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Philippe Bougerol, Manon Defosseux

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PITMAN TRANSFORMS AND BROWNIAN MOTION IN THE INTERVAL VIEWED AS AN AFFINE ALCOVE

PHILIPPE BOUGEROL AND MANON DEFOSSEUX

ABSTRACT. Pitman's theorem states that if $\{B_t, t \geq 0\}$ is a one-dimensional Brownian motion, then $\{B_t - 2 \inf_{s \leq t} B_s, t \geq 0\}$ is a three dimensional Bessel process, i.e. a Brownian motion conditioned in Doob sense to remain forever positive. In this paper one gives a similar representation for the Brownian motion in an interval. Due to the double barrier, it is much more involved and only asymptotic. This uses the fact that the interval is an alcove of the Affine Lie algebra A_1^1 .

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1. INTRODUCTION

The probability transition of the Brownian motion conditioned to stay positive $R_t, t \geq 0$, is the difference of two heat kernels. This is a consequence of the reflection principle at zero. Pitman's theorem ([32]) of 1994 gives the path representation $R = \mathcal{P}B$ where B is a standard Brownian motion and

$$\mathcal{P}B_t = B_t - 2 \min_{0 \leq s \leq t} B_s.$$

The transformation $\mathcal{P}B$ is written with the reflection at 0. The Brownian motion Z_t conditioned to stay in the interval $[0, 1]$ forever can be seen at the Doob transform of the Brownian motion with Dirichlet condition at 0 and 1. It is classical that its probability transition is an alternating infinite sum which can be obtained by applying successive principles of reflection at 0 and 1 (method of images). It is therefore natural to ask oneself if Pitman's theorem as an analogue written with an infinite number of transformations at 0 and 1. The main result of this article is to show that, at our surprise, this is not exactly true. A small correction (a Lévy transform) has to be added. We give the precise statement in Section 2.

The proof of our main result is given in Section 3. It uses only basic tools (no knowledge on Kac Moody algebra is necessary) but is somewhat intricate. Many notions that are used come from representation theory. We first now explain the ideas behind the scene in this perspective because this theory has been a source of inspiration and also as a reading guide for the paper.

An interval can be viewed as a fundamental alcove for the action of an affine Weyl group. A key remark is that one can write $Z_t = tB(1/t)$ where (t, B_t) is the space time real Brownian motion conditioned to $0 \leq B_s \leq s$ for any $s \geq 0$. This can be interpreted as a condition to stay in a Weyl chamber of the affine Lie algebra A_1^1 .

Let us elaborate on this point of view, by first recalling the link between the classical Pitman's theorem and the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, through Littelmann path theory. Let α be a positive root and $V = \mathbb{R}\alpha$ (identified with the real part of the dual of the Cartan subalgebra).

Fix $T > 0$. A path $\pi : [0, T] \rightarrow V$ is a continuous function such that $\pi(0) = 0$. It can be written as $\pi(t) = f(t)\alpha$ with $f(t) \in \mathbb{R}$. Fix $T > 0$. An integral path is a path such that $2f(T), 2 \min_{t \leq T} f(t) \in \mathbb{Z}$. A dominant path is a path with values in the Weyl chamber, which is here $\mathbb{R}_+\alpha$. For an irreducible highest weight $\mathfrak{sl}_2(\mathbb{C})$ -module, one chooses an integral dominant path π defined on $[0, T]$, such that $\pi(T)$ is its highest weight. A path realization of the crystal associated to this module is the Littelmann module $B\pi$, which is the set of integral paths η defined on $[0, T]$ such that $\mathcal{P}_\alpha \eta = \pi$, where \mathcal{P}_α is the path transformation defined by

$$(1.1) \quad \mathcal{P}_\alpha \eta(t) = (\varphi(t) - 2 \inf_{s \leq t} \varphi(s))\alpha,$$

where $\eta(t) = \varphi(t)\alpha$. One recognizes the Pitman's transform.

In the case of the A_1^1 affine Lie algebra, a realization of a real Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}$ and of its dual $\mathfrak{h}_{\mathbb{R}}^*$ is given by three dimensional vector spaces defined by

$$\mathfrak{h}_{\mathbb{R}} = \text{Span}_{\mathbb{R}}\{\tilde{\alpha}_1, c, d\}, \quad \text{and} \quad \mathfrak{h}_{\mathbb{R}}^* = \text{Span}_{\mathbb{R}}\{\Lambda_0, \alpha_1, \delta\},$$

where $\tilde{\alpha}_1 = (0, 1, 0)$, $c = (1, 0, 0)$, $d = (0, 0, 1)$, and $\Lambda_0 = (1, 0, 0)$, $\alpha_1 = (0, 2, 0)$, $\delta = (0, 0, 1)$. We let $\tilde{\alpha}_0 = (1, -1, 0)$ and $\alpha_0 = (0, -2, 1)$ so that $\delta = \alpha_0 + \alpha_1$. In the following one identifies the quotient space $\mathfrak{h}_{\mathbb{R}}^*/\mathbb{R}\delta$, in which $\alpha_0 + \alpha_1 = 0$, with $\mathbb{R}\Lambda_0 \oplus \mathbb{R}\alpha_1$. The Weyl group is generated by the fundamental reflections s_{α_i} , $i = 0, 1$, defined on $\mathfrak{h}_{\mathbb{R}}^*/\mathbb{R}\delta$ by $s_{\alpha_i}(v) = v - \tilde{\alpha}_i(v)\alpha_i$. A fundamental chamber for its action is the convex cone

$$C_{\text{aff}} = \{t\Lambda_0 + x\alpha_1/2 : t, x \in \mathbb{R}, \text{ s.t. } 0 \leq x \leq t\}.$$

A path η is now a continuous function defined on an interval $[0, T]$, with values in $\mathfrak{h}_{\mathbb{R}}^*/\mathbb{R}\delta$, such that $\eta(0) = 0$. One defines path transformations \mathcal{P}_{α_i} , $i \in \{0, 1\}$, by

$$\mathcal{P}_{\alpha_i}\eta(t) = \eta(t) - \min_{s \leq t} \tilde{\alpha}_i(\eta(s))\alpha_i, \quad t \in [0, T].$$

A dominant path is a path with values in C_{aff} . As above, one can define integral paths. For an integral dominant path π defined on $[0, T]$, the Littelmann paths module generated by π is the set of integral paths η defined on $[0, T]$ such that it exists $n \in \mathbb{N}$ such that

$$\mathcal{P}_{\alpha_n}\mathcal{P}_{\alpha_{n-1}} \cdots \mathcal{P}_{\alpha_1}\mathcal{P}_{\alpha_0}\eta = \pi,$$

where $\alpha_n = \alpha_0$ when n is even and $\alpha_n = \alpha_1$ when n is odd. This gives a description of the crystal of highest weight $\pi(T)$ (see [29], [3]).

For an integral path η , and $t \in [0, T]$, and more generally for continuous piecewise C^1 path (see Proposition 5.6), there is a $k \in \mathbb{N}$ such that for all $n \geq k$,

$$\mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_0}\eta(t) = \mathcal{P}_{\alpha_k} \cdots \mathcal{P}_{\alpha_0}\eta(t).$$

We let

$$(1.2) \quad \mathcal{P}\eta = \lim_{n \rightarrow \infty} \mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_0}\eta.$$

A Brownian motion conditioned to remain in an interval $[0, 1]$ is equal in law, up to a time inversion, to the space component of the space time Brownian motion $B^{(0)} = \{t\Lambda_0 + B_t\alpha_1/2, t \geq 0\}$, conditioned to remain in C_{aff} , where B is a standard real Brownian motion (see Section 5). Thus it is natural to ask if the limit of (1.2) exists when η is replaced by the space-time Brownian motion $B^{(0)}$. We prove in this paper that the answer is no. Nevertheless one proves that a slight modification converges to the space-time Brownian motion $B^{(0)}$ conditioned to remain in the affine Weyl chamber C_{aff} .

Let us explain the strategy of the proof. We have to understand the sequence

$$(1.3) \quad \mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_0}B^{(0)}.$$

One can associate to any path η defined on $[0, T]$ a sequence of positive real numbers called the affine string parameters of η , which when η is integral coincide with the string parameters of Kashiwara or Littelmann of the element corresponding in the crystal, ([26],[29]). Let $\xi(T) = (\xi_i(T))_{i \in \mathbb{N}}$ be the sequence of affine string parameters of the space-time Brownian motion with drift $B^{(\mu)} = \{t\Lambda_0 + (B_t + \mu t)\alpha_1/2, 0 \leq t \leq T\}$. One has

$$\mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_0}B^{(\mu)}(T) = B^{(\mu)}(T) + \sum_{i=0}^n \xi_i(T)\alpha_i.$$

The point is the following: when $0 < \mu < 1$ the limits $\xi_i = \lim_{T \rightarrow +\infty} \xi_i(T)$ are well defined (we call them the Verma affine string parameters as they correspond to the crystal $B(\infty)$). Moreover their law can be expressed easily with independent exponential random variables. The law of $\xi(T)$ can be deduced by a truncation argument. Finally we prove that a.s.

$$(1.4) \quad \mathcal{P}B^{(\mu)}(t) = B_t^{(\mu)} + \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \xi_i(t) \alpha_i + \frac{1}{2} \xi_n(t) \alpha_n, \quad t \geq 0,$$

and identify the limit.

Actually to prove these results we approximate the space time Brownian motion $B^{(\mu)}$ by planar Brownian motions with proper drifts and the affine Weyl group by finite dihedral groups $I(m), m \geq 1$, whose positive Weyl chambers are a wedge in \mathbb{R}^2 of dihedral angle π/m and use deeply the results of [3] in this situation. Due to the need of the correction term the approximation is not immediate.

The paper is organized as follows. In Section 2 we state the main result of the paper. In Section 3.1 we recall the Pitman representation theorem for a planar Brownian motion in a dihedral cone and give a precise description of the law of the string and Verma string parameters. One studies their asymptotic behaviour in Section 3.2 and obtains a description of the Verma affine string parameters $\xi = (\xi_k)$ of a space time Brownian motion. In Section 3.3 one proves that the series $\sum_{i=0}^n \xi_i \alpha_i$ up to a correcting term, converges. In Section 3.5 we prove the almost sure convergence of (1.4). The limit, called the highest weight process is shown to be the space time Brownian motion conditioned to remain in the affine Weyl chamber C_{aff} . One obtains in Section 3.6 a representation theorem for this conditioned space-time Brownian motion by applying a last Lévy transform to the iterates of Pitman's ones. In Section 3.8 one makes some comments on the distributions of the first string parameter ξ_1 of the space time Brownian motion and of its affine Verma weight.

In Section 4, we show how our results are related to highest weight representation of the affine Lie algebra A_1^1 . We consider a kind of Duistermaat Heckman measure appearing as a semiclassical limit, and study the asymptotic of large weights for the infinity crystal $B(\infty)$. Section 5 is an appendix where we define the conditioned Brownian motion in the interval $[0, 1]$ and the conditioned space-time Brownian motion in the affine Weyl chamber and prove that the first one is equal in law, up to a time inversion, to the space component of the second one. Notice that approximation by finite dihedral group is very natural but unusual in the study of A_1^1 .

2. REPRESENTATION OF THE CONDITIONED BROWNIAN MOTION IN $[0, 1]$

In this section we state the main result of this paper for the Brownian motion in the interval $[0, 1]$. We consider, for a continuous path $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$, such that $\varphi(0) = 0$,

$$\begin{aligned} \mathcal{L}_1 \varphi(t) &= \varphi(t) - \inf_{0 \leq s \leq t} \varphi(s) \\ \mathcal{P}_1 \varphi(t) &= \varphi(t) - 2 \inf_{0 \leq s \leq t} \varphi(s), \end{aligned}$$

that we call the classical Lévy and and Pitman transform. The first occurs in the representation of the reflected Brownian motion, the second in the representation of the Brownian motion conditioned to remain positive.

Let $T\varphi(t) = t - \varphi(t)$. We introduce the transformations \mathcal{L}_0 and \mathcal{P}_0 given by $T\mathcal{L}_0 = \mathcal{L}_1T$ and $T\mathcal{P}_0 = \mathcal{P}_1T$, namely

$$\begin{aligned}\mathcal{L}_0\varphi(t) &= \varphi(t) + \inf_{0 \leq s \leq t} (s - \varphi(s)) \\ \mathcal{P}_0\varphi(t) &= \varphi(t) + 2 \inf_{0 \leq s \leq t} (s - \varphi(s)).\end{aligned}$$

The Brownian motion with drift is $B_t^\mu = B_t + t\mu, t \geq 0$, where B_t is the standard Brownian motion with $B_0 = 0$. We let $\mathcal{P}_n = \mathcal{P}_0$ and $\mathcal{L}_n = \mathcal{L}_0$ when n is even and $\mathcal{P}_n = \mathcal{P}_1$ and $\mathcal{L}_n = \mathcal{L}_1$ when n is odd. The main result of this paper is the following representation theorem.

Theorem 2.1. *Let $\mu \in [0, 1]$. For any $t > 0$, almost surely,*

$$\lim_{n \rightarrow \infty} t\mathcal{L}_n\mathcal{P}_{n-1} \cdots \mathcal{P}_1\mathcal{P}_0 B^\mu(1/t) = \lim_{n \rightarrow \infty} t\mathcal{L}_n\mathcal{P}_{n-1} \cdots \mathcal{P}_2\mathcal{P}_1 B^\mu(1/t) = Z_t$$

where $\{Z_t, t \geq 0\}$ is the standard Brownian motion starting from μ , conditioned to remain in $[0, 1]$ forever (see Definition 5.1).

The proof is the juxtaposition of Theorem 3.37 and Theorem 5.5. We also show that, a.s.,

$$\lim_{n \rightarrow \infty} t\mathcal{P}_{n-1} \cdots \mathcal{P}_1\mathcal{P}_0 B^\mu(1/t) + (-1)^n = Z(t)$$

which clearly shows the need of a correction for the Pitman transforms. As an illustration we prove that (see Proposition 3.45),

Proposition 2.2. *When $0 < \mu < 1$, there is a standard Brownian motion W , a stopping time $\sigma > 0$ for W and a random variable D^μ such that for $0 \leq t \leq \sigma$,*

$$Z_t = \mu + W_t + tD^\mu.$$

3. REPRESENTATION OF THE CONDITIONED SPACE-TIME BROWNIAN MOTION IN THE AFFINE WEYL CONE

In order to represent the Brownian motion Z conditioned to stay in $[0, 1]$, we will linearize the problem and use the fact that Z is the space component of the time inverted process of $A^{(\mu)}$, where $A^{(\mu)}$ is the space-time Brownian motion conditioned to stay in an affine cone C_{aff} (see Appendix 5.3). This process $A^{(\mu)}$ will be approached by the planar Brownian motion conditioned to remain in a dihedral cone.

3.1. Pitman representation for the dihedral cone C_m . In this section we describe the representation of the planar Brownian motion conditioned to stay in a dihedral cone, using the results of [3],[4].

3.1.1. Dihedral Coxeter system. The dihedral group $I(m)$, where $m \in \mathbb{N}$, is the finite group generated by two involutions s_0, s_1 with the only relation $(s_0s_1)^m = 1$. In the two dimensional vector space $V = \mathbb{R}^2$ identified with its dual \tilde{V} , one chooses two pairs $(v_0^m, \tilde{v}_0^m), (v_1^m, \tilde{v}_1^m)$ in $V \times \tilde{V}$, such that

$$\begin{aligned}\tilde{v}_0^m(v_0^m) &= \tilde{v}_1^m(v_1^m) = 2, \\ \tilde{v}_0^m(v_1^m) &= \tilde{v}_1^m(v_0^m) = -2 \cos(\pi/m).\end{aligned}$$

Namely, one takes

$$v_0^m = (2 \sin(\pi/m), -2 \cos(\pi/m)), v_1^m = (0, 2),$$

and $\tilde{v}_0^m = v_0^m/2, \tilde{v}_1^m = v_1^m/2$. The following two reflections s_0^m, s_1^m of \mathbb{R}^2 ,

$$s_i^m(v) = v - \tilde{v}_i^m(v)v_i^m, \quad v \in \mathbb{R}^2,$$

generate the group $I(m)$. The usual scalar product on \mathbb{R}^2 is denoted by $\langle \cdot, \cdot \rangle$, with our convention $\tilde{v}_i^m(v) = \langle v_i^m, v \rangle$.

Definition 3.1. *The convex dihedral cone C_m , with closure \bar{C}_m , is*

$$\begin{aligned} C_m &= \{v \in \mathbb{R}^2; \tilde{v}_i^m(v) > 0, i = 0, 1\} \\ &= \{(r \cos \theta, r \sin \theta) \in \mathbb{R}^2, r > 0, 0 < \theta < \pi/m\}. \end{aligned}$$

Let $\mathcal{C}_0(\mathbb{R}^2)$ be the set of continuous path $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^2$ such that $\eta(0) = 0$. The following path transformation is introduced in [3].

Definition 3.2. *The Pitman transform $\mathcal{P}_{s_i^m}^m, i = 0, 1$, is defined on $\mathcal{C}_0(\mathbb{R}^2)$ by the formula, for $\eta \in \mathcal{C}_0(\mathbb{R}^2), t \geq 0$,*

$$\mathcal{P}_{s_i^m}^m \eta(t) = \eta(t) - \inf_{t \geq s \geq 0} \tilde{v}_i^m(\eta(s))v_i^m.$$

Notice that

$$\mathcal{P}_{s_i^m}^m \eta(t) = \eta(t) - \inf_{t \geq s \geq 0} (id - s_i^m)\eta(s).$$

Theorem 3.3 ([3]). *Let $w = s_{i_r}^m \cdots s_{i_1}^m$ be a reduced decomposition of $w \in I(m)$ with $s_{i_1}^m, \dots, s_{i_r}^m \in \{s_0^m, s_1^m\}$. Then*

$$\mathcal{P}_w^m := \mathcal{P}_{s_{i_r}^m}^m \cdots \mathcal{P}_{s_{i_1}^m}^m$$

depends only on w and not on the chosen decomposition.

For each $i \in \mathbb{N}$, when we write $v_i^m, \tilde{v}_i^m, s_i^m, \dots$, we take i modulo 2. In $I(m)$ there is a unique longest word, namely

$$(3.1) \quad w_0 = s_{m-1}^m \cdots s_1^m s_0^m = s_m^m \cdots s_2^m s_1^m.$$

Proposition 3.4 ([3]). *For any path $\eta \in \mathcal{C}_0(\mathbb{R}^2)$, the path $\mathcal{P}_{w_0}^m \eta$ takes values in the closed dihedral cone \bar{C}_m .*

Let $\gamma \in \mathbb{R}^2$ and let $W^{(\gamma)}$ be the standard planar Brownian motion in \mathbb{R}^2 with drift γ and identity covariance matrix, starting from the origin. The conditioned planar Brownian motion in the cone C_m with drift $\gamma \in \bar{C}_m$ rigorously defined in [3], [4], is intuitively given by $W^{(\gamma)}$ conditioned to stay in C_m forever (see Section 5.2). We first recall:

Theorem 3.5 ([3]). *Let $W^{(\gamma)}$ be the planar Brownian motion with drift γ , where $\gamma \in \bar{C}_m$. Then $\mathcal{P}_{w_0}^m W^{(\gamma)}$ is the conditioned planar Brownian motion in the cone C_m with drift γ .*

3.1.2. *String parameters in $\mathcal{C}_0(\mathbb{R}^2)$.* We fix an integer $m \geq 1$. For simplicity of notations, one chooses the decomposition

$$w_0 = s_{m-1}^m \cdots s_1^m s_0^m$$

of the longest word w_0 in $I(m)$. Notice that here the roles of s_0^m and s_1^m are symmetric. When $\eta \in \mathcal{C}_0(\mathbb{R}^2)$, one defines in [3] new paths $\eta_k \in \mathcal{C}_0(\mathbb{R}^2)$, $0 \leq k \leq m-1$, by $\eta_0 = \eta$, and, for $k \geq 1$,

$$\eta_k = \mathcal{P}_{s_{k-1}^m}^m \cdots \mathcal{P}_{s_1^m}^m \mathcal{P}_{s_0^m}^m \eta,$$

called Pitman's transforms of η . Let

$$x_k^m(t) = - \min_{0 \leq s \leq t} \tilde{v}_k^m(\eta_k(s)).$$

Definition 3.6. *Let η be a path in $\mathcal{C}_0(\mathbb{R}^2)$ and*

$$x^m(t) = (x_0^m(t), x_1^m(t), \dots, x_{m-1}^m(t)).$$

We call $x_0^m(t), \dots, x_{m-1}^m(t)$ the string parameters of the path $\{\eta(s), 0 \leq s \leq t\}$. When

$$(3.2) \quad \lim_{s \rightarrow +\infty} \tilde{v}_0^m(\eta(s)) = \lim_{s \rightarrow +\infty} \tilde{v}_1^m(\eta(s)) = +\infty,$$

we define the Verma string parameters x_0^m, \dots, x_{m-1}^m of η by

$$x_k^m = - \min_{0 \leq s < +\infty} \tilde{v}_k^m(\eta_k(s)).$$

Notice that

$$(3.3) \quad \mathcal{P}_{w_0}^m \eta(t) = \eta(t) + \sum_{k=0}^{m-1} x_k^m(t) v_k^m.$$

When (3.2) holds, then the Pitman transforms of η have the same property and for all $t \geq 0$ and $0 \leq k \leq m-1$,

$$(3.4) \quad 0 \leq x_k^m(t) \leq x_k^m < +\infty.$$

Remark 3.7. *For $t \geq 0$, for all $k = 0, \dots, m-1$,*

$$2x_k^m(t) = \tilde{v}_k^m(\mathcal{P}_{s_k^m}^m \cdots \mathcal{P}_{s_0^m}^m \eta(t) - \mathcal{P}_{s_{k-1}^m}^m \cdots \mathcal{P}_{s_0^m}^m \eta(t)).$$

Let, for $1 \leq k < m$,

$$a_k^m = \sin(k\pi/m).$$

Definition 3.8. *The cone Γ_m in \mathbb{R}^m is defined as*

$$\Gamma_m = \{(x_0, \dots, x_{m-1}) \in \mathbb{R}^m; \frac{x_1}{a_1^m} \geq \frac{x_2}{a_2^m} \geq \dots \geq \frac{x_{m-1}}{a_{m-1}^m} \geq 0, x_0 \geq 0\}.$$

For $\lambda \in \bar{C}_m$, the polytope $\Gamma_m(\lambda)$ is

$$\Gamma_m(\lambda) = \{(x_0, \dots, x_{m-1}) \in \Gamma_m; 0 \leq x_r \leq \tilde{v}_r^m(\lambda - \sum_{n=r+1}^{m-1} x_n v_n^m), 0 \leq r \leq m-1\}.$$

Notice that the particular role of x_0 . The following proposition is proved in [4], Section 5.1, for the string parameters, its extension to Verma ones is immediate.

Proposition 3.9 ([4], 5.1). *The set of string parameters of paths in $\mathcal{C}_0(\mathbb{R}^2)$ is contained in Γ_m and the set of Verma string parameters is equal to Γ_m .*

3.1.3. String parameters of the planar Brownian motion. When $\gamma \in C_m$, the planar Brownian motion $W^{(\gamma)}$ with drift γ satisfies (3.2) by the law of large numbers, so its Verma string parameters are well defined:

Definition 3.10. *Let $\gamma \in \bar{C}_m$. We denote with $\xi^m(t) = (\xi_0^m(t), \dots, \xi_{m-1}^m(t))$, the string parameters of $\{W_s^{(\gamma)}, 0 \leq s \leq t\}$ and with $\xi^m = (\xi_0^m, \dots, \xi_{m-1}^m)$ the Verma string parameters of $W^{(\gamma)}$ when $\gamma \in C_m$.*

Theorem 3.11. *(i) When $\gamma \in \bar{C}_m$, for any $t > 0$, conditionally on the σ -algebra $\sigma(\mathcal{P}_{w_0}^m W^{(\gamma)}(s), s \leq t)$ the random variables $\xi_k^m(t), 0 \leq k < m$, are independent exponentials with parameters $\gamma_k^m = \langle \gamma, v_k^m \rangle$ conditioned to be in $\Gamma_m(\mathcal{P}_{w_0}^m W^{(\gamma)}(t))$.*

(ii) When $\gamma \in C_m$, the random variables $\xi_k^m, 0 \leq k < m$, are independent exponentials with parameters $\gamma_k^m = \langle \gamma, v_k^m \rangle$ conditioned to be in Γ_m .

Proof. It is shown in [4], Theorem 5.2, that when $\gamma = 0$, the law of $\xi^m(t)$ conditionally on $\mathcal{F}_t = \sigma(\mathcal{P}_{w_0}^m W^{(0)}(s), s \leq t)$ is, when $\mathcal{P}_{w_0}^m W^{(0)}(t) = \lambda$, the normalized Lebesgue measure on $\Gamma_m(\lambda)$. It follows from (3.3) that

$$(3.5) \quad W_t^{(0)} = \mathcal{P}_{w_0}^m W^{(0)}(t) - \sum_{k=0}^{m-1} \xi_k^m(t) v_k^m.$$

By the Bayes and the Cameron-Martin formulas, for any bounded measurable $\psi : \mathcal{C}_0(\mathbb{R}^2) \rightarrow \mathbb{R}$,

$$\begin{aligned} & \mathbb{E}(\psi(W_s^{(\gamma)}, 0 \leq s \leq t) | \mathcal{F}_t) \\ &= \frac{\mathbb{E}(\psi(W_s^{(0)}, 0 \leq s \leq t) e^{\langle \gamma, W_t^{(0)} \rangle} | \mathcal{F}_t)}{\mathbb{E}(e^{\langle \gamma, W_t^{(0)} \rangle} | \mathcal{F}_t)} \\ &= \frac{\mathbb{E}(\psi(W_s^{(0)}, 0 \leq s \leq t) e^{\langle \gamma, \mathcal{P}_{w_0}^m W^{(0)}(t) - \sum_{k=0}^{m-1} \xi_k^m(t) v_k^m \rangle} | \mathcal{F}_t)}{\mathbb{E}(e^{\langle \gamma, \mathcal{P}_{w_0}^m W^{(0)}(t) - \sum_{k=0}^{m-1} \xi_k^m(t) v_k^m \rangle} | \mathcal{F}_t)}. \end{aligned}$$

If we take $\psi(W_s^{(\gamma)}, 0 \leq s \leq t) = F(\xi^m(t))$ where $F : \mathbb{R}^m \rightarrow \mathbb{R}$, and if dx is the Lebesgue measure on \mathbb{R}^m we obtain that

$$\mathbb{E}(F(\xi^m(t)) | \mathcal{F}_t) = \frac{\int 1_{\Gamma_m(\mathcal{P}_{w_0}^m W_t^{(\gamma)})}(x) F(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx}{\int 1_{\Gamma_m(\mathcal{P}_{w_0}^m W_t^{(\gamma)})}(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx},$$

which proves (i). We suppose that $\gamma \in C_m$, so for any $0 \leq k < m$, $\xi_k^m = \lim_{t \rightarrow +\infty} \xi_k^m(t)$. Therefore, when F is bounded and continuous,

$$\begin{aligned} \mathbb{E}(F(\xi^m)) &= \lim_{t \rightarrow +\infty} \mathbb{E}(F(\xi^m(t))) \\ &= \lim_{t \rightarrow +\infty} \mathbb{E}(\mathbb{E}(F(\xi^m(t)) | \mathcal{F}_t)) \\ &= \lim_{t \rightarrow +\infty} \mathbb{E}\left(\frac{\int 1_{\Gamma_m(\mathcal{P}_{w_0}^m W_t^{(\gamma)})}(x) F(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx}{\int 1_{\Gamma_m(\mathcal{P}_{w_0}^m W_t^{(\gamma)})}(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx}\right) \\ &= \frac{\int 1_{\Gamma_m}(x) F(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx}{\int 1_{\Gamma_m}(x) e^{-\sum_{k=0}^{m-1} \gamma_k^m x_k} dx}, \end{aligned}$$

since, a.s., $\mathcal{P}_{w_0}^m W_t^{(\gamma)}/t$ converges to γ as $t \rightarrow +\infty$ and thus each $\tilde{v}_r^m(\mathcal{P}_{w_0}^m W_t^{(\gamma)})$ tends to $+\infty$. This proves (ii). \square

Proposition 3.12. *Let $\varepsilon_n, n \in \mathbb{N}$, be independent exponential random variables with parameter 1. When $\gamma \in C_m$, in law, $\xi_0^m = \varepsilon_0/\gamma_0^m$, and for $k = 1, \dots, m-1$ and $\gamma_k^m = \langle \gamma, v_k^m \rangle$,*

$$\xi_k^m = a_k^m \sum_{l=k}^{m-1} \frac{\varepsilon_l}{\gamma_1^m a_1^m + \dots + \gamma_l^m a_l^m} a_l^m.$$

Proof. Theorem 3.11 says that

$$\begin{aligned} & \mathbb{P}(\xi_0^m \in dx_0, \dots, \xi_{m-1}^m \in dx_{m-1}) \\ &= C e^{-\gamma_0^m x_0} dx_0 \prod_{k=2}^m e^{-\gamma_{k-1}^m x_{k-1}} 1_{\{\frac{x_{k-1}}{a_{k-1}^m} \geq \frac{x_k}{a_k^m}\}} dx_{k-1} \\ &= C e^{-\gamma_0^m x_0} dx_0 \prod_{k=2}^m e^{-(\gamma_1^m a_1^m + \dots + \gamma_{k-1}^m a_{k-1}^m)(\frac{x_{k-1}}{a_{k-1}^m} - \frac{x_k}{a_k^m})} 1_{\{\frac{x_{k-1}}{a_{k-1}^m} \geq \frac{x_k}{a_k^m}\}} dx_{k-1} \end{aligned}$$

where by convention $x_m/a_m^m = 0$ and C is a normalizing constant. Thus

$$(\xi_0^m, \frac{\xi_1^m}{a_1^m} - \frac{\xi_2^m}{a_2^m}, \frac{\xi_2^m}{a_2^m} - \frac{\xi_3^m}{a_3^m}, \dots, \frac{\xi_{m-1}^m}{a_{m-1}^m})$$

has the same law as

$$(\frac{\varepsilon_0}{\gamma_0^m}, \frac{\varepsilon_1}{\gamma_1^m a_1^m}, \frac{\varepsilon_2}{\gamma_1^m a_1^m + \gamma_2^m a_2^m}, \dots, \frac{\varepsilon_{m-1}}{\gamma_1^m a_1^m + \dots + \gamma_{m-1}^m a_{m-1}^m}),$$

which proves the claim. \square

Notice that (1) is similar to Renyi's representation of order statistics ([34]).

Warning 3.13. *In all this paper, when $0 < \mu < 1$, we suppose that m is large enough to have $\frac{m}{\pi} \tan \frac{\pi}{m} > \mu$ to insure that $(\frac{m}{\pi}, \mu) \in C_m$.*

3.2. Affine string parameter ξ of the space-time Brownian motion. We will use a terminology inspired by the Kac-Moody affine algebra A_1^1 (see in particular [25, 26] and Section 4). The infinite dihedral group $I(\infty)$ is the infinite group generated by two involutions s_0, s_1 with no relation. In $V = \mathbb{R}^2$ identified with its dual \tilde{V} , let $(\alpha_0, \tilde{\alpha}_0), (\alpha_1, \tilde{\alpha}_1)$ in $V \times \tilde{V}$, given by

$$\alpha_1 = (0, 2), \alpha_0 = (0, -2),$$

$$\tilde{\alpha}_1 = (0, 1), \tilde{\alpha}_0 = (1, -1),$$

then $\tilde{\alpha}_0(\alpha_0) = \tilde{\alpha}_1(\alpha_1) = 2$, and $\tilde{\alpha}_0(\alpha_1) = \tilde{\alpha}_1(\alpha_0) = -2$. The two reflections s_0, s_1 of \mathbb{R}^2 ,

$$s_i(v) = v - \tilde{\alpha}_i(v)\alpha_i, \quad v \in \mathbb{R}^2,$$

generate the group $I(\infty)$. Notice that s_0 is a non orthogonal reflection. For each $k \in \mathbb{N}$, as above when we write $\alpha_k, \tilde{\alpha}_k, s_k, \dots$, we take k modulo 2. Thus

$$\alpha_k = (-1)^k \alpha_0.$$

Definition 3.14. *The affine Weyl cone (or chamber) is*

$$C_{\text{aff}} = \{v \in \mathbb{R}^2; \tilde{\alpha}_i(v) > 0, i = 0, 1\} = \{(t, x) \in \mathbb{R}^2; 0 < x < t\},$$

and \bar{C}_{aff} is its closure.

As in the dihedral case, we define the Pitman transform \mathcal{P}_{s_i} , $i = 0, 1$, associated with $(\alpha_i, \tilde{\alpha}_i)$ on $\mathcal{C}_0(\mathbb{R}^2)$ by the formula:

$$(3.6) \quad \mathcal{P}_{s_i}\eta(t) = \eta(t) - \inf_{t \geq s \geq 0} \tilde{\alpha}_i(\eta(s))\alpha_i,$$

where $\eta \in \mathcal{C}_0(\mathbb{R}^2)$. We will use space-time paths which can be written as $\eta(t) = (t, \varphi(t))$, $t \geq 0$, in this case,

$$\mathcal{P}_{s_0}\eta(t) = (t, \varphi(t) + 2 \inf_{s \leq t} (s - \varphi(s)))$$

$$\mathcal{P}_{s_1}\eta(t) = (t, \varphi(t) - 2 \inf_{s \leq t} \varphi(s)).$$

One recognizes in the second component the transformations $\mathcal{P}_0, \mathcal{P}_1$ defined in Section 2. One defines $\eta_0 = \eta$, and, for $k \geq 1$,

$$\eta_k = \mathcal{P}_{s_{k-1}} \dots \mathcal{P}_{s_1} \mathcal{P}_{s_0} \eta.$$

Definition 3.15. Let η a path in $\mathcal{C}_0(\mathbb{R}^2)$, and $x(t) = \{x_k(t), k \geq 0\}$ given by

$$x_k(t) = - \min_{0 \leq s \leq t} \tilde{\alpha}_k(\eta_k(s)).$$

We call $x_k(t), k \geq 0$, the affine string parameters of the path $\{\eta(s), 0 \leq s \leq t\}$. When $\lim_{s \rightarrow +\infty} \tilde{\alpha}_i(\eta(s)) = +\infty$, $i = 0, 1$, one defines the Verma affine string parameters of η as $x = \{x_k, k \geq 0\}$ where

$$x_k = - \min_{0 \leq s \leq +\infty} \tilde{\alpha}_k(\eta_k(s)).$$

We fix a real μ such that $0 \leq \mu \leq 1$ and let

$$B_t^{(\mu)} = (t, B_t + t\mu)$$

where B_t is a standard real Brownian motion starting from 0.

Definition 3.16. We let $\xi(t) = \{\xi_k(t), k \geq 0\}$ be the affine string parameters of $\{B_s^{(\mu)}, s \leq t\}$, and, when $0 < \mu < 1$, $\xi = \{\xi_k, k \geq 0\}$ be the Verma affine string parameters of $B^{(\mu)}$.

Let $\tau_m : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$(3.7) \quad \tau_m(t, x) = \left(\frac{\pi t}{m}, x\right),$$

for $(t, x) \in \mathbb{R}^2$. We will frequently use that for $v \in \mathbb{R}^2$,

$$(3.8) \quad \tau_m v = \left(\frac{\pi}{m \sin \frac{\pi}{m}} (\tilde{v}_0^m(v) + \tilde{v}_1^m(v) \cos \frac{\pi}{m}), \tilde{v}_1^m(v)\right),$$

so the asymptotics of $\tau_m v$ and $(\tilde{v}_0^m(v) + \tilde{v}_1^m(v), \tilde{v}_1^m(v))$ are the same as $m \rightarrow \infty$.

Lemma 3.17. For $i = 0, 1$,

$$\begin{aligned} \lim_{m \rightarrow +\infty} \tau_m v_i^m &= \alpha_i \\ \lim_{m \rightarrow +\infty} \tau_m \circ s_i^m \circ \tau_m^{-1} &= s_i. \end{aligned}$$

Proof. The first statement is clear. The second one is trivial when $i = 1$ and for $(t, x) \in \mathbb{R}^2$ and $i = 0$,

$$(\tau_m \circ (id - s_0^m) \circ \tau_m^{-1})(t, x) = \tau_m(\tilde{v}_0^m(\frac{mt}{\pi}, x)v_0^m) = \tilde{v}_0^m(\frac{mt}{\pi}, x)\tau_m v_0^m$$

converges to $(t - x)(0, -2) = (id - s_0)(t, x)$. \square

One considers for $t \geq 0$, the string parameters $\xi^m(t)$ of $\{W_s^{(\frac{m}{\pi}, \mu)}, s \leq t\}$, and when $0 < \mu < 1$, and m large enough to have $\frac{m}{\pi} \tan \frac{\pi}{m} > \mu$, the Verma string parameters ξ^m of $W^{(\frac{m}{\pi}, \mu)}$.

Proposition 3.18. *Almost surely, for all $t > 0$,*

$$\lim_{m \rightarrow +\infty} \xi^m(t) = \xi(t),$$

and when $0 < \mu < 1$,

$$\lim_{m \rightarrow +\infty} \xi^m = \xi.$$

Proof. Let, for all $k \in \mathbb{N}$, $w_k^m = s_k^m \cdots s_0^m \in I(m)$ and $w_k = s_k \cdots s_0 \in I(\infty)$. For any path η^m ,

$$(\tau_m \circ \mathcal{P}_{s_i^m}^m \circ \tau_m^{-1})(\eta^m)(t) = \eta^m(t) + \inf_{s \leq t} (\tau_m \circ (id - s_i^m) \circ \tau_m^{-1}) \eta^m(s),$$

so it follows from Lemma 3.17 and by iteration that

$$\lim_{m \rightarrow +\infty} (\tau_m \circ \mathcal{P}_{w_k^m}^m \circ \tau_m^{-1}) = \mathcal{P}_{w_k}.$$

Hence, if $\eta^m \in C_0(\mathbb{R}^2)$ is a sequence of paths converging uniformly on compact sets to a path η ,

$$\lim_{m \rightarrow +\infty} (\tau_m \circ (\mathcal{P}_{w_k^m}^m - \mathcal{P}_{w_{k-1}^m}^m) \circ \tau_m^{-1}) \eta^m(t) = (\mathcal{P}_{w_k} - \mathcal{P}_{w_{k-1}}) \eta(t).$$

This implies that

$$\lim_{m \rightarrow +\infty} \tilde{v}_k^m (\tau_m (\mathcal{P}_{w_k^m}^m - \mathcal{P}_{w_{k-1}^m}^m) \tau_m^{-1} \eta^m(t)) = \tilde{\alpha}_k ((\mathcal{P}_{w_k} - \mathcal{P}_{w_{k-1}}) \eta(t))$$

since $\tilde{v}_1 = \tilde{\alpha}_1$ and since the first coordinate of $(\mathcal{P}_{w_k} - \mathcal{P}_{w_{k-1}}) \eta(t)$ is 0 when k is even. We choose $\eta^m(t) = (\frac{\pi \beta t}{m} + t, B_t + \mu t)$. Then $\eta(t) = B_t^{(\mu)}$ and $\tau_m^{-1} \eta^m = W^{(m/\pi, \mu)}$. One has

$$(\mathcal{P}_{w_k^m}^m - \mathcal{P}_{w_{k-1}^m}^m) \tau_m^{-1} \eta^m(t) = \xi_k^m(t) v_k^m.$$

Since, when $m \rightarrow \infty$, $\tilde{v}_k^m (\tau_m v_k^m) \rightarrow 2$, we see that $\xi_k^m(t) \rightarrow \xi_k(t)$ using the analogue of Remark 3.7 in the affine case. When $0 < \mu < 1$, this drift implies that, a.s., $\xi_k^m = \xi_k^m(t)$ and $\xi_k = \xi_k(t)$ for t large enough, so $\xi_k^m \rightarrow \xi_k$. \square

As before, we let $\varepsilon_n, n \in \mathbb{N}$, be independent exponential random variables with parameter 1.

Theorem 3.19. *We suppose that $0 < \mu < 1$. In law $\xi_0 = \varepsilon_0/2(1 - \mu)$, and for $k \geq 1$,*

$$\frac{\xi_k}{k} = \sum_{n=k}^{+\infty} \frac{2\varepsilon_n}{n(n+1) + (1-2\mu)\nu_n},$$

where $\nu_n = n1_{2\mathbb{N}}(n) - (n+1)1_{2\mathbb{N}}(n+1)$.

Proof. The drift of $W^{(m/\pi, \mu)}$ is $(m/\pi, \mu)$, thus

$$\gamma_0^m = \langle (\frac{m}{\pi}, \mu), v_0^m \rangle = 2(\frac{m}{\pi} \sin(\frac{\pi}{m}) - \mu \cos(\frac{\pi}{m})), \gamma_1^m = \langle (\frac{m}{\pi}, \mu), v_1^m \rangle = 2\mu.$$

By Proposition 3.18 and Theorem 3.11

$$\frac{\xi_k}{k} = \lim_{m \rightarrow +\infty} \frac{\pi \xi_k^m}{m a_k^m} = \lim_{m \rightarrow +\infty} \frac{\pi}{m} \sum_{n=k}^{m-1} \frac{\varepsilon_n}{a_1^m \gamma_1^m + \cdots + a_n^m \gamma_n^m}.$$

Let $c = \max\{1/\gamma_i^m, m \in \mathbb{N}, i = 0, 1\}$. Since $0 \leq 2t \leq \sin \pi t$, when $0 \leq t \leq 1/2$, one has for $k \leq [m/2]$,

$$\sum_{n=1}^k a_n^m \nu_n^m \geq \frac{1}{c} \sum_{n=1}^k \sin\left(\frac{n\pi}{m}\right) \geq \sum_{n=1}^k \frac{2n}{cm} \geq \frac{k^2}{cm}.$$

For $k \geq [m/2]$,

$$\sum_{n=1}^k a_n^m \nu_n^m \geq \sum_{n=1}^{[m/2]} a_n^m \nu_n^m \geq \frac{m}{4c}.$$

Therefore, by the law of large numbers, when N is large enough,

$$\begin{aligned} \sup_{m \geq N} \frac{\pi}{m} \sum_{n=N}^{m-1} \frac{\varepsilon_n}{a_1^m \gamma_1^m + \cdots + a_n^m \gamma_n^m} &\leq \sup_{m \geq N} \left(2c\pi \sum_{n=N}^{[m/2]} \frac{\varepsilon_n}{n^2} + 4c\pi \sum_{n=[m/2]}^m \frac{\varepsilon_n}{m^2} \right) \\ &\leq \sup_{m \geq N} \left(2c\pi \sum_{n=N}^{\infty} \frac{\varepsilon_n}{n^2} + \frac{4c\pi}{m} \sum_{n=0}^m \frac{\varepsilon_n}{m} \right) \end{aligned}$$

is a.s. as small as we want. We finishes the proof by using that, for any fixed n ,

$$\lim_{m \rightarrow +\infty} \frac{\pi}{m} \frac{\varepsilon_n}{a_1^m \gamma_1^m + \cdots + a_n^m \gamma_n^m} = \frac{2\varepsilon_n}{n(n+1) + (1-2\mu)\nu_n}$$

since $\lim_{m \rightarrow +\infty} m a_k^m = k\pi$ and for any $r \in \mathbb{N}$,

$$\lim_{m \rightarrow +\infty} \gamma_{2r}^m = \lim_{m \rightarrow +\infty} \gamma_0^m = 2(1-\mu), \quad \lim_{m \rightarrow +\infty} \gamma_{2r+1}^m = \lim_{m \rightarrow +\infty} \gamma_1^m = 2\mu.$$

□

An important for us and maybe unexpected result is:

Theorem 3.20. *Almost surely, $\lim_{k \rightarrow +\infty} \xi_k = 2$.*

Proof. It is easy to see that $\mathbb{E}(\xi_k) \rightarrow 2$ and that $\sum_{k=1}^{\infty} \mathbb{E}((\xi_k - \mathbb{E}(\xi_k))^4)$ is finite, which implies the almost sure convergence. □

3.3. Affine Verma weight $L^{(\mu)}$. In this subsection we introduce the affine Verma weights. We consider for $0 < \mu < 1$, the affine Verma string parameters $\xi = (\xi_k)$ of $B^{(\mu)}$. For $k \geq 1$, let

$$M_k = \sum_{n=0}^{k-1} \xi_n \alpha_n + \frac{1}{2} \xi_k \alpha_k.$$

Notice that, more simply, since $\alpha_n = (-1)^n \alpha_0$,

$$M_k = (0, (-1)^{k+1} \xi_k + 2 \sum_{n=0}^{k-1} (-1)^{n+1} \xi_n),$$

the notation with the α_n 's, maybe strange for the reader, is explained by its natural interpretation in A_1^1 .

Proposition 3.21. *Let $\mu \in (0, 1)$. When k goes to infinity, M_k converges almost surely and in L^2 towards*

$$L^{(\mu)} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{\varepsilon_{2n}}{n+1-\mu} \alpha_0 + \frac{\varepsilon_{2n+1}}{n+\mu} \alpha_1 \right).$$

Proof. One has

$$M_{2p+2} - M_{2p} = (0, -\xi_{2p} + 2\xi_{2p+1} - \xi_{2p+2}) = (0, \frac{\varepsilon_{2p+1}}{p+\mu} - \frac{\varepsilon_{2p}}{p+1-\mu}),$$

hence $\{M_{2p}, p \geq 0\}$ is a martingale, bounded in L_2 and its limit is $L^{(\mu)}$. As

$$M_{2p+1} = M_{2p} + \frac{1}{2}(\xi_{2p} - \xi_{2p+1})\alpha_0$$

and $\xi_k \rightarrow 2$ by Theorem 3.20, M_{2p+1} has the same limit. \square

Definition 3.22. We call $L^{(\mu)}$ the affine Verma weight of $B^{(\mu)}$.

The terminology is due to the fact, that we will see in Section 4, that $L^{(\mu)}$ plays the role of an asymptotic weight in the Verma module of highest weight 0 in the case of the affine Lie algebra A_1^1 (more correctly of minus this weight modulo δ). Remark that, in coordinates, $L^{(\mu)} = (0, D^\mu)$ where

$$(3.9) \quad D^\mu = \sum_{n=0}^{\infty} \left(\frac{\varepsilon_{2n+1}}{n+\mu} - \frac{\varepsilon_{2n}}{n+1-\mu} \right).$$

Notice that $L^{(\mu)}$ is not the limit of the series $\sum_{k=0}^n \xi_k \alpha_k$, since $\lim_{k \rightarrow \infty} \xi_k = 2$ (Theorem 3.20).

3.4. Some technical results. In this subsection we prove some results which will be used to conclude the proof of our main theorem. We take here $\mu \in (0, 1)$ and m large enough to have $\frac{m}{\pi} \tan \frac{\pi}{m} > \mu$. Let for $0 \leq k \leq m$, with the convention that $\xi_m^m = 0$,

$$M_k^m = \sum_{n=0}^{k-1} \xi_n^m v_n^m + \frac{1}{2} \xi_k^m v_k^m.$$

Proposition 3.23. For any $k \geq 0$, $\tau_m M_k^m$ converges to M_k a.s. when m goes to infinity.

Proof. This follows from Lemma 3.17 and Proposition 3.18. \square

Proposition 3.24. One has, in probability

$$(3.10) \quad \lim_{m \rightarrow +\infty} \sup_{1 \leq k \leq m} \|\tau_m M_k^m - M_k\| = 0.$$

Proof. We use Theorem 3.11. For any $1 \leq 2p+1 \leq m$,

$$M_{2p+2}^m - M_{2p}^m = \frac{1}{2} \xi_{2p+2}^m v_{2p+2}^m + \xi_{2p+1}^m v_{2p+1}^m + \frac{1}{2} \xi_{2p}^m v_{2p}^m.$$

Therefore,

$$\begin{aligned} \tilde{v}_0^m (M_{2p+2}^m - M_{2p}^m) &= \xi_{2p+2}^m - 2 \cos\left(\frac{\pi}{m}\right) \xi_{2p+1}^m + \xi_{2p}^m \\ &= -2 \cos\left(\frac{\pi}{m}\right) a_{2p+1}^m \frac{\varepsilon_{2p+1}}{\gamma_1^m a_1^m + \gamma_0^m a_2^m + \cdots + \gamma_1^m a_{2p+1}^m} \\ &\quad + a_{2p}^m \left(\frac{\varepsilon_{2p+1}}{\gamma_1^m a_1^m + \cdots + \gamma_1^m a_{2p+1}^m} + \frac{\varepsilon_{2p}}{\gamma_1^m a_1^m + \cdots + \gamma_0^m a_{2p}^m} \right), \end{aligned}$$

since

$$a_{2p+2}^m - 2 \cos\left(\frac{\pi}{m}\right) a_{2p+1}^m + a_{2p}^m = 0.$$

One deduces that

$$\tilde{v}_0^m(M_{2p}^m) = 2 \sin \frac{\pi}{m} \sum_{l=0}^{p-1} \left(\frac{\varepsilon_{2l} \cos \frac{l\pi}{m}}{\gamma_1^m \sin \frac{l}{m}\pi + \gamma_0^m \sin \frac{l+1}{m}\pi} - \frac{\varepsilon_{2l+1} \cos \frac{(l+1)\pi}{m}}{\gamma_1^m \sin \frac{l+1}{m}\pi + \gamma_0^m \sin \frac{l}{m}\pi} \right),$$

by using the relations

$$\sum_{k=1}^n a_{2k}^m = \frac{\sin \frac{n\pi}{m} \sin \frac{(n+1)\pi}{m}}{\sin \frac{\pi}{m}}, \quad \sum_{k=1}^n a_{2k-1}^m = \frac{\sin^2 \frac{n\pi}{m}}{\sin \frac{\pi}{m}}, \quad \sum_{k=1}^n a_k^m = \frac{\sin \frac{n\pi}{2m} \sin \frac{(n+1)\pi}{2m}}{\sin \frac{\pi}{2m}}.$$

Similarly, one finds

$$\begin{aligned} \tilde{v}_1^m(M_{2p}^m) &= -2 \cos \frac{\pi}{m} \xi_0^m + \sin \frac{\pi}{m} \tan \frac{p\pi}{m} \xi_{2p}^m \\ &+ 2 \sin \frac{\pi}{m} \sum_{l=1}^{p-1} \left(\frac{\varepsilon_{2l+1} \cos \frac{l\pi}{m}}{\gamma_1^m \sin \frac{l+1}{m}\pi + \gamma_0^m \sin \frac{l}{m}\pi} - \frac{\varepsilon_{2l} \cos \frac{(l+1)\pi}{m}}{\gamma_1^m \sin \frac{l}{m}\pi + \gamma_0^m \sin \frac{l+1}{m}\pi} \right). \end{aligned}$$

On the other hand, one has

$$M_{2p+1}^m - M_{2p}^m = \frac{1}{2} (\xi_{2p}^m v_0^m + \xi_{2p+1}^m v_1^m).$$

Thus the proposition follows from the next lemma. \square

Lemma 3.25. *Let for $p \in \{0, \dots, [m/2]\}$, $0 < \mu < 1$, and γ_1^m, γ_0^m , which converge respectively towards 2μ and $2(1-\mu)$,*

$$\begin{aligned} S_p^m &= \sin \frac{\pi}{m} \sum_{l=0}^{p-1} \left(\frac{\varepsilon_{2l} \cos \frac{l\pi}{m}}{\gamma_1^m \sin \frac{l}{m}\pi + \gamma_0^m \sin \frac{l+1}{m}\pi} - \frac{\varepsilon_{2l+1} \cos \frac{(l+1)\pi}{m}}{\gamma_1^m \sin \frac{l+1}{m}\pi + \gamma_0^m \sin \frac{l}{m}\pi} \right), \\ S_p &= \sum_{l=0}^{p-1} \left(\frac{\varepsilon_{2l}}{2l + 2(1-\mu)} - \frac{\varepsilon_{2l+1}}{2l + 2\mu} \right). \end{aligned}$$

Then in probability

$$\lim_{m \rightarrow \infty} \sup_{1 \leq p \leq [m/2]} |S_p^m - S_p| = 0.$$

Proof. $\{S_p^m - S_p - \mathbb{E}(S_p^m - S_p), p = 1, \dots, [m/2]\}$ is a martingale. Let $\varepsilon > 0$. We have

$$\begin{aligned} \text{Var}(S_{[m/2]}^m - S_{[m/2]}) &= \sum_{l=0}^{[m/2]-1} \left(\frac{\sin \pi/m \cos l\pi/m}{\gamma_1^m \sin \frac{l}{m}\pi + \gamma_0^m \sin \frac{l+1}{m}\pi} - \frac{1}{2l + 2(1-\mu)} \right)^2 \\ &+ \left(\frac{\sin \pi/m \cos(l+1)\pi/m}{\gamma_1^m \sin \frac{l+1}{m}\pi + \gamma_0^m \sin \frac{l}{m}\pi} - \frac{1}{2l + 2\mu} \right)^2 \end{aligned}$$

As for $x \in [0, \frac{\pi}{2}]$, one has $\frac{2}{\pi}x \leq \sin x \leq x$, one has the majorations

$$\frac{\sin \frac{\pi}{m}}{\gamma_1^m \sin \frac{l}{m}\pi + \gamma_0^m \sin \frac{l+1}{m}\pi} \leq \frac{\pi/2}{\gamma_1^m l + \gamma_0^m (l+1)},$$

and

$$\frac{\sin \frac{\pi}{m}}{\gamma_1^m \sin \frac{l+1}{m}\pi + \gamma_0^m \sin \frac{l}{m}\pi} \leq \frac{\pi/2}{\gamma_1^m (l+1) + \gamma_0^m l},$$

for $0 \leq l \leq [m/2] - 1$. Thus there is a $p_0 \in \mathbb{N}$ such that for all $m \geq 2(p_0 + 1)$, the sum on the right hand side above for $l \geq p_0$ is smaller than ε . Since each

term tends to 0, we obtain that for m large enough this sum is smaller than ε . Thus we have by Doob's martingale inequality

$$\mathbb{E}\left(\sup_{1 \leq p \leq [m/2]} (S_p^m - S_p - \mathbb{E}(S_p^m - S_p))^2\right) \leq 4\varepsilon.$$

Thus $\sup_{1 \leq p \leq [m/2]} |S_p^m - S_p - \mathbb{E}(S_p^m - S_p)|$ converges in probability toward 0. Besides one has for $p, q \in \{0, \dots, [m/2]\}$, $p \geq q$,

$$|\mathbb{E}(S_p^m - S_q^m)| \leq \sum_{l=q}^{p-1} \frac{\pi^2(\gamma_0^m + \gamma_1^m)}{4(\gamma_1 l + \gamma_0(l+1))(\gamma_1(l+1) + \gamma_0 l)}$$

and

$$\mathbb{E}(S_p - S_q) = \sum_{l=q}^{p-1} \frac{2(2\mu - 1)}{(2l + 2(1 - \mu))(2l + 2\mu)}.$$

As for a fixed k , $\mathbb{E}(S_k^m)$ converges towards $\mathbb{E}(S_k)$ when m goes to infinity, one obtains that $\sup_{0 \leq p \leq [m/2]} |\mathbb{E}(S_p^m - S_p)|$ converges towards 0 when m goes to infinity, which finishes the proof of the lemma. \square

For a sequence $x = (x_k) \in \mathbb{R}_+^{\mathbb{N}}$ we let

$$(3.11) \quad \sigma(x) = \lim_{n \rightarrow +\infty} \sum_{k=0}^{n-1} x_k \alpha_k + \frac{1}{2} x_n \alpha_n,$$

when this limit exists in \mathbb{R}^2 (recall that $\alpha_k = (-1)^k \alpha_0$).

Definition 3.26. One defines, for $\lambda \in \bar{C}_{\text{aff}}$,

$$\Gamma = \{x = (x_k) \in \mathbb{R}^{\mathbb{N}} : \frac{x_k}{k} \geq \frac{x_{k+1}}{k+1} \geq 0, \text{ for all } k \geq 1, x_0 \geq 0, \sigma(x) \in \mathbb{R}^2\},$$

$$\Gamma(\lambda) = \{x \in \Gamma : x_k \leq \tilde{\alpha}_k(\lambda - \sigma(x) + \sum_{i=0}^k x_i \alpha_i), \text{ for every } k \geq 0\}.$$

We remark that $x \in \Gamma$ is in $\Gamma(\lambda)$ if and only if for all $k \geq 0$,

$$\tilde{\alpha}_k(\lambda - \sigma(x) + \sum_{i=0}^{k-1} x_i \alpha_i + \frac{1}{2} x_k \alpha_k) \geq 0.$$

When $0 < \mu < 1$, the Verma string parameters ξ of B^μ are a.s. in Γ , and

$$\sigma(\xi) = L^{(\mu)}.$$

The following proposition will allow us to pass from Verma affine string parameters to affine string parameters in the next subsection.

Proposition 3.27. Let (λ_m) be a sequence of \mathbb{R}^2 such that $\lambda_m \in C_m$ and such that $\tau_m \lambda_m \rightarrow \lambda$ when $m \rightarrow \infty$ where $\lambda \in C_{\text{aff}}$. The random sets $\{\xi^m \in \Gamma_m(\lambda_m)\}$ converge in probability to $\{\xi \in \Gamma(\lambda)\}$.

Proof. To prove convergence in probability, it is enough to show that each subsequence has a subsequence which converges almost surely. Therefore, by Propositions 3.24 and 3.21, working with a subsequence, we can suppose that the set of $\omega \in \Omega$ for which

$$\sup_{1 \leq k \leq m} \|\tau_m M_k^m(\omega) - M_k(\omega)\| \rightarrow 0, M_k(\omega) \rightarrow L^{(\mu)}(\omega)$$

is of probability one. So we see that if,

$$X_k^m = M_m^m - M_k^m, \quad X_k = L^{(\mu)} - M_k.$$

then (see (3.8))

$$(3.12) \quad \sup_{0 \leq k \leq m} |\tilde{v}_k^m(X_k^m(\omega)) - \tilde{\alpha}_k(X_k(\omega))| \rightarrow 0, \tilde{\alpha}_k(X_k(\omega)) \rightarrow 0.$$

For these ω we will show that

$$(3.13) \quad \limsup_{m \rightarrow \infty} \{\xi^m \in \Gamma_m(\lambda_m)\} \subset \{\xi \in \Gamma(\lambda)\} \subset \liminf_{m \rightarrow \infty} \{\xi^m \in \Gamma_m(\lambda_m)\}.$$

Notice that $\tilde{v}_k^m(\lambda_m) \rightarrow \tilde{\alpha}_k(\lambda)$ for $k = 0, 1$. One has $\xi^m \in \Gamma_m(\lambda_m)$ if and only if $\tilde{v}_k^m(\lambda_m - X_k^m) \geq 0$ for $0 \leq k < m$, and $\xi \in \Gamma(\lambda)$ if and only if $\tilde{\alpha}_k(\lambda - X_k) \geq 0$ for every $k \in \mathbb{N}$. The left inclusion in (3.13) follows from (3.12). Now, suppose that $\xi(\omega) \in \Gamma(\lambda)$. Since λ is fixed and the distributions of X_k are continuous,

$$\mathbb{P}(\forall k \geq 0, \tilde{\alpha}_k(\lambda - X_k) = 0) = 0,$$

and one can suppose that

$$\forall k \geq 0, \tilde{\alpha}_k(\lambda - X_k(\omega)) > 0.$$

We choose $\varepsilon > 0$ such that

$$\tilde{\alpha}_i(\lambda) > \varepsilon, i \in \{0, 1\}.$$

Using (3.12), one can choose m_0 such that $m \geq m_0$ implies that

$$\sup_{0 \leq k \leq m} \tilde{v}_k^m(X_k^m(\omega)) - \tilde{\alpha}_k(X_k(\omega)) < \varepsilon.$$

and then we choose k_0 such that $k \geq k_0$ implies that

$$\tilde{\alpha}_k(X_k(\omega)) < \tilde{\alpha}_k(\lambda) - \varepsilon.$$

As for each k , $\tilde{v}_k^m(X_k^m)$ converges towards $\tilde{\alpha}_k(X_k)$ when m goes to infinity, one takes m_1 such that when $m \geq m_1$,

$$\tilde{v}_k^m(X_k^m(\omega)) < \tilde{\alpha}_k(\lambda),$$

for $k = 1, \dots, k_0$. Then for $m \geq m_0, m_1$ one has for $k \geq k_0$

$$\tilde{v}_k^m(X_k^m(\omega)) = \tilde{v}_k^m(X_k^m(\omega)) - \tilde{\alpha}_k(X_k(\omega)) + \tilde{\alpha}_k(X_k(\omega)) < \tilde{\alpha}_k(\lambda),$$

and for $k \leq k_0$,

$$\tilde{v}_k^m(X_k^m(\omega)) < \tilde{\alpha}_k(\lambda),$$

which proves the right inclusion, and finishes the proof of the proposition. \square

3.5. The highest weight process $\Lambda^{(\mu)}(t)$. In this subsection we introduce the highest process which will appear as the limit of our Pitman's transform (with a correction) and show that it coincide in law with the space-time Brownian motion conditioned in C_{aff} .

Let $0 \leq \mu \leq 1$. We define for $k \geq 0$,

$$M_k(t) = \sum_{n=0}^{k-1} \xi_n(t) \alpha_n + \frac{1}{2} \xi_k(t) \alpha_k,$$

where $\xi(t) = \{\xi_n(t), n \geq 0\}$ are the affine string parameters of $\{B_s^{(\mu)}, s \leq t\}$. We consider the planar Brownian motion $W^{(m/\pi, \mu)}$ with drift $(m/\pi, \mu)$ and let

$$\Lambda_m(t) = \mathcal{P}_{w_0}^m W^{(m/\pi, \mu)}(t),$$

and

$$M_k^m(t) = \sum_{n=0}^{k-1} \xi_n^m(t) v_n^m + \frac{1}{2} \xi_k^m(t) v_k^m.$$

Proposition 3.28. *For every $k \geq 1$, $\tau_m M_k^m(t)$ converges to $M_k(t)$ almost surely when m goes to infinity.*

Proof. By the Cameron-Martin theorem, it is enough to prove the proposition for $\mu \in (0, 1)$. In that case, it follows from Proposition 3.18 and Lemma 3.17. \square

Theorem 3.29. *When $\mu \in [0, 1]$, for each $t \geq 0$, $M_k(t)$ converges almost surely when k goes to infinity. We denote by $L^{(\mu)}(t)$ the limit.*

Proof. By the Cameron-Martin theorem, it is enough to prove the proposition for $\mu \in (0, 1)$. Let $\varepsilon > 0$. By Lemma 3.17 and Proposition 3.18, $\tau_m M_p^m(t)$ converges a.s. to $M_p(t)$ as $m \rightarrow \infty$. Hence, for $p, q \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}(\|M_{p+q}(t) - M_p(t)\| \geq \varepsilon) &= \lim_{m \rightarrow \infty} \mathbb{P}(\|\tau_m M_{p+q}^m(t) - \tau_m M_p^m(t)\| \geq \varepsilon) \\ &= \lim_{m \rightarrow \infty} \mathbb{E}(f_m(\Lambda_m(t))) \end{aligned}$$

where

$$f_m(\Lambda_m(t)) = \mathbb{E}(1_{\|\tau_m M_{p+q}^m(t) - \tau_m M_p^m(t)\| \geq \varepsilon} | \Lambda_m(t)).$$

One has, by Theorem 3.11,

$$f_m(\lambda) = \mathbb{P}(\|\tau_m M_{p+q}^m - \tau_m M_p^m\| \geq \varepsilon | \xi_m \in \Gamma_m(\lambda)).$$

If $\tau_m \lambda_m \rightarrow \lambda$ as in Proposition 3.27 then $f_m(\lambda_m) \rightarrow f(\lambda)$ where

$$f(\lambda) = \mathbb{P}(\|M_{p+q} - M_p\| \geq \varepsilon | \xi \in \Gamma(\lambda)).$$

On the other hand we will show in Theorem 3.48, independently of this proof, that when m tends to infinity, $\tau_m \Lambda_m(t)$ converges in law to $A_t^{(\mu)}$ therefore $\mathbb{E}(f_m(\Lambda_m(t))) \rightarrow \mathbb{E}(f(A_t^{(\mu)}))$ by Proposition 3.27. This shows that

$$\mathbb{P}(\|M_{p+q}(t) - M_p(t)\| \geq \varepsilon) = \int_{\Omega} \mathbb{P}(\|M_{p+q} - M_p\| \geq \varepsilon | \xi \in \Gamma(A_t^{(\mu)}(\omega))) d\mathbb{P}(\omega).$$

Since (M_k) converges a.s. to $L^{(\mu)}$, we see that $M_p(t), p \in \mathbb{N}$, is a Cauchy sequence for the convergence in probability, and thus converges in probability. We will prove the almost sure convergence at the end of this subsection. \square

Proposition 3.30. *In probability, as $m \rightarrow \infty$, $\tau_m M_m^m(t)$ converges to $L^{(\mu)}(t)$.*

Proof. By Cameron-Martin theorem, it is enough to prove the proposition for $\mu \in (0, 1)$. In that case, one has for $\epsilon > 0$,

$$\begin{aligned} \mathbb{P}(|\tau_m M_m^m(t) - L^{(\mu)}(t)| > \epsilon) &\leq \mathbb{P}(|\tau_m M_m^m(t) - M_m(t)| > \frac{\epsilon}{2}) \\ &\quad + \mathbb{P}(|M_m(t) - L^{(\mu)}(t)| > \frac{\epsilon}{2}). \end{aligned}$$

For the first term, we condition by $\Lambda_m(t)$ as in the proof of Theorem 3.29 and use Proposition 3.24, and for the second we use Theorem 3.29. \square

We had (see (3.3))

$$\Lambda_m(t) = \mathcal{P}_{w_0}^m W^{(m/\pi, \mu)}(t) = W^{(m/\pi, \mu)}(t) + M_m^m(t),$$

therefore it is natural to introduce the following definition.

Definition 3.31. For $0 \leq \mu \leq 1$, we define the highest weight process of $\{B_t^{(\mu)}, t \geq 0\}$, by, for $t \geq 0$,

$$\Lambda^{(\mu)}(t) = B_t^{(\mu)} + L^{(\mu)}(t).$$

Proposition 3.32. In probability,

$$\begin{aligned} \lim_{m \rightarrow +\infty} \tau_m W^{(m/\pi, \mu)}(t) &= B_t^{(\mu)}, \\ \lim_{m \rightarrow +\infty} \tau_m \Lambda_m(t) &= \Lambda^{(\mu)}(t). \end{aligned}$$

Proof. The first statement is obvious. The second is then a consequence of Proposition 3.30 and the relation $\Lambda_m(t) = W_t^{(m/\pi, \mu)} + M_m^m(t)$. \square

Let $\{A_t^{(\mu)}, t \geq 0\}$ be the conditioned space-time Brownian motion in the affine Weyl chamber C_{aff} , with drift μ , starting from the origin defined in 5.3.

Theorem 3.33. For any $\mu \in [0, 1]$, in law

$$\{\Lambda^{(\mu)}(t), t \geq 0\} = \{A_t^{(\mu)}, t \geq 0\}.$$

Proof. This follows from Proposition 3.32, and Theorem 3.48. \square

In order to prove the almost sure convergence in Theorem 3.29, let us show:

Proposition 3.34. For $0 < \mu < 1$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ bounded and continuous,

$$\mathbb{E}(f(B_t^{(\mu)}, \xi(t)) | \sigma(\{\Lambda^{(\mu)}(s), s \leq t\})) = g(\Lambda^{(\mu)}(t))$$

for each $t > 0$ where

$$g(\lambda) = \mathbb{E}(f(\lambda - L^{(\mu)}, \xi) 1_{\{\xi \in \lambda\}}) / \mathbb{P}(\xi \in \lambda).$$

Proof. By Theorem 3.11, the conditional distribution of $(\tau_m M_m^m(t), \xi_m(t))$ knowing $\sigma(\{\Lambda_m(s), s \leq t\})$ is the one of $(\tau_m M_m^m, \xi_m)$ knowing $\{\xi_m \in \Gamma_m(\Lambda_m(t))\}$. Hence

$$\mathbb{E}(f(\Lambda^{(\mu)}(t) - \tau_m M_m^m(t), \xi_m(t)) | \sigma(\Lambda_m(s), s \leq t)) = g_m(\Lambda_m(t))$$

where

$$g_m(\lambda) = \mathbb{E}(f(\lambda - \tau_m M_m^m, \xi_m) 1_{\{\xi_m \in \lambda\}}) / \mathbb{P}(\xi_m \in \lambda).$$

Using the convergences in probability obtained before, if β is bounded, continuous, and depends only on a finite number of variables,

$$\begin{aligned} & \mathbb{E}(f(\Lambda^{(\mu)}(t) - L^{(\mu)}(t), \xi(t))\beta(\{\Lambda^{(\mu)}(s), s \leq t\})) \\ &= \lim_{m \rightarrow +\infty} \mathbb{E}(f(\Lambda^{(\mu)}(t) - \tau_m M_m^m(t), \xi_m(t))\beta(\{\tau_m \Lambda_m(s), s \leq t\})) \\ &= \lim_{m \rightarrow +\infty} \mathbb{E}(g_m(\Lambda_m(t))\beta(\{\tau_m \Lambda_m(s), s \leq t\})). \end{aligned}$$

Since $\Lambda_m(t)$ tends to $\Lambda^{(\mu)}(t)$, we know from Proposition 3.27 that, in probability, $\{\xi_m \in \Gamma_m(\Lambda_m(t))\}$ tends to $\{\xi \in \Gamma(\Lambda^{(\mu)}(t))\}$. Therefore the limit above is equal to

$$\mathbb{E}(g(\Lambda^{(\mu)}(t))\beta(\{\Lambda^{(\mu)}(s), s \leq t\}))$$

which proves the theorem. \square

End of the proof of Theorem 3.29. By the Cameron-Martin theorem it is enough to prove the proposition for $\mu \in (0, 1)$. In that case, this follows from

$$\mathbb{P}(\lim M_k(t) = L^{(\mu)}(t)) = \mathbb{E}(\mathbb{E}(1_{\{\lim M_k(t)=L^{(\mu)}(t)\}} | \sigma(\Lambda^{(\mu)}(t)))) = 1$$

since $M_k \rightarrow L^{(\mu)}$ a.s. and by Proposition 3.34 which implies that

$$\mathbb{E}(1_{\{\lim M_k(t)=L^{(\mu)}(t)\}} | \sigma(\Lambda^{(\mu)}(t))) = \mathbb{E}(1_{\{\lim M_k=L\}} | \xi \in \Gamma(\Lambda^{(\mu)}(t))) = 1$$

\square

3.6. Representation using Pitman and Lévy's transform. Let us remind where we stand. For $B_t^{(\mu)} = (t, B_t + t\mu)$, we have written, see (3.3),

$$\mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_1} \mathcal{P}_{\alpha_0} B^{(\mu)}(t) = B_t^{(\mu)} + \sum_{i=0}^n \xi_i(t) \alpha_i,$$

and

$$M_{n+1}(t) = \sum_{n=0}^n \xi_i(t) \alpha_i + \frac{1}{2} \xi_{n+1}(t) \alpha_{n+1}.$$

We have seen that when $\mu \in [0, 1]$, a.s., for $t > 0$, $\lim_{k \rightarrow +\infty} M_k(t) = L^{(\mu)}(t)$ and that the process $\Lambda_t^{(\mu)} = B^{(\mu)}(t) + L^{(\mu)}(t)$ has the same distribution as $A^{(\mu)}$. Hence,

Theorem 3.35. *When $0 \leq \mu \leq 1$, almost surely, the process*

$$\lim_{n \rightarrow +\infty} \mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_1} \mathcal{P}_{\alpha_0} B^{(\mu)}(t) + \frac{1}{2} \xi_{n+1}(t) \alpha_{n+1}, \quad t \geq 0,$$

exists and has the same distribution as $\{A^{(\mu)}(t), t \geq 0\}$.

To interpret this correction term it is worthwhile to introduce the Lévy's transform. Lévy's Theorem (see [35], VI.2) that if B_t is the Brownian motion

$$\mathcal{L}B(t) = B_t - \inf_{0 \leq s \leq t} B_s$$

has the same law as $|B_t|$ and that $-\inf_{0 \leq s \leq t} B_s$ is the local time of $\mathcal{L}B(t)$ at 0. We introduce here the following Lévy's transform (sometimes the Lévy's transform of B is defined as $\int_0^t \text{sign}(B_s) dB_s$, this is related but different from our transform).

Definition 3.36. For $\eta \in \mathcal{C}_0(\mathbb{R}^2)$ and $i = 0, 1$, the Lévy transform $\mathcal{L}_{\alpha_i}\eta$ of η is

$$\mathcal{L}_{\alpha_i}\eta(t) = \eta(t) - \frac{1}{2} \inf_{s \leq t} \tilde{\alpha}_i(\eta(s))\alpha_i.$$

Another way to state the former theorem is

Theorem 3.37. *Almost surely,*

$$\lim_{n \rightarrow +\infty} \mathcal{L}_{\alpha_{n+1}} \mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_1} \mathcal{P}_{\alpha_0} B^{(\mu)}(t)$$

exists and has the same distribution as $A_t^{(\mu)}$.

The following proposition indicates that the presence of this Lévy transform is due to the bad behaviour of $A_t^{(\mu)}$ for t near 0 (recall that each $\xi_k(t)$ is non decreasing in t).

Proposition 3.38. *For all $t > 0$, a.s.*

$$\lim_{k \rightarrow +\infty} \xi_k(t) = 2.$$

Proof. Since $\xi_k \rightarrow 2$ a.s. (Theorem 3.20), the proof is the same as the end of the one of Theorem 3.29. \square

It implies that

$$\lim_{n \rightarrow +\infty} \mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_1} \mathcal{P}_{\alpha_0} B^{(\mu)}(t) + (-1)^n \alpha_0 = A_t^{(\mu)}.$$

So without correction the iterates of Pitman's transform do not converge.

In the whole paper one has chosen to begin by the transformation \mathcal{P}_{α_0} . Let us notice that we obtain the same results if we begin with \mathcal{P}_{α_1} . More precisely, if we denote with a tilde the quantities previously defined when we begin by \mathcal{P}_{α_1} rather than \mathcal{P}_{α_0} , one has, for each $\mu \in [0, 1]$,

Theorem 3.39. (1). *Almost surely,*

$$\lim_{k \rightarrow +\infty} \mathcal{L}_{\alpha_k} \mathcal{P}_{\alpha_{k-1}} \cdots \mathcal{P}_{\alpha_1} \mathcal{P}_{\alpha_0} B^{(\mu)}(t) = \lim_{k \rightarrow +\infty} \mathcal{L}_{\alpha_{k+1}} \mathcal{P}_{\alpha_k} \cdots \mathcal{P}_{\alpha_2} \mathcal{P}_{\alpha_1} B^{(\mu)}(t).$$

(2) $\tilde{\xi}(t)$ defined for $B^{(\mu)}$ has the same law as $\xi(t)$ defined for $B^{(1-\mu)}$.

Proof. For (1) we have to prove that $L^{(\mu)}(t) = \tilde{L}^{(\mu)}(t)$. We have seen in Proposition 3.30 that $\tau_m M_m^m(t)$ converges in probability to $L^{(\mu)}(t)$. By the same proof $\tau_m \tilde{M}_m^m(t)$ converges in probability to $\tilde{L}^{(\mu)}(t)$. Now we use the non trivial fact, proven in [3], that $M_m^m(t) = \tilde{M}_m^m(t)$ by (3.1). Thus $L^{(\mu)}(t) = \tilde{L}^{(\mu)}(t)$ a.s..

(2) follows from the fact that $Id - B^\mu$ has the same law as $B^{1-\mu}$. \square

3.7. The law of $B^{(\mu)}(t)$ conditionally to $\{\Lambda^{(\mu)}(s), s \leq t\}$. Let $\gamma \in \mathbb{R}^2$ and let $W^{(\gamma)}$ be the standard Brownian motion in \mathbb{R}^2 with drift γ . There is a root system associated with the group $I(m)$ and a so-called alternating polynomial (see [15], 6.2.3) defined as the product of roots and given explicitly by, for $v = (x, y) \in \mathbb{R}^2$,

$$h_m(v) = h_m(x, y) = \Im[(x + iy)^m].$$

For $w \in I(m)$ let $l(w)$ be the length of its shortest expression with s_0^m and s_1^m . Let, for $v \in \mathbb{R}^2, \gamma \in \bar{C}_m$,

$$\psi_v(\gamma) = \sum_{w \in I(m)} (-1)^{l(w)} e^{\langle w(\gamma), v \rangle - \langle \gamma, v \rangle}.$$

Lemma 3.40. For $z \in \mathbb{R}^2$ and $v = \mathcal{P}_{w_0}^m W^{(\gamma)}(t)$

$$\mathbb{E}(e^{\langle z, W_t^{(\gamma)} \rangle} | \sigma(\mathcal{P}_{w_0}^m W^{(\gamma)}(s), 0 \leq s \leq t)) = \frac{\psi_v(z + \gamma)}{\psi_v(\gamma)} \frac{h_m(\gamma)}{h_m(z + \gamma)} e^{\langle z, v \rangle}.$$

Proof. Theorem 5.5 in [4] gives, for $\mathcal{F}_t = \sigma(\mathcal{P}_{w_0}^m W^{(0)}(s), 0 \leq s \leq t)$,

$$\mathbb{E}(e^{\langle z, W_t^{(0)} \rangle} | \mathcal{F}_t) = k \frac{\psi_v(z)}{h_m(v) h_m(z)} e^{\langle v, z \rangle},$$

for $v = \mathcal{P}_{w_0}^m W^{(\gamma)}(t)$ and k a constant independent of v and z . We conclude with the Bayes and the Cameron-Martin formulas which give

$$\mathbb{E}(e^{\langle z, W_t^{(\gamma)} \rangle} | \mathcal{F}_t) = \frac{\mathbb{E}(e^{\langle z + \gamma, W_t^{(0)} \rangle} | \mathcal{F}_t)}{\mathbb{E}(e^{\langle \gamma, W_t^{(0)} \rangle} | \mathcal{F}_t)}.$$

□

For $\alpha \in (0, 1)$, we define a function φ_α on $\mathbb{R}_+^* \times \mathbb{R}$ by,

$$(3.14) \quad \varphi_\alpha(t, x) = e^{-\alpha x} \sum_{k \in \mathbb{Z}} \frac{\pi \sinh(\alpha(2kt + x))}{\sin(\alpha\pi)} e^{-2(kx + k^2 t)}, \quad t > 0, x \in \mathbb{R}.$$

Notice that the definition of φ_α makes sense by continuity for $\alpha = 0$. Besides, the following lemma implies that $\varphi_\alpha(t, x) = \varphi_{1-\alpha}(t, t - x)$ and allows to extend the definition of φ_α to all $\alpha \in \mathbb{R}$.

Lemma 3.41. For $\alpha \in (0, 1)$, $t > 0$, $x \in \mathbb{R}$,

$$\begin{aligned} \varphi_\alpha(t, x) &= e^{\frac{1}{2t}(x - \alpha t)^2} \sum_{k \in \mathbb{Z}} \sqrt{\frac{\pi}{2t}} e^{-\frac{1}{2t} k^2 \pi^2} \frac{\pi \sin(k\pi\alpha) \sin(k\pi x/t)}{\sin(\alpha\pi)}, \\ \varphi_0(t, x) &= e^{\frac{x^2}{2t}} \sum_{k \in \mathbb{Z}} \sqrt{\frac{\pi}{2t}} e^{-\frac{1}{2t} k^2 \pi^2} k\pi \sin(k\pi x/t). \end{aligned}$$

Proof. The first identity follows from the Poisson summation formula (see [1]) for the $2t$ -periodic function $x \mapsto e^{-\frac{x^2}{2t}} \varphi_\alpha(t, x)$. Letting α goes to zero in the first identity leads to the second one. □

Recall that $\tau_m(t, x) = (\pi t/m, x)$.

Lemma 3.42. Let $v^m \in \mathbb{R}^2, \gamma^m \in \bar{C}_m$ such that, as $m \rightarrow +\infty$, $\tau_m(\gamma^m) \rightarrow (1, \alpha)$ and $\tau_m(v^m) \rightarrow (t, x)$, then

$$(3.15) \quad \lim_{m \rightarrow \infty} \psi_{v^m}(\gamma^m) = \frac{\sin(\alpha\pi)}{\pi} \varphi_\alpha(t, x),$$

$$(3.16) \quad \lim_{m \rightarrow \infty} \left(\frac{\pi}{m}\right)^m h_m(\gamma^m) = \sin(\pi\alpha).$$

Proof. We use complex notations. Let $\omega = e^{2i\pi/m}$, the dihedral group $I(m)$ of order $2m$ is equal to the set of the m reflections $r_k : z \mapsto \bar{z}\omega^k$ and the m rotations $\theta_k : z \mapsto z\omega^k$, $1 \leq k \leq m$. When $m \rightarrow +\infty$,

$$\langle \theta_j(\gamma^m), v^m \rangle = \Re\left(\left(\frac{m}{\pi} + i\alpha\right)\omega^j\left(\frac{mt}{\pi} - ix\right)\right) = \frac{m^2}{\pi^2}t - 2k^2t + 2kx\pi + (x - 2kt)\alpha + o(1),$$

$$\langle r_j(\gamma^m), v^m \rangle = \Re\left(\left(\frac{m}{\pi} - i\alpha\right)\omega^j\left(\frac{mt}{\pi} - ix\right)\right) = \frac{m^2}{\pi^2}t - 2k^2t + 2kx\pi - (x - 2kt)\alpha + o(1),$$

When $[\frac{p}{2}] + 1 \leq |k| \leq p$, $\cos(\frac{2k}{m}\pi) \leq 0$, so one can choose $m_0 \in \mathbb{N}$ such that for $m \geq m_0$,

$$e^{-\langle \gamma^m, v^m \rangle} \sum_{k: [p/2]+1 \leq |k| \leq p} (e^{\langle r^k \gamma^m, v^m \rangle} - e^{\langle r^k s_1 \gamma^m, v^m \rangle}) \leq \epsilon.$$

Since

$$e^{-\langle \gamma^m, v^m \rangle} \sum_{k=-N}^N (e^{\langle r^k \gamma^m, v^m \rangle} - e^{\langle r^k s_1 \gamma^m, v^m \rangle}) \sim \sum_{k=-N}^N \sinh(\mu(y - 2kt))e^{-2(k^2t - kx) - \mu x},$$

this gives (3.15). The relation (3.16) is immediate. \square

Recall that $B^{(\mu)}(t) = (t, B_t + t\mu)$.

Theorem 3.43. *One has, for $s \in \mathbb{R}$, when $L^{(\mu)}(t) = (0, D_t^\mu)$*

$$\mathbb{E}(e^{sB_t} | \sigma(\Lambda^{(\mu)}(s), s \leq t)) = \frac{e^{s(\Lambda^{(\mu)}(t) - t\mu)} \varphi_{s+\mu}(\Lambda^{(\mu)}(t))}{\varphi_\mu(\Lambda^{(\mu)}(t))}$$

$$\mathbb{E}(e^{-sD_t^\mu} | \sigma(\Lambda^{(\mu)}(s), s \leq t)) = \frac{\varphi_{s+\mu}(\Lambda^{(\mu)}(t))}{\varphi_\mu(\Lambda^{(\mu)}(t))}.$$

Proof. The first formula is obtained by letting m goes to infinity in the Lemma 3.40 and using Proposition 3.32 and Lemma 3.42. It gives the second one since $\Lambda^{(\mu)}(t) = B_t^{(\mu)} + L^{(\mu)}(t)$. \square

3.8. Remarks on the Verma weight $L^{(\mu)}$ and string parameter ξ_1 .

3.8.1. *Law of $L^{(\mu)} = (0, D^\mu)$.* The Laplace transform of D^μ is, by (3.9),

$$\mathbb{E}(e^{-sD^\mu}) = \prod_{n=0}^{\infty} \left(\left(1 + \frac{s}{n+\mu}\right) \left(1 - \frac{s}{n+1-\mu}\right) \right)^{-1}.$$

Using

$$(3.17) \quad \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\gamma)\Gamma(\beta-\gamma)} = \prod_{n=0}^{+\infty} \left(1 + \frac{\gamma}{n+\alpha}\right) \left(1 - \frac{\gamma}{n+\beta}\right)$$

(Formula 8.325.1 of [20]) and $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ we obtain that

$$(3.18) \quad \mathbb{E}(e^{-sD^\mu}) = \frac{\sin(\pi\mu)}{\sin(\pi(\mu+s))},$$

in particular,

$$(3.19) \quad \mathbb{E}(D^\mu) = \pi \cot(\pi\mu).$$

By taking the expectation in Theorem 3.43 one sees that, in law,

$$\lim_{t \rightarrow \infty} D_t^\mu = D^\mu.$$

Corollary 3.44. *The density of $D^{1/2}$ is $1/(\pi \cosh x)$.*

Proof. One uses that the Fourier transform of $1/\pi \cosh(x)$ is $1/\cosh(\lambda\pi/2)$. \square

Notice that the law of $\sum_{n=1}^{\infty} (\frac{\varepsilon_{2n+1}}{n} - \frac{\varepsilon_{2n}}{n+1})$ appears in [10].

3.8.2. *A representation of Z_t before a small random time.*

Proposition 3.45. *When $0 < \mu < 1$ and $Z_0 = \mu$, there is a standard Brownian motion W and a stopping time $\sigma > 0$ a.s. for W such that for $0 \leq t \leq \sigma$,*

$$Z_t = W_t + \mu + tD^\mu.$$

Notice that $\mathbb{E}(D^\mu) = \pi \cot \pi\mu$ in agreement with the generator $\frac{1}{2}d^2/dx^2 + (\pi \cot \pi x)d/dx$ of Z_t .

Proof. Let $\tau_0 = \max\{t \geq 0, B_t^{(\mu)} \notin C_{\text{aff}}\}$, and

$$\tau_{n+1} = \max\{t \geq \tau_n, B_t^{(\mu)} + \sum_{k=0}^n \xi_k(t)\alpha_k \notin C_{\text{aff}}\}$$

for $n \geq 0$. Then, a.s., $\tau_0 < +\infty$ and for $t > \tau_0$, $\xi_0(t) = \xi_0$, hence

$$\tau_1 = \max\{t \geq \tau_0, B_t^{(\mu)} + \xi_0\alpha_0 \notin C_{\text{aff}}\}.$$

Recursively, one has $\tau_n < +\infty$ and

$$\tau_{n+1} = \max\{t \geq \tau_n, B_t^{(\mu)} + \sum_{k=0}^n \xi_k\alpha_k \notin C_{\text{aff}}\}.$$

We know that $\sum_{k=0}^n \xi_k\alpha_k$ is bounded, a.s., therefore $\tau = \sup \tau_n < +\infty$ and for $t > \tau$, $\xi_k(t) = \xi_k$ for all $k \geq 0$. This shows that for $t > \tau$, $L^{(\mu)}(t) = L^{(\mu)}$ hence $A_t^{(\mu)} = B_t^{(\mu)} + L^{(\mu)}$, which implies the representation since $(t, tZ_{1/t}) = A_t^{(\mu)}$. \square

3.8.3. *Law de ξ_1 .* The Bessel 3 with drift ν can be defined as

$$\varrho_t^{(\nu)} = \|W_1^2(t) + W_2^2(t) + W_3^2(t) + t\nu\|$$

where W_1, W_2, W_3 are three standard independent Brownian motions.

Corollary 3.46. *For $0 < \mu \leq 1$, ξ_1 has the same law as $\sup_{t \geq 0} (\varrho_t^{(1-\mu)} - t)$.*

Proof. By [33], $\varrho_t^{(1-\mu)}$ has the same law as $\mathcal{P}_{\alpha_1} B_t^{(1-\mu)}$, so the claim follows from (2) of Theorem 3.39 when $0 < \mu < 1$, and by continuity also when $\mu = 1$. \square

One has, when $\mu = 1$,

$$\xi_1 = \frac{1}{2} \sum_{n=1}^{+\infty} \frac{\varepsilon_{2n-1} + \varepsilon_{2n}}{n^2}$$

Its distribution is studied in [2] where it is symbolized $\frac{\pi^2}{4}\Sigma_2$. Its Laplace transform is

$$\mathbb{E}(e^{-2\lambda\xi_1}) = \frac{\pi^2\lambda}{\sinh^2(\pi\sqrt{\lambda})}$$

and its distribution function is (cf. Table 1 in [2])

$$F(x) = 1 + 2 \sum_{k=1}^{+\infty} (1 - 4k^2 x) e^{-2k^2 x}$$

so in this case, Corollary 3.46 is also given in the exemple 20 of [38].

When $\mu = 1/2$,

$$\xi_1 = \sum_{n=1}^{+\infty} \frac{2\varepsilon_n}{n(n+1)}$$

Proposition 3.47. *When $\mu = 1/2$, the Laplace transform of ξ_1 is*

$$\mathbb{E}(e^{-\lambda \xi_1}) = \frac{2\pi\lambda}{\cosh(\pi\sqrt{2\lambda - 1/4})},$$

its density is

$$\sum_{n=0}^{+\infty} (-1)^{n+1} n(n+1)(2n+1) e^{-n(n+1)x/2},$$

and, in law,

$$\xi_1 = \sup_{n>0, i=1,2,3} \frac{\varepsilon_n^{(i)}}{n},$$

where the $\varepsilon_n^{(i)}$ are exponentials independent random variables with parameter 1.

Proof. It is easy to see that, using the formula

$$\cosh \pi z = (1 + 4z^2) \prod_{n=1}^{+\infty} \left(1 + \frac{z^2}{(n + \frac{1}{2})^2}\right),$$

one has

$$\mathbb{E}(e^{-\lambda \xi_1}) = \prod_{n=1}^{+\infty} \left(1 + \frac{2\lambda}{n(n+1)}\right)^{-1} = \frac{2\pi\lambda}{\cosh(\pi\sqrt{2\lambda - 1/4})}.$$

Let

$$g(x) = 2 \sum_{n=0}^{+\infty} (-1)^n (2n+1) e^{-\frac{x}{2}(n+\frac{1}{2})^2}.$$

One knows (e.g. [2]) that $1/\cosh\sqrt{2\lambda}$ is the Laplace transform of $\pi g(\pi^2 x)/2$. Hence

$$\frac{2\lambda\pi}{\cosh \pi\sqrt{2\lambda - 1/4}} = \int_0^\infty \lambda e^{-\lambda x} e^{x/8} g(x) dx = \int_0^\infty e^{-\lambda x} (e^{x/8} g(x))' dx,$$

by an integration by parts. Computing the derivative $(e^{x/8} g(x))'$ gives the density. By integration, the distribution function is

$$\sum_{n=0}^{+\infty} (-1)^n (2n+1) e^{-n(n+1)x/2}$$

which is equal to $\prod_{n=1}^\infty (1 - e^{-nx})^3$ by a formula of Jacobi (Thm 357 of [19]). \square

3.9. Convergence of the dihedral highest weight. In this subsection, we prove the convergence of the dihedral highest weight to the conditioned space-time Brownian motion $\{A_t^{(\mu)}, t \geq 0\}$ that we have used before. Let $0 \leq \mu \leq 1$ and $m \geq 1$. We consider the planar Brownian motion $W^{(m/\pi, \mu)}$ and

$$\Lambda_m(t) = \mathcal{P}_{w_0}^m W^{(m/\pi, \mu)}(t).$$

Theorem 3.48. *As a process, $\{\tau_m \Lambda_m(t), t \geq 0\}$ converges in law to $\{A_t^{(\mu)}, t \geq 0\}$ when $m \rightarrow +\infty$.*

We will prove this theorem after the following proposition. Let $\{Z_t, t \geq 0\}$, be the conditioned Brownian motion in $[0, 1]$ starting from μ (as defined in 5.1).

Proposition 3.49. *For $m \in \mathbb{N}$, let $X_t^m, t \geq 0$, be the \mathbb{R}^2 -valued continuous process such that $X_{m^2 t}^m$ is the conditioned planar Brownian motion in the cone C_m , with drift $\gamma = m^2 e^{i\pi\mu/m}$ where $0 \leq \mu \leq 1$. One writes in polar parameters*

$$X_t^m = R_t^m \exp i\pi\theta_t^m,$$

$R_t^m > 0, \theta_t^m \in [0, 1/m]$. Then, when $m \rightarrow +\infty$, as processes,

$$m\theta_t^m \rightarrow Z_{1/\pi^2 t}, R_t^m \rightarrow t.$$

Proof. The process $X_{m^2 t}^m, t \geq 0$, is an example of Dunkl process introduced in [36, 37]. More precisely it is a radial multidimensional Dunkl process with drift (see also [7], [14] and for the case with drift [18], here we are in the case where the multiplicity $k \equiv 1$). Therefore, the process $R_{m^2 t}^m$ is a Bessel process of dimension $2(m+1)$ with drift m^2 , starting from 0. In other words, one can write

$$(R_{m^2 t}^m)^2 = (m^2 t + B_t(1))^2 + \dots + (B_t(2(m+1)))^2$$

where $B_t(1), \dots, B_t(2(m+1))$ are independent standard real Brownian motions. Since

$$\mathbb{E}\left(\sum_{k=1}^{2(m+1)} (B_{t/m^2}(k))^2\right) = 2t(m+1)/m^2$$

tends to 0 as $m \rightarrow +\infty$, R_t^m converges to t in L^1 . It is shown in [18] that the process $Y_t = X_{m^2 t}^m$ has the time inversion property, in the sense that $tY_{1/t}$ is the conditioned planar Brownian motion in \bar{C}_m without drift but starting from the drift γ . Using its skew product decomposition, one can write

$$\pi\theta_{m^2 t}^m = \sigma_{\gamma_t^m}^m, \quad \text{with } \gamma_t^m = \int_t^{+\infty} \frac{1}{(R_{m^2 s}^m)^2} ds,$$

where the process σ_t^m is a solution of the following stochastic differential equation

$$d\sigma_t^m = dB_t + m \cot(m\sigma_t^m) dt$$

where B is a Brownian motion independent of R^m and $\sigma_0^m = \mu\pi/m$ (see [14]). One remarks that $Z_t = \frac{m}{\pi} \sigma_{t\pi^2/m^2}^m$ satisfies to

$$(3.20) \quad dZ_t = d\beta_t + \pi \cot(\pi Z_t) dt$$

for another Brownian motion β , and is therefore the conditioned Brownian motion in $[0, 1]$ starting from μ (see (5.1)). As $m \rightarrow +\infty$,

$$\gamma_{t/m^2}^m = \frac{1}{m^2} \int_t^{+\infty} \frac{1}{(R_{u/m^2}^m)^2} du$$

is equivalent to $1/tm^2$. Therefore $m\theta_t^m = \frac{m}{\pi}\sigma_{\gamma_{t/m^2}}^m$ converges to $Z_{1/t\pi^2}$. \square

Proof of Theorem 3.48. Let $X_t^m = \frac{1}{m\pi}\Lambda_m(\pi^2 t)$. Since

$$\frac{1}{m\pi}W_{\pi^2 m^2 t}^{(m/\pi, \mu)} = (\beta_t + m^2 t, B_t + m\mu\pi t),$$

one sees that $X_{m^2 t}^m$ is the conditioned planar Brownian motion in the cone C_m with a drift equivalent to $m^2 e^{i\pi\mu/m}$. One writes its polar decomposition as $X_t^m = R_t^m \exp i\pi\theta_t^m$. Using the continuity of the solution of (3.20) with respect to the initial condition, we see that Proposition 3.49 also holds when the drift γ is only equivalent to $m^2 e^{i\pi\mu/m}$. Therefore $m\theta_t^m \rightarrow Z_{1/t\pi^2}$ and $R_t^m \rightarrow t$ and

$$\tau_m \Lambda_m(t) = (\pi^2 R_{t/\pi^2}^m \cos(\pi\theta_{t/\pi^2}^m), m\pi R_{t/\pi^2}^m \sin(\pi\theta_{t/\pi^2}^m))$$

converges in law, as a process, to $(t, tZ_{1/t})$ which has the same law as $A_t^{(\mu)}$ \square

4. SOME ASYMPTOTICS FOR REPRESENTATIONS OF THE AFFINE ALGEBRA A_1^1

As we said in the introduction, there are strong links between what we have done and the affine Lie algebra A_1^1 . In this section we will see that some quantities we have met occur in the description of some semi-classical limits of highest representations of A_1^1 . In particular a kind of Duistermaat Heckman measure is given by the conditional law of the space time Brownian motion. The asymptotic behaviour of the infinity crystal $B(\infty)$ of Kashiwara, for large weights is given by our Verma string coordinates ξ .

4.1. The Kac Moody algebra A_1^1 . We consider the affine Lie algebra A_1^1 . For our purpose, we only need to consider a realization of a real Cartan subalgebra. We introduce, as in the introduction, $\mathfrak{h}_{\mathbb{R}} = \text{Span}_{\mathbb{R}}\{\tilde{\alpha}_1, c, d\}$, and $\mathfrak{h}_{\mathbb{R}}^* = \text{Span}_{\mathbb{R}}\{\Lambda_0, \alpha_1, \delta\}$, where $\tilde{\alpha}_1 = (0, 1, 0)$, $c = (1, 0, 0)$, $d = (0, 0, 1)$, and $\Lambda_0 = (1, 0, 0)$, $\alpha_1 = (0, 2, 0)$, $\delta = (0, 0, 1)$. We let $\tilde{\alpha}_0 = (1, -1, 0)$ and $\alpha_0 = (0, -2, 1)$, so that $c = \tilde{\alpha}_0 + \tilde{\alpha}_1$ and $\delta = \alpha_0 + \alpha_1$. Notice that these α_0 and α_1 project on the ones given in 3.2 by the projection on $\mathfrak{h}_{\mathbb{R}}^*/\mathbb{R}\delta$ which we identified with V . Also $\tilde{\alpha}_0, \tilde{\alpha}_1$ project on the ones given by their projections on $\mathfrak{h}_{\mathbb{R}}/\mathbb{R}d$, identified with \tilde{V} . Usually α_1 and α_0 are called the two positive simple roots and $\tilde{\alpha}_1$ and $\tilde{\alpha}_0$ their coroots. One considers the set of integral weights

$$P = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : \lambda(\tilde{\alpha}_i) \in \mathbb{Z}, i = 0, 1\},$$

and the set of dominant integral weights

$$P_+ = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : \lambda(\tilde{\alpha}_i) \in \mathbb{N}, i = 0, 1\}.$$

For a dominant integral weight λ one defines the character of an irreducible highest-weight representation $V(\lambda)$ of A_1^1 with highest weight λ , as a formal series

$$(4.1) \quad \text{ch}_{\lambda} = \sum_{\beta \in P} \dim(V(\lambda)_{\beta}) e^{\beta},$$

where $V(\lambda)_{\beta}$ is the weight space of $V(\lambda)$ corresponding to the weight β . If we let $e^{\beta}(h) = e^{\beta(h)}$, for $h \in \mathfrak{h}_{\mathbb{R}}$, and evaluate this formal series at h , the series converges absolutely or diverges, and it converges if and only if $\delta(h) > 0$. For more details about affine Lie algebras and their representations, we refer to [25].

4.2. A Duistermaat Heckman measure. A way to define the Duistermaat Heckman measure for a semi-simple complex Lie algebra, is an approximation of the distribution of the weights of an irreducible representation when its highest weight is large (see [21]). For a compact connected Lie group, this measure also appears as a measure associated to a coadjoint orbit. Actually this is the pushforward measure of the Liouville measure on this coadjoint orbit by the projection on a Cartan subalgebra.

In an affine framework it is also possible to consider a semiclassical approximation of the distribution of the weights of an highest weight irreducible module when the highest weight is large. Let us explain how. Let $r \geq 1$, for a dominant weight λ_r , and $h_r \in \mathfrak{h}_{\mathbb{R}}$ such that $\delta(h_r) > 0$, we define a measure γ_r on $\mathfrak{h}_{\mathbb{R}}^*$, letting

$$(4.2) \quad \gamma_r = \sum_{\beta \in P} \dim(V(\lambda_r)_{\beta}) e^{\beta(h_r)} \delta_{\beta/r},$$

where $\delta_{\beta/r}$ is the Dirac measure at β/r .

Proposition 4.1. *Let $(t, x) \in C_{\text{aff}}$, and $\{\lambda_r, r \in \mathbb{N}^*\}$ be a sequence of dominant weights such that λ_r is equivalent to $r(t\Lambda_0 + x\frac{\alpha_1}{2})$ when r goes to infinity. If for $\mu \in [0, 1]$ one chooses $h_r = \frac{1}{r}(\mu\tilde{\alpha}_1 + 2d)$ in (4.2), then, as $r \rightarrow \infty$,*

$$\int_{\mathfrak{h}_{\mathbb{R}}^*} e^{s\beta(\tilde{\alpha}_1)} \gamma_r(d\beta) \sim \frac{1}{\pi\sqrt{\pi r}} e^{r\pi^2/4} e^{(s+\mu)x} \varphi_{s+\mu}(t, x)$$

Proof. One has for any $v \in \mathfrak{h}$,

$$\int e^{\beta(v)} \gamma_r(d\beta) = \text{ch}_{\lambda_r}\left(\frac{v}{r} + h_r\right).$$

The Weyl-Kac character formula implies that

$$\text{ch}_{\lambda_r}(a\tilde{\alpha}_1 + bd) = \frac{\sum_{k \in \mathbb{Z}} \sinh(a(m+1) + 2ak(n+2)) e^{-b(k(m+1) + k^2(n+2))}}{\sum_{k \in \mathbb{Z}} \text{sh}(a + 4ak) e^{-b(k+2k^2)}},$$

for $\lambda_r = n\Lambda_0 + m\frac{\alpha_1}{2}$, with $(m, n) \in \mathbb{N}^2$ such that $0 \leq m \leq n$, $a \in \mathbb{R}$, and $b \in \mathbb{R}_+^*$ (see [11]). For λ_r equivalent to $r(t\Lambda_0 + x\alpha_1/2)$, $v = s\tilde{\alpha}_1$, and $h_r = \frac{1}{r}(\mu\alpha_1 + 2d)$, one has immediately that the numerator of the character $\text{ch}_{\lambda}(\frac{v}{r} + h_r)$ converges towards

$$e^{(s+\mu)x} \frac{\sin(s+\mu)\pi}{\pi} \varphi_{s+\mu}(t, x),$$

when r goes to infinity. Besides Lemma 3.41 implies that the denominator of the character is equivalent to

$$\sqrt{\pi r} e^{-\frac{1}{4}r\pi^2} \sin \pi(s + \mu),$$

which finishes the proof. □

We denote by ν_r the probability measure on $\mathfrak{h}_{\mathbb{R}}^*$

$$\nu_r = \frac{1}{\text{ch}_{\lambda_r}(h_r)} \gamma_r.$$

The theorem below states that the conditional measure of Theorem 3.43 can be interpreted as a kind of normalized Duistermaat Heckman measure.

Theorem 4.2. *Under the hypotheses of Proposition 4.1, the sequence of the push forward probabilities of $\{\nu_r, r \geq 1\}$ by the quotient map from $\mathfrak{h}_{\mathbb{R}}^*$ to $\mathfrak{h}_{\mathbb{R}}^*/\mathbb{R}\delta$ converges to the law of $B_t^{(\mu)}$ conditionally to $\Lambda_t^{(\mu)} = (t, x)$ when r goes to infinity.*

Proof. This follows immediately from Theorem 3.43 and Proposition 4.1. \square

Remark 4.3. *Frenkel has shown in [17] that the law of a Brownian motion on $\mathfrak{su}(2)$ indexed by $[0, 1]$ given the orbit of the endpoint of its wrapping on $SU(2)$ plays the role of normalized Liouville measure on a coadjoint orbit. Identifying $\mathfrak{su}(2)$ and its dual, the Duistermaat Heckman probability measure appearing as a limit measure in Theorem 4.2 is actually the projection of this law on a Cartan subalgebra of $\mathfrak{su}(2)$.*

4.3. Asymptotics for the crystal $B(\infty)$ of A_1^1 . The infinity crystal $B(\infty)$ of Kashiwara ([26]) is the crystal associated with the Verma module with highest weight 0. It is important since any irreducible highest weight crystal may be obtained from $B(\infty)$. It is shown in [31] that using string parametrizations, a realization of $B(\infty)$ is given by

$$B(\infty) = \{x \in \mathbb{N}^{\mathbb{N}}; \text{ for some } n \in \mathbb{N}, \frac{x_1}{1} \geq \frac{x_2}{2} \geq \dots \geq \frac{x_n}{n} > 0, x_k = 0 \text{ for } k > n\}.$$

Notice that the only condition on x_0 is $x_0 \in \mathbb{N}$. For $x \in B(\infty)$, we let

$$\sigma(x) = \sum_{k=0}^n x_k \alpha_k \in \mathfrak{h}_{\mathbb{R}}^*,$$

when $x_k = 0$ for $k > n$. Then $-\sigma(x)$ is the weight of x in the crystal $B(\infty)$ (see [26]). Notice that (3.11) holds modulo δ . For $\tilde{\rho} = 2d + \tilde{\alpha}_1/2$, we define

$$s(x) = \sum_{k=0}^{+\infty} x_k = \sigma(x)(\tilde{\rho}).$$

The character ch_{∞} of the Verma module is defined as in (4.1), and one has that (see chapter 9 of [25]),

$$(4.3) \quad \text{ch}_{\infty} = \prod_{\beta \in R_+} (1 - e^{-\beta})^{-1},$$

where

$$(4.4) \quad R_+ = \{\alpha_i + n\delta, i = 0, 1, n \in \mathbb{N}\} \cup \{n\delta, n \in \mathbb{N}^*\}$$

are the so called positive roots. As previously, if we let $e^{\beta}(h) = e^{\beta(h)}$, for $h \in \mathfrak{h}_{\mathbb{R}}$, and evaluate the formal character at h , it converges if and only if $\delta(h) > 0$. Let $r \in \mathbb{R}_+^*$. On each element x of the crystal $B(\infty)$ we put the Boltzman weight $e^{-s(x)/r}$. Then

$$Z_r = \sum_{x \in B(\infty)} e^{-s(x)/r} = \text{ch}_{\infty}(\tilde{\rho}/r)$$

and we introduce the probability distribution β_r on $B(\infty)$ given by

$$\beta_r(x) = \frac{e^{-s(x)/r}}{Z_r}, \quad x \in B(\infty).$$

Since $B(\infty)$ is a subset of Γ we consider β_r as a probability on Γ . By the convergence in distribution below we mean the convergence of finite coordinates.

The following theorem indicates that the affine Brownian model describes an asymptotic (or continuous) version of the infinity crystal $B(\infty)$ for the affine Lie algebra A_1^1 .

Theorem 4.4. *Let $\chi^r \in \Gamma, r \in \mathbb{N}$, be random variable with distribution β_r . Then χ^r/r converges in distribution on Γ to the law of the Verma parameter ξ of $B^{(1/2)}$.*

Proof. We use the results on anti-lecture Hall compositions recalled in 5.5. Let us first prove that χ_1^r/r converges in law to ξ_1 . For $q = e^{-1/r}$, it follows from (5.9), (5.10) that

$$\begin{aligned} \mathbb{P}(\chi_1^r \leq ar) &= \frac{\sum_{x \in B(\infty); x_1 \leq ar} q^{s(x)}}{\sum_{x \in B(\infty)} q^{s(x)}} = \frac{\sum_{\lambda \in A_\infty; \lambda_1 \leq ar} q^{|\lambda|}}{\sum_{\lambda \in A_\infty} q^{|\lambda|}} \\ &= \frac{1}{1-q} (q; e^{-a}q^2)_\infty (e^{-a}q; e^{-a}q^2)_\infty (e^{-a}q^2; e^{-a}q^2)_\infty. \end{aligned}$$

Since $(q; e^{-a}q^2)_\infty$ is equivalent to $(1-q)(e^{-a}; e^{-a})$ one has, by Proposition 3.47

$$\lim_{r \rightarrow \infty} \mathbb{P}\left(\frac{\chi_1^r}{r} \leq a\right) = (e^{-a}; e^{-a})_\infty^3 = \prod_{n=1}^{\infty} (1 - e^{-na})^3 = \mathbb{P}(\xi_1 \leq a).$$

So χ_1^r/r converges to ξ_1 .

We now consider the full sequence $\chi_k^r/r, k \in \mathbb{N}$. First it is clear that χ_0^r/r converges in law to ξ_0 . For any r and $n \geq 1$, one has

$$\frac{\chi_1^r}{1} \geq \frac{\chi_2^r}{2} \geq \dots \geq \frac{\chi_n^r}{n},$$

which implies that for any $n \in \mathbb{N}$, the collection of laws of $(\frac{1}{r}\chi_1^r, \dots, \frac{1}{r}\chi_n^r)_{r>0}$ is tight. By Cantor's diagonal argument, we construct an increasing sequence $\varphi(r) \in \mathbb{N}, r \in \mathbb{N}$, such that the sequence of random variables $(\frac{1}{\varphi(r)}\chi_k^{\varphi(r)}, k \geq 0)$ converges in finite dimensional distribution when r goes to infinity. Let us denote by $(\chi_k, k \geq 0)$ the limit, and let us first prove that $\{\chi_k/k - \chi_{k+1}/(k+1), k \geq 1\}$ has the same distribution as $\{2\varepsilon_k/k(k+1), k \geq 1\}$. For $x \in B(\infty)$, one has

$$\begin{aligned} \mathbb{P}(\chi^r = x) &= \frac{1}{\text{ch}_\infty(\frac{1}{r}\tilde{\rho})} e^{-\frac{1}{r}x_0} \prod_{k=2}^{\infty} e^{-\frac{1}{2r}(kx_{k-1} - (k-1)x_k)} 1_{\{\frac{x_{k-1}}{k-1} \geq \frac{x_k}{k}\}} \\ &= \frac{1}{\text{ch}_\infty(\frac{1}{r}\tilde{\rho})} e^{-\frac{1}{r}x_0} \prod_{k=1}^{\infty} e^{-\frac{k+1}{2r}(x_k - \lceil \frac{k}{k+1}x_{k+1} \rceil)} e^{-\frac{1}{2r}(k\lceil \frac{k-1}{k}x_k \rceil - (k-1)x_k)} 1_{\{x_k \geq \lceil k\frac{x_{k+1}}{k+1} \rceil\}}. \end{aligned}$$

As for $k = 1, \dots, n$,

$$e^{-\frac{k}{2r}} \leq e^{-\frac{1}{2r}(k\lceil \frac{k-1}{k}x_k \rceil - (k-1)x_k)} \leq 1,$$

one obtains for $n \in \mathbb{N}$ and $t_0, \dots, t_n \in \mathbb{R}_+$,

$$C(r)(1 + o(\frac{1}{r})) \leq \frac{\mathbb{P}(\chi_0^r \geq t_0, \dots, \chi_n^r - \lceil \frac{n}{n+1}\chi_{n+1}^r \rceil \geq t_n)}{\mathbb{P}(Y_0^r \geq t_0, \dots, Y_n^r \geq t_n)} \leq C(r),$$

where $C(r)$ is independent of t_0, \dots, t_n , and where Y_0^r and $Y_k^r, k = 1, \dots, n$, are independent geometric random variables with values in \mathbb{N} , and parameters 1

and $e^{-\frac{k+1}{2r}}$, $k = 1, \dots, n$. This proves that for any n ,

$$\frac{1}{r}\chi_0^r, \frac{1}{r}(\chi_k^r - \lceil \frac{k}{k+1}\chi_{k+1}^r \rceil), \quad 1 \leq k \leq n,$$

converges jointly towards $\varepsilon_0, \frac{2}{k+1}\varepsilon_k, 1 \leq k \leq n$, when r goes to infinity. Besides

$$\lim_{r \rightarrow \infty} \frac{1}{\varphi(r)}(\chi_k^{\varphi(r)} - \lceil \frac{k}{k+1}\chi_{k+1}^{\varphi(r)} \rceil) = \chi_k - \frac{k}{k+1}\chi_{k+1}.$$

Thus, for $k \geq 1$, $\frac{1}{k}\chi_k - \frac{1}{k+1}\chi_{k+1}$, are independent random variables with the same law as $\frac{2}{k(k+1)}\varepsilon_k$. The positive sequence χ_k/k is decreasing. Let S be its limit. We have the identity in law, for all $k \geq 1$,

$$\frac{\chi_k}{k} = \sum_{n=k}^{+\infty} \frac{2\varepsilon_n}{n(n+1)} + S.$$

We have proved that in law $\chi_1 = \xi_1$, so $S = 0$ which finishes the proof. \square

Recall (Corollary 3.44) that $L^{(1/2)} = (0, D^{1/2})$ where the density of $D^{1/2}$ is $1/(\pi \cosh x)$.

Proposition 4.5. *When r goes to infinity, in law,*

- (1) *The normalized weights $\sigma(\chi^r)/r$ converges to $L^{(1/2)}$ in the quotient space $\mathfrak{h}_{\mathbb{R}}^*/\mathbb{R}\delta$.*
- (2) *The coordinate of $\sigma(\chi^r)/r$ along δ goes to infinity.*

Proof. In view of (4.4), the expression (4.3) of the character shows that, in distribution,

$$\sigma(\chi^r) = \sum_{n \geq 0} (G_0(n)(\alpha_0 + n\delta) + G_1(n)(\alpha_1 + n\delta) + G_2(n)(n+1)\delta),$$

where $G_i(n)$, $i = 0, 1, 2$, and $n \in \mathbb{N}$, are independent random variables such that $G_0(n)$, $G_1(n)$ and $G_2(n)$, are geometrically distributed with respective parameter $e^{-(\alpha_0+n\delta)(\tilde{\rho})}$, $e^{-(\alpha_1+n\delta)(\tilde{\rho})}$, and $e^{-(n+1)\delta(\tilde{\rho})}$, i.e. with respective parameter $e^{-2(n+1/2)/r}$, $e^{-2(n+1/2)/r}$ and $e^{-2(n+1)/r}$. The proposition follows easily. \square

5. APPENDIX

5.1. The conditioned Brownian motion in $[0, 1]$. We recall some well known facts about the Brownian motion conditioned to stay forever in the interval $[0, 1]$. It first appeared in Knight ([28]), called there the Taboo process, as the limit when $t \rightarrow \infty$ of the standard Brownian motion conditioned to reach the boundary after time t . To define it rigorously, consider the Brownian motion in $[0, 1]$ killed at the boundary (its generator is half the Laplacian with Dirichlet boundary condition). Its maximal eigenvalue is $\pi^2/2$ with eigenvector

$$h(x) = \sin(\pi x),$$

called the ground state. This function h is $\pi^2/2$ -harmonic, and we consider the h -Doob process $\{Z_t, t \geq 0\}$. It is the Markov process with transition probability density $q_t(x, y)$ given when $x, y \in (0, 1)$ by

$$q_t(x, y) = \frac{\sin(\pi y)}{\sin(\pi x)} e^{\pi^2 t/2} u_t(x, y),$$

where $u_t(x, y)$ is the transition probability density of the killed Brownian motion.

Definition 5.1. We call Z_t the conditioned Brownian motion process in $[0, 1]$.

It can also be viewed as the diffusion with generator

$$\frac{1}{2} \frac{d^2}{dx^2} + \pi \cot(\pi x) \frac{d}{dx}$$

in $[0, 1]$. It is well known (and follows from the reflection principle), that, for $x, y \in (0, 1)$,

$$(5.1) \quad u_t(x, y) = \sum_{k \in \mathbb{Z}} (p_t(x + 2k, y) - p_t(-x - 2k, y)),$$

where p_t is the standard heat kernel ([24]). Using the Poisson formula, or the spectral decomposition of the generator,

$$(5.2) \quad u_t(x, y) = \sum_{n \in \mathbb{Z}} \sin(n\pi x) \sin(n\pi y) e^{-\pi^2 n^2 t/2}.$$

By scale function techniques, one sees that 0 and 1 are entrance non-exit boundaries. In other words, Z_t can be started from the boundaries and do not touch them at positive time. The entrance density measure from 0 is the limit of $q_t(x, y)$ when $x \rightarrow 0$, which is for $y \in (0, 1)$, by (5.2),

$$(5.3) \quad q_t(0, y) = \sin(\pi y) \sum_{n \in \mathbb{Z}} n \sin(n\pi y) e^{-\frac{t}{2} \pi^2 (n^2 - 1)}.$$

Let us remark that Z_t can also be defined by the latitude of the Brownian motion on the 3-dimensional sphere ([24] Section 7.15) or the argument of an eigenvalue of the Brownian motion in $SU(2)$. The boundaries behaviour is also clear from this description. Another interpretation is given by the distance between two non colliding Brownian motions on a circle ([22]). A natural question is to extend our results to more than two points on a circle, or in other words to consider the affine Lie algebra $A_n^1, n \geq 2$.

5.2. The conditioned planar Brownian motion in the dihedral cone C_m .

It is given by the two dimensional Brownian motion starting from 0 conditioned to stay in the cone C_m forever. It is rigorously defined in 5.1 of [3].

5.3. The conditioned space-time Brownian motion in C_{aff} .

We fix some $\mu \in [0, 1]$ and define rigorously the process $A_t^{(\mu)}, t \geq 0$, which is the space time Brownian motion with drift μ starting from $(0, 0)$, namely

$$B_t^{(\mu)} = (t, B_t + \mu t)$$

(where (B_t) is the standard real Brownian motion), conditioned to stay forever in the affine cone

$$C_{\text{aff}} = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 < x < t\}.$$

Let $\{K_t^\mu, t \geq 0\}$ be the space-time process $B_t^{(\mu)}$ killed at the boundary of \bar{C}_{aff} . This is the process in the cone with generator $\frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x}$ and Dirichlet

boundary condition, and let u_t^μ be its transition probability density. Using the Cameron-Martin formula and [12], Proposition 2.2,

$$(5.4) \quad u_t^\mu((s, x), (t + s, y)) = e^{\mu(y-x) - \frac{t}{2}\mu^2} \hat{p}_t^0((s, x), (t + s, y))$$

$$(5.5) \quad = \sum_{k \in \mathbb{Z}} e^{-2(kx+k^2s)} (e^{\mu(2ks+x)} p_t^\mu(x + 2ks, y) - e^{-\mu(2ks+x)} p_t^\mu(-x - 2ks, y)),$$

where p_t^μ is the standard heat kernel with a drift μ . The function φ_μ defined on $\mathbb{R}_+^* \times \mathbb{R}$ by

$$(5.6) \quad \varphi_\mu(t, x) = e^{-\mu x} \sum_{k \in \mathbb{Z}} \frac{\pi \sinh(\mu(2kt + x))}{\sin(\mu\pi)} e^{-2(kx+k^2t)}, \quad t > 0, x \in \mathbb{R},$$

when $0 < \mu < 1$ and by continuity for $\mu = 0, 1$, is a space-time harmonic for the killed process $K^{(\mu)}$, non negative on C_{aff} and vanishing on its boundary [11].

Definition 5.2. For $0 \leq \mu \leq 1$, the continuous Markov process $\{A_t^{(\mu)}, t \geq 0\}$ in C_{aff} , with transition densities

$$(5.7) \quad \hat{q}_t^\mu((s, x), (s + t, y)) = \frac{\varphi_\mu(s + t, y)}{\varphi_\mu(s, x)} \hat{u}_t^\mu((s, x), (s + t, y)),$$

for $(s, x), (s + t, y) \in C_{\text{aff}}$ and entrance measure

$$(5.8) \quad \hat{q}_t^\mu((0, 0), (t, y)) = \varphi_\mu(t, y) \sin\left(\frac{y}{t}\pi\right) e^{-\frac{1}{2t}(y-\mu t)^2}$$

for $(t, y) \in C_{\text{aff}}$ is the space-time Brownian motion conditioned to stay in C_{aff} .

Lemma 5.3. For $0 \leq s \leq t$,

$$u_{1/s-1/t}^0(y/t, x/s) e^{-\frac{1}{2t}y^2} = u_{t-s}^0((s, x), (t, y)) e^{-\frac{1}{2s}x^2}.$$

Proof.

$$\begin{aligned} u_{1/s-1/t}^0(y/t, x/s) &= \sum_{k \in \mathbb{Z}} (p_{1/s-1/t}(y/t + 2k, x/s) - p_{1/s-1/t}(-y/t - 2k, x/s)) \\ &= \sum_{k \in \mathbb{Z}} \frac{p_s(0, x)}{p_t(0, y + 2kt)} (p_{t-s}(x, y + 2kt) - p_{t-s}(x, -y - 2kt)) \\ &= \frac{p_s(0, x)}{p_t(0, y)} \sum_{k \in \mathbb{Z}} e^{-2kx-2k^2s} (p_{t-s}(x + 2kt, y) - p_{t-s}(-x - 2ks, y)), \end{aligned}$$

which gives the lemma, by identity (5.5). \square

Lemma 3.41 and (5.2) imply the following lemma.

Lemma 5.4.

$$q_t(x, y) = c_t \sin(\pi y) e^{-\frac{1}{2t}(y-x)^2} \varphi_x(1/t, y/t).$$

Theorem 5.5. For $\mu \in [0, 1]$, the processes $\{A_t^{(\mu)}, t > 0\}$ and $\{(t, tZ_{1/t}), t > 0\}$, where Z starts from μ , are equal in law.

Proof. We let $X_t = tZ_{1/t}$, $t > 0$. For $0 < t_1 < \dots < t_n$, the quantity $\mathbb{P}(X_{t_1} \in dx_1, \dots, X_{t_n} \in dx_n)$ is equal to

$$q_{\frac{1}{t_n}}(\mu, \frac{x_n}{t_n}) q_{\frac{1}{t_{n-1}} - \frac{1}{t_n}}(\frac{x_n}{t_n}, \frac{x_{n-1}}{t_{n-1}}) \cdots q_{\frac{1}{t_1} - \frac{1}{t_2}}(\frac{x_2}{t_2}, \frac{x_1}{t_1}) dx_1 \cdots dx_n.$$

We conclude by Lemma 5.4 for the first factor and Lemma 5.3 for the others. \square

Since we have described in a precise way the boundary behaviour of Z_t , the one of $A_t^{(\mu)}$ follows from this theorem.

5.4. A property of Pitman transform for piecewise C^1 paths. As mentioned in the introduction, the need of an infinite number of Pitman transforms to represent the space-time Brownian motion in C_{aff} is due to its wild behaviour. The situation is much simpler for regular curves, as shown by the following proposition. Let $\eta(t) = (t, \pi(t))$ be a continuous piecewise C^1 path such that $\pi(0) = 0$.

Proposition 5.6. *For any $T > 0$, there is an n such that for all $t \in [0, T]$,*

$$\mathcal{P}_{\alpha_n} \cdots \mathcal{P}_{\alpha_2} \mathcal{P}_{\alpha_1} \eta(t) \in C_{\text{aff}}$$

We use the notations of Section 2. It is equivalent to prove that there is an $n > 0$ such that

$$0 \leq \mathcal{P}_n \cdots \mathcal{P}_2 \mathcal{P}_1 \pi(t) \leq t,$$

for all $t \in [0, T]$. Let $\tau_1(\pi) = \inf\{t > 0; \pi(t) < 0\}$ and $\tau_0(\pi) = \inf\{t > 0; \pi(t) > t\}$, and let $|\pi'|$ be the supremum of the left and right derivatives of π on $[0, T]$.

Lemma 5.7. (1) *Si $|\pi'| \leq M$, then*

$$|\mathcal{P}_1 \pi'| \leq M, \text{ and } |\mathcal{P}_0 \pi'| \leq 2 + M.$$

(2) *Pour $0 \leq t \leq \tau_1(\pi) \wedge \tau_0(\pi)$*

$$\mathcal{P}_1 \pi(t) = \mathcal{P}_0 \pi(t) = \pi(t).$$

(3) *There is an $n > 0$ such that*

$$0 \leq (\mathcal{P}_n \cdots \mathcal{P}_2 \mathcal{P}_1 \pi)'(0) \leq 1.$$

Proof. (1) and (2) are straightforward. For (3): if $s_0(x) = -x$ and $s_1(x) = 2 - x$, for $i = 0, 1$, $\mathcal{P}_i \pi'(0) = s_i \pi'(0)$, and it is well known that one can bring any real into $[0, 1]$ by the actions of s_0 and s_1 . \square

Lemma 5.8. *Theorem 5.6 holds when $\tau_1(\pi) \wedge \tau_0(\pi) > 0$.*

Proof. By symmetry it is enough to consider the case where $\tau_1(\pi) < \tau_0(\pi)$. Let $\pi_1 = \mathcal{P}_1 \pi$, $\pi_2 = \mathcal{P}_0 \pi_1$, $\pi_3 = \mathcal{P}_1 \pi_2$, \dots and $a_1 = \tau_1(\pi)$, $a_2 = \tau_0(\pi_1)$, $a_3 = \tau_1(\pi_2)$, \dots . By (1) of Lemma 5.7, $|\pi'_n| \leq M + 2n$ so, by the mean value theorem,

$$\begin{aligned} |\pi_{2n}(a_{2n+1}) - \pi_{2n}(a_{2n})| &\leq (M + 4n)(a_{2n+1} - a_{2n}) \\ |\pi_{2n+1}(a_{2n+2}) - \pi_{2n+1}(a_{2n+1})| &\leq (M + 4n + 2)(a_{2n+2} - a_{2n+1}) \end{aligned}$$

One has $\pi_{2n}(a_{2n}) = a_{2n}$, $\pi_{2n}(a_{2n+1}) = 0$, $\pi_{2n+1}(a_{2n+1}) = 0$, $\pi_{2n+1}(a_{2n+2}) = a_{2n+2}$, thus

$$a_{2n+1} \geq \frac{M + 4n + 1}{M + 4n} a_{2n}, a_{2n+2} \geq \frac{M + 4n + 2}{M + 4n + 1} a_{2n+1},$$

and $a_n \geq T$ for n large enough. \square

Proof of Proposition 5.6. Using (3) of Lemma 5.7 one can suppose that $0 \leq \pi'(0) \leq 1$. When $0 < \pi'(0) < 1$, then $\tau_1(\pi) \wedge \tau_0(\pi) > 0$ and we can apply Lemma 5.8. If for instance $\pi'(0) = 0$, let $\eta = \mathcal{P}_1\pi$. Then $\eta'(0) = 0$, and $\eta(t) \geq 0$ for $t \geq 0$. In that case $\tau_1(\eta) \geq 1$, and $\tau_0(\eta) > 0$. Indeed otherwise there is a sequence t_n decreasing to 0 such that $\eta(t_n) \geq t_n$ and $\eta'(0)$ is not 0. So we can also apply Lemma 5.8. \square

5.5. Anti-lecture Hall compositions. The set A_∞ of anti-lecture Hall compositions is defined in [8] as the sets of sequence of integers $\lambda_1, \lambda_2, \dots$ such that, for some $n \in \mathbb{N}$,

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \dots \geq \frac{\lambda_n}{n} > 0,$$

and $\lambda_p = 0$ when $p > n$. So we see that

$$A_\infty = \{(\lambda_1, \lambda_2, \dots); (0, \lambda_1, \lambda_2, \dots) \in B(\infty)\}.$$

For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n, 0, 0, \dots) \in A_\infty$, let $|\lambda| = \sum_{k=1}^n \lambda_k$. It is shown, see [8, 9] and (1.2) and (1.3) in [6], that for $0 \leq q < 1$,

$$(5.9) \quad \sum_{\lambda \in A_\infty} q^{|\lambda|} = \frac{(-q; q)_\infty}{(q^2; q)_\infty},$$

$$(5.10) \quad \sum_{\lambda \in A_\infty, \lambda_1 \leq k} q^{|\lambda|} = \frac{(-q; q)_\infty (q; q^{k+2})_\infty (q^{k+1}; q^{k+2})_\infty (q^{k+2}; q^{k+2})_\infty}{(q; q)_\infty},$$

where $(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)$.

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