rightarrow \mathbb{R}$ be a continuous piecewise C^1 function such that $\varphi(0) = 0$ and $\eta(t) = (t, \varphi(t))$. There is an n such that for all $t \in [0, T]$,*

$$\mathcal{P}_{s_n} \cdots \mathcal{P}_{s_1} \mathcal{P}_{s_0} \eta(t) \in \bar{C}_{\text{aff}}.$$

We use the notations of 1.2 and Section 3. It is equivalent to prove that there is an $n > 0$ such that

$$0 \leq \mathcal{P}_n \cdots \mathcal{P}_1 \mathcal{P}_0 \varphi(t) \leq t,$$

for all $t \in [0, T]$. Let $\tau_1(\varphi) = \inf\{t > 0; \varphi(t) < 0\}$, $\tau_0(\varphi) = \inf\{t > 0; \varphi(t) > t\}$, and let $|\varphi'|$ be the supremum of the left and right derivatives of φ on $[0, T]$.

Lemma 5.9. (1) $|(\mathcal{P}_1 \varphi)'| \leq |\varphi'|$, and $|(\mathcal{P}_0 \varphi)'| \leq 2 + |\varphi'|$.

(2) When $0 \leq t \leq \tau_1(\varphi) \wedge \tau_0(\varphi)$, $\mathcal{P}_1 \varphi(t) = \mathcal{P}_0 \varphi(t) = \varphi(t)$.

(3) There is an $n > 0$ such that $0 \leq (\mathcal{P}_n \cdots \mathcal{P}_2 \mathcal{P}_1 \varphi)'(0) \leq 1$.

Proof. (1) and (2) are straightforward. For (3): if σ_0 and σ_1 are given by $\sigma_0(x) = -x$ and $\sigma_1(x) = 2-x$, then $(\mathcal{P}_i \varphi)'(0) = \sigma_i(\varphi'(0))$, for $i = 0, 1$ and it is well known that one can bring any real into $[0, 1]$ by repeated actions of σ_0 and σ_1 . \square

Lemma 5.10. *Proposition 5.8 holds when $\tau_1(\varphi) \wedge \tau_0(\varphi) > 0$.*

Proof. We first suppose that $\tau_0(\varphi) < \tau_1(\varphi)$. Let $\varphi_1 = \mathcal{P}_0 \varphi$, $\varphi_2 = \mathcal{P}_1 \varphi_1$, $\varphi_3 = \mathcal{P}_0 \varphi_2, \dots$ and $a_1 = \tau_0(\varphi)$, $a_2 = \tau_1(\varphi_1)$, $a_3 = \tau_0(\varphi_2), \dots$. By (1) of Lemma 5.9, $|\varphi'_n| \leq |\varphi'| + 2n$. Since $\varphi_{2n}(a_{2n}) = 0$, $\varphi_{2n}(a_{2n+1}) = a_{2n+1}$, we obtain by the mean value theorem that

$$a_{2n+1} - a_{2n} \geq \frac{a_{2n+1}}{|\varphi'| + 4n},$$

and so the increasing sequence a_n is bigger than T for n large enough. When $\tau_0(\varphi) > \tau_1(\varphi)$, then $\tau_0(\varphi_1) > \tau_1(\varphi_1)$ and we apply the same proof to φ_1 , beginning by applying \mathcal{P}_1 . \square

Proof of Proposition 5.8. Using (3) of Lemma 5.9 one can suppose that $0 \leq \varphi'(0) \leq 1$. When $0 < \varphi'(0) < 1$, then $\tau_1(\varphi) \wedge \tau_0(\varphi) > 0$ and we can apply Lemma 5.10. If $\varphi'(0) = 1$, let $\gamma = \mathcal{P}_0 \varphi$. Then $\gamma'(0) = 1$, and $\gamma(t) \leq t$ for $t \geq 0$. In that case $\tau_0(\gamma) \geq T$, and $\tau_1(\gamma) > 0$. Indeed otherwise there is a sequence $t_n > 0$ decreasing to 0 such that $\gamma(t_n) \leq 0$ and $\gamma'(0)$ can not be 1. So we can also apply Lemma 5.10. The case $\varphi'(0) = 0$ is similar. \square

5.5. Anti-lecture hall compositions. The set A_∞ of anti-lecture hall compositions is defined in Corteel and Savage [7] as the set of sequence of integers $\lambda_1, \lambda_2, \dots$ such that, for some $n \in \mathbb{N}$,

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \cdots \geq \frac{\lambda_n}{n} > 0,$$

and $\lambda_p = 0$ when $p > n$. So we see that, if $B(\infty)$ is the crystal defined in 4.3,

$$A_\infty = \{(\lambda_1, \lambda_2, \dots); (0, \lambda_1, \lambda_2, \dots) \in B(\infty)\}.$$

For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n, 0, 0, \dots) \in A_\infty$, let $|\lambda| = \sum_{k=1}^n \lambda_k$. Then, see Corteel et al. [7, 8] and (1.2) and (1.3) in Chen et al. [6], for $0 \leq q < 1$,

$$(5.9) \quad \sum_{\lambda \in A_\infty} q^{|\lambda|} = \frac{(-q; q)_\infty}{(q^2; q)_\infty},$$

$$(5.10) \quad \sum_{\lambda \in A_\infty, \lambda_1 \leq k} q^{|\lambda|} = \frac{(-q; q)_\infty (q; q^{k+2})_\infty (q^{k+1}; q^{k+2})_\infty (q^{k+2}; q^{k+2})_\infty}{(q; q)_\infty},$$

where $(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)$.

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