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# PITMAN TRANSFORMS AND BROWNIAN MOTION IN THE INTERVAL VIEWED AS AN AFFINE ALCOVE

PHILIPPE BOUGEROL AND MANON DEFOSSEUX

ABSTRACT. Pitman's theorem states that if  $\{B_t, t \geq 0\}$  is a one dimensional Brownian motion, then  $\{B_t - 2 \inf_{s \leq t} B_s, t \geq 0\}$  is a three dimensional Bessel process, i.e. a Brownian motion conditioned to remain forever positive. This paper gives a similar representation for the Brownian motion conditioned to remain in an interval. Due to the double barrier condition, it is more involved and only asymptotic. One uses the fact that the interval is an alcove of the Kac-Moody affine Lie algebra  $A_1^{(1)}$  and the Littelmann path approach of representation theory.

couleurs de la corde  
dépôt de cette image  
cristaux du temps  
traces d'espace

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Raymond Queneau

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## 1. INTRODUCTION

**1.1.** The probability transition of the Brownian motion conditioned to stay positive is the difference of two heat kernels. This is a consequence of the reflection principle at 0. Pitman's theorem ([32]) of 1975 gives the path representation of this process as

$$\mathcal{P}B(t) = B_t - 2 \inf_{0 \leq s \leq t} B_s.$$

The transform  $\mathcal{P}B$  is written with the reflection at 0. We consider  $Z_t, t \geq 0$ , the Brownian motion conditioned to stay in the interval  $[0, 1]$  forever (see Definition 5.1). It can be seen as the Doob transform of the Brownian motion with Dirichlet condition at 0 and 1. Its probability transition is an alternating infinite sum which can be obtained by applying successive principles of reflection at 0 and 1 (method of images). It is therefore natural to ask if Pitman's theorem has an analogue for  $Z$ , written with an infinite number of transforms at 0 and 1. The main result of this article is to show that, to our surprise, this is not exactly true. A small correction (a Lévy transform) has to be added. The same correction occurs in some asymptotics of the highest weight representations of the affine Lie algebra  $A_1^{(1)}$ .

**1.2.** Let us state our main probabilistic result. We consider, for a continuous real path  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ , such that  $\varphi(0) = 0$ ,

$$\begin{aligned} \mathcal{L}_1\varphi(t) &= \varphi(t) - \inf_{0 \leq s \leq t} \varphi(s), \\ \mathcal{P}_1\varphi(t) &= \varphi(t) - 2 \inf_{0 \leq s \leq t} \varphi(s), \end{aligned}$$

that we call the classical Lévy and Pitman transforms of  $\varphi$ . We introduce

$$\begin{aligned} \mathcal{L}_0\varphi(t) &= \varphi(t) + \inf_{0 \leq s \leq t} (s - \varphi(s)), \\ \mathcal{P}_0\varphi(t) &= \varphi(t) + 2 \inf_{0 \leq s \leq t} (s - \varphi(s)). \end{aligned}$$

The Brownian motion with drift  $\mu$  is  $B_t^\mu = B_t + t\mu, t \geq 0$ , where  $B$  is the standard Brownian motion with  $B_0 = 0$ . For  $n \in \mathbb{N}$ , we let  $\mathcal{P}_{2n} = \mathcal{P}_0, \mathcal{L}_{2n} = \mathcal{L}_0$  and  $\mathcal{P}_{2n+1} = \mathcal{P}_1, \mathcal{L}_{2n+1} = \mathcal{L}_1$ . The aim of this paper is the following representation theorem (see Theorem 3.1).

**Theorem.** *Let  $\mu \in [0, 1]$ . For any  $t > 0$ , almost surely,*

$$\lim_{n \rightarrow \infty} t \mathcal{L}_{n+1} \mathcal{P}_n \cdots \mathcal{P}_1 \mathcal{P}_0 B^\mu(1/t) = \lim_{n \rightarrow \infty} t \mathcal{L}_{n+1} \mathcal{P}_n \cdots \mathcal{P}_2 \mathcal{P}_1 B^\mu(1/t) = Z_t,$$

where  $Z$  is the Brownian motion conditioned to stay in  $[0, 1]$  forever, and  $Z_0 = \mu$ .

**1.3.** Briefly, the strategy of the proof is the following. We will use the fact that, when  $Z_0 = \mu$ ,  $Z$  is the space component of the time inverted process of  $A^{(\mu)}$ ,

where  $A^{(\mu)}$  is the space-time Brownian motion  $B_t^{(\mu)} = (t, B_t^\mu), t \geq 0$ , conditioned to stay in the affine cone

$$C_{\text{aff}} = \{(t, x) \in \mathbb{R}^2; 0 < x < t\},$$

(see Appendix 5.3). We define a sequence of non-negative functions  $\xi_n(t), t \geq 0, n \in \mathbb{N}$ , by

$$\xi_n(t) = - \inf_{0 \leq s \leq t} \{s 1_{2\mathbb{N}}(n) + (-1)^{n-1} \mathcal{P}_{n-1} \cdots \mathcal{P}_0 B^\mu(s)\}$$

Then

$$\mathcal{P}_n \cdots \mathcal{P}_0 B^\mu(t) = B_t^\mu + 2 \sum_{k=0}^n (-1)^{k+1} \xi_k(t).$$

When  $0 < \mu < 1$ , the random variables  $\xi_k(\infty) = \lim_{t \rightarrow +\infty} \xi_k(t), k \in \mathbb{N}$ , are finite and we show that their law have a simple explicit representation with independent exponential random variables. The law of  $\xi_k(t), k \in \mathbb{N}$ , can be deduced by a truncation argument. This allows us to show that for all  $t > 0$  the limit of

$$(1.1) \quad \mathcal{P}_n \cdots \mathcal{P}_0 B^\mu(t) + (-1)^n \xi_{n+1}(t)$$

exists a.s. and has the law of the space component of  $A^{(\mu)}$ . We also prove that  $\xi_n(t)$  tends to 2 when  $n$  tends to  $\infty$ , which shows the need of the correction.

Actually, to prove these results, we approximate the space-time Brownian motion  $B^{(\mu)}$  by planar Brownian motions with proper drifts and  $A^{(\mu)}$  by these planar Brownian motions conditioned to remain in a wedge in  $\mathbb{R}^2$  of dihedral angle  $\pi/m$ . We use the results of Biane et al. [3] in this situation. Due to the need of the correction term, the approximation is not immediate.

**1.4.** Before describing the plan of the paper, let us make an observation. At the heart of our approach is the fact that the interval  $[0, 1]$  is an alcove for the Kac-Moody affine Lie algebra  $A_1^{(1)}$  and that  $A^{(\mu)}$  can be seen as a process conditioned to remain in a Weyl chamber of  $A_1^{(1)}$ . This is linked to highest weight representations of  $A_1^{(1)}$  through Littelmann path approach. We have chosen to present the proof of our main probabilistic result without explicit reference to Kac-Moody algebra, so that it can be read by a probabilist. But let us now explain the ideas from representation theory behind the scenes because this has been a source of inspiration. This may be helpful for some readers.

We first recall the link between Littelmann path theory for the Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$  (see [28]) and the classical Pitman's theorem as explained in Biane et al. [3]. Consider a real line  $V = \mathbb{R}\alpha$  where  $\alpha$  is a positive root. A path  $\pi$  in  $V$  is a continuous function  $\pi : \mathbb{R}^+ \rightarrow V$  such that  $\pi(0) = 0$ . It can be written, for  $s \geq 0$ , as  $\pi(s) = f(s)\alpha$  with  $f(s) \in \mathbb{R}$ . A dominant path is a path with values in the Weyl chamber, which is here  $\mathbb{R}_+\alpha$ , so that  $\pi$  is dominant when  $f(s) \geq 0$  for all  $s \geq 0$ . We fix a  $t > 0$ . An integral path on  $[0, t]$  is a path such that  $2f(t)$  and  $2\min_{s \leq t} f(s)$  are in  $\mathbb{Z}$ . For an irreducible highest weight  $\mathfrak{sl}_2(\mathbb{C})$ -module, one chooses a dominant integral path  $\pi$  on  $[0, t]$ , such that  $\pi(t)$  is its highest weight. A path realization of the Kashiwara crystal associated to this module

is the Littelmann module  $B\pi$ , which is the set of integral paths  $\eta$  on  $[0, t]$  such that  $\mathcal{P}_\alpha\eta(s) = \pi(s), 0 \leq s \leq t$ , where  $\mathcal{P}_\alpha$  is the path transform defined by

$$(1.2) \quad \mathcal{P}_\alpha\eta(s) = (\varphi(s) - 2 \inf_{0 \leq u \leq s} \varphi(u))\alpha,$$

when  $\eta(s) = \varphi(s)\alpha$ . One recognizes the Pitman's transform.

Let us consider now the Littelmann path theory for the affine Lie algebra  $A_1^{(1)}$ . A realization of a real Cartan subalgebra  $\mathfrak{h}_\mathbb{R}$  is given by  $\mathbb{R}^3$  and we write the canonical basis as  $\{c, \tilde{\alpha}_1, d\}$ , and the dual canonical basis of  $\mathfrak{h}_\mathbb{R}^*$  as  $\{\Lambda_0, \alpha_1/2, \delta\}$ . We choose  $\tilde{\alpha}_0 = c - \tilde{\alpha}_1$  and  $\alpha_0 = \delta - \alpha_1$ . The Weyl group is generated by the reflections  $s_{\alpha_0}, s_{\alpha_1}$  on  $\mathfrak{h}_\mathbb{R}^*$  defined by, for  $v \in \mathfrak{h}_\mathbb{R}^*$  and  $i = 0, 1$ ,

$$s_{\alpha_i}(v) = v - \tilde{\alpha}_i(v)\alpha_i.$$

which are reflections along the walls of the Weyl chamber

$$C_W = \{t\Lambda_0 + x\alpha_1/2 + y\delta, (t, x) \in C_{\text{aff}}, y \in \mathbb{R}\} = C_{\text{aff}} \times \mathbb{R}.$$

In Littelmann's theory, a path  $\eta : \mathbb{R}^+ \rightarrow \mathfrak{h}_\mathbb{R}^*$  is now a continuous map  $\eta(0) = 0$ . One defines path transforms  $\mathcal{P}_{\alpha_i}, i \in \{0, 1\}$ , by

$$\mathcal{P}_{\alpha_i}\eta(t) = \eta(t) - \inf_{0 \leq s \leq t} \tilde{\alpha}_i(\eta(s))\alpha_i.$$

A dominant path is a path with values in the closure of  $C_W$  and as above, one can define integral paths on  $[0, t]$ . For a fixed  $t > 0$  and an integral dominant path  $\pi$  on  $[0, t]$ , the Littelmann paths module generated by  $\pi$  is the set of integral paths  $\eta$  on  $[0, t]$  for which it exists  $n \in \mathbb{N}$  such that

$$\mathcal{P}_{\alpha_n}\mathcal{P}_{\alpha_{n-1}} \cdots \mathcal{P}_{\alpha_1}\mathcal{P}_{\alpha_0}\eta(s) = \pi(s)$$

when  $0 \leq s \leq t$ , where  $\alpha_{2k} = \alpha_0$  and  $\alpha_{2k+1} = \alpha_1$  for any  $k \in \mathbb{N}$ . This gives a description of the Kashiwara crystal of highest weight  $\pi(t)$  (see Kashiwara [26], Littelmann [28]). For an integral path  $\eta$  on  $[0, t]$  and more generally for a continuous piecewise  $C^1$  path (see Proposition 5.8), there is a  $k \in \mathbb{N}$  such that for all  $n \geq k$ ,

$$\mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_0}\eta(s) = \mathcal{P}_{\alpha_k} \cdots \mathcal{P}_{\alpha_0}\eta(s),$$

for  $0 \leq s \leq t$ . This new path is dominant. One can ask if, similarly to Pitman's theorem, at least the limit when  $n$  tends to infinity, of  $\mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_0}\eta$ , exists when  $\eta$  is replaced by a space-time Brownian motion.

We show in this paper that the answer is no. Nevertheless one proves that a slight modification converges in  $\mathfrak{h}_\mathbb{R}^*/\mathbb{R}\delta$  to the space-time Brownian motion  $B^{(\mu)}$  conditioned to remain in the affine cone  $C_{\text{aff}}$ . Actually it will be enough for us to work in the space  $V = \mathfrak{h}_\mathbb{R}^*/\mathbb{R}\delta$ , so our paths and transforms will be defined on  $V$ .

We call the sequence  $\{\xi_k(t), k \in \mathbb{N}\}$ , defined above, the affine string parameters, by analogy with the string parameters in the Littelmann model because, for integral paths, they are the string parameters of the element corresponding in the highest weight crystal (see Kashiwara [25], Littelmann [29]). Likewise, we call  $\xi_k(\infty), k \in \mathbb{N}$ , the Verma affine string parameters, by analogy with the string parameters of the crystal  $B(\infty)$  of Kashiwara [25] associated with the Verma module of highest weight 0.

**1.5.** The paper is organized as follows. As said before the process  $Z$  is the space component of the time inverse of the space time process  $A^{(\mu)}$ . Dealing with the two dimensional process  $A^{(\mu)}$  linearizes the situation and so is easier. So we first study it in Section 2, by using approximations by processes in dihedral cones.

In Section 2.1 one recalls the Pitman representation theorem for a planar Brownian motion in a dihedral cone and gives a precise description of the law of their string and Verma string parameters. One studies their asymptotic behaviour in Section 2.2 and obtains a description of the Verma affine string parameters  $\xi(\infty)$  of the space-time Brownian motion  $B^{(\mu)}$ . In Section 2.3 one proves that the series  $\sum_{k=0}^n (-1)^k \xi_k(\infty)$  converges up to a correcting term. In Section 2.5 one shows the almost sure convergence of (1.1) and identifies the limit. One obtains in Section 2.6 a representation theorem for the conditioned space-time Brownian motion by applying at last a Lévy transform to the iterates of Pitman's ones. In Section 2.8 one makes some comments on the distributions of the first string parameter  $\xi_1(\infty)$  of the space-time Brownian motion and of its affine Verma weight.

In Section 3 one proves the main result of the paper on the Brownian motion in the interval. As mentioned we only use probabilistic notions and arguments up to this point.

In Section 4, we now consider the affine Lie algebra  $A_1^{(1)}$  and show how our results are related to its highest weight representations. First we show that the conditional law of the Brownian motion is a Duistermaat Heckman measure for a circle action, describing the semiclassical behaviour of the weights of the representation its highest weight is large. Secondly, we show that the Lévy correction term also occurs in the behaviour of the elements of large weight of the Kashiwara crystal  $B(\infty)$ .

Section 5 is an appendix where we define rigoursosly the conditioned Brownian motion in the interval  $[0, 1]$  and the conditioned space-time Brownian motion in the affine Weyl chamber  $C_{\text{aff}}$ . We prove in Theorem 5.7 that the first one is equal in law, up to a time inversion, to the space component of the second one.

**1.6.** In conclusion we see that the, a priori simple, Brownian motion in the interval can be studied thanks to its links with the affine Lie algebra  $A_1^{(1)}$ . It is an example of integrable probability in the sense of Borodin and Petrov ([5]). It would be interesting to study the higher rank case  $A_n^{(1)}$  which occurs in the analysis of  $n + 1$  non colliding Brownian motions on a circle (see Hobson and Werner [21]). This requires new ideas.

## 2. REPRESENTATION OF THE CONDITIONED SPACE-TIME REAL BROWNIAN MOTION IN THE AFFINE WEYL CONE

In order to represent the Brownian motion  $Z$  conditioned to stay in  $[0, 1]$ , we will linearize the problem and use the fact that  $Z$  is the space component of the time inverted process of  $A^{(\mu)}$ , where  $A^{(\mu)}$  is the space-time Brownian motion conditioned to stay in the affine cone  $C_{\text{aff}} = \{(t, x) \in \mathbb{R}^2; 0 < x < t\}$ , rigoursosly defined in Definition 5.5. This process  $A^{(\mu)}$  will be approached by planar Brownian motions conditioned to remain in dihedral cones.

**2.1. Pitman representation for the planar Brownian motion in the dihedral cone  $C_m$ .** In this section we describe the representation of the planar Brownian motion conditioned to stay in a dihedral cone, using the results of Biane et al. [3],[4].

*2.1.1. Dihedral Coxeter system.* The dihedral group  $I(m)$ , where  $m \in \mathbb{N}$ , is the finite group generated by two involutions  $s_0, s_1$  with the only relation  $(s_0 s_1)^m = 1$ . In the two dimensional vector space  $V = \mathbb{R}^2$  identified with its dual  $\tilde{V}$ , one chooses two pairs  $(v_0^m, \tilde{v}_0^m), (v_1^m, \tilde{v}_1^m)$  in  $V \times \tilde{V}$ , associated with the matrix  $(\tilde{v}_i^m(v_j^m))_{0 \leq i, j \leq 1}$  given by

$$\begin{pmatrix} 2 & -2 \cos(\pi/m) \\ -2 \cos(\pi/m) & 2 \end{pmatrix},$$

Namely, one takes

$$v_0^m = (2 \sin(\pi/m), -2 \cos(\pi/m)), v_1^m = (0, 2),$$

and  $\tilde{v}_0^m = v_0^m/2, \tilde{v}_1^m = v_1^m/2$ . The following two reflections  $s_0^m, s_1^m$  of  $\mathbb{R}^2$ ,

$$s_i^m(v) = v - \tilde{v}_i^m(v) v_i^m, \quad v \in \mathbb{R}^2,$$

generate the group  $I(m)$ . The usual scalar product on  $\mathbb{R}^2$  is denoted by  $\langle \cdot, \cdot \rangle$ , and with our convention  $\tilde{v}_i^m(v) = \langle v_i^m, v \rangle$ .

**Definition 2.1.** *The convex dihedral cone  $C_m$ , with closure  $\bar{C}_m$ , is*

$$C_m = \{v \in \mathbb{R}^2; \tilde{v}_i^m(v) > 0, i = 0, 1\} = \{(r \cos \theta, r \sin \theta) \in \mathbb{R}^2, r > 0, 0 < \theta < \pi/m\}.$$

Let  $\mathcal{C}_0(\mathbb{R}^2)$  be the set of continuous path  $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^2$  such that  $\eta(0) = 0$ . The following path transforms are introduced in [3].

**Definition 2.2.** *The Pitman transforms  $\mathcal{P}_{s_i^m}^m, i = 0, 1$ , are defined on  $\mathcal{C}_0(\mathbb{R}^2)$  by the formula, for  $\eta \in \mathcal{C}_0(\mathbb{R}^2)$ ,  $t \geq 0$ ,*

$$\mathcal{P}_{s_i^m}^m \eta(t) = \eta(t) - \inf_{0 \leq s \leq t} \tilde{v}_i^m(\eta(s)) v_i^m.$$

Notice that

$$\mathcal{P}_{s_i^m}^m \eta(t) = \eta(t) - \inf_{0 \leq s \leq t} (id - s_i^m)(\eta(s)).$$

For each  $i \in \mathbb{N}$ , when we write  $v_i^m, \tilde{v}_i^m, s_i^m, \dots$ , we take  $i$  modulo 2.

**Theorem 2.3** (Biane et al. [3]). *Let  $w = s_{i_r}^m \cdots s_{i_1}^m$  be a reduced decomposition of  $w \in I(m)$  where  $i_1, \dots, i_r \in \{0, 1\}$ . Then*

$$\mathcal{P}_w^m := \mathcal{P}_{s_{i_r}^m}^m \cdots \mathcal{P}_{s_{i_1}^m}^m$$

*depends only on  $w$  and not on the chosen decomposition.*

In  $I(m)$  there is a unique longest word, namely

$$(2.1) \quad w_0 = s_{m-1}^m \cdots s_1^m s_0^m = s_m^m \cdots s_2^m s_1^m.$$

**Proposition 2.4** ([3]). *For any path  $\eta \in \mathcal{C}_0(\mathbb{R}^2)$ , the path  $\mathcal{P}_{w_0}^m \eta$  takes values in the closed dihedral cone  $\bar{C}_m$ .*

Let  $\gamma \in \bar{C}_m$  and let  $W^{(\gamma)}$  be the standard planar Brownian motion in  $\mathbb{R}^2$  with drift  $\gamma$  and identity covariance matrix, starting from the origin. The conditioned planar Brownian motion in the cone  $C_m$  with drift is intuitively given by  $W^{(\gamma)}$  conditioned to stay in  $C_m$  forever. It is rigorously defined in Appendix 5.2. The following is proved in Biane et al. [3] when there is no drift, the case with drift follows easily and proved below in Proposition 5.3.

**Theorem 2.5** (Biane et al. [3]). *Let  $W^{(\gamma)}$  be the planar Brownian motion with drift  $\gamma$ , where  $\gamma \in \bar{C}_m$ . Then  $\mathcal{P}_{w_0}^m W^{(\gamma)}$  is the conditioned planar Brownian motion in the cone  $C_m$  with drift  $\gamma$ .*

*2.1.2. String parameters in  $\mathcal{C}_0(\mathbb{R}^2)$ .* We fix an integer  $m \geq 1$ . For simplicity of notations, and without loss of generality, one chooses one the two decompositions of the longest word  $w_0$  in  $I(m)$ , namely,

$$w_0 = s_{m-1}^m \cdots s_1^m s_0^m.$$

For  $\eta \in \mathcal{C}_0(\mathbb{R}^2)$ ,  $0 \leq k \leq m-1$  and  $0 \leq t \leq +\infty$ , let

$$x_k^m(t) = - \inf_{0 \leq s \leq t} \tilde{v}_k^m(\mathcal{P}_{s_{k-1}}^m \cdots \mathcal{P}_{s_1}^m \mathcal{P}_{s_0}^m \eta(s)).$$

**Definition 2.6.** *We call*

$$x^m(t) = (x_0^m(t), x_1^m(t), \cdots, x_{m-1}^m(t))$$

*the string parameters of the path  $\eta$  on  $[0, t]$  and we call  $x^m(\infty)$  its Verma string parameters.*

Notice that

$$(2.2) \quad \mathcal{P}_{w_0}^m \eta(t) = \eta(t) + \sum_{k=0}^{m-1} x_k^m(t) v_k^m.$$

and that, for  $t \geq 0$ , for all  $k = 0, \cdots, m-1$ ,

$$(2.3) \quad 2x_k^m(t) = \tilde{v}_k^m(\mathcal{P}_{s_k}^m \cdots \mathcal{P}_{s_0}^m \eta(t)) - \mathcal{P}_{s_{k-1}}^m \cdots \mathcal{P}_{s_0}^m \eta(t).$$

When

$$(2.4) \quad \lim_{s \rightarrow +\infty} \tilde{v}_0^m(\eta(s)) = \lim_{s \rightarrow +\infty} \tilde{v}_1^m(\eta(s)) = +\infty$$

holds, then the Pitman transforms  $\mathcal{P}_w^m \eta$ ,  $w \in I(m)$ , have the same property and for all  $t \geq 0$  and  $0 \leq k \leq m-1$ ,

$$(2.5) \quad 0 \leq x_k^m(t) \leq x_k^m(\infty) < +\infty,$$

and  $x_k^m(t) = x_k^m(\infty)$  for  $t$  large enough. Let, for  $1 \leq k < m$ ,

$$a_k^m = \sin(k\pi/m).$$

**Definition 2.7.** *The cone  $\Gamma_m$  in  $\mathbb{R}^m$  is defined as*

$$\Gamma_m = \{(x_0, \cdots, x_{m-1}) \in \mathbb{R}^m; \frac{x_1}{a_1^m} \geq \frac{x_2}{a_2^m} \geq \cdots \geq \frac{x_{m-1}}{a_{m-1}^m} \geq 0, x_0 \geq 0\}.$$

*For  $\lambda \in \bar{C}_m$ , the polytope  $\Gamma_m(\lambda)$  is*

$$\Gamma_m(\lambda) = \{(x_0, \cdots, x_{m-1}) \in \Gamma_m; 0 \leq x_r \leq \tilde{v}_r^m(\lambda - \sum_{n=r+1}^{m-1} x_n v_n^m), 0 \leq r \leq m-1\}.$$



Remark the particular role of  $x_0$ . The following proposition is proved in [4], Propositions 4.4 and 4.7.

**Proposition 2.8.** *The set of string parameters of paths on  $[0, t]$  is  $\Gamma_m$ .*

*2.1.3. String parameters of the planar Brownian motion.* When  $\gamma \in C_m$ , the planar Brownian motion  $W^{(\gamma)}$  with drift  $\gamma$  satisfies (2.4) by the law of large numbers, so its Verma string parameters are finite.

**Theorem 2.9.** *Let  $x^m(t)$  be the string parameters of  $W^{(\gamma)}$  on  $[0, t]$  and  $x^m(\infty)$  be its Verma string parameters.*

(i) *When  $\gamma \in C_m$ , for any  $t > 0$ , conditionally on  $\sigma(\mathcal{P}_{w_0}^m W^{(\gamma)}(s), s \leq t)$  the random variables  $x_k^m(t), 0 \leq k < m$ , are independent exponentials with parameters  $\gamma_k^m = \langle \gamma, v_k^m \rangle$  conditioned on the event*

$$\{x^m(t) \in \Gamma_m(\mathcal{P}_{w_0}^m W^{(\gamma)}(t))\}.$$

(ii) *When  $\gamma \in C_m$ , the random variables  $x_k^m(\infty), 0 \leq k < m$ , are independent exponentials with parameters  $\gamma_k^m = \langle \gamma, v_k^m \rangle$  conditioned on the event*

$$\{x^m(\infty) \in \Gamma_m\}.$$

*Proof.* Let  $\mathcal{F}_t^\gamma = \sigma(\mathcal{P}_{w_0}^m W^{(\gamma)}(s), s \leq t)$ . It is shown in Biane et al. [4], Theorem 5.2, that when  $\gamma = 0$ , the law of  $x^m(t)$  conditionally on  $\mathcal{F}_t^0$  is the normalized Lebesgue measure on  $\Gamma_m(\mathcal{P}_{w_0}^m W^{(0)}(t))$ . Let  $\psi : \mathcal{C}_0(\mathbb{R}^2) \rightarrow \mathbb{R}$  be a bounded measurable function. Is is easy to see by applying the definition of the conditional expectation and the Cameron Martin formula twice that we have the following Bayes formula,

$$(2.6) \quad \mathbb{E}(\psi(W_s^{(\gamma)}, 0 \leq s \leq t) | \mathcal{F}_t^\gamma) = \varphi(\mathcal{P}_{w_0}^m W^{(\gamma)}(t))$$

where

$$\varphi(\mathcal{P}_{w_0}^m W^{(0)}(t)) = \frac{\mathbb{E}(\psi(W_s^{(0)}, 0 \leq s \leq t) e^{\langle \gamma, W_t^{(0)} \rangle} | \mathcal{F}_t^0)}{\mathbb{E}(e^{\langle \gamma, W_t^{(0)} \rangle} | \mathcal{F}_t^0)}.$$

It follows from (2.2) that  $W_t^{(0)} = \mathcal{P}_{w_0}^m W^{(0)}(t) - \sum_{k=0}^{m-1} x_k^m(t) v_k^m$ , where  $x^m$  are the string parameters of  $W^{(0)}$ , so that

$$\varphi(\mathcal{P}_{w_0}^m W^{(0)}(t)) = \frac{\mathbb{E}(\psi(W_s^{(0)}, 0 \leq s \leq t) e^{-\sum_{k=0}^{m-1} x_k^m(t) v_k^m} | \mathcal{F}_t^0)}{\mathbb{E}(e^{-\sum_{k=0}^{m-1} x_k^m(t) v_k^m} | \mathcal{F}_t^0)}.$$

If we take  $\psi(W_s^{(\gamma)}, 0 \leq s \leq t) = F(x^m(t))$  where  $F : \mathbb{R}^m \rightarrow \mathbb{R}$  is bounded and continuous, and now  $x^m$  are the string parameters of  $W^{(\gamma)}$ , and if  $dx$  is the Lebesgue measure on  $\mathbb{R}^m$  we obtain that

$$\mathbb{E}(F(x^m(t)) | \mathcal{F}_t^\gamma) = \frac{\int 1_{\Gamma_m(\mathcal{P}_{w_0}^m W^{(\gamma)}(t))}(x) F(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx}{\int 1_{\Gamma_m(\mathcal{P}_{w_0}^m W^{(\gamma)}(t))}(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx},$$

which proves (i). We suppose that  $\gamma \in C_m$ , so

$$\begin{aligned} \mathbb{E}(F(x^m(\infty))) &= \lim_{t \rightarrow +\infty} \mathbb{E}(F(x^m(t))) \\ &= \lim_{t \rightarrow +\infty} \mathbb{E}(\mathbb{E}(F(x^m(t)) | \mathcal{F}_t)) \\ &= \lim_{t \rightarrow +\infty} \mathbb{E}\left(\frac{\int 1_{\Gamma_m(\mathcal{P}_{w_0}^m W^{(\gamma)}(t))}(x) F(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx}{\int 1_{\Gamma_m(\mathcal{P}_{w_0}^m W^{(\gamma)}(t))}(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx}\right) \\ &= \frac{\int 1_{\Gamma_m}(x) F(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx}{\int 1_{\Gamma_m}(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx}, \end{aligned}$$

since, a.s.,  $\mathcal{P}_{w_0}^m W^{(\gamma)}(t)/t$  converges to  $\gamma$  as  $t$  tends to  $+\infty$  and thus  $\Gamma_m(\mathcal{P}_{w_0}^m W^{(\gamma)}(t))$  tends to  $\Gamma_m$  for  $\gamma \in C_m$ . This proves (ii).  $\square$

**Proposition 2.10.** *We suppose that  $\gamma \in C_m$ . Let  $\varepsilon_n, n \in \mathbb{N}$ , be independent exponential random variables with parameter 1. Then, in law,  $x_0^m(\infty) = \varepsilon_0/\gamma_0^m$ , and for  $k = 1, \dots, m-1$ , and  $\gamma_k^m = \langle \gamma, v_k^m \rangle$ ,*

$$x_k^m(\infty) = a_k^m \sum_{l=k}^{m-1} \frac{\varepsilon_l}{\gamma_1^m a_1^m + \dots + \gamma_l^m a_l^m}.$$

where  $x^m(\infty)$  are the Verma string parameters of  $W^{(\gamma)}$ .

*Proof.* Theorem 2.9 says that

$$\begin{aligned} \mathbb{P}(x_0^m(\infty) \in dx_0, \dots, x_{m-1}^m(\infty) \in dx_{m-1}) \\ &= C e^{-\gamma_0^m x_0} dx_0 \prod_{k=2}^m e^{-\gamma_{k-1}^m x_{k-1}} 1_{\{\frac{x_{k-1}}{a_{k-1}^m} \geq \frac{x_k}{a_k^m}\}} dx_{k-1} \\ &= C e^{-\gamma_0^m x_0} dx_0 \prod_{k=2}^m e^{-(\gamma_1^m a_1^m + \dots + \gamma_{k-1}^m a_{k-1}^m)(\frac{x_{k-1}}{a_{k-1}^m} - \frac{x_k}{a_k^m})} 1_{\{\frac{x_{k-1}}{a_{k-1}^m} \geq \frac{x_k}{a_k^m}\}} dx_{k-1} \end{aligned}$$

where by convention  $x_m/a_m^m = 0$  and  $C$  is a normalizing constant. Thus

$$(x_0^m(\infty), \frac{x_1^m(\infty)}{a_1^m} - \frac{x_2^m(\infty)}{a_2^m}, \frac{x_2^m(\infty)}{a_2^m} - \frac{x_3^m(\infty)}{a_3^m}, \dots, \frac{x_{m-1}^m(\infty)}{a_{m-1}^m})$$

has the same law as

$$\left(\frac{\varepsilon_0}{\gamma_0^m}, \frac{\varepsilon_1}{\gamma_1^m a_1^m}, \frac{\varepsilon_2}{\gamma_1^m a_1^m + \gamma_2^m a_2^m}, \dots, \frac{\varepsilon_{m-1}}{\gamma_1^m a_1^m + \dots + \gamma_{m-1}^m a_{m-1}^m}\right),$$

which proves the claim.  $\square$

Notice that this is similar to Renyi's representation of order statistics ([34]).

**2.2. Affine string parameter  $x$  of the space-time Brownian motion.** We will use a terminology inspired by the Kac-Moody affine algebra  $A_1^{(1)}$  (see in particular Kac [24], Kashiwara [25] and Section 4). The infinite dihedral group  $I(\infty)$  is the infinite group generated by two involutions  $s_0, s_1$  with no relation.

In  $V = \mathbb{R}^2$  identified with its dual  $\tilde{V}$ , let  $(\alpha_0, \tilde{\alpha}_0), (\alpha_1, \tilde{\alpha}_1)$  in  $V \times \tilde{V}$ , given by

$$\begin{cases} \alpha_0 = (0, -2), \\ \alpha_1 = (0, 2), \\ \tilde{\alpha}_0 = (1, -1), \\ \tilde{\alpha}_1 = (0, 1). \end{cases}$$

The matrix  $(\tilde{\alpha}_i(\alpha_j))_{0 \leq i, j \leq 1}$  is the Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The two reflections  $s_0, s_1$  of  $\mathbb{R}^2$ ,

$$s_i(v) = v - \tilde{\alpha}_i(v)\alpha_i, \quad v \in \mathbb{R}^2,$$

generate the group  $I(\infty)$ . Notice that  $s_0$  is a non orthogonal reflection. For each  $k \in \mathbb{N}$ , when we write  $\alpha_k, \tilde{\alpha}_k, s_k, \dots$ , we take  $k$  modulo 2, as above. Thus

$$\alpha_k = (-1)^k \alpha_0.$$

**Definition 2.11.** *The affine Weyl cone (or chamber) is*

$$C_{\text{aff}} = \{v \in \mathbb{R}^2; \tilde{\alpha}_i(v) > 0, i = 0, 1\} = \{(t, x) \in \mathbb{R}^2; 0 < x < t\},$$

and  $\bar{C}_{\text{aff}}$  is its closure.

As in the dihedral case, we define the Pitman transform  $\mathcal{P}_{s_i}, i = 0, 1$ , associated with  $(\alpha_i, \tilde{\alpha}_i)$  on  $\mathcal{C}_0(\mathbb{R}^2)$  by the formula:

$$(2.7) \quad \mathcal{P}_{s_i}\eta(t) = \eta(t) - \inf_{0 \leq s \leq t} \tilde{\alpha}_i(\eta(s))\alpha_i,$$

where  $\eta \in \mathcal{C}_0(\mathbb{R}^2)$ . We will use mainly space-time paths, i.e. paths which can be written as  $\eta(t) = (t, \varphi(t)), t \geq 0$ . In this case,

$$\begin{aligned} \mathcal{P}_{s_0}\eta(t) &= (t, \varphi(t) + 2 \inf_{s \leq t} (s - \varphi(s))) \\ \mathcal{P}_{s_1}\eta(t) &= (t, \varphi(t) - 2 \inf_{s \leq t} \varphi(s)). \end{aligned}$$

One recognizes in the second component the transforms  $\mathcal{P}_0, \mathcal{P}_1$  defined in Section 3. One defines  $\eta_0 = \eta$ , and, for  $k \geq 1$ ,

$$\eta_k = \mathcal{P}_{s_{k-1}} \dots \mathcal{P}_{s_1} \mathcal{P}_{s_0} \eta.$$

Let  $\eta$  a path in  $\mathcal{C}_0(\mathbb{R}^2)$ , and  $x(t) = \{x_k(t), k \geq 0\}$  given by

$$x_k(t) = - \inf_{0 \leq s \leq t} \tilde{\alpha}_k(\eta_k(s)).$$

We call  $x_k(t), k \geq 0$ , the affine string parameters of the path  $\eta$  on  $[0, t]$  and we call  $x_k(\infty), k \geq 0$ , its Verma affine string parameters.

We fix a real  $\mu$  such that  $0 \leq \mu \leq 1$  and we consider the space time process

$$B_t^{(\mu)} = (t, B_t + t\mu)$$

where  $B$  is a standard real Brownian motion starting from 0.

**Definition 2.12.** *We let  $\xi(t) = \{\xi_k(t), k \geq 0\}$  be the affine string parameters of  $B^{(\mu)}$  on  $[0, t]$ , and  $\xi(\infty) = \{\xi_k(\infty), k \geq 0\}$  be its Verma affine string parameters.*

One has, for any  $n \geq 0$ ,

$$(2.8) \quad \mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_0} B^{(\mu)}(t) = B_t^{(\mu)} + \sum_{k=0}^n \xi_k(t) \alpha_k.$$

When  $0 < \mu < 1$ ,  $\xi_k(\infty) < +\infty$  for each  $k \geq 0$  by the law of large numbers. We will approach  $B_t^{(\mu)}$  by  $W^{(\frac{m}{\pi}, \mu)}$  which is the planar Brownian motion with drift  $(\frac{m}{\pi}, \mu)$  starting from the origin.

**Definition 2.13.** We define  $\xi^m(t)$  as the string parameters of  $W^{(\frac{m}{\pi}, \mu)}$  on  $[0, t]$  and  $\xi^m(\infty)$  as its Verma string parameters. The dihedral highest weight process is

$$\Lambda_m(t) = \mathcal{P}_{w_0}^m W^{(m/\pi, \mu)}(t), \quad t \geq 0.$$

Let  $\tau_m : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by

$$(2.9) \quad \tau_m(t, x) = \left( \frac{\pi t}{m}, x \right),$$

for  $(t, x) \in \mathbb{R}^2$ . We will frequently use that for  $v \in \mathbb{R}^2$ ,

$$(2.10) \quad \tau_m v = \left( \frac{\pi}{m \sin \frac{\pi}{m}} (\tilde{v}_0^m(v) + \tilde{v}_1^m(v) \cos \frac{\pi}{m}), \tilde{v}_1^m(v) \right),$$

so the asymptotics of  $\tau_m v$  and  $(\tilde{v}_0^m(v) + \tilde{v}_1^m(v), \tilde{v}_1^m(v))$  are the same as  $m$  tends to  $+\infty$ .

**Lemma 2.14.** For  $i = 0, 1$ ,

$$\begin{aligned} \lim_{m \rightarrow +\infty} \tau_m v_i^m &= \alpha_i, \\ \lim_{m \rightarrow +\infty} \tau_m \circ s_i^m \circ \tau_m^{-1} &= s_i. \end{aligned}$$

*Proof.* The first statement is clear. The second one is trivial when  $i = 1$ . For  $(t, x) \in \mathbb{R}^2$  and  $i = 0$ ,

$$(\tau_m \circ (id - s_0^m) \circ \tau_m^{-1})(t, x) = \tau_m(\tilde{v}_0^m(\frac{mt}{\pi}, x) v_0^m) = \tilde{v}_0^m(\frac{mt}{\pi}, x) \tau_m v_0^m$$

converges to  $(t - x)(0, -2) = (id - s_0)(t, x)$ .  $\square$

**Proposition 2.15.** Almost surely, for all  $t > 0$ ,  $k \in \mathbb{N}$ ,

$$\lim_{m \rightarrow +\infty} \xi_k^m(t) = \xi_k(t),$$

and when  $0 < \mu < 1$ ,

$$\lim_{m \rightarrow +\infty} \xi_k^m(\infty) = \xi_k(\infty).$$

*Proof.* Let, for all  $k \in \mathbb{N}$ ,  $w_k^m = s_k^m \cdots s_0^m \in I(m)$  and  $w_k = s_k \cdots s_0 \in I(\infty)$ . For any path  $\eta^m \in \mathcal{C}_0(\mathbb{R}^2)$ ,

$$(\tau_m \circ \mathcal{P}_{s_i^m}^m \circ \tau_m^{-1})(\eta^m)(t) = \eta^m(t) - \inf_{s \leq t} (\tau_m \circ (id - s_i^m) \circ \tau_m^{-1}) \eta^m(s),$$

so it follows from Lemma 2.14 and by iteration that

$$\lim_{m \rightarrow +\infty} (\tau_m \circ \mathcal{P}_{w_k^m}^m \circ \tau_m^{-1}) = \mathcal{P}_{w_k}.$$

The sequence of paths  $\eta^m(t) = (\frac{\pi\beta_t}{m} + t, B_t + \mu t)$  converge to the path  $\eta(t) = B_t^{(\mu)}$ , hence

$$\lim_{m \rightarrow +\infty} (\tau_m \circ (\mathcal{P}_{w_k^m}^m - \mathcal{P}_{w_{k-1}^m}^m) \circ \tau_m^{-1}) \eta^m(t) = (\mathcal{P}_{w_k} - \mathcal{P}_{w_{k-1}}) \eta(t).$$

This implies that

$$\lim_{m \rightarrow +\infty} \tilde{v}_k^m (\tau_m (\mathcal{P}_{w_k^m}^m - \mathcal{P}_{w_{k-1}^m}^m) \tau_m^{-1} \eta^m(t)) = \tilde{\alpha}_k ((\mathcal{P}_{w_k} - \mathcal{P}_{w_{k-1}}) \eta(t)).$$

On the other hand,  $\tau_m^{-1} \eta^m = W^{(m/\pi, \mu)}$ , one has

$$(\mathcal{P}_{w_k^m}^m - \mathcal{P}_{w_{k-1}^m}^m) \tau_m^{-1} \eta^m(t) = \xi_k^m(t) v_k^m.$$

When  $m$  tends to  $\infty$ ,  $\tilde{v}_k^m (\tau_m v_k^m)$  tends to 2, so we see that  $\xi_k^m(t)$  tends to  $\xi_k(t)$  using the analogue of Remark 2.3 in the affine case. When  $0 < \mu < 1$ , a.s.,  $\xi_k^m(\infty) = \xi_k^m(t)$  and  $\xi_k(\infty) = \xi_k(t)$  for  $t$  large enough. As a consequence  $\xi_k^m(\infty)$  tends to  $\xi_k(\infty)$ .  $\square$

As before, we let  $\varepsilon_n, n \in \mathbb{N}$ , be independent exponential random variables with parameter 1.

**Theorem 2.16.** *We suppose that  $0 < \mu < 1$ . In law  $\xi_0(\infty) = \varepsilon_0/2(1 - \mu)$ , and for  $k \geq 1$ ,*

$$\frac{\xi_k(\infty)}{k} = \sum_{n=k}^{+\infty} \frac{2\varepsilon_n}{n(n+1) + (1-2\mu)\nu_n},$$

where  $\nu_n = n$  when  $n$  is even  $\nu_n = -(n+1)$  when  $n$  is odd.

*Proof.* The drift of  $W^{(m/\pi, \mu)}$  is  $(m/\pi, \mu)$ , thus

$$\gamma_0^m = \langle (\frac{m}{\pi}, \mu), v_0^m \rangle = 2(\frac{m}{\pi} \sin(\frac{\pi}{m}) - \mu \cos(\frac{\pi}{m})), \gamma_1^m = \langle (\frac{m}{\pi}, \mu), v_1^m \rangle = 2\mu.$$

By Propositions 2.15 and 2.10, since  $\lim_{m \rightarrow +\infty} m a_k^m = k\pi$ ,

$$\frac{\xi_k(\infty)}{k} = \lim_{m \rightarrow +\infty} \frac{\xi_k^m(\infty)}{k} = \lim_{m \rightarrow +\infty} \frac{\pi}{m} \sum_{n=k}^{m-1} \frac{\varepsilon_n}{a_1^m \gamma_1^m + \dots + a_n^m \gamma_n^m}.$$

Let  $c = \max\{1/\gamma_i^m, m \in \mathbb{N}, i = 0, 1\}$ . Since  $0 \leq 2t \leq \sin \pi t$  when  $0 \leq t \leq 1/2$ , one has, for  $k \leq [m/2]$ ,

$$\sum_{n=1}^k a_n^m \gamma_n^m \geq \frac{1}{c} \sum_{n=1}^k \sin(\frac{n\pi}{m}) \geq \sum_{n=1}^k \frac{2n}{cm} \geq \frac{k^2}{cm},$$

and, for  $k > [m/2]$ ,

$$\sum_{n=1}^k a_n^m \gamma_n^m \geq \sum_{n=1}^{[m/2]+1} a_n^m \gamma_n^m \geq \frac{m}{4c}.$$

Therefore, by the law of large numbers, a.s., when  $N$  is large enough,

$$\begin{aligned} \sup_{m \geq N} \frac{\pi}{m} \sum_{n=N}^{m-1} \frac{\varepsilon_n}{a_1^m \gamma_1^m + \dots + a_n^m \gamma_n^m} &\leq \sup_{m \geq N} (c\pi \sum_{n=N}^{[m/2]} \frac{\varepsilon_n}{n^2} + 4c\pi \sum_{n=[m/2]+1}^m \frac{\varepsilon_n}{m^2}) \\ &\leq \sup_{m \geq N} (c\pi \sum_{n=N}^{\infty} \frac{\varepsilon_n}{n^2} + \frac{4c\pi}{m} \sum_{n=0}^m \frac{\varepsilon_n}{m}) \end{aligned}$$

is as small as we want. We finish the proof by using that, for any fixed  $n$ ,

$$\lim_{m \rightarrow +\infty} \frac{\pi}{m} \frac{\varepsilon_n}{a_1^m \gamma_1^m + \cdots + a_n^m \gamma_n^m} = \frac{2\varepsilon_n}{n(n+1) + (1-2\mu)\nu_n}$$

since for any  $r \in \mathbb{N}$ ,

$$\lim_{m \rightarrow +\infty} \gamma_{2r}^m = \lim_{m \rightarrow +\infty} \gamma_0^m = 2(1-\mu), \quad \lim_{m \rightarrow +\infty} \gamma_{2r+1}^m = \lim_{m \rightarrow +\infty} \gamma_1^m = 2\mu.$$

□

An important for us and maybe unexpected result is:

**Theorem 2.17.** *Almost surely,  $\lim_{k \rightarrow +\infty} \xi_k(\infty) = 2$ .*

*Proof.* It is easy to see that  $\mathbb{E}(\xi_k(\infty))$  tends to 2 and that  $\sum_{k=1}^{\infty} \mathbb{E}((\xi_k(\infty) - \mathbb{E}(\xi_k(\infty)))^4)$  is finite, which implies the almost sure convergence. □

**2.3. Affine Verma weight  $L^{(\mu)}(\infty)$ .** In this subsection we introduce the affine Verma weights. We consider for  $0 < \mu < 1$ , the affine Verma string parameters  $\xi(\infty) = \{\xi_k(\infty), k \geq 0\}$  of  $B^{(\mu)}$ . For  $k \geq 1$ , let

$$M_k = \frac{1}{2} \xi_k(\infty) \alpha_k + \sum_{n=0}^{k-1} \xi_n(\infty) \alpha_n.$$

Notice that, more simply, since  $\alpha_n = (0, (-1)^{n+1}2)$ ,

$$M_k = (0, (-1)^{k+1} \xi_k(\infty) + 2 \sum_{n=0}^{k-1} (-1)^{n+1} \xi_n(\infty)),$$

the notation with the  $\alpha_n$ 's may be strange for the reader. It is explained by its natural interpretation in  $A_1^{(1)}$  (see 4.1).

**Proposition 2.18.** *Let  $\mu \in (0, 1)$ . When  $k$  goes to infinity,  $M_k$  converges almost surely and in  $L^2$  towards*

$$L^{(\mu)}(\infty) = \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{\varepsilon_{2n}}{n+1-\mu} \alpha_0 + \frac{\varepsilon_{2n+1}}{n+\mu} \alpha_1 \right).$$

*Proof.* One has the key relation

$$M_{2p+2} - M_{2p} = (0, -\xi_{2p}(\infty) + 2\xi_{2p+1}(\infty) - \xi_{2p+2}(\infty)) = (0, \frac{\varepsilon_{2p+1}}{p+\mu} - \frac{\varepsilon_{2p}}{p+1-\mu}),$$

hence  $\{M_{2p}, p \geq 0\}$  is a martingale, bounded in  $L_2$  and its limit is  $L^{(\mu)}(\infty)$ . As

$$M_{2p+1} = M_{2p} + \frac{1}{2} (\xi_{2p}(\infty) - \xi_{2p+1}(\infty)) \alpha_0$$

and  $\xi_k(\infty)$  tends to 2 by Theorem 2.17,  $M_{2p+1}$  has the same limit. □

**Definition 2.19.** *We call  $L^{(\mu)}(\infty)$  the affine Verma weight of  $B^{(\mu)}$ .*

The terminology is due to the fact, that we will see in Section 4, that  $L^{(\mu)}(\infty)$  plays the role of an asymptotic weight in the Verma module of highest weight

0 in the case of the affine Lie algebra  $A_1^{(1)}$  (more correctly of minus this weight modulo  $\delta$ ). Remark that, in coordinates,  $L^{(\mu)}(\infty) = (0, D^\mu(\infty))$  where

$$(2.11) \quad D^\mu(\infty) = \sum_{n=0}^{\infty} \left( \frac{\varepsilon_{2n+1}}{n+\mu} - \frac{\varepsilon_{2n}}{n+1-\mu} \right).$$

Notice that  $L^{(\mu)}(\infty)$  is not the limit of the series  $\sum_{k=0}^n \xi_k(\infty) \alpha_k$ , since  $\xi_k(\infty)$  tends to 2 (Theorem 2.17). For a sequence  $x = (x_k) \in \mathbb{R}_+^{\mathbb{N}}$  we let

$$(2.12) \quad \sigma(x) = \lim_{n \rightarrow +\infty} \sum_{k=0}^{n-1} x_k \alpha_k + \frac{1}{2} x_n \alpha_n,$$

when this limit exists in  $\mathbb{R}^2$  (recall that  $\alpha_k = (-1)^k \alpha_0$ ).

**Definition 2.20.** *One defines, for  $\lambda \in \bar{C}_{\text{aff}}$ ,*

$$\Gamma = \{x = (x_k) \in \mathbb{R}^{\mathbb{N}} : \frac{x_k}{k} \geq \frac{x_{k+1}}{k+1} \geq 0, \text{ for all } k \geq 1, x_0 \geq 0, \sigma(x) \in \mathbb{R}^2\},$$

$$\Gamma(\lambda) = \{x \in \Gamma : x_k \leq \tilde{\alpha}_k(\lambda - \sigma(x) + \sum_{i=0}^k x_i \alpha_i), \text{ for every } k \geq 0\}.$$

We remark that  $x \in \Gamma$  is in  $\Gamma(\lambda)$  if and only if for all  $k \geq 0$ ,

$$\tilde{\alpha}_k(\lambda - \sigma(x) + \sum_{i=0}^{k-1} x_i \alpha_i + \frac{1}{2} x_k \alpha_k) \geq 0.$$

Notice the occurrence of the coefficient  $1/2$  here which will explain the correction term in the representation theorem. The same coefficient already occurs in the crystal  $B(\lambda)$  of  $A_1^{(1)}$  given by Nakashima [31] which is the discrete analogue of  $\Gamma(\lambda)$ . When  $0 < \mu < 1$ , the Verma string parameters  $\xi(\infty)$  of  $B^{(\mu)}$  are a.s. in  $\Gamma$ , and

$$\sigma(\xi(\infty)) = L^{(\mu)}(\infty).$$

In a sense  $\Gamma$  will have the role of a continuous Verma crystal and  $\sigma(x)$  has a role of weight of  $x \in \Gamma$ . In the same way  $\Gamma(\lambda)$  will be a kind of continuous crystal of highest weight  $\lambda$ .

**2.4. Some technical results.** In this subsection we prove some results which will be used to conclude the proof of our main theorem. We take here  $\mu \in (0, 1)$ . Notice that for  $m$  large enough,  $(\frac{m}{\pi}, \mu) \in C_m$  so that the Verma string parameters of  $W^{(\frac{m}{\pi}, \mu)}$  are finite. Let, for  $0 \leq k \leq m$ , with the convention that  $\xi_m^m(\infty) = 0$ ,

$$M_k^m = \frac{1}{2} \xi_k^m(\infty) v_k^m + \sum_{n=0}^{k-1} \xi_n^m(\infty) v_n^m.$$

**Proposition 2.21.** *For any  $k \geq 0$ ,  $\tau_m M_k^m$  converges to  $M_k$  a.s. when  $m$  goes to infinity.*

*Proof.* This follows from Lemma 2.14 and Proposition 2.15.  $\square$

**Proposition 2.22.** *One has, in probability,*

$$(2.13) \quad \lim_{m \rightarrow +\infty} \sup_{1 \leq k \leq m} \|\tau_m M_k^m - M_k\| = 0.$$

*Proof.* We use Theorem 2.9. For  $1 \leq 2p+1 \leq m$ ,

$$M_{2p+2}^m - M_{2p}^m = \frac{1}{2}\xi_{2p+2}^m(\infty)v_{2p+2}^m + \xi_{2p+1}^m(\infty)v_{2p+1}^m + \frac{1}{2}\xi_{2p}^m(\infty)v_{2p}^m.$$

Therefore,

$$\begin{aligned} \tilde{v}_0^m(M_{2p+2}^m - M_{2p}^m) &= \xi_{2p+2}^m(\infty) - 2\cos\left(\frac{\pi}{m}\right)\xi_{2p+1}^m(\infty) + \xi_{2p}^m(\infty) \\ &= -2\cos\left(\frac{\pi}{m}\right)a_{2p+1}^m \frac{\varepsilon_{2p+1}}{\gamma_1^m a_1^m + \gamma_0^m a_2^m + \cdots + \gamma_1^m a_{2p+1}^m} \\ &\quad + a_{2p}^m \left( \frac{\varepsilon_{2p+1}}{\gamma_1^m a_1^m + \cdots + \gamma_1^m a_{2p+1}^m} + \frac{\varepsilon_{2p}}{\gamma_1^m a_1^m + \cdots + \gamma_0^m a_{2p}^m} \right), \end{aligned}$$

since

$$a_{2p+2}^m - 2\cos\left(\frac{\pi}{m}\right)a_{2p+1}^m + a_{2p}^m = 0.$$

One deduces that

$$\tilde{v}_0^m(M_{2p}^m) = 2\sin\frac{\pi}{m} \sum_{l=0}^{p-1} \left( \frac{\varepsilon_{2l} \cos \frac{l\pi}{m}}{\gamma_1^m \sin \frac{l}{m}\pi + \gamma_0^m \sin \frac{l+1}{m}\pi} - \frac{\varepsilon_{2l+1} \cos \frac{(l+1)\pi}{m}}{\gamma_1^m \sin \frac{l+1}{m}\pi + \gamma_0^m \sin \frac{l}{m}\pi} \right),$$

by using the relations

$$\sum_{k=1}^n a_{2k}^m = \frac{\sin \frac{n\pi}{m} \sin \frac{(n+1)\pi}{m}}{\sin \frac{\pi}{m}}, \quad \sum_{k=1}^n a_{2k-1}^m = \frac{\sin^2 \frac{n\pi}{m}}{\sin \frac{\pi}{m}}, \quad \sum_{k=1}^n a_k^m = \frac{\sin \frac{n\pi}{2m} \sin \frac{(n+1)\pi}{2m}}{\sin \frac{\pi}{2m}}.$$

Similarly, one finds

$$\begin{aligned} \tilde{v}_1^m(M_{2p}^m) &= -2\cos\frac{\pi}{m}\xi_0^m(\infty) + \sin\frac{\pi}{m}\tan\frac{p\pi}{m}\xi_{2p}^m(\infty) \\ &\quad + 2\sin\frac{\pi}{m} \sum_{l=1}^{p-1} \left( \frac{\varepsilon_{2l+1} \cos \frac{l\pi}{m}}{\gamma_1^m \sin \frac{l+1}{m}\pi + \gamma_0^m \sin \frac{l}{m}\pi} - \frac{\varepsilon_{2l} \cos \frac{(l+1)\pi}{m}}{\gamma_1^m \sin \frac{l}{m}\pi + \gamma_0^m \sin \frac{l+1}{m}\pi} \right). \end{aligned}$$

On the other hand, one has

$$M_{2p+1}^m - M_{2p}^m = \frac{1}{2}(\xi_{2p}^m(\infty)v_0^m + \xi_{2p+1}^m(\infty)v_1^m).$$

Thus the proposition follows from the next lemma.  $\square$

**Lemma 2.23.** *Let  $\gamma_0^m, \gamma_1^m, m \in \mathbb{N}$ , be real numbers such that  $\gamma_0^m$  tends to  $2(1-\mu)$  and  $\gamma_1^m$  tends to  $2\mu$ . Let for  $p \in \{0, \dots, [m/2]\}$ ,*

$$\begin{aligned} S_p^m &= \sin\frac{\pi}{m} \sum_{l=0}^{p-1} \left( \frac{\varepsilon_{2l} \cos \frac{l\pi}{m}}{\gamma_1^m \sin \frac{l}{m}\pi + \gamma_0^m \sin \frac{l+1}{m}\pi} - \frac{\varepsilon_{2l+1} \cos \frac{(l+1)\pi}{m}}{\gamma_1^m \sin \frac{l+1}{m}\pi + \gamma_0^m \sin \frac{l}{m}\pi} \right), \\ S_p &= \sum_{l=0}^{p-1} \left( \frac{\varepsilon_{2l}}{2l + 2(1-\mu)} - \frac{\varepsilon_{2l+1}}{2l + 2\mu} \right). \end{aligned}$$

Then in probability

$$\lim_{m \rightarrow \infty} \sup_{1 \leq p \leq [m/2]} |S_p^m - S_p| = 0.$$



*Proof.* For each  $m \in \mathbb{N}$ ,  $\{S_p^m - S_p - \mathbb{E}(S_p^m - S_p), p = 1, \dots, [m/2]\}$  is a martingale. Let  $\varepsilon > 0$ . We have

$$\begin{aligned} \text{Var}(S_{[m/2]}^m - S_{[m/2]}) &= \sum_{l=0}^{[m/2]-1} \left( \frac{\sin \pi/m \cos l\pi/m}{\gamma_1^m \sin \frac{l}{m}\pi + \gamma_0^m \sin \frac{l+1}{m}\pi} - \frac{1}{2l + 2(1-\mu)} \right)^2 \\ &\quad + \left( \frac{\sin \pi/m \cos(l+1)\pi/m}{\gamma_1^m \sin \frac{l+1}{m}\pi + \gamma_0^m \sin \frac{l}{m}\pi} - \frac{1}{2l + 2\mu} \right)^2. \end{aligned}$$

As  $\frac{2x}{\pi} \leq \sin x \leq x$  for  $0 \leq x \leq \frac{\pi}{2}$ , one has the majorations

$$\frac{\sin \frac{\pi}{m}}{\gamma_1^m \sin \frac{l}{m}\pi + \gamma_0^m \sin \frac{l+1}{m}\pi} \leq \frac{\pi/2}{\gamma_1^m l + \gamma_0^m (l+1)},$$

and

$$\frac{\sin \frac{\pi}{m}}{\gamma_1^m \sin \frac{l+1}{m}\pi + \gamma_0^m \sin \frac{l}{m}\pi} \leq \frac{\pi/2}{\gamma_1^m (l+1) + \gamma_0^m l},$$

for  $0 \leq l \leq [m/2] - 1$ . Thus there is a  $p_0 \in \mathbb{N}$  such that for all  $m \geq 2(p_0 + 1)$ , the sum on the right hand side of the variance above is smaller than  $\varepsilon$  for  $l \geq p_0$ . Since each term tends to 0, we obtain that for  $m$  large enough this sum is smaller than  $\varepsilon$ . Thus we have by Doob's martingale inequality

$$\mathbb{E} \left( \sup_{1 \leq p \leq [m/2]} (S_p^m - S_p - \mathbb{E}(S_p^m - S_p))^2 \right) \leq 4\varepsilon.$$

Hence  $\sup_{1 \leq p \leq [m/2]} |S_p^m - S_p - \mathbb{E}(S_p^m - S_p)|$  converges in probability to 0. Besides one has for  $p, q \in \{0, \dots, [m/2]\}$ ,  $p \geq q$ ,

$$|\mathbb{E}(S_p^m - S_q^m)| \leq \sum_{l=q}^{p-1} \frac{\pi^2(\gamma_0^m + \gamma_1^m)}{4(\gamma_1^m l + \gamma_0^m (l+1))(\gamma_1^m (l+1) + \gamma_0^m l)}$$

and

$$\mathbb{E}(S_p - S_q) = \sum_{l=q}^{p-1} \frac{2(2\mu - 1)}{(2l + 2(1-\mu))(2l + 2\mu)}.$$

As for a fixed  $k$ , when  $m$  goes to infinity,  $\mathbb{E}(S_k^m)$  converges towards  $\mathbb{E}(S_k)$  when  $m$  goes to infinity, one obtains that  $\sup_{0 \leq p \leq [m/2]} |\mathbb{E}(S_p^m - S_p)|$  converges towards 0, which finishes the proof of the lemma.  $\square$

Recall that  $\xi^m(\infty)$ , resp.  $\xi(\infty)$ , are the Verma string parameters of  $W^{(m/\pi, \mu)}$ , resp.  $B^{(\mu)}$ . The following proposition will allow us to pass from Verma affine string parameters to affine string parameters in the next subsection.

**Proposition 2.24.** *Let  $(\lambda_m)$  be a sequence of  $\mathbb{R}^2$  such that  $\lambda_m \in C_m$  and such that  $\tau_m \lambda_m$  tends to  $\lambda$  when  $m$  tends to  $\infty$  where  $\lambda \in C_{\text{aff}}$ . The random sets  $\{\xi^m(\infty) \in \Gamma_m(\lambda_m)\}$  converge in probability to  $\{\xi(\infty) \in \Gamma(\lambda)\}$ .*

*Proof.* To prove convergence in probability of a sequence, it is enough to show that each subsequence has a subsequence which converges almost surely. Therefore, by Propositions 2.22 and 2.18, working with a subsequence, we can suppose that the set of  $\omega \in \Omega$  for which

$$\sup_{1 \leq k \leq m} \|\tau_m M_k^m(\omega) - M_k(\omega)\| \rightarrow 0,$$

and  $M_k(\omega)$  tends to  $L^{(\mu)}(\infty)(\omega)$  is of probability one. So we see that if,

$$X_k^m = M_m^m - M_k^m, \quad X_k = L^{(\mu)}(\infty) - M_k.$$

then (see (2.10))

$$(2.14) \quad \sup_{0 \leq k \leq m} |\tilde{v}_k^m(X_k^m(\omega)) - \tilde{\alpha}_k(X_k(\omega))| \rightarrow 0, \quad \tilde{\alpha}_k(X_k(\omega)) \rightarrow 0.$$

For these  $\omega$  we will show that

$$\limsup_{m \rightarrow \infty} \{\xi^m(\infty) \in \Gamma_m(\lambda_m)\} \subset \{\xi(\infty) \in \Gamma(\lambda)\} \subset \liminf_{m \rightarrow \infty} \{\xi^m(\infty) \in \Gamma_m(\lambda_m)\}.$$

Notice that  $\tilde{v}_k^m(\lambda_m)$  tends to  $\tilde{\alpha}_k(\lambda)$  for  $k = 0, 1$ . One has  $\xi^m(\infty) \in \Gamma_m(\lambda_m)$  if and only if  $\tilde{v}_k^m(\lambda_m - X_k^m) \geq 0$  for  $0 \leq k < m$ , and  $\xi(\infty) \in \Gamma(\lambda)$  if and only if  $\tilde{\alpha}_k(\lambda - X_k) \geq 0$  for every  $k \in \mathbb{N}$ . The left inclusion above follows from (2.14). Now, suppose that  $\xi(\infty)(\omega) \in \Gamma(\lambda)$ . Since  $\lambda$  is fixed and the distributions of  $X_k$  are continuous, for any  $k \geq 0$ ,

$$\mathbb{P}(\tilde{\alpha}_k(\lambda - X_k) = 0) = 0,$$

and one can suppose that

$$\forall k \geq 0, \quad \tilde{\alpha}_k(\lambda - X_k(\omega)) > 0.$$

We choose  $\varepsilon > 0$  such that  $\tilde{\alpha}_i(\lambda) > \varepsilon$ ,  $i \in \{0, 1\}$ . Using (2.14), one can choose  $m_0$  such that  $m \geq m_0$  implies that

$$\sup_{0 \leq k \leq m} \tilde{v}_k^m(X_k^m(\omega)) - \tilde{\alpha}_k(X_k(\omega)) < \varepsilon,$$

and then we choose  $k_0$  such that  $k \geq k_0$  implies that

$$\tilde{\alpha}_k(X_k(\omega)) < \tilde{\alpha}_k(\lambda) - \varepsilon.$$

As for each  $k$ ,  $\tilde{v}_k^m(X_k^m)$  converges towards  $\tilde{\alpha}_k(X_k)$  when  $m$  goes to infinity, one takes  $m_1$  such that when  $m \geq m_1$ ,

$$\tilde{v}_k^m(X_k^m(\omega)) < \tilde{\alpha}_k(\lambda),$$

for  $k = 1, \dots, k_0$ . Then for  $m \geq m_0, m_1$  one has for  $k \geq k_0$

$$\tilde{v}_k^m(X_k^m(\omega)) = \tilde{v}_k^m(X_k^m(\omega)) - \tilde{\alpha}_k(X_k(\omega)) + \tilde{\alpha}_k(X_k(\omega)) < \tilde{\alpha}_k(\lambda),$$

and for  $k \leq k_0$ ,

$$\tilde{v}_k^m(X_k^m(\omega)) < \tilde{\alpha}_k(\lambda),$$

which proves the right inclusion above, and shows the proposition.  $\square$

**2.5. The highest weight process  $\Lambda^{(\mu)}$ .** In this subsection we introduce the highest process associated with the space-time Brownian motion  $B^{(\mu)}$  which will appear as the limit of its Pitman's transforms (with a correction) and show that it coincides in law with  $B^{(\mu)}$  conditioned to remain in  $C_{\text{aff}}$  forever. Let  $0 \leq \mu \leq 1$ . We define for  $k \geq 0$ ,

$$M_k(t) = \frac{1}{2} \xi_k(t) \alpha_k + \sum_{n=0}^{k-1} \xi_n(t) \alpha_n,$$

where  $\xi(t) = \{\xi_n(t), n \geq 0\}$  are the affine string parameters of  $B^{(\mu)}$  on  $[0, t]$  and

$$M_k^m(t) = \frac{1}{2} \xi_k^m(t) v_k^m + \sum_{n=0}^{k-1} \xi_n^m(t) v_n^m,$$

where  $\xi^m(t) = \{\xi_n^m(t), 0 \leq n < m\}$  are the string parameters of  $W^{(m/\pi, \mu)}$  on  $[0, t]$  (by convention  $\xi_n^m(t) = 0$  when  $n \geq m$ ). In particular,  $M_m^m(t) = \sum_{n=0}^{m-1} \xi_n^m(t) v_n^m$ .

**Proposition 2.25.** *For every  $k \geq 1$  and  $t \geq 0$ ,  $\tau_m M_k^m(t)$  converges to  $M_k(t)$  almost surely when  $m$  goes to infinity.*

*Proof.* By the Cameron-Martin theorem, it is enough to prove the proposition for  $\mu \in (0, 1)$ . In that case, it follows from Proposition 2.15 and Lemma 2.14.  $\square$

**Theorem 2.26.** *When  $\mu \in [0, 1]$ , for each  $t \geq 0$ ,  $M_k(t)$  converges almost surely when  $k$  goes to infinity. We denote by  $L^{(\mu)}(t)$  the limit.*

*Proof.* By the Cameron-Martin theorem, one can suppose that  $\mu \in (0, 1)$ . Let  $\varepsilon > 0$ . By Lemma 2.14 and Proposition 2.15,  $\tau_m M_p^m(t)$  converges a.s. to  $M_p(t)$  as  $m$  tends to  $\infty$ . Hence, for  $p, q \in \mathbb{N}$ ,

$$\begin{aligned} \mathbb{P}(\|M_{p+q}(t) - M_p(t)\| \geq \varepsilon) &= \lim_{m \rightarrow \infty} \mathbb{P}(\|\tau_m M_{p+q}^m(t) - \tau_m M_p^m(t)\| \geq \varepsilon) \\ &= \lim_{m \rightarrow \infty} \mathbb{E}(f_m(\Lambda_m(t))) \end{aligned}$$

where

$$f_m(\Lambda_m(t)) = \mathbb{E}(1_{\|\tau_m M_{p+q}^m(t) - \tau_m M_p^m(t)\| \geq \varepsilon} | \Lambda_m(t)).$$

One has, by Theorem 2.9,

$$f_m(\lambda) = \mathbb{P}(\|\tau_m M_{p+q}^m - \tau_m M_p^m\| \geq \varepsilon | \xi^m(\infty) \in \Gamma_m(\lambda)).$$

If  $\lambda \in C_{\text{aff}}$  and  $\tau_m \lambda_m$  tends to  $\lambda$ , then by Proposition 2.24,  $f_m(\lambda_m)$  tends to  $f(\lambda)$  where

$$f(\lambda) = \mathbb{P}(\|M_{p+q} - M_p\| \geq \varepsilon | \xi(\infty) \in \Gamma(\lambda)).$$

On the other hand we will show in Theorem 2.45, independently of this proof, that when  $m$  tends to infinity,  $\tau_m \Lambda_m(t)$  converges in law to  $A_t^{(\mu)}$ . Therefore  $\mathbb{E}(f_m(\Lambda_m(t)))$  tends to  $\mathbb{E}(f(A_t^{(\mu)}))$ . This shows that

$$\mathbb{P}(\|M_{p+q}(t) - M_p(t)\| \geq \varepsilon) = \int_{\Omega} \mathbb{P}(\|M_{p+q} - M_p\| \geq \varepsilon | \xi(\infty) \in \Gamma(A_t^{(\mu)}(\omega))) d\mathbb{P}(\omega).$$

Since  $M_k$  converges a.s. to  $L^{(\mu)}(\infty)$  (Proposition 2.18) we see that  $M_p(t), p \in \mathbb{N}$ , is a Cauchy sequence for the convergence in probability, and thus converges in probability. We will prove the almost sure convergence at the end of this subsection.  $\square$

Notice that the first component of  $L^{(\mu)}(t)$  is 0, so one can write

$$L^{(\mu)}(t) = (0, D^\mu(t)),$$

where  $D^\mu(t) \in \mathbb{R}$ .

**Proposition 2.27.** *When  $0 < \mu < 1$  there is an almost surely finite  $B^{(\mu)}$ -stopping time  $\sigma$  such that, for  $t \geq \sigma$ ,  $L^{(\mu)}(t) = L^{(\mu)}(\infty)$ .*

*Proof.* Let  $\sigma_0 = \max\{t \geq 0, B_t^{(\mu)} \notin C_{\text{aff}}\}$ , and

$$\sigma_{n+1} = \max\{t \geq \sigma_n, B_t^{(\mu)} + \sum_{k=0}^n \xi_k(t) \alpha_k \notin C_{\text{aff}}\}$$

for  $n \geq 0$ . Then, a.s.,  $\sigma_0 < +\infty$  and for  $t > \sigma_0$ ,  $\xi_0(t) = \xi_0(\infty)$ , hence

$$\sigma_1 = \max\{t \geq \sigma_0, B_t^{(\mu)} + \xi_0(\infty)\alpha_0 \notin C_{\text{aff}}\}.$$

Recursively, one has  $\sigma_n < +\infty$  and

$$\sigma_{n+1} = \max\{t \geq \sigma_n, B_t^{(\mu)} + \sum_{k=0}^n \xi_k(\infty)\alpha_k \notin C_{\text{aff}}\}.$$

We know that  $\sum_{k=0}^n \xi_k(\infty)\alpha_k$  is bounded, a.s., therefore  $\sigma = \sup \sigma_n < +\infty$  and for  $t > \sigma$ ,  $\xi_k(t) = \xi_k(\infty)$  for all  $k \geq 0$ . So, for  $t \geq \sigma$ ,  $L^{(\mu)}(t) = L^{(\mu)}(\infty)$ .  $\square$

**Proposition 2.28.** *In probability, as  $m$  tends to  $\infty$ ,  $\tau_m M_m^m(t)$  converges to  $L^{(\mu)}(t)$ .*

*Proof.* By Cameron-Martin's theorem, it is enough to prove the proposition for  $\mu \in (0, 1)$ . In that case, one has for  $\epsilon > 0$ ,

$$\begin{aligned} \mathbb{P}(|\tau_m M_m^m(t) - L^{(\mu)}(t)| > \epsilon) &\leq \mathbb{P}\left(\sup_{1 \leq k \leq m} |\tau_m(M_m^m(t) - M_k^m(t))| > \frac{\epsilon}{3}\right) \\ &\quad + \mathbb{P}(|\tau_m M_p^m(t) - M_p(t)| > \frac{\epsilon}{3}) + \mathbb{P}(|M_p(t) - L^{(\mu)}(t)| > \frac{\epsilon}{3}). \end{aligned}$$

For the first term, we condition by  $\Lambda_m(t)$  as in the proof of Theorem 2.26 and use covergences (2.14) which imply that  $\sup_{0 \leq k \leq m} |\tau_m(M_m^m - M_k^m)| \xrightarrow{m \rightarrow \infty} 0$  in probability, and proposition 2.22. For the second one we use proposition 2.25 and Theorem 2.26 for the third one.  $\square$

Notice that

$$(2.15) \quad \Lambda_m(t) = \mathcal{P}_{w_0}^m W^{(m/\pi, \mu)}(t) = W_t^{(m/\pi, \mu)} + M_m^m(t).$$

We introduce the following definition,

**Definition 2.29.** *For  $0 \leq \mu \leq 1$ , we define the (affine) highest weight process of  $\{B_t^{(\mu)}, t \geq 0\}$  by, for  $t \geq 0$ ,*

$$\Lambda^{(\mu)}(t) = B_t^{(\mu)} + L^{(\mu)}(t).$$

In the analogy with the Littelmann model, for  $t > 0$  fixed,  $\{B_s^{(\mu)}, 0 \leq s \leq t\}$  is a path with weight  $B_t^{(\mu)}$ , with highest weight  $\Lambda^{(\mu)}(t)$ , and  $L^{(\mu)}(t)$  is the weight seen from the highest weight.

**Proposition 2.30.** *In probability,*

$$\begin{aligned} \lim_{m \rightarrow +\infty} \tau_m W_t^{(m/\pi, \mu)} &= B_t^{(\mu)}, \\ \lim_{m \rightarrow +\infty} \tau_m \Lambda_m(t) &= \Lambda^{(\mu)}(t). \end{aligned}$$

*Proof.* The first statement is obvious. The second is then a consequence of Proposition 2.28 and (2.15).  $\square$

Let  $\{A_t^{(\mu)}, t \geq 0\}$  be the conditioned space-time Brownian motion in the affine Weyl chamber  $C_{\text{aff}}$ , with drift  $\mu$ , starting from the origin defined in 5.3.

**Theorem 2.31.** *For any  $\mu \in [0, 1]$ , in law,*

$$\{\Lambda^{(\mu)}(t), t \geq 0\} = \{A_t^{(\mu)}, t \geq 0\}.$$

*Proof.* This follows from Proposition 2.30, and Theorem 2.45.  $\square$

This shows in particular that  $\Lambda^{(\mu)}(t), t \geq 0$ , has a continuous version. In order to prove the almost sure convergence in Theorem 2.26, let us show:

**Proposition 2.32.** *For  $0 < \mu < 1$  and  $f : \mathbb{R}^2 \times \Gamma \rightarrow \mathbb{R}$ , bounded and measurable,*

$$\mathbb{E}(f(B_t^{(\mu)}, \xi(t)) | \sigma(\Lambda^{(\mu)}(s), s \leq t)) = g(\Lambda^{(\mu)}(t))$$

for each  $t > 0$ , where

$$g(\lambda) = \mathbb{E}(f(\lambda - L^{(\mu)}(\infty), \xi(\infty)) 1_{\{\xi(\infty) \in \Gamma(\lambda)\}}) / \mathbb{P}(\xi(\infty) \in \Gamma(\lambda)).$$

*Proof.* By Theorem 2.9, the conditional distribution of  $\xi^m(t)$  knowing the sigma-algebra  $\sigma(\Lambda_m(s), s \leq t)$  is the one of  $\xi^m(\infty)$  knowing  $\{\xi^m(\infty) \in \Gamma_m(\Lambda_m(t))\}$ . Hence

$$\mathbb{E}(f(\tau_m \Lambda_m(t) - \tau_m M_m^m(t), \xi^m(t)) | \sigma(\Lambda_m(s), s \leq t)) = g_m(\Lambda_m(t))$$

where

$$g_m(\lambda) = \mathbb{E}(f(\tau_m \lambda - \tau_m M_m^m, \xi^m) 1_{\{\xi^m(\infty) \in \Gamma_m(\lambda)\}}) / \mathbb{P}(\xi^m(\infty) \in \Gamma_m(\lambda)).$$

Using the convergences in probability of the propositions 2.28 and 2.30, if  $h$  is bounded, measurable, and depends only on a finite number of variables,

$$\begin{aligned} & \mathbb{E}(f(\Lambda^{(\mu)}(t) - L^{(\mu)}(t), \xi(t)) h(\Lambda^{(\mu)}(s), s \leq t)) \\ &= \lim_{m \rightarrow +\infty} \mathbb{E}(f(\tau_m \Lambda_m(t) - \tau_m M_m^m(t), \xi^m(t)) h(\tau_m \Lambda_m(s), s \leq t)) \\ &= \lim_{m \rightarrow +\infty} \mathbb{E}(g_m(\Lambda_m(t)) h(\tau_m \Lambda_m(s), s \leq t)). \end{aligned}$$

Since  $\tau_m \Lambda_m(t)$  tends to  $\Lambda^{(\mu)}(t)$ , we know from Proposition 2.24 that, in probability,  $\{\xi^m(\infty) \in \Gamma_m(\Lambda_m(t))\}$  tends to  $\{\xi(\infty) \in \Gamma(\Lambda^{(\mu)}(t))\}$ . Therefore the limit above is equal to

$$\mathbb{E}(g(\Lambda^{(\mu)}(t)) h(\Lambda^{(\mu)}(s), s \leq t))$$

which proves the proposition.  $\square$

**Corollary 2.33.** *Almost surely,  $\xi(\infty)$  is in  $\Gamma$ , and when the highest weight  $\Lambda^{(\mu)}(t)$  is equal to  $\lambda$ ,  $\xi(t) \in \Gamma(\lambda)$ .*

*End of the proof of Theorem 2.26.* By the Cameron-Martin theorem it is enough to prove the proposition for  $\mu \in (0, 1)$ . In that case, the theorem follows from

$$\mathbb{P}(\lim M_k(t) = L^{(\mu)}(t)) = \mathbb{E}(\mathbb{E}(1_{\{\lim M_k(t) = L^{(\mu)}(t)\}} | \sigma(\Lambda^{(\mu)}(t)))) = 1$$

since  $M_k$  tends to  $L^{(\mu)}(\infty)$  a.s., by Proposition 2.32 which implies that

$$\mathbb{E}(1_{\{\lim M_k(t) = L^{(\mu)}(t)\}} | \sigma(\Lambda^{(\mu)}(t))) = \mathbb{E}(1_{\{\lim M_k = L^{(\mu)}(\infty)\}} | \xi(\infty) \in \Gamma(\Lambda^{(\mu)}(t))) = 1.$$

$\square$

**2.6. Representation using Pitman and Lévy's transforms.** Let us remind where we stand. For  $B_t^{(\mu)} = (t, B_t + t\mu)$  we have written, see (2.8),

$$\mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_1} \mathcal{P}_{\alpha_0} B^{(\mu)}(t) = B_t^{(\mu)} + \sum_{i=0}^n \xi_i(t) \alpha_i,$$

and

$$M_{n+1}(t) = \frac{1}{2} \xi_{n+1}(t) \alpha_{n+1} + \sum_{i=0}^n \xi_i(t) \alpha_i.$$

We have seen that when  $\mu \in [0, 1]$ , for  $t > 0$ ,  $\lim_{k \rightarrow +\infty} M_k(t) = L^{(\mu)}(t)$  a.s. and that the process  $\Lambda^{(\mu)}(t) = B_t^{(\mu)} + L^{(\mu)}(t)$ ,  $t \geq 0$ , has the same distribution as the process  $A_t^{(\mu)}$ ,  $t \geq 0$ . Hence,

**Theorem 2.34.** *When  $0 \leq \mu \leq 1$ , for  $t \geq 0$ , almost surely,*

$$\lim_{n \rightarrow +\infty} \mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_1} \mathcal{P}_{\alpha_0} B^{(\mu)}(t) + \frac{1}{2} \xi_{n+1}(t) \alpha_{n+1}$$

*exists, and the limiting process has the same distribution as  $\{A^{(\mu)}(t), t \geq 0\}$ .*

To interpret the correction term, it is worthwhile to introduce the Lévy's transform (sometimes called Skorokhod's transform). Lévy's theorem (see Revuz and Yor [35], VI.2) states that if  $\beta$  is the standard Brownian motion, then

$$\mathcal{L}\beta(t) = \beta_t - \inf_{0 \leq s \leq t} \beta_s$$

has the same law as  $|\beta_t|$  and that  $-\inf_{0 \leq s \leq t} \beta_s$  is the local time of  $\mathcal{L}\beta$  at 0. We introduce here the following Lévy's transform (sometimes the Lévy's transform of  $\beta$  is defined as  $\int_0^t \text{sign}(\beta_s) d\beta_s$ , this is related to our transform, but different).

**Definition 2.35.** *For  $\eta \in \mathcal{C}_0(\mathbb{R}^2)$  and  $i = 0, 1$ , the Lévy transform  $\mathcal{L}_{\alpha_i} \eta$  of  $\eta$  is*

$$\mathcal{L}_{\alpha_i} \eta(t) = \eta(t) - \frac{1}{2} \inf_{0 \leq s \leq t} \bar{\alpha}_i(\eta(s)) \alpha_i.$$

Another way to state the former theorem is

**Theorem 2.34 bis.** *For  $t \geq 0$ ,*

$$\lim_{n \rightarrow +\infty} \mathcal{L}_{\alpha_{n+1}} \mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_1} \mathcal{P}_{\alpha_0} B^{(\mu)}(t)$$

*exists a.s. and the limiting process has the same distribution as  $A^{(\mu)}$ .*

The following proposition indicates that the presence of this Lévy transform is due to the bad behavior of  $A_t^{(\mu)}$  for  $t$  near the origin.

**Proposition 2.36.** *For all  $t > 0$ , a.s.*

$$\lim_{k \rightarrow +\infty} \xi_k(t) = 2.$$

*Proof.* Since  $\xi_k(\infty)$  tends to 2 almost surely (Theorem 2.17), this follows from Proposition 2.32 by conditioning by  $\sigma(\Lambda_t^{(\mu)})$ .  $\square$

It implies that, for  $t > 0$ ,

$$\lim_{n \rightarrow +\infty} \mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_1} \mathcal{P}_{\alpha_0} B^{(\mu)}(t) + (-1)^n \alpha_1 = A_t^{(\mu)}.$$

So without correction the iterates of Pitman's transform do not converge.

In the whole paper we have chosen, in the iterations of Pitman's transforms, to begin by first applying  $\mathcal{P}_{\alpha_0}$  to  $B^{(\mu)}$ . Let us show the non trivial fact that we obtain the same limits if we begin with  $\mathcal{P}_{\alpha_1}$ . More precisely, if we denote with a tilde the quantities previously defined when we begin by  $\mathcal{P}_{\alpha_1}$  rather than by  $\mathcal{P}_{\alpha_0}$ , one has, for each  $\mu \in [0, 1]$ ,

**Theorem 2.37.** (1). *Almost surely,*

$$\lim_{n \rightarrow +\infty} \mathcal{L}_{\alpha_{n+1}} \mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_1} \mathcal{P}_{\alpha_0} B^{(\mu)}(t) = \lim_{n \rightarrow +\infty} \mathcal{L}_{\alpha_{n+1}} \mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_2} \mathcal{P}_{\alpha_1} B^{(\mu)}(t).$$

(2)  $\tilde{\xi}(t)$  defined for  $B^{(\mu)}$  has the same law as  $\xi(t)$  defined for  $B^{(1-\mu)}$ .

*Proof.* For (1) we have to prove that  $L^{(\mu)}(t) = \tilde{L}^{(\mu)}(t)$ . We have seen in Proposition 2.28 that  $\tau_m M_m^m(t)$  converges in probability to  $L^{(\mu)}(t)$ . By the same proof  $\tau_m \tilde{M}_m^m(t)$  converges in probability to  $\tilde{L}^{(\mu)}(t)$ . Now we use the equality  $M_m^m(t) = \tilde{M}_m^m(t)$  which follows from Theorem 2.3 and (2.1). Thus  $L^{(\mu)}(t) = \tilde{L}^{(\mu)}(t)$  a.s..

(2) follows from the fact that  $Id - B^\mu$  has the same law as  $B^{1-\mu}$ .  $\square$

**2.7. The law of  $B_t^{(\mu)}$  conditionally to  $\{\Lambda^{(\mu)}(s), s \leq t\}$ .** We will compute the conditional law of  $B_t^{(\mu)}$  by approaching it by the dihedral case. The alternating polynomial associated to the dihedral group  $I(m)$  is given by (see Dunkl and Xu [14], 6.2.3), for  $v = (x, y) \in \mathbb{R}^2$ ,

$$h_m(v) = h_m(x, y) = \Im((x + iy)^m).$$

It is equal to the product of roots for the root system of  $I(m)$ . For  $w \in I(m)$  let  $l(w)$  be the length of its shortest expression with  $s_0^m$  and  $s_1^m$ . Let, for  $v \in \mathbb{R}^2, \gamma \in \bar{C}_m$ ,

$$(2.16) \quad \psi_v^m(\gamma) = \sum_{w \in I(m)} (-1)^{l(w)} e^{\langle w(\gamma) - \gamma, v \rangle},$$

and let  $W^{(\gamma)}$  be the standard planar Brownian motion in  $\mathbb{R}^2$  with drift  $\gamma$ .

**Lemma 2.38.** For  $\zeta \in \mathbb{R}^2$  and  $v = \mathcal{P}_{w_0}^m W^{(\gamma)}(t)$ ,

$$\mathbb{E}(e^{\langle \zeta, W_t^{(\gamma)} \rangle} | \sigma(\mathcal{P}_{w_0}^m W^{(\gamma)}(s), 0 \leq s \leq t)) = \frac{\psi_v^m(\zeta + \gamma)}{\psi_v^m(\gamma)} \frac{h_m(\gamma)}{h_m(\zeta + \gamma)} e^{\langle \zeta, v \rangle}.$$

*Proof.* Theorem 5.5 in [4] gives, for  $\mathcal{F}_t = \sigma(\mathcal{P}_{w_0}^m W^{(0)}(s), 0 \leq s \leq t)$ ,

$$\mathbb{E}(e^{\langle \zeta, W_t^{(0)} \rangle} | \mathcal{F}_t) = k \frac{\psi_v^m(\zeta)}{h_m(v) h_m(\zeta)} e^{\langle \zeta, v \rangle},$$

for  $v = \mathcal{P}_{w_0}^m W^{(0)}(t)$  and a constant  $k$  independent of  $v$  and  $\zeta$ . We conclude with the Bayes formula (2.6).  $\square$

For  $\alpha \in \mathbb{R}$ , we define a theta function  $\varphi_\alpha$  on  $\mathbb{R}_+^* \times \mathbb{R}$  first, when  $\alpha \notin \mathbb{Z}$ , by

$$(2.17) \quad \varphi_\alpha(t, x) = \frac{e^{-\alpha x}}{\sin(\alpha\pi)} \sum_{k \in \mathbb{Z}} \sinh(\alpha(2kt + x)) e^{-2(kx + k^2t)},$$

for  $t > 0, x \in \mathbb{R}$ , and then by continuity for all  $\alpha \in \mathbb{R}$  using Lemma 2.41 which implies that  $\varphi_\alpha(t, x) = \varphi_{1-\alpha}(t, t-x)$ . Its importance for us is in particular due to its harmonicity (see Proposition 5.4).

**Lemma 2.39.** *Let  $v_m \in \mathbb{R}^2, \gamma_m \in \bar{C}_m$  such that, as  $m$  tends to  $+\infty$ ,  $\tau_m(\gamma_m)$  tends to  $(1, \alpha)$  and  $\tau_m(v_m)$  tends to  $(t, x)$ , then*

$$(2.18) \quad \lim_{m \rightarrow \infty} \psi_{v_m}^m(\gamma_m) = \frac{\sin(\alpha\pi)}{2} \varphi_\alpha(t, x),$$

$$(2.19) \quad \lim_{m \rightarrow \infty} \left(\frac{\pi}{m}\right)^m h_m(\gamma_m) = \sin(\alpha\pi).$$

*Proof.* Let  $r$  be the rotation of  $\mathbb{R}^2$  of angle  $2\pi/m$  and  $s$  be the symmetry  $s(x, y) = (x, -y)$ ,  $(x, y) \in \mathbb{R}^2$ . Let  $J(m) = \{-[m/2], \dots, [m/2]\}$  when  $m$  is odd and  $J(m) = \{-[m/2] + 1, \dots, [m/2] - 1\} \cup \{m/2\}$  when  $m$  is even. The dihedral group  $I(m)$  is  $I(m) = \{r^k, k \in J(m)\} \rtimes \{Id, s\}$ , and  $l(r) = 2, l(s) = 1$ . Therefore

$$\psi_{v_m}^m(\gamma_m) = e^{-\langle \gamma_m, v_m \rangle} \sum_{k \in J(m)} (e^{\langle r^k(\gamma_m), v_m \rangle} - e^{\langle r^k s(\gamma_m), v_m \rangle}) = 2e^{-\gamma_m^2 v_m^2} \sum_{k \in J(m)} I(m, k)$$

where

$$I(m, k) = e^{v_m^1 \gamma_m^1 (\cos \frac{2k\pi}{m} - 1) + \gamma_m^1 v_m^2 \sin \frac{2k\pi}{m}} \sinh(\gamma_m^2 (-v_m^1 \sin \frac{2k\pi}{m} + v_m^2 \cos \frac{2k\pi}{m}))$$

and where we write  $v_m = (v_m^1, v_m^2), \gamma_m = (\gamma_m^1, \gamma_m^2)$  with  $v_m$  equivalent to  $(mt/\pi, x)$  and  $\gamma_m$  equivalent to  $(m/\pi, \alpha)$ . For  $\varepsilon > 0$ , we choose  $m_0 \geq 0$  such that, for  $m \geq m_0$ ,

$$\sum_{m/4 \leq |k| \leq m/2} |I(m, k)| \leq \varepsilon.$$

by using the inequality  $\cos(\frac{2k}{m}\pi) \leq 0$  when  $\frac{m}{4} \leq |k| \leq \frac{m}{2}$ . Besides, when  $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ , then  $\cos t \leq 1 - \frac{t^2}{\pi}$  and  $|\sin(t)| \leq |t|$ , Hence, for  $|k| \leq \frac{m}{4}$ ,

$$|I(m, k)| \leq e^{-c_1 k^2 + c_2 k}$$

where  $c_1, c_2 > 0$  do not depend on  $m$ . So, one can choose  $N$  such that for  $m \geq 1$

$$\sum_{\frac{m}{4} \geq |k| \geq N} |I(m, k)| \leq \varepsilon.$$

Since, for  $N$  fixed,

$$\lim_{m \rightarrow +\infty} \sum_{k=-N}^N I(m, k) = e^{-\alpha x} \sum_{k=-N}^N \sinh(\alpha(2kt + x)) e^{-2(kx + k^2t)},$$

we obtain (2.18) The relation (2.19) is immediate.  $\square$

**Theorem 2.40.** *One has, when  $\zeta = (0, \tau) \in \mathbb{R}^2$ ,*

$$\mathbb{E}(e^{\langle \zeta, B_t^{(\mu)} \rangle} | \sigma(\Lambda^{(\mu)}(s), s \leq t)) = \frac{e^{\langle \zeta, \Lambda^{(\mu)}(t) \rangle} \varphi_{\tau + \mu}(\Lambda^{(\mu)}(t))}{\varphi_\mu(\Lambda^{(\mu)}(t))}.$$



*Proof.* The first formula is obtained by letting  $m$  goes to infinity in the Lemma 2.38 and using Proposition 2.30 and Lemma 2.39.  $\square$

We have used the following lemma which follows from the Poisson summation formula (see Bellman [1]) for the  $2t$ -periodic function  $x \mapsto e^{-\frac{x^2}{2t}} \varphi_\alpha(t, x)$ .

**Lemma 2.41.** *For  $\alpha \in (0, 1)$ ,  $t > 0$ ,  $x \in \mathbb{R}$ ,*

$$\varphi_\alpha(t, x) = \frac{\sqrt{\pi} e^{-\frac{(x-\alpha t)^2}{2t}}}{\sqrt{2t} \sin(\alpha\pi)} \sum_{k \in \mathbb{Z}} \sin(k\pi\alpha) \sin(k\pi x/t) e^{-\frac{k^2 \pi^2}{2t}}.$$

## 2.8. Remarks on the laws of $L^{(\mu)}(\infty)$ and $\xi_1(\infty)$ .

*2.8.1. Law of  $L^{(\mu)}(\infty)$ .* Recall that  $L^{(\mu)}(\infty) = (0, D^\mu(\infty))$ . The Laplace transform of  $D^\mu(\infty)$  is, by (2.11), for  $\tau > 0$ ,

$$\mathbb{E}(e^{-\tau D^\mu(\infty)}) = \prod_{n=0}^{\infty} \left( \left(1 + \frac{\tau}{n+\mu}\right) \left(1 - \frac{\tau}{n+1-\mu}\right) \right)^{-1}.$$

Using

$$(2.20) \quad \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\gamma)\Gamma(\beta-\gamma)} = \prod_{n=0}^{+\infty} \left(1 + \frac{\gamma}{n+\alpha}\right) \left(1 - \frac{\gamma}{n+\beta}\right),$$

(Formula 8.325.1 of [19]) and  $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ , we obtain that

$$(2.21) \quad \mathbb{E}(e^{-\tau D^\mu(\infty)}) = \frac{\sin(\pi\mu)}{\sin(\pi(\mu+\tau))}.$$

In particular,

$$(2.22) \quad \mathbb{E}(D^\mu(\infty)) = \pi \cot(\pi\mu).$$

**Corollary 2.42.** *The density of  $D^{1/2}(\infty)$  is  $1/(\pi \cosh x)$ .*

*Proof.* One uses that the Fourier transform of  $1/(\pi \cosh x)$  is  $1/\cosh(\lambda\pi/2)$ .  $\square$

Notice that the law of  $\sum_{n=1}^{\infty} \left(\frac{\varepsilon_{2n+1}}{n} - \frac{\varepsilon_{2n}}{n+1}\right)$  appears in Diaconis et al. [9].

*2.8.2. Law of  $\xi_1(\infty)$ .* The three dimensional Bessel process  $\rho^{(\nu)}$  with drift  $\nu \geq 0$  is the norm of a Brownian motion in  $\mathbb{R}^3$  with a drift of length  $\nu$ .

**Corollary 2.43.** *For  $0 \leq \mu \leq 1$ ,  $\xi_1(\infty)$  has the same law as  $\sup_{t \geq 0} (\varrho_t^{(1-\mu)} - t)$ .*

*Proof.* By Pitman and Rogers [33],  $\varrho_t^{(1-\mu)}$  has the same law as  $\mathcal{P}_{\alpha_1} B_t^{(1-\mu)}$ , so the claim follows from (2) of Theorem 2.37 when  $0 < \mu < 1$ , and by continuity also when  $\mu = 1$ . For  $\mu = 0$ ,  $\xi_1(\infty) = +\infty$ .  $\square$

When  $\mu = 1$ ,  $\xi_0(\infty) = +\infty$  and

$$\xi_1(\infty) = \frac{1}{2} \sum_{n=1}^{+\infty} \frac{\varepsilon_{2n-1} + \varepsilon_{2n}}{n^2}.$$

Its distribution is studied in Biane et al. [2] where it is symbolized  $\pi^2 S_2/4$ . Its Laplace transform is given by, for  $\tau \geq 0$ ,

$$\mathbb{E}(e^{-2\tau\xi_1(\infty)}) = \frac{\pi^2 \tau}{\sinh^2(\pi\sqrt{\tau})},$$

and its distribution function is (cf. Table 1 in [2])

$$F(x) = 1 + 2 \sum_{k=1}^{+\infty} (1 - 4k^2 x) e^{-2k^2 x},$$

so in this case the corollary is also given in Exemple 20 of Salminen and Yor [37].

When  $\mu = 1/2$ ,

$$\xi_1(\infty) = \sum_{n=1}^{+\infty} \frac{2\varepsilon_n}{n(n+1)}.$$

**Proposition 2.44.** *When  $\mu = 1/2$ , the Laplace transform of  $\xi_1(\infty)$  is, for  $\tau \geq 0$*

$$\mathbb{E}(e^{-\tau \xi_1(\infty)}) = \frac{2\pi\tau}{\cosh(\pi\sqrt{2\tau - 1/4})},$$

its density is

$$\sum_{n=0}^{+\infty} (-1)^{n+1} n(n+1)(2n+1) e^{-n(n+1)x/2},$$

and, in law,

$$\xi_1(\infty) = \sup_{n>0, i=1,2,3} \frac{\varepsilon_n^{(i)}}{n},$$

where the  $\varepsilon_n^{(i)}$  are exponential independent random variables with parameter 1.

*Proof.* It is easy to see that, using the formula

$$\cosh \pi z = (1 + 4z^2) \prod_{n=1}^{+\infty} \left(1 + \frac{z^2}{(n + 1/2)^2}\right),$$

one has

$$\mathbb{E}(e^{-\tau \xi_1(\infty)}) = \prod_{n=1}^{+\infty} \left(1 + \frac{2\tau}{n(n+1)}\right)^{-1} = \frac{2\pi\tau}{\cosh(\pi\sqrt{2\tau - 1/4})}.$$

Let

$$g(x) = 2 \sum_{n=0}^{+\infty} (-1)^n (2n+1) e^{-\frac{x}{2}(n+\frac{1}{2})^2}.$$

Since  $1/\cosh \sqrt{2\tau}$  is the Laplace transform of  $\pi g(\pi^2 x)/2$ , (e.g. [2]),

$$\frac{2\pi\tau}{\cosh \pi\sqrt{2\tau - 1/4}} = \int_0^\infty \tau e^{-\tau x} e^{x/8} g(x) dx = \int_0^\infty e^{-\tau x} (e^{x/8} g(x))' dx,$$

by an integration by parts. Computing the derivative  $(e^{x/8} g(x))'$  gives the density. By integration, the distribution function is

$$\sum_{n=0}^{+\infty} (-1)^n (2n+1) e^{-n(n+1)x/2}$$

which is equal to  $\prod_{n=1}^\infty (1 - e^{-nx})^3$  by a formula of Jacobi (Theorem 357 of [18]).  $\square$

**2.9. Convergence of the dihedral highest weight to the affine one.** In this subsection, we prove the convergence of the dihedral highest weight to the conditioned space-time Brownian motion  $\{A_t^{(\mu)}, t \geq 0\}$  that we have used before. Let  $0 \leq \mu \leq 1$  and  $m \geq 1$ . We consider the planar Brownian motion  $W^{(m/\pi, \mu)}$  and

$$\Lambda_m(t) = \mathcal{P}_{w_0}^m W^{(m/\pi, \mu)}(t).$$

**Theorem 2.45.** *As a process,  $\{\tau_m \Lambda_m(t), t \geq 0\}$  converges in law to  $\{A_t^{(\mu)}, t \geq 0\}$  when  $m$  tends to  $+\infty$ .*

We will prove this theorem after the following proposition. Let  $\{Z_t, t \geq 0\}$ , be the conditioned Brownian motion in  $[0, 1]$  starting from  $\mu$  (as defined in 5.1).

**Proposition 2.46.** *For  $m \in \mathbb{N}$ , let  $X_t^m, t \geq 0$ , be the  $\mathbb{R}^2$ -valued continuous process such that  $X_{m^2 t}^m$  is the conditioned planar Brownian motion in the cone  $C_m$ , with drift  $\gamma_m = m^2 e^{i\pi\mu/m}$  where  $0 \leq \mu \leq 1$ . One writes in polar coordinates*

$$X_t^m = R_t^m \exp i\pi\theta_t^m,$$

$R_t^m > 0, \theta_t^m \in [0, 1/m]$ . Then, when  $m$  tends to  $+\infty$ , as processes,  $m\theta_t^m$  tends to  $Z_{1/\pi^2 t}, R_t^m \rightarrow t$ .

*Proof.* As shown in Appendix 5.2,  $X_{m^2 t}^m, t \geq 0$ , is a radial multidimensional Dunkl process with drift. and  $R_{m^2 t}^m$  is a Bessel process of dimension  $2(m+1)$  with drift  $m^2$ , starting from 0. In other words, one can write

$$(R_{m^2 t}^m)^2 = (m^2 t + B_t^{(1)})^2 + \dots + (B_t^{(2(m+1))})^2$$

where  $B_t^{(1)}, \dots, B_t^{(2(m+1))}$  are independent standard real Brownian motions. Since

$$\mathbb{E}\left(\sum_{k=1}^{2(m+1)} (B_{t/m^2}^{(k)})^2\right) = 2t(m+1)/m^2$$

tends to 0 as  $m$  tends to  $+\infty$ ,  $R_t^m$  converges to  $t$  in  $L^1$ . It is shown in Gallardo and Yor [17] that the process  $Y_t = X_{m^2 t}^m$  has the time inversion property, in the sense that  $tY_{1/t}$  is the conditioned planar Brownian motion in  $\bar{C}_m$  without drift but starting from the drift  $\gamma_m$ . Using its skew product decomposition, one can write

$$\pi\theta_{m^2 t}^m = \sigma_{a_t^m}^m, \quad \text{with } a_t^m = \int_t^{+\infty} \frac{1}{(R_{m^2 s}^m)^2} ds,$$

where the process  $\sigma_t^m$  is a solution of the following stochastic differential equation

$$d\sigma_t^m = dB_t + m \cot(m\sigma_t^m) dt$$

where  $B$  is a Brownian motion independent of  $R^m$  and  $\sigma_0^m = \mu\pi/m$  (see Demni [13]). One remarks that  $Z_t = \frac{m}{\pi} \sigma_{t\pi^2/m^2}^m$  satisfies to

$$(2.23) \quad dZ_t = d\beta_t + \pi \cot(\pi Z_t) dt$$

for another Brownian motion  $\beta$ , and is therefore the conditioned Brownian motion in  $[0, 1]$  starting from  $\mu$  (see Appendix 5.1). As  $m$  tends to  $+\infty$ ,  $a_{t/m^2}^m$  is equivalent to  $1/tm^2$ . Therefore  $m\theta_t^m = \frac{m}{\pi} \sigma_{a_{t/m^2}^m}^m$  converges to  $Z_{1/t\pi^2}$ .  $\square$

*Proof of Theorem 2.45.* Let  $X_t^m = \frac{1}{m\pi} \Lambda_m(\pi^2 t)$ . Since

$$\frac{1}{m\pi} W_{\pi^2 m^2 t}^{(m/\pi, \mu)} = (\beta_t + m^2 t, B_t + m\mu\pi t),$$

one sees that  $X_{m^2 t}^m$  is the conditioned planar Brownian motion in the cone  $C_m$  with a drift equivalent to  $m^2 e^{i\pi\mu/m}$ . One writes its polar decomposition as  $X_t^m = R_t^m \exp i\pi\theta_t^m$ . Using the continuity of the solution of (2.23) with respect to the initial condition, we see that Proposition 2.46 also holds when the drift  $\gamma_m$  is only equivalent to  $m^2 e^{i\pi\mu/m}$ . Therefore  $m\theta_t^m$  tends to  $Z_{1/t\pi^2}$  and  $R_t^m$  tends to  $t$  and the process

$$\tau_m \Lambda_m(t) = (\pi^2 R_{t/\pi^2}^m \cos(\pi\theta_{t/\pi^2}^m), m\pi R_{t/\pi^2}^m \sin(\pi\theta_{t/\pi^2}^m)), t \geq 0,$$

converges in law to  $(t, tZ_{1/t}), t \geq 0$ , equal in law to  $A^{(\mu)}$  by Theorem 5.7.  $\square$

### 3. REPRESENTATION OF THE CONDITIONED BROWNIAN MOTION IN $[0, 1]$

Let us first recall some notions of the introduction. For a continuous real path  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ , such that  $\varphi(0) = 0$ , we have defined

$$\begin{aligned} \mathcal{L}_1 \varphi(t) &= \varphi(t) - \inf_{0 \leq s \leq t} \varphi(s), \\ \mathcal{P}_1 \varphi(t) &= \varphi(t) - 2 \inf_{0 \leq s \leq t} \varphi(s), \end{aligned}$$

and  $\mathcal{L}_0 = T\mathcal{L}_1T, \mathcal{P}_0 = T\mathcal{P}_1T$ , where  $T\varphi(t) = t - \varphi(t)$ . Let  $B_t^\mu = B_t + t\mu, t \geq 0$ . The theorem stated in the introduction is

**Theorem 3.1.** *Let  $\mu \in [0, 1]$ . For any  $t > 0$ , almost surely,*

$$\lim_{n \rightarrow \infty} t \mathcal{L}_{n+1} \mathcal{P}_n \cdots \mathcal{P}_1 \mathcal{P}_0 B^\mu(1/t) = \lim_{n \rightarrow \infty} t \mathcal{L}_{n+1} \mathcal{P}_n \cdots \mathcal{P}_2 \mathcal{P}_1 B^\mu(1/t) = Z_t,$$

where  $Z_t, t \geq 0$ , is the Brownian motion conditioned to stay in the interval  $[0, 1]$  forever, starting from  $Z_0 = \mu$ .

*Proof.* The proof is just the juxtaposition of the theorems ??, 2.37 and 5.7.  $\square$

Remark that one also have that, for  $t > 0$ , a.s.,

$$\lim_{n \rightarrow \infty} t \mathcal{P}_n \cdots \mathcal{P}_1 \mathcal{P}_0 B^\mu(1/t) + (-1)^n 2 = Z_t,$$

which clearly shows the need of a correction for the Pitman transforms. As an illustration let us show that:

**Proposition 3.2.** *When  $0 < \mu < 1$  and  $Z_0 = \mu$ , there is a standard Brownian motion  $\beta$  and a stopping time  $\tau > 0$  a.s. for  $\beta$  such that for  $0 \leq t \leq \tau$ ,*

$$Z_t = \beta_t + \mu + tD^\mu(\infty).$$

*Proof.* For  $\sigma$  given by Proposition 2.27, for  $t > \sigma$ ,  $L^{(\mu)}(t) = L^{(\mu)}(\infty)$  hence  $A_t^{(\mu)} = B_t^{(\mu)} + L^{(\mu)}(\infty)$ , which implies the representation for  $\tau = 1/\sigma$ , since  $(t, tZ_{1/t}) = A_t^{(\mu)}$ .  $\square$

Notice that, by (2.22),  $\mathbb{E}(D^\mu(\infty)) = \pi \cot \pi\mu$  as expected from the generator  $\frac{1}{2}d^2/dx^2 + (\pi \cot \pi x)d/dx$  of  $Z_t$ .

#### 4. SOME ASYMPTOTICS FOR REPRESENTATIONS OF THE AFFINE ALGEBRA $A_1^{(1)}$

As we said in the introduction, there are strong links between what we have done and the affine Kac-Moody Lie algebra  $A_1^{(1)}$ . In this section we will see that quantities we have met also occur in the description of some semi-classical limits of highest representations of  $A_1^{(1)}$ . In particular a Duistermaat Heckman measure is given by the conditional law of the space-time Brownian motion knowing the highest weight process. The asymptotic behaviour of the infinity crystal  $B(\infty)$  of Kashiwara, for large weights is given by the Verma affine string coordinates  $\xi(\infty)$ .

**4.1. The Kac-Moody algebra  $A_1^{(1)}$ .** We consider the affine Lie algebra  $A_1^{(1)}$ . For our purpose, we only need to consider a realization of a real Cartan subalgebra. We introduce, as in the introduction

$$\mathfrak{h}_{\mathbb{R}} = \text{Span}_{\mathbb{R}}\{c, \tilde{\alpha}_1, d\}, \quad \mathfrak{h}_{\mathbb{R}}^* = \text{Span}_{\mathbb{R}}\{\Lambda_0, \alpha_1, \delta\},$$

where  $c = (1, 0, 0)$ ,  $\tilde{\alpha}_1 = (0, 1, 0)$ ,  $d = (0, 0, 1)$ , and  $\Lambda_0 = (1, 0, 0)$ ,  $\alpha_1 = (0, 2, 0)$ ,  $\delta = (0, 0, 1)$ . We let  $\tilde{\alpha}_0 = (1, -1, 0)$  and  $\alpha_0 = (0, -2, 1)$ , so that  $c = \tilde{\alpha}_0 + \tilde{\alpha}_1$  and  $\delta = \alpha_0 + \alpha_1$ . Notice that these  $\alpha_0$  and  $\alpha_1$  project on the ones given in 2.2 by the projection on  $\mathfrak{h}_{\mathbb{R}}^*/\mathbb{R}\delta$ , identified with  $\mathbb{R}\Lambda_0 \oplus \mathbb{R}\alpha_1$ , and thus also with the space  $V = \mathbb{R}^2$  of Section 2. Usually  $\alpha_0, \alpha_1$  are called the two positive simple roots of  $A_1^{(1)}$  and  $\tilde{\alpha}_0, \tilde{\alpha}_1$  their coroots. One considers the set of integral weights

$$P = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : \lambda(\tilde{\alpha}_i) \in \mathbb{Z}, i = 0, 1\},$$

and the set of dominant integral weights

$$P_+ = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : \lambda(\tilde{\alpha}_i) \in \mathbb{N}, i = 0, 1\}.$$

For a dominant integral weight  $\lambda$  one defines the character of the irreducible highest-weight representation  $V(\lambda)$  of  $A_1^{(1)}$  with highest weight  $\lambda$ , as a formal series

$$(4.1) \quad \text{char}_{\lambda} = \sum_{\beta \in P} \dim(V_{\beta}(\lambda)) e^{\beta},$$

where  $V_{\beta}(\lambda)$  is the weight space of  $V(\lambda)$  corresponding to the weight  $\beta$ . If we let  $e^{\beta}(h) = e^{\beta(h)}$ , for  $h \in \mathfrak{h}_{\mathbb{R}}$ , and evaluate this formal series at  $h$ , the series converges absolutely or diverges, and it converges when  $\delta(h) > 0$ . For more details about affine Lie algebras and their representations, we refer to Kac [24].

**4.2. A Duistermaat Heckman measure.** A way to define the Duistermaat Heckman measure for a semi-simple complex Lie algebra, is as an approximation of the distribution of the weights of an irreducible representation when its highest weight is large (see Heckman [20]). Let us explain how the same approach is possible in  $A_1^{(1)}$ . Let  $r \geq 1$ , for a dominant integral weight  $\lambda_r$ , and  $h_r \in \mathfrak{h}_{\mathbb{R}}$  such that  $\delta(h_r) > 0$ , we define a measure  $\gamma_r$  on  $\mathfrak{h}_{\mathbb{R}}^*$ , letting

$$(4.2) \quad \gamma_r = \sum_{\beta \in P} \dim(V(\lambda_r)_{\beta}) e^{\beta(h_r)} \delta_{\beta/r},$$

where  $\delta_{\beta/r}$  is the Dirac measure at  $\beta/r$ .

**Proposition 4.1.** *Let  $\mu \in [0, 1]$ ,  $(t, x) \in C_{\text{aff}}$ , and  $\{\lambda_r, r \in \mathbb{N}^*\}$  be a sequence of dominant weights such that*

$$\lim_{r \rightarrow +\infty} \lambda_r / r = t\Lambda_0 + x \frac{\alpha_1}{2},$$

and  $h_r = \frac{1}{r}(\mu\tilde{\alpha}_1 + 2d)$ . As  $r \rightarrow \infty$ , for  $\tau \in \mathbb{R}$ ,  $u \geq 0$ .

$$\int_{\mathfrak{h}_{\mathbb{R}}^*} e^{\beta(\tau\tilde{\alpha}_1 + ud)} \gamma_r(d\beta) \sim \sqrt{\frac{2(2+u)}{\pi r}} e^{\frac{r\pi^2}{2(2+u)} + (\tau+\mu)x} \varphi_{\frac{2(\tau+\mu)}{2+u}}\left(\frac{(2+u)t}{2}, \frac{(2+u)x}{2}\right).$$

*Proof.* One has for any  $v \in \mathfrak{h}$ ,

$$\int_{\mathfrak{h}_{\mathbb{R}}^*} e^{\beta(v)} \gamma_r(d\beta) = \text{char}_{\lambda_r}\left(\frac{v}{r} + h_r\right).$$

When  $\lambda = n\Lambda_0 + m\frac{\alpha_1}{2}$ , with  $(m, n) \in \mathbb{N}^2$  such that  $0 \leq m \leq n$ ,  $a \in \mathbb{R}$ , and  $b \in \mathbb{R}_+$ , the Weyl-Kac character formula implies that

$$\text{char}_{\lambda}(a\tilde{\alpha}_1 + bd) = \frac{\sum_{k \in \mathbb{Z}} \sinh(a(m+1) + 2ak(n+2)) e^{-b(k(m+1) + k^2(n+2))}}{\sum_{k \in \mathbb{Z}} \sinh(a + 4ak) e^{-b(k+2k^2)}},$$

(see Kac [24]). So for  $v = \tau\tilde{\alpha}_1 + ud$ , one has immediately that the numerator of the character  $\text{ch}_{\lambda_r}(\frac{v}{r} + h_r)$  converges, when  $r$  goes to infinity, towards

$$e^{(\tau+\mu)x} \sin\left(\frac{2\pi(\tau+\mu)}{2+u}\right) \varphi_{\frac{2(\tau+\mu)}{2+u}}\left(\frac{(2+u)t}{2}, \frac{(2+u)x}{2}\right).$$

Besides Lemma 2.41 implies that the denominator of the character is equivalent to

$$\frac{\sqrt{\pi r}}{\sqrt{2(2+u)}} e^{-\frac{1}{2(u+2)} r \pi^2} \sin\left(\frac{2\pi(\tau+\mu)}{u+2}\right),$$

which finishes the proof.  $\square$

By taking  $u = 0$  we obtain

**Corollary 4.2.** *For  $\tau \in \mathbb{R}$ ,*

$$\lim_{r \rightarrow +\infty} \frac{1}{\text{char}_{\lambda_r}(h_r)} \int_{\mathfrak{h}_{\mathbb{R}}^*} e^{\tau\beta(\tilde{\alpha}_1)} \gamma_r(d\beta) = e^{\tau x} \frac{\varphi_{\tau+\mu}(t, x)}{\varphi_{\mu}(t, x)}.$$

We denote by  $\nu_r$  the probability measure on  $\mathfrak{h}_{\mathbb{R}}^*$  given by

$$\nu_r = \frac{1}{\text{char}_{\lambda_r}(h_r)} \gamma_r.$$

From the next theorem we see that the conditional measure of Theorem 2.40 can be interpreted as a kind of normalized Duistermaat Heckman measure.

**Theorem 4.3.** *Under the hypotheses of Proposition 4.1, the sequence of the push forward probabilities of  $\{\nu_r, r \geq 1\}$  by the quotient map from  $\mathfrak{h}_{\mathbb{R}}^*$  to  $\mathfrak{h}_{\mathbb{R}}^*/\mathbb{R}\delta$  converges to the law of  $B_t^{(\mu)}$  conditionally to  $\Lambda_t^{(\mu)} = (t, x)$  when  $r$  goes to infinity.*

*Proof.* This follows from the corollary and Theorem 2.40.  $\square$

**Remark 4.4.** *Let  $F_r\Lambda_0 + G_r\alpha_1 + H_r\delta$  be a random variable in  $\mathfrak{h}_{\mathbb{R}}^*$  with law  $\nu_r$ . From the theorem, as  $r$  tends to infinity,  $F_r$  tends to the constant  $t$ , and  $G_r$  converges in law. But  $H_r$  tends to  $-\infty$ , as follows from Proposition 4.1 with  $s = 0$  and  $u \geq 0$ .*

For a compact connected Lie group, the Duistermaat Heckman measure also appears as the pushforward measure of the Liouville measure on a coadjoint orbit by the projection on a Cartan subalgebra. Frenkel has shown in [16] that the law of a Brownian motion on  $\mathfrak{su}(2)$  indexed by the time in  $[0, 1]$ , given the orbit of the endpoint of its wrapping on  $SU(2)$  under conjugacy, plays the role of a normalized Liouville measure on a coadjoint orbit of the loop group  $L(SU(2))$ . Identifying  $\mathfrak{su}(2)$  and its dual, the probability measure appearing as a limit measure in Theorem 4.3 is actually the projection of this law on a Cartan subalgebra of  $\mathfrak{su}(2)$  (see Frenkel [16] and also Defosseux [12]). It is the Duistermaat-Heckman measure corresponding to the action of the torus of  $SU(2)$  on the coadjoint orbit through  $(t, x)$ .

**4.3. Asymptotics for the crystal  $B(\infty)$  of  $A_1^{(1)}$ .** The infinity crystal  $B(\infty)$  of Kashiwara ([25]) is the crystal of the Verma module with highest weight 0 of  $A_1^{(1)}$ . This crystal is important since any irreducible highest weight crystal may be obtained from  $B(\infty)$ . It is shown in Nakashima and Zelevinski ([30]) that using string parametrizations, a realization of  $B(\infty)$  is given by

$$B(\infty) = \{x \in \mathbb{N}^{\mathbb{N}}; \text{ for some } n \in \mathbb{N}^*, \frac{x_1}{1} \geq \frac{x_2}{2} \geq \dots \geq \frac{x_n}{n} > 0, x_k = 0 \text{ for } k > n\}.$$

Notice that the only condition on  $x_0$  is  $x_0 \in \mathbb{N}$ . For  $x \in B(\infty)$ , we let

$$\sigma(x) = \sum_{k=0}^n x_k \alpha_k \in \mathfrak{h}_{\mathbb{R}}^*,$$

when  $x_k = 0$  for  $k > n$ . Then  $-\sigma(x)$  is the weight of  $x$  in the crystal  $B(\infty)$  (see [30]). For  $\tilde{\rho} = 2d + \tilde{\alpha}_1/2$ , we define

$$s(x) = \sum_{k=0}^{+\infty} x_k = \sigma(x)(\tilde{\rho}).$$

The character  $\text{char}_{\infty}$  of the Verma module of highest weight 0 is defined as in (4.1), and

$$(4.3) \quad \text{char}_{\infty} = \prod_{\beta \in R_+} (1 - e^{-\beta})^{-1},$$

where

$$(4.4) \quad R_+ = \{\alpha_i + n\delta, i = 0, 1, n \in \mathbb{N}\} \cup \{n\delta, n \in \mathbb{N}^*\}$$

are the so called positive roots of  $A_1^{(1)}$  (see Kac [24], (9.7.2)). As previously, if we let  $e^{\beta}(h) = e^{\beta(h)}$ , for  $h \in \mathfrak{h}_{\mathbb{R}}$ , and evaluate the formal character at  $h$ , it converges if and only if  $\delta(h) > 0$ . Let  $r \in \mathbb{R}_+^*$ . On each element  $x$  of the crystal  $B(\infty)$  we put the Boltzman weight  $e^{-s(x)/r}$ . We introduce the probability distribution  $\beta_r$  on  $B(\infty)$  by

$$\beta_r(\{x\}) = \frac{e^{-s(x)/r}}{Z_r}, \quad x \in B(\infty),$$

where

$$Z_r = \sum_{x \in B(\infty)} e^{-s(x)/r} = \text{char}_{\infty}(\tilde{\rho}/r).$$

Since  $B(\infty)$  is a Borel subset of  $\Gamma$  we may consider  $\beta_r$  as a probability on  $\Gamma$ . The following theorem indicates that the affine Brownian model describes a kind of continuous version of the infinity crystal  $B(\infty)$  for the affine Lie algebra  $A_1^{(1)}$ .

**Theorem 4.5.** *Let  $X^{(r)} \in \Gamma, r \in \mathbb{N}$ , be random variables with distribution  $\beta_r$ . Then  $X^{(r)}/r$  converges in distribution on  $\Gamma$  to the Verma parameter  $\xi(\infty)$  of  $B^{(1/2)}$ .*

*Proof.* We use the results on anti-lecture hall compositions recalled in 5.5. Let us first prove that  $X_1^{(r)}/r$  converges in law to  $\xi_1(\infty)$ . For  $q = e^{-1/r}$ , it follows from (5.8), (5.9) that, for  $a \geq 0$ ,

$$\begin{aligned} \mathbb{P}(X_1^{(r)} \leq ar) &= \frac{\sum_{\{x \in B(\infty); x_1 \leq ar\}} q^{s(x)}}{\sum_{\{x \in B(\infty)\}} q^{s(x)}} = \frac{\sum_{\{\lambda \in A_\infty; \lambda_1 \leq ar\}} q^{|\lambda|}}{\sum_{\{\lambda \in A_\infty\}} q^{|\lambda|}} \\ &= \frac{1}{1-q} (q; e^{-a}q^2)_\infty (e^{-a}q; e^{-a}q^2)_\infty (e^{-a}q^2; e^{-a}q^2)_\infty. \end{aligned}$$

where  $(p; q)_\infty = \prod_{n=0}^{\infty} (1-pq^n)$ . Since  $(q; e^{-a}q^2)_\infty$  is equivalent to  $(1-q)(e^{-a}; e^{-a})_\infty$  one has, by Proposition 2.44

$$\lim_{r \rightarrow \infty} \mathbb{P}(X_1^{(r)} \leq ra) = (e^{-a}; e^{-a})_\infty^3 = \prod_{n=1}^{\infty} (1 - e^{-na})^3 = \mathbb{P}(\xi_1(\infty) \leq a).$$

So  $X_1^{(r)}/r$  converges to  $\xi_1(\infty)$ . We now consider the full sequence  $X_k^{(r)}/r, k \in \mathbb{N}$ . First it is clear that  $X_0^{(r)}/r$  converges in law to  $\xi_0(\infty)$ . For any  $r$  and  $n \geq 1$ , one has

$$\frac{X_1^{(r)}}{1} \geq \frac{X_2^{(r)}}{2} \geq \dots \geq \frac{X_n^{(r)}}{n},$$

which implies that for any  $n \in \mathbb{N}$ , the collection of laws of  $(\frac{1}{r}X_1^{(r)}, \dots, \frac{1}{r}X_n^{(r)})_{r>0}$  is tight since  $X_1^{(r)}/r$  converges in law. By Cantor's diagonal argument, we construct an increasing sequence  $\varphi(r) \in \mathbb{N}, r \in \mathbb{N}$ , such that the sequence of random variables  $(\frac{1}{\varphi(r)}X_k^{(\varphi(r))}, k \geq 0)$  converges in finite dimensional distribution when  $r$  goes to infinity. Let us denote by  $(R_k, k \geq 0)$  the limit, and let us first prove that  $\{\frac{R_k}{k} - \frac{R_{k+1}}{k+1}, k \geq 1\}$  has the same distribution as  $\{2\varepsilon_k/k(k+1), k \geq 1\}$  where  $(\varepsilon_k)$  are independent exponential random variables with parameter 1. For  $x \in B(\infty)$ , one has, since  $s(x) = x_0 + \frac{1}{2} \sum_{k=2}^{+\infty} (kx_{k-1} - (k-1)x_k)$ ,

$$\begin{aligned} \mathbb{P}(X^{(r)} = x) &= \frac{1}{\text{char}_\infty(\frac{1}{r}\tilde{\rho})} e^{-\frac{1}{r}x_0} \prod_{k=2}^{\infty} e^{-\frac{1}{2r}(kx_{k-1} - (k-1)x_k)} \mathbf{1}_{\{\frac{x_{k-1}}{k-1} \geq \frac{x_k}{k}\}} \\ &= \frac{1}{\text{char}_\infty(\frac{1}{r}\tilde{\rho})} e^{-\frac{1}{r}x_0} \prod_{k=1}^{\infty} e^{-\frac{k+1}{2r}(x_k - \lceil \frac{k}{k+1}x_{k+1} \rceil)} e^{-\frac{1}{2r}(k \lceil \frac{k-1}{k}x_k \rceil - (k-1)x_k)} \mathbf{1}_{\{x_k \geq \lceil k \frac{x_{k+1}}{k+1} \rceil\}}, \end{aligned}$$

where  $\lceil \cdot \rceil$  is the ceiling function. As for  $k = 1, \dots, n$ ,

$$e^{-\frac{k}{2r}} \leq e^{-\frac{1}{2r}(k \lceil \frac{k-1}{k}x_k \rceil - (k-1)x_k)} \leq 1,$$



one obtains for  $n \in \mathbb{N}$  and  $t_0, \dots, t_n \in \mathbb{R}_+$ ,

$$C(r, n)(1 + o(\frac{1}{r})) \leq \frac{\mathbb{P}(X_0^{(r)} \leq t_0, \dots, X_n^{(r)} - \lceil \frac{n}{n+1} X_{n+1}^{(r)} \rceil \leq t_n)}{\mathbb{P}(Y_0^{(r)} \leq t_0, \dots, Y_n^{(r)} \leq t_n)} \leq C(r, n),$$

where  $C(r, n)$  is independent of  $t_0, \dots, t_n$ , and tends to 1 when  $r$  tends to  $+\infty$ , where  $Y_0^{(r)}$  and  $Y_k^{(r)}$ ,  $k = 1, \dots, n$ , are independent geometric random variables with values in  $\mathbb{N}$ , and  $Y_0^{(r)}$  with parameter 1 and  $Y_k^{(r)}$  with parameter  $e^{-\frac{k+1}{2r}}$ , when  $k \geq 1$ . This proves that for any  $n$ ,

$$\frac{1}{r} X_0^{(r)}, \frac{1}{r} (X_k^{(r)} - \lceil \frac{k}{k+1} X_{k+1}^{(r)} \rceil), \quad 1 \leq k \leq n,$$

converges jointly towards  $\varepsilon_0, \frac{2}{k+1} \varepsilon_k, 1 \leq k \leq n$ , when  $r$  goes to infinity. Besides

$$\lim_{r \rightarrow \infty} \frac{1}{\varphi(r)} (X_k^{(\varphi(r))} - \lceil \frac{k}{k+1} X_{k+1}^{(\varphi(r))} \rceil) = R_k - \frac{k}{k+1} R_{k+1}.$$

Thus, for  $k \geq 1$ ,  $R_k - \frac{k}{k+1} R_{k+1}$  are independent random variables with the same law as  $\frac{2}{k+1} \varepsilon_k$ . The positive sequence  $R_k/k$  is decreasing. Let  $S$  be its limit. We have the identity in law, for all  $k \geq 1$ ,

$$\frac{R_k}{k} = \sum_{n=k}^{+\infty} \frac{2\varepsilon_n}{n(n+1)} + S.$$

We have proved that in law  $R_1 = \xi_1(\infty)$ , so  $S = 0$  which finishes the proof.  $\square$

Recall (Corollary 2.42) that  $L^{(1/2)}(\infty) = (0, D^{1/2}(\infty))$  where the density of  $D^{1/2}(\infty)$  is  $1/(\pi \cosh x)$ .

**Proposition 4.6.** *When  $r$  goes to infinity, in law,*

- (1) *The normalized weights  $\sigma(X^{(r)})/r$  converges to  $L^{(1/2)}(\infty)$  in the quotient space  $\mathfrak{h}_{\mathbb{R}}^*/\mathbb{R}\delta$ .*
- (2) *The coordinate of  $\sigma(X^{(r)})/r$  along  $\delta$  goes to  $+\infty$ .*

*Proof.* One has for any  $u \in \mathfrak{h}_{\mathbb{R}}$ ,

$$\mathbb{E}(e^{-\sigma(X^{(r)})(u)}) = \frac{\text{char}_{\infty}(u + \tilde{\rho}/r)}{\text{char}_{\infty}(\tilde{\rho}/r)}.$$

In view of (4.4), the expression (4.3) of the character gives the Laplace transform of  $\sigma(X^{(r)})$  and shows that, in distribution,

$$\sigma(X^{(r)}) = \sum_{n \geq 0} (G_0(n)(\alpha_0 + n\delta) + G_1(n)(\alpha_1 + n\delta) + G_2(n)(n+1)\delta),$$

where  $G_i(n)$ ,  $i = 0, 1, 2$ , and  $n \in \mathbb{N}$ , are independant random variables such that  $G_0(n)$ ,  $G_1(n)$  and  $G_2(n)$ , are geometrically distributed with respective parameter  $e^{-(\alpha_0+n\delta)(\tilde{\rho}/r)}$ ,  $e^{-(\alpha_1+n\delta)(\tilde{\rho}/r)}$ , and  $e^{-(n+1)\delta(\tilde{\rho}/r)}$ , i.e. with respective parameter  $e^{-2(n+1/2)/r}$ ,  $e^{-2(n+1/2)/r}$  and  $e^{-2(n+1)/r}$ . The proposition follows easily.  $\square$

Notice that, due to the need of the Lévy correction, the formal expression  $\sigma(\lim_{r \rightarrow +\infty} \frac{X^{(r)}}{r})$  is not equal to  $\lim_{r \rightarrow +\infty} \sigma(\frac{X^{(r)}}{r})$ , mod  $\delta$ .

5. APPENDIX

**5.1. The conditioned Brownian motion in  $[0, 1]$ .** We recall some well known facts about the Brownian motion conditioned to stay forever in the interval  $[0, 1]$ . It first appeared in Knight ([27]), called there the Taboo process, as the limit when  $t$  tends to  $\infty$  of the standard Brownian motion starting in  $(0, 1)$ , conditioned to reach the boundary after time  $t$ . To define it rigorously, consider the Brownian motion in  $[0, 1]$  killed at the boundary. Its generator is  $\frac{1}{2} \frac{d^2}{dx^2}$  with Dirichlet boundary condition. Its maximal eigenvalue is  $-\pi^2/2$  with positive eigenvector

$$h(x) = \sin(\pi x),$$

called the ground state. We consider the associated  $h$ -Doob process  $\{Z_t, t \geq 0\}$ . It is the Markov process with transition probability density  $q_t(x, y)$  given when  $x, y \in (0, 1)$  by

$$(5.1) \quad q_t(x, y) = \frac{\sin(\pi y)}{\sin(\pi x)} e^{\pi^2 t/2} u_t(x, y),$$

where  $u_t(x, y)$  is the transition probability density of the killed Brownian motion.

**Definition 5.1.** We call  $(Z_t)$  the conditioned Brownian motion process in  $[0, 1]$ .

It can also be viewed as the diffusion in  $[0, 1]$  with generator

$$\frac{1}{2} \frac{d^2}{dx^2} + \pi \cot(\pi x) \frac{d}{dx}.$$

It is well known (and follows from the reflection principle), that, for  $x, y \in (0, 1)$ ,

$$(5.2) \quad u_t(x, y) = \sum_{k \in \mathbb{Z}} (p_t(x + 2k, y) - p_t(-x - 2k, y)),$$

where  $p_t$  is the standard heat kernel ([23]). Using the Poisson formula, or the spectral decomposition of the generator, one has,

$$(5.3) \quad u_t(x, y) = \sum_{n \in \mathbb{Z}} \sin(n\pi x) \sin(n\pi y) e^{-\pi^2 n^2 t/2}.$$

By scale function techniques, one sees that 0 and 1 are entrance non-exit boundaries. In other words,  $(Z_t)$  can be started from the boundaries and does not touch them at positive time. Let us remark that  $(Z_t)$  can also be defined by the latitude of the Brownian motion on the 3-dimensional sphere (see Ito and McKean [23], Section 7.15) or by the argument of an eigenvalue of the Brownian motion in  $SU(2)$ . The boundaries behaviour is also clear from this description.

The entrance density measure starting from 0 is the limit of  $q_t(x, y)$  when  $x$  tends to 0, which is for  $y \in (0, 1)$ , by (5.3),

$$(5.4) \quad q_t(0, y) = \sin(\pi y) \sum_{n \in \mathbb{Z}} n \sin(n\pi y) e^{-\frac{t}{2} \pi^2 (n^2 - 1)}.$$

Another interpretation of  $Z$  is given by the distance between two non colliding Brownian motions on a circle (see Hobson and Werner [21]).

**5.2. The conditioned planar Brownian motion in a dihedral cone.** It is intuitively given by the two dimensional Brownian motion with drift starting from 0 conditioned to stay in the cone  $C_m$  forever. It is rigorously defined in 5.1 of [4] for a planar Brownian motion without drift. The definition when there is a drift is the following. We recall (see Section 2.7) that for  $v = (x, y) \in \mathbb{R}^2$ ,

$$h_m(v) = h_m(x, y) = \Im((x + iy)^m).$$

and, for  $\gamma \in \bar{C}_m$

$$\psi_v^m(\gamma) = \sum_{w \in I(m)} (-1)^{l(w)} e^{\langle w(\gamma) - \gamma, v \rangle}.$$

For  $x, y \in C_m$ , let

$$r_t(x, y) = \frac{\psi_\gamma^m(y)}{\psi_\gamma^m(x)} v_t^\gamma(x, y),$$

where  $v_t^\gamma$  is the semi group density of the standard planar Brownian motion with drift  $\gamma$  killed at the boundary of  $C_m$ .

**Definition 5.2.** *The conditioned planar Brownian motion  $A_m^{(\gamma)}$  in the dihedral cone  $C_m$  with drift  $\gamma \in \bar{C}_m$ , is the continuous Markov process with values in  $C_m \cup \{0\}$  with transition probability density  $r_t(x, y)$  for  $x, y \in C_m$ , such that  $A_m^{(\gamma)}(0) = 0$ , and with entrance probability density at time  $t$  proportional to, for  $x \in C_m$ ,*

$$h_m(x) \frac{\psi_\gamma^m(x)}{h_m(\gamma)} e^{-\frac{1}{2t} \langle x, x \rangle}.$$

In the definition above,  $\psi_\gamma^m(x)/h_m(\gamma)$  is obtained by analytical continuation when  $h_m(\gamma) = 0$ .

**Proposition 5.3.** *Let  $W^{(\gamma)}$  be the planar Brownian motion with drift  $\gamma \in \bar{C}_m$  starting from the origin. The process  $\mathcal{P}_{w_0}^m W^{(\gamma)}$  has the same law as  $A_m^{(\gamma)}$ .*

*Proof.* This is proved in Biane et al. [4] when  $\gamma = 0$ . For  $0 < t_1 < \dots < t_n$ , and a bounded measurable function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$ , the Cameron–Martin formula gives

$$\mathbb{E}(F(A_m^{(\gamma)}(t_1), \dots, A_m^{(\gamma)}(t_n))) = \mathbb{E}(F(A_m^{(0)}(t_1), \dots, A_m^{(0)}(t_n)) e^{-\frac{1}{2} \langle \gamma, \gamma \rangle t_n + \langle \gamma, W_{t_n}^{(0)} \rangle}).$$

By Theorem 5.5 of [4], this is equal to

$$k \mathbb{E}(F(A_m^{(0)}(t_1), \dots, A_m^{(0)}(t_n)) e^{-\frac{1}{2} \langle \gamma, \gamma \rangle t_n} \frac{\psi_\gamma^m(A_m^{(0)}(t_n))}{h_m(\gamma) h_m(A_m^{(0)}(t_n))} e^{\langle \gamma, A_m^{(0)}(t_n) \rangle}),$$

where  $k$  is a constant. This implies easily the proposition.  $\square$

Another quick manner to define rigorously  $A_m^{(\gamma)}$  is to use Dunkl processes with multiplicity one and the approach given by Gallardo and Yor in [17]. One considers the Dunkl Laplacian associated with  $I(m)$  given, for  $f \in C^2(\mathbb{R}^2)$ , by

$$\tilde{\Delta} f = \Delta f + 2 \sum_{\alpha \in R^+} \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle^2}$$

where  $\sigma_\alpha$  is the reflection with respect to  $\{x \in \mathbb{R}^2, \langle x, \alpha \rangle = 0\}$ ,  $\Delta$  is the Laplacian and  $\nabla$  the gradient. Roesler and Voit ([36], Section 3) show that for any  $\gamma \in \mathbb{R}^2$ , there exists a cad-lag process  $X_\gamma(t), t \geq 0$ , with generator  $\tilde{\Delta}/2$  such that  $X_\gamma(0) =$

$\gamma$ . Let  $p : \mathbb{R}^2 \rightarrow \bar{C}_m$  be the projection where  $\bar{C}_m$  is identified with  $\mathbb{R}^2/I(m)$ . Then (Roesler Voit, Theorems 4.10, 4.11)  $\bar{X}_\gamma(t) = p(X_\gamma(t))$  is a continuous process and its norm is a Bessel process of dimension  $2(m+1)$ . According to Gallardo and Yor ([17], Section 3.3), the conditioned Brownian motion in  $C_m$  with drift  $\gamma \in \bar{C}_m$ , starting from 0 is defined as  $t\bar{X}_\gamma(1/t)$ . Its probability transitions are given by (25) of [17], which shows that it coincides in law with  $A_m^{(\gamma)}$ .

**5.3. The conditioned space-time Brownian motion in  $C_{\text{aff}}$ .** Let  $(B_t)$  be the standard real Brownian motion and  $B_t^{(\mu)} = (t, B_t + \mu t)$  be the space time Brownian motion with drift  $\mu$ . We will define rigorously the process  $A_t^{(\mu)}, t \geq 0$ , which is the process  $B^{(\mu)}$  conditioned to stay forever in the affine cone  $C_{\text{aff}} = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 < x < t\}$  starting from  $(0, 0)$ . It has been introduced and studied in Defosseux [10, 11, 12].

We suppose that  $\mu \in [0, 1]$ . Let  $\{K_t^{(\mu)}, t \geq 0\}$  be the space-time process  $B_t^{(\mu)}$  killed at the boundary of  $\bar{C}_{\text{aff}}$ . This is the process in the cone with generator  $\frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x}$  and Dirichlet boundary condition.

**Proposition 5.4.** *The function  $\varphi_\mu$  defined in (2.17) is a space-time non negative harmonic function for the killed process  $K^{(\mu)}$  on  $C_{\text{aff}}$ , vanishing on its boundary.*

*Proof.* The harmonicity is clear by computation. The boundary condition  $\varphi_\mu(t, 0) = 0$ , resp.  $\varphi_\mu(t, t) = 0$ , follows from the change of variable from  $k$  in  $-k$ , resp.  $k$  in  $-1 - k$ . Positivity follows for instance from the lemmas 2.38 and 2.39.  $\square$

Using the Cameron-Martin formula and Defosseux ([11], Proposition 2.2), one obtains that, if we write  $K_t^\mu = (t, K_t^\mu)$ , the density  $w_t^\mu((r, x), (t+r, \cdot))$  of  $K_{t+r}^\mu$  given  $K_r^\mu = x$  is

$$(5.5) \quad \begin{aligned} w_t^\mu((r, x), (t+r, y)) &= e^{\mu(y-x) - \frac{t}{2}\mu^2} w_t^0((r, x), (t+r, y)) \\ &= \sum_{k \in \mathbb{Z}} e^{-2(kx+k^2r)} (e^{\mu(2kr+x)} p_t^\mu(x+2kr, y) \\ &\quad - e^{-\mu(2kr+x)} p_t^\mu(-x-2kr, y)), \end{aligned}$$

where  $p_t^\mu$  is the standard heat kernel with a drift  $\mu$ . For  $0 \leq \mu \leq 1$ , let  $\{A_t^{(\mu)}, t \geq 0\}$  be the Markov process in  $C_{\text{aff}} \cup \{(0, 0)\}$  such that, if we write  $A_t^{(\mu)} = (t, A_t^{(\mu)})$ , the law of  $A_{t+r}^{(\mu)}$  given  $A_r^{(\mu)} = x$  has the density

$$(5.6) \quad s_t^\mu((r, x), (r+t, y)) = \frac{\varphi_\mu(r+t, y)}{\varphi_\mu(r, x)} w_t^\mu((r, x), (r+t, y)),$$

for  $(r, x), (r+t, y) \in C_{\text{aff}}$ , such that  $A_0^{(\mu)} = (0, 0)$  and with entrance density given for  $(t, y) \in C_{\text{aff}}$  by

$$(5.7) \quad s_t^\mu((0, 0), (t, y)) = \varphi_\mu(t, y) \sin\left(\frac{y}{t}\pi\right) e^{-\frac{1}{2t}(y-\mu t)^2}.$$

**Definition 5.5.** *For  $0 \leq \mu \leq 1$ , the process  $\{A_t^{(\mu)}, t \geq 0\}$  is the space-time Brownian motion conditioned to stay in  $C_{\text{aff}}$ .*

Recall that  $u_t$ , resp.  $q_t$ , is the transition probability density of the killed Brownian motion in  $[0, 1]$ , resp. of  $Z$  (see (5.1)).

**Lemma 5.6.** For  $0 < r \leq t$ ,

$$u_{1/r-1/t}(y/t, x/r)e^{-\frac{1}{2t}y^2} = w_{t-r}^0((r, x), (t, y))e^{-\frac{1}{2r}x^2}, \text{ for } 0 \leq x \leq r, 0 \leq y \leq t,$$

$$q_t(x, y) = c_t \sin(\pi y)e^{-\frac{1}{2t}(y-x)^2} \varphi_x(1/t, y/t), \text{ for } 0 \leq x, y \leq 1.$$

*Proof.*

$$\begin{aligned} u_{1/r-1/t}(y/t, x/r) &= \sum_{k \in \mathbb{Z}} (p_{1/r-1/t}(y/t + 2k, x/r) - p_{1/r-1/t}(-y/t - 2k, x/r)) \\ &= \sum_{k \in \mathbb{Z}} \frac{p_r(0, x)}{p_t(0, y + 2kt)} (p_{t-r}(x, y + 2kt) - p_{t-r}(x, -y - 2kt)) \\ &= \frac{p_r(0, x)}{p_t(0, y)} \sum_{k \in \mathbb{Z}} e^{-2kx - 2k^2r} (p_{t-r}(x + 2kt, y) - p_{t-r}(-x - 2kr, y)), \end{aligned}$$

which gives the first equality, by identity (5.5). The second one follows from Lemma 2.41 and (5.3).  $\square$

**Theorem 5.7.** For  $\mu \in [0, 1]$ , the processes  $\{A_t^{(\mu)}, t > 0\}$  and  $\{(t, tZ_{1/t}), t > 0\}$ , where  $Z$  starts from  $\mu$ , are equal in law.

*Proof.* We let  $X_t = tZ_{1/t}$ ,  $t > 0$ . For  $0 < t_1 < \dots < t_n$ , the quantity  $\mathbb{P}(X_{t_1} \in dx_1, \dots, X_{t_n} \in dx_n)$  is equal to

$$q_{\frac{1}{t_n}}(\mu, \frac{x_n}{t_n}) q_{\frac{1}{t_{n-1}} - \frac{1}{t_n}}(\frac{x_n}{t_n}, \frac{x_{n-1}}{t_{n-1}}) \cdots q_{\frac{1}{t_1} - \frac{1}{t_2}}(\frac{x_2}{t_2}, \frac{x_1}{t_1}) dx_1 \cdots dx_n.$$

We conclude by using the Lemma 5.6.  $\square$

This implies that  $A_t^{(\mu)}$  is really in the interior of the cone for  $t > 0$ .

**5.4. A property of Pitman transform for piecewise  $C^1$  paths.** As mentioned in the introduction, the need of an infinite number of Pitman transforms to represent the space-time Brownian motion in  $C_{\text{aff}}$  is due to its wild behaviour. The situation is much simpler for regular curves, as shown by the following proposition (recall that paths in  $C_{\text{aff}}$  are fixed under the Pitman transforms).

**Proposition 5.8.** Let  $\pi : [0, T] \rightarrow \mathbb{R}$  and  $\eta(t) = (t, \pi(t))$  be a continuous piecewise  $C^1$  path such that  $\pi(0) = 0$ . There is an  $n$  such that for all  $t \in [0, T]$ ,

$$\mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_1} \mathcal{P}_{\alpha_0} \eta(t) \in \bar{C}_{\text{aff}}.$$

We use the notations of Section 3. It is equivalent to prove that there is an  $n > 0$  such that

$$0 \leq \mathcal{P}_n \cdots \mathcal{P}_1 \mathcal{P}_0 \pi(t) \leq t,$$

for all  $t \in [0, T]$ . Let  $\tau_1(\pi) = \inf\{t > 0; \pi(t) < 0\}$ ,  $\tau_0(\pi) = \inf\{t > 0; \pi(t) > t\}$ , and let  $|\pi'|$  be the supremum of the left and right derivatives of  $\pi$  on  $[0, T]$ .

**Lemma 5.9.** (1)  $|(\mathcal{P}_1 \pi)'| \leq |\pi'|$ , and  $|(\mathcal{P}_0 \pi)'| \leq 2 + |\pi'|$ .

(2) When  $0 \leq t \leq \tau_1(\pi) \wedge \tau_0(\pi)$ ,  $\mathcal{P}_1 \pi(t) = \mathcal{P}_0 \pi(t) = \pi(t)$ .

(3) There is an  $n > 0$  such that  $0 \leq (\mathcal{P}_n \cdots \mathcal{P}_2 \mathcal{P}_1 \pi)'(0) \leq 1$ .

*Proof.* (1) and (2) are straightforward. For (3): if  $s_0$  and  $s_1$  are given by  $s_0(x) = -x$  and  $s_1(x) = 2 - x$ , then  $(\mathcal{P}_i \pi)'(0) = s_i \pi'(0)$ , for  $i = 0, 1$  and it is well known that one can bring any real into  $[0, 1]$  by the actions of  $s_0$  and  $s_1$ .  $\square$

**Lemma 5.10.** *Proposition 5.8 holds when  $\tau_1(\pi) \wedge \tau_0(\pi) > 0$ .*

*Proof.* We first suppose that  $\tau_0(\pi) < \tau_1(\pi)$ . Let  $\pi_1 = \mathcal{P}_0\pi, \pi_2 = \mathcal{P}_1\pi_1, \pi_3 = \mathcal{P}_0\pi_2, \dots$  and  $a_1 = \tau_0(\pi), a_2 = \tau_1(\pi_1), a_3 = \tau_0(\pi_2), \dots$ . By (1) of Lemma 5.9,  $|\pi'_n| \leq |\pi'| + 2n$ . Since  $\pi_{2n}(a_{2n}) = 0, \pi_{2n}(a_{2n+1}) = a_{2n+1}$ , we obtain by the mean value theorem that

$$a_{2n+1} - a_{2n} \geq \frac{a_{2n+1}}{|\pi'| + 4n}$$

and so the increasing sequence  $a_n$  is bigger than  $T$  for  $n$  large enough. When  $\tau_0(\pi) > \tau_1(\pi)$ , then  $\tau_0(\pi_1) > \tau_1(\pi_1)$  and we apply the same proof to  $\pi_1$ , beginning by applying  $\mathcal{P}_1$ .  $\square$

*Proof of Proposition 5.8.* Using (3) of Lemma 5.9 one can suppose that  $0 \leq \pi'(0) \leq 1$ . When  $0 < \pi'(0) < 1$ , then  $\tau_1(\pi) \wedge \tau_0(\pi) > 0$  and we can apply Lemma 5.10. If  $\pi'(0) = 1$ , let  $\gamma = \mathcal{P}_0\pi$ . Then  $\gamma'(0) = 1$ , and  $\gamma(t) \leq t$  for  $t \geq 0$ . In that case  $\tau_0(\gamma) \geq T$ , and  $\tau_1(\gamma) > 0$ . Indeed otherwise there is a sequence  $t_n$  decreasing to 0 such that  $\gamma(t_n) \geq 0n$  and  $\gamma'(0)$  is not 1. So we can also apply Lemma 5.10. The case  $\pi'(0) = 0$  is similar.  $\square$

**5.5. Anti-lecture hall compositions.** The set  $A_\infty$  of anti-lecture hall compositions is defined in Corteel and Savage [7] as the set of sequence of integers  $\lambda_1, \lambda_2, \dots$  such that, for some  $n \in \mathbb{N}$ ,

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \dots \geq \frac{\lambda_n}{n} > 0,$$

and  $\lambda_p = 0$  when  $p > n$ . So we see that

$$A_\infty = \{(\lambda_1, \lambda_2, \dots); (0, \lambda_1, \lambda_2, \dots) \in B(\infty)\}.$$

For  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n, 0, 0, \dots) \in A_\infty$ , let  $|\lambda| = \sum_{k=1}^n \lambda_k$ . It is known, see Corteel et al. [7, 8] and (1.2) and (1.3) in Chen et al. [6], that for  $0 \leq q < 1$ ,

$$(5.8) \quad \sum_{\lambda \in A_\infty} q^{|\lambda|} = \frac{(-q; q)_\infty}{(q^2; q)_\infty},$$

$$(5.9) \quad \sum_{\lambda \in A_\infty, \lambda_1 \leq k} q^{|\lambda|} = \frac{(-q; q)_\infty (q; q^{k+2})_\infty (q^{k+1}; q^{k+2})_\infty (q^{k+2}; q^{k+2})_\infty}{(q; q)_\infty},$$

where  $(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)$ .

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