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BEHAVIOUR OF SOME HODGE INVARIANTS BY MIDDLE CONVOLUTION

BY NICOLAS MARTIN

ABSTRACT. — Following an article of Dettweiler and Sabbah, this article studies the behaviour of various Hodge invariants by middle additive convolution with a Kummer module. The main result gives the behaviour of the nearby cycle local Hodge numerical data at infinity. We also give expressions for Hodge numbers and degrees of some Hodge bundles without making the hypothesis of scalar monodromy at infinity.

The initial motivation to study the behaviour of various Hodge invariants by middle additive convolution is the Katz algorithm [Kat96], which makes possible to reduce a rigid irreducible local system \mathcal{L} on a punctured projective line to a rank-one local system. This algorithm is a successive application of tensor products with a rank-one local system and middle additive convolutions with a Kummer local system, and terminates with a rank-one local system. We assume that the monodromy at infinity of \mathcal{L} is scalar, so this property is preserved throughout the algorithm.

If we assume that eigenvalues of local monodromies of \mathcal{L} have absolute value one, we get at each step of the algorithm a variation of polarized complex Hodge structure unique up to a shift of the Hodge filtration [Sim90, Del87]. The work of Dettweiler and Sabbah [DS13] is devoted to computing the behaviour of Hodge invariants at each step of the algorithm.

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Our purpose in this article is to complement the previous work of Dettweiler and Sabbah without assuming that the monodromy at infinity is scalar, and to do that, we take up the notations introduced in [DS13, §2.2] and recalled in §1.1. More precisely, our main result consists in making explicit the behaviour of the nearby cycle local Hodge numerical data at infinity by middle additive convolution with the Kummer module \mathcal{K}_{λ_0} , with $\gamma_0 \in (0, 1)$ such that $\exp(-2i\pi\gamma_0) = \lambda_0$. Considering a regular holonomic $\mathcal{D}_{\mathbb{A}^1}$ -module M verifying various assumptions, whose singularities at finite distance belong to $\mathbf{x} = \{x_1, \dots, x_r\}$, we denote by $\text{MC}_{\lambda_0}(M)$ this convolution and show the following theorem (see §1.1 for the notation and assumptions).

THEOREM 1. — *Let \mathcal{M}^{\min} be the $\mathcal{D}_{\mathbb{P}^1}$ -module minimal extension of M at infinity. Given $\gamma \in [0, 1)$ and $\lambda = \exp(-2i\pi\gamma)$, we have:*

$$\nu_{\infty, \lambda, \ell}^p(\text{MC}_{\lambda_0}(M)) = \begin{cases} \nu_{\infty, \lambda \lambda_0, \ell}^{p-1}(M) & \text{if } \gamma \in (0, 1 - \gamma_0) \\ \nu_{\infty, \lambda \lambda_0, \ell}^p(M) & \text{if } \gamma \in (1 - \gamma_0, 1) \\ \nu_{\infty, \lambda_0, \ell+1}^p(M) & \text{if } \lambda = 1 \\ \nu_{\infty, 1, \ell-1}^{p-1}(M) & \text{if } \lambda = \overline{\lambda_0}, \ell \geq 1 \\ h^p H^1(\mathbb{P}^1, \text{DR } \mathcal{M}^{\min}) & \text{if } \lambda = \overline{\lambda_0}, \ell = 0. \end{cases}$$

This result has applications beyond the Katz algorithm since it enables us to give another proof of a theorem of Fedorov [Fed17] computing the Hodge invariants of hypergeometric equations, this work is developed in [Mar18b]. In addition, we get general expressions for Hodge numbers h^p of the variation and degrees δ^p of some Hodge bundles (recalled in §1.1) which generalize those of Dettweiler and Sabbah. The results are the following.

THEOREM 2. — *The local invariants $h^p(\text{MC}_{\lambda_0}(M))$ are given by:*

$$h^p(\text{MC}_{\lambda_0}(M)) = \sum_{\gamma \in [0, \gamma_0)} \nu_{\infty, \lambda}^p(M) + \sum_{\gamma \in [\gamma_0, 1)} \nu_{\infty, \lambda}^{p-1}(M) + h^p H^1(\mathbb{A}^1, \text{DR } M) - \nu_{\infty, \lambda_0, \text{prim}}^{p-1}(M).$$

THEOREM 3. — *The global invariants $\delta^p(\text{MC}_{\lambda_0}(M))$ are given by:*

$$\delta^p(\text{MC}_{\lambda_0}(M)) = \delta^p(M) + \sum_{\gamma \in [\gamma_0, 1)} \nu_{\infty, \lambda}^p(M) - \sum_{i=1}^r \left(\mu_{x_i, 1}^p(M) + \sum_{\gamma \in (0, 1 - \gamma_0)} \mu_{x_i, \lambda}^{p-1}(M) \right).$$

1. Hodge numerical data and modules of normal crossing type

1.1. Hodge invariants. — In this section, we recall the definition of local and global invariants introduced in [DS13, §2.2]. Let Δ be a disc centered in 0 with coordinate t and $(V, F^\bullet V, \nabla)$ be a variation of polarizable Hodge structure on Δ^* . We denote by M the corresponding \mathcal{D}_Δ -module minimal extension at 0.

Nearby cycles. For $a \in (-1, 0]$ and $\lambda = e^{-2i\pi a}$, the nearby cycle space at the origin $\psi_\lambda(M)$ is equipped with the nilpotent endomorphism $N = -2i\pi(t\partial_t - a)$ and the Hodge filtration is such that $NF^p\psi_\lambda(M) \subset F^{p-1}\psi_\lambda(M)$. The monodromy filtration induced by N enables us to define the spaces $P_\ell\psi_\lambda(M)$ of primitive vectors, equipped with a polarizable Hodge structure. The nearby cycle local Hodge numerical data are defined by

$$\nu_{\lambda,\ell}^p(M) := h^p(P_\ell\psi_\lambda(M)) = \dim \operatorname{gr}_F^p P_\ell\psi_\lambda(M),$$

with the relation $\nu_\lambda^p(M) := h^p\psi_\lambda(M) = \sum_{\ell \geq 0} \sum_{k=0}^{\ell} \nu_{\lambda,\ell}^{p+k}(M)$. We set

$$\nu_{\lambda,\text{prim}}^p(M) := \sum_{\ell \geq 0} \nu_{\lambda,\ell}^p(M) \quad \text{and} \quad \nu_{\lambda,\text{coprim}}^p(M) := \sum_{\ell \geq 0} \nu_{\lambda,\ell}^{p+\ell}(M).$$

Vanishing cycles. For $\lambda \neq 1$, the vanishing cycle space at the origin is given by $\phi_\lambda(M) = \psi_\lambda(M)$ and comes with N and F^p as before. For $\lambda = 1$, the Hodge filtration on $\phi_1(M)$ is such that $F^p P_\ell\phi_1(M) = N(F^p P_{\ell+1}\psi_1(M))$. Similarly to nearby cycles, the vanishing cycle local Hodge numerical data is defined by $\mu_{\lambda,\ell}^p(M) := h^p(P_\ell\phi_\lambda(M)) = \dim \operatorname{gr}_F^p P_\ell\phi_\lambda(M)$.

Degrees δ^p . For a variation of polarizable Hodge structure $(V, F^\bullet V, \nabla)$ on $\mathbb{A}^1 \setminus \mathbf{x}$, we set M the underlying $\mathcal{D}_{\mathbb{A}^1}$ -module minimal extension at each point of \mathbf{x} . The Deligne extension V^0 of (V, ∇) on \mathbb{P}^1 is contained in M , and we set $\delta^p(M) = \deg \operatorname{gr}_F^p V^0$.

In this paper, we are mostly interested in the behaviour of the nearby cycle local Hodge numerical data at infinity by middle convolution with the Kummer module $\mathcal{K}_{\lambda_0} = \mathcal{D}_{\mathbb{A}^1} / \mathcal{D}_{\mathbb{A}^1} \cdot (t\partial_t - \gamma_0)$, with $\gamma_0 \in (0, 1)$ such that $\exp(-2i\pi\gamma_0) = \lambda_0$. This operation is denoted by MC_{λ_0} .

Assumptions. As in [DS13], we assume in what follows that M is an irreducible regular holonomic $\mathcal{D}_{\mathbb{A}^1}$ -module, not isomorphic to $(\mathbb{C}[t], d)$ and not supported on a point.

1.2. Modules of normal crossing type. — Let us consider X a polydisc of \mathbb{C}^n with analytic coordinates x_1, \dots, x_n , $D = \{x_1 \cdots x_n = 0\}$ and M a coherent \mathcal{D}_X -module of normal crossing type (notion defined in [Sai90, §3.2]). For every $\alpha \in \mathbb{R}^n$, we define the sub-object $M_\alpha = \bigcap_{i=1}^n \bigcup_{k_i \geq 0} \ker(x_i \partial_{x_i} - \alpha_i)^{k_i}$ of M . There exists $A \subset [-1, 0]^n$ finite such that $M_\alpha = 0$ for $\alpha \notin A + \mathbb{Z}^n$. If we set $M^{\text{alg}} := \bigoplus_\alpha M_\alpha$, the natural morphism $M^{\text{alg}} \otimes_{\mathbb{C}[x_1, \dots, x_n]} \langle \partial_{x_1}, \dots, \partial_{x_n} \rangle \mathcal{D}_X \rightarrow M$ is an isomorphism.

To be precise, only the case $n = 2$ will occur in this paper. In the situations that we will consider, it will be possible to make explicit the decomposition and then apply the general theory of Hodge modules of M. Saito.

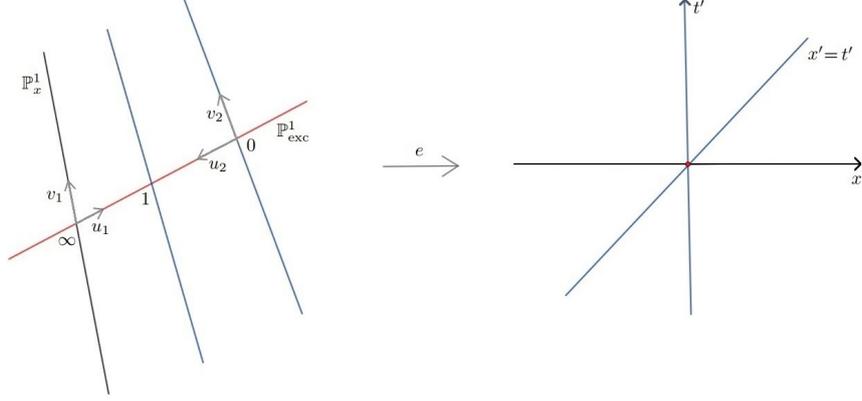
2. Proof of Theorem 1

Steps of the proof. Let us begin to itemize the different steps of the proof:

1. We write the middle convolution $\text{MC}_{\lambda_0}(M)$ as an intermediate direct image by the sum map. By changing coordinates and projectivizing, we can consider the case of a proper projection.
2. We use a property of commutation between nearby cycles and projective direct image in the theory of Hodge modules of M. Saito, in order to carry out the local study of a nearby cycle sheaf.
3. To be in a normal crossing situation and use the results of the theory of Hodge modules, we realize a blow-up and make completely explicit the nearby cycle sheaf previously introduced (Lemma 2.3).
4. We take into account monodromy and Hodge filtrations, using the degeneration at E_1 of the Hodge de Rham spectral sequence and Riemann-Roch theorem to get the expected theorem.

Geometric situation. Let $s : \mathbb{A}_x^1 \times \mathbb{A}_y^1 \rightarrow \mathbb{A}_t^1$ be the sum map. We can change the coordinates so that s becomes the projection onto the second factor and projectivize to get $\tilde{s} : \mathbb{P}_x^1 \times \mathbb{P}_t^1 \rightarrow \mathbb{P}_t^1$. We set $x' = 1/x$ and $t' = 1/t$ coordinates at the neighbourhood of $(\infty, \infty) \in \mathbb{P}_x^1 \times \mathbb{P}_t^1$, $M_{\lambda_0} = M \boxtimes \mathcal{K}_{\lambda_0}$ and $\mathcal{M}_{\lambda_0} = (M_{\lambda_0})_{\min(x'=0)}$ the minimal extension of M_{λ_0} along the divisor $\{x' = 0\}$. A reasoning similar to that of [DS13, Prop 1.1.10] gives $\text{MC}_{\lambda_0}(M) = \tilde{s}_+ \mathcal{M}_{\lambda_0}$.

Let us specify the geometric situation that we will consider in the following, in which we blow up the point (∞, ∞) in $\mathbb{P}_x^1 \times \mathbb{P}_t^1$. We set $X = \text{Bl}_{(\infty, \infty)}(\mathbb{P}_x^1 \times \mathbb{P}_t^1)$, $e : X \rightarrow \mathbb{P}_x^1 \times \mathbb{P}_t^1$ and $j : \mathbb{A}_x^1 \times \mathbb{P}_t^1 \hookrightarrow X$ the natural inclusion. There are two charts : one given by coordinates $(u_1, v_1) \mapsto (t' = u_1 v_1, x' = v_1)$ and the other one by $(u_2, v_2) \mapsto (t' = v_2, x' = u_2 v_2)$. The strict transform of the line $\{t' = 0\}$ is naturally called \mathbb{P}_x^1 , and the exceptional divisor is called $\mathbb{P}_{\text{exc}}^1$. We denote by $0 \in \mathbb{P}_{\text{exc}}^1$ the point given by $u_2 = 0$, $1 \in \mathbb{P}_{\text{exc}}^1$ the point given by $u_2 = 1$ and $\infty \in \mathbb{P}_x^1 \cap \mathbb{P}_{\text{exc}}^1$. We have the following picture:



On $\mathbb{A}_x^1 \times \mathbb{A}_t^1$, we have

$$\begin{aligned} M \boxtimes \mathcal{K}_{\lambda_0} &= M[t] \otimes \left(\mathbb{C}[x, t, (t-x)^{-1}], d_{(x,t)} + \gamma_0 \frac{d(t-x)}{t-x} \right) \\ &= \left(M[t, (t-x)^{-1}], \nabla_{(x,t)} + \gamma_0 \frac{d(t-x)}{t-x} \right). \end{aligned}$$

If we denote by $M(*\infty)$ the localization of M at infinity, we have

$$M_{\lambda_0} = \left(M(*\infty)[t', t'^{-1}, (x' - t')^{-1}], \nabla_{(x',t')} + \gamma_0 \left(-\frac{dx'}{x'} - \frac{dt'}{t'} + \frac{d(x' - t')}{x' - t'} \right) \right).$$

NOTATION 2.1. — Let us set $N_{\lambda_0} = e^+ M_{\lambda_0}$, $\mathcal{N}_{\lambda_0} = (N_{\lambda_0})_{\min(x' \circ e=0)}$ and $T = \psi_{t' \circ e, \lambda} \mathcal{N}_{\lambda_0}$ equipped of a nilpotent endomorphism denoted by N .

LEMMA 2.2. — $\mathcal{M}_{\lambda_0}[t'^{-1}] = e_+ \mathcal{N}_{\lambda_0}[t'^{-1}]$.

Proof. — By definition of the minimal extension, \mathcal{N}_{λ_0} is the image of the map $j_{\dagger} j^+ N_{\lambda_0} \rightarrow j_+ j^+ N_{\lambda_0}$. For $i \neq 0$, $H^i e_+ \mathcal{N}_{\lambda_0}$ is supported on (∞, ∞) , thereby $H^i e_+ \mathcal{N}_{\lambda_0}[t'^{-1}] = 0$. As the kernel of $H^0 e_+ \mathcal{N}_{\lambda_0} \rightarrow M_{\lambda_0}$ is similarly supported on (∞, ∞) , we deduce that $e_+ \mathcal{N}_{\lambda_0}[t'^{-1}]$ is a submodule of M_{λ_0} .

We have $e_+ j_+ j^+ N_{\lambda_0} = (e \circ j)_+ (e \circ j)^+ M_{\lambda_0}$ and, as e is proper, we can write $e_+ j_{\dagger} j^+ N_{\lambda_0} = e_{\dagger} j_{\dagger} j^+ N_{\lambda_0} = (e \circ j)_{\dagger} (e \circ j)^+ M_{\lambda_0}$. Then $\mathcal{M}_{\lambda_0}[t'^{-1}]$ is the image of the map $e_+ j_{\dagger} j^+ N_{\lambda_0}[t'^{-1}] \rightarrow e_+ j_+ j^+ N_{\lambda_0}[t'^{-1}]$. Outside of (∞, ∞) , $\mathcal{M}_{\lambda_0}[t'^{-1}]$ and $e_+ \mathcal{N}_{\lambda_0}[t'^{-1}]$ are submodules of M_{λ_0} which are isomorphic. Now, we can consider the intersection of these two submodules of M_{λ_0} , with two morphisms from the intersection to each of them. The kernel and the cokernel of these two morphisms are a priori supported on (∞, ∞) , but as t' is invertible, they are zero. Then $\mathcal{M}_{\lambda_0}[t'^{-1}]$ and $e_+ \mathcal{N}_{\lambda_0}[t'^{-1}]$ are isomorphic. \square

Let us fix $\gamma \in [0, 1)$ and $\lambda = \exp(-2i\pi\gamma)$. As \tilde{s} and e are proper, the nearby cycles functor is compatible with $(\tilde{s} \circ e)_+$ [Sai88, Prop 3.3.17], so we get

$$\psi_{\infty, \lambda}(\text{MC}_{\lambda_0}(M)) = \psi_{t', \lambda}(\tilde{s}_+ \mathcal{M}_{\lambda_0}) = \psi_{t', \lambda}(\tilde{s}_+ e_+ \mathcal{N}_{\lambda_0}) = \tilde{s}_+ e_+ T.$$

LEMMA 2.3. — We set $M(*\infty)_\gamma = \ker(x'\partial_{x'} - \gamma)^r$ acting on $\psi_{x'}M(*\infty)$ for $r \gg 0$ and $M(*\infty)^\lambda = M(*\infty)_\gamma[x', x'^{-1}]$. Let us set

$$T_0^\lambda = \left(M(*\infty)^{\lambda\lambda_0}[(x' - 1)^{-1}], \nabla + \gamma_0 \left(-\frac{dx'}{x'} + \frac{dx'}{x' - 1} \right) \right)$$

in the chart $\mathbb{P}_{\text{exc}}^1 \setminus \{\infty\}$ (with the coordinate x' instead of u_2) and denote similarly its meromorphic extension at infinity, with the action of the nilpotent endomorphism $x'\partial_{x'} - (\gamma + \gamma_0)$. Then:

(Case 1) For $\lambda \notin \{1, \overline{\lambda_0}\}$, T is supported on $\mathbb{P}_{\text{exc}}^1$ and $T = T_0^\lambda$.

(Case 2) For $\lambda = 1$, T is supported on $\mathbb{P}_{\text{exc}}^1$ and is isomorphic to the minimal extension of T_0^1 at $x' = 0$.

(Case 3) For $\lambda = \overline{\lambda_0}$, T is supported on $\mathbb{P}_x^1 \cup \mathbb{P}_{\text{exc}}^1$ and comes in an exact sequence

$$0 \rightarrow (T_0^{\overline{\lambda_0}})' \rightarrow T \rightarrow T_1 \rightarrow 0$$

compatible with the nilpotent endomorphism, where $(T_0^{\overline{\lambda_0}})'$ is the extension by zero of $T_0^{\overline{\lambda_0}}$ at infinity (instead of meromorphic), and T_1 is supported on \mathbb{P}_x^1 and is isomorphic to the meromorphic extension of M at infinity with the action by 0 of the nilpotent endomorphism.

Proof. — We realize a local study of the problem, reasoning in the three following charts:

- (i) in the chart (u_2, v_2) , called *Chart 1*;
- (ii) in the neighbourhood of $\mathbb{P}_x^1 \setminus \{\infty\}$, called *Chart 2*;
- (iii) in the neighbourhood of ∞ , called *Chart 3*.

The cases of Charts 1 and 2 do not contain any significant problem and are treated in [Mar18a, 4.2.4]. In Chart 1, we find

$$\psi_{v_2, \lambda} \mathcal{N}_{\lambda_0} = \begin{cases} T_0^\lambda & \text{if } \lambda \neq 1 \\ (T_0^\lambda)_{\min(\{0\})} & \text{if } \lambda = 1. \end{cases}$$

In Chart 2, in which one can use the coordinates (x, t') , we find

$$\psi_{t', \lambda} \mathcal{M}_{\lambda_0} = \begin{cases} 0 & \text{if } \lambda \neq \overline{\lambda_0} \\ M & \text{if } \lambda = \overline{\lambda_0}. \end{cases}$$

Let us now make precise the case of Chart 3. For $\alpha \in \mathbb{R}^2$, we set

$$(N_{\lambda_0})_\alpha = \bigcup_{r_1 \geq 0} \ker(u_1 \partial_{u_1} - \alpha_1)^{r_1} \cap \bigcup_{r_2 \geq 0} \ker(v_1 \partial_{v_1} - \alpha_2)^{r_2}.$$

By writing the expression of the connection in coordinates (u_1, v_1) , we get that $(N_{\lambda_0})_{(-\gamma_0, \alpha - \gamma_0)}$ can be identified with $M(*\infty)_\alpha$ with actions of $u_1 \partial_{u_1}$ and $v_1 \partial_{v_1}$ respectively expressed as $-\gamma_0 \text{Id}$ and $x' \partial_{x'} - \gamma_0 \text{Id}$. Here $\psi_{t' \circ e, \lambda} \mathcal{N}_{\lambda_0} = \psi_{g, \lambda} \mathcal{N}_{\lambda_0}$ where $g = u_1 v_1$, and we are in the situation of a calculation of nearby cycles of a

coherent $\mathbb{C}[u_1, v_1]\langle \partial_{u_1}, \partial_{v_1} \rangle$ -module of normal crossing type along $D = \{g = 0\}$ where g is a monomial function. The general question is developed in [Sai90, §3.a] (see also [SS17, §11.3]), let us make precise it in our particular case. If we consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{i_g} & X \times \mathbb{C}_z \\ & \searrow g & \downarrow p_2 \\ & & \mathbb{C}_z \end{array}$$

we can see that $(i_g)_+ \mathcal{N}_{\lambda_0} = \mathcal{N}_{\lambda_0}[\partial_z] = \bigoplus_{k \geq 0} (\mathcal{N}_{\lambda_0} \otimes \partial_z^k)$ is a left $\mathbb{C}[u_1, v_1, z]\langle \partial_{u_1}, \partial_{v_1}, \partial_z \rangle$ -module equipped of the following actions:

- (i) action of $\mathbb{C}[u_1, v_1] : f(u_1, v_1) \cdot (m \otimes \partial_z^k) = (f(u_1, v_1)m) \otimes \partial_z^k$.
- (ii) action of $\partial_z : \partial_z(m \otimes \partial_z^k) = m \otimes \partial_z^{k+1}$.
- (iii) action of $\partial_{u_1} : \partial_{u_1}(m \otimes \partial_z^k) = (\partial_{u_1}m) \otimes \partial_z^k - v_1m \otimes \partial_z^{k+1}$.
- (iv) action of $\partial_{v_1} : \partial_{v_1}(m \otimes \partial_z^k) = (\partial_{v_1}m) \otimes \partial_z^k - u_1m \otimes \partial_z^{k+1}$.
- (v) action of $z : z \cdot (m \otimes \partial_z^k) = gm \otimes \partial_z^k - km \otimes \partial_z^{k-1}$.

Let us denote by S_u (resp. S_v) the action defined by $S_u(m \otimes \partial_z^k) = (u_1 \partial_{u_1} m) \otimes \partial_z^k$ (resp. $S_v(m \otimes \partial_z^k) = (v_1 \partial_{v_1} m) \otimes \partial_z^k$). With $E = z \partial_z$, we get the relations

$$u_1 \partial_{u_1}(m \otimes \partial_z^k) = (S_u - E - (k+1))(m \otimes \partial_z^k)$$

and

$$v_1 \partial_{v_1}(m \otimes \partial_z^k) = (S_v - E - (k+1))(m \otimes \partial_z^k).$$

With $V^\bullet(\mathcal{N}_{\lambda_0}[\partial_z])$ the V -filtration with respect to z , we have $T = \psi_{g,\lambda} \mathcal{N}_{\lambda_0} = \text{gr}_V^\gamma(\mathcal{N}_{\lambda_0}[\partial_z])$. We have the decompositions $\mathcal{N}_{\lambda_0}^{\text{alg}} = \bigoplus_{\alpha \in \mathbb{R}^2} (\mathcal{N}_{\lambda_0})_\alpha$ and $T^{\text{alg}} = \bigoplus_{\beta \in \mathbb{R}^2} T_\beta$, and by arguing in a way similar to that of [SS17, 11.3.11] (with left modules), we show that the only indices β that appear are those such that $\alpha = \beta + (\gamma + k + 1)(1, 1)$ for α in the decomposition of $\mathcal{N}_{\lambda_0}^{\text{alg}}$ and $k \in \mathbb{Z}$. In particular, we cannot have $\alpha_2 \in \mathbb{Z}$ and having a minimal extension along $\{v_1 = 0\}$ does not play any role here. In other words, we can identify in this part \mathcal{N}_{λ_0} and N_{λ_0} .

More precisely, for $\beta_1, \beta_2 \geq -1$, we deduce from [Sai90, Th. 3.3] (or [SS17, Cor. 11.3.16]) the following expressions for T_β :

$$T_\beta = \begin{cases} 0 & \text{if } \beta_1 \neq -1, \beta_2 \neq -1 \\ \text{coker}(S_u - E \in \text{End}((N_{\lambda_0})_{\gamma, \beta_2 + \gamma + 1}[E])) & \text{if } \beta_1 = -1, \beta_2 \neq -1 \\ \text{coker}(S_v - E \in \text{End}((N_{\lambda_0})_{\beta_1 + \gamma + 1, \gamma}[E])) & \text{if } \beta_1 \neq -1, \beta_2 = -1 \\ \text{coker}((S_u - E)(S_v - E) \in \text{End}((N_{\lambda_0})_{\gamma, \gamma}[E])) & \text{if } \beta = (-1, -1). \end{cases}$$

Let us set $A \subset (-1, 0]$ the (finite) set of $\alpha \in (-1, 0]$ such that $M(*\infty)_\alpha \neq 0$ and look at the different cases for $\beta \in [-1, 0]^2$:

- (i) For $\beta_2 \neq -1$, we have $T_{(-1, \beta_2)} \neq 0$ iff $\gamma = -\gamma_0$ and $\beta_2 \in A$.
- (ii) For $\beta_1 \neq -1$, we have $T_{(\beta_1, -1)} \neq 0$ iff $\gamma = \alpha - \gamma_0$ with $\alpha \in A \bmod \mathbb{Z}$ and $\beta_1 = -\alpha \bmod \mathbb{Z}$.
- (iii) $T_{(-1, -1)} \neq 0$ iff $\gamma = -\gamma_0$ and $0 \in A$.

We deduce from these relations that:

(Case 1+2) If $\gamma \neq -\gamma_0$ then T is supported on $\mathbb{P}_{\text{exc}}^1$ and, according to (ii), is determined by the only data of $\text{coker}(S_v - E \in \text{End}((N_{\lambda_0})_{-\gamma_0, \gamma}[E]))$ equipped with an action of $E - \gamma$, that we can identify with $(N_{\lambda_0})_{-\gamma_0, \gamma}$ where the action of $E - \gamma$ can be identified with $S_v - \gamma$. Consequently, T is determined by $M(*\infty)_{\gamma + \gamma_0}$ with an action of $x' \partial_{x'} - (\gamma + \gamma_0)$.

(Case 3) If $\gamma = -\gamma_0$ then T is determined:

- by the first data, according to (i), of $\text{coker}(S_u - E \in \text{End}((N_{\lambda_0})_{-\gamma_0, \alpha - \gamma_0}[E]))$ for $\alpha \in A \bmod \mathbb{Z}$, $\alpha \notin \mathbb{Z}$, supported on \mathbb{P}_x^1 and equipped with an action of $E + \gamma_0$, that we can identify with $(N_{\lambda_0})_{-\gamma_0, \alpha - \gamma_0}$ where the action of $E + \gamma_0$ is identified with $S_u + \gamma_0$, that we identify with $M(*\infty)_\alpha$ with an action by 0.
- by the second data of the biquiver

$$\begin{array}{ccc} T_{(-1, -1)} & \begin{array}{c} \xrightarrow{v_1} \\ \xleftarrow{\partial_{v_1}} \end{array} & T_{(-1, 0)} \\ \partial_{u_1} \updownarrow u_1 & & \\ T_{(0, -1)} & & \end{array}$$

As u_1^{-1} acts on $(N_{\lambda_0})_{-\gamma_0+1, -\gamma_0}[E]$ and v_1^{-1} on $(N_{\lambda_0})_{-\gamma_0, -\gamma_0+1}$, it is possible to assume that $T_{(-1, -1)}$, $T_{(0, -1)}$ and $T_{(-1, 0)}$ are all three cokernels of applications of $\text{End}((N_{\lambda_0})_{-\gamma_0, -\gamma_0}[E])$. Setting $C_{uv} = \text{coker}(S_u - E)(S_v - E)$, $C_u = \text{coker}(S_u - E)$ and $C_v = \text{coker}(S_v - E)$, the previous biquiver can be identified with the following

$$\begin{array}{ccc} C_{uv} & \begin{array}{c} \xrightarrow{\varphi_v} \\ \xleftarrow{S_v - E} \end{array} & C_u \\ S_u - E \updownarrow \varphi_u & & \\ C_v & & \end{array}$$

where $\varphi_u : C_{uv} \rightarrow C_v$ is induced by the inclusion $\text{im}(S_u - E)(S_v - E) \subset \text{im}(S_v - E)$, and the same for φ_v . As $S_u - E \in \text{End}((N_{\lambda_0})_{-\gamma_0, -\gamma_0}[E])$ is injective (because S_u is identified on $M(*\infty)_0$ with $-\gamma_0 \text{Id}$) and

$$\text{im}(S_u - E : C_v \rightarrow C_{uv}) = \frac{\text{im}(S_u - E)}{\text{im}(S_u - E)(S_v - E)} = \ker \varphi_v,$$

we deduce the following exact sequence:

$$0 \rightarrow C_v \rightarrow C_{uv} \rightarrow C_u \rightarrow 0.$$

Therefore, we have an exact sequence of biquivers:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C_v & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} & 0 & \longrightarrow & C_{uv} & \begin{array}{c} \xrightarrow{\varphi_v} \\ \xleftarrow{S_v-E} \end{array} & C_u & \longrightarrow & C_u & \begin{array}{c} \xrightarrow{\text{Id}} \\ \xleftarrow{S_v-E} \end{array} & C_u & \longrightarrow & 0 \\
 & & \updownarrow \text{Id} & & \updownarrow S_u-E & & \updownarrow S_u-E & & \updownarrow \varphi_u & & \updownarrow & & \updownarrow & & \\
 & & C_v & & C_v & & C_v & & C_v & & 0 & & & &
 \end{array}$$

- The left biquiver is a quiver of extension by zero supported on $\mathbb{P}_{\text{exc}}^1$ and identified with $(N_{\lambda_0})_{-\gamma_0, -\gamma_0}$ where the action of $E + \gamma_0$ can be identified with $S_v + \gamma_0$, in other words $M(*\infty)_0$ with the action of $x'\partial_{x'}$. This is the following biquiver:

$$\begin{array}{ccc}
 M(*\infty)_0 & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} & 0 \\
 \updownarrow \text{Id} & & \\
 M(*\infty)_0 & &
 \end{array}$$

- The right biquiver is a quiver of meromorphic extension supported on \mathbb{P}_x^1 and identified with $(N_{\lambda_0})_{-\gamma_0, -\gamma_0}$ where the action of $E + \gamma_0$ is equal to 0, in other words $M(*\infty)_0$ with the action by 0. This is the following biquiver:

$$\begin{array}{ccc}
 M(*\infty)_0 & \begin{array}{c} \xrightarrow{\text{Id}} \\ \xleftarrow{x'\partial_{x'}} \end{array} & M(*\infty)_0 \\
 \updownarrow & & \\
 0 & &
 \end{array}$$

- It is possible to make explicit the central biquiver in terms of $M(*\infty)_0$, insofar as we can identify C_{uv} with $(M(*\infty)_0)^2$ with an action of

$$E + \gamma_0 = \left(\begin{array}{c|c} \gamma_0 \text{Id} & \gamma_0(x'\partial_{x'} - \gamma_0 \text{Id}) \\ \hline \text{Id} & x'\partial_{x'} - \gamma_0 \text{Id} \end{array} \right),$$

and we get the following biquiver:

$$\begin{array}{ccc}
 (M(*\infty)_0)^2 & \begin{array}{c} \xrightarrow{p_v} \\ \xleftarrow{(x'\partial_{x'} - \gamma_0 \text{Id}, -\text{Id})} \end{array} & M(*\infty)_0 \\
 \updownarrow (-\gamma_0 \text{Id}, -\text{Id}) & & \\
 M(*\infty)_0 & &
 \end{array}$$

where $p_u = p_1 + (x'\partial_{x'} - \gamma_0)p_2$ and $p_v = p_1 - \gamma_0 p_2$, with p_1, p_2 the projections onto the first and second factors.

Finally, for $\lambda = \overline{\lambda_0}$ we have an exact sequence

$$0 \rightarrow (T_0^{\overline{\lambda_0}})' \rightarrow T \rightarrow T_1 \rightarrow 0$$

where $(T_0^{\overline{\lambda_0}})'$ is the extension by zero of $T_0^{\overline{\lambda_0}}$ at infinity equipped with the nilpotent endomorphism $x'\partial_{x'}$, and T_1 is supported on \mathbb{P}_x^1 and given by the meromorphic extension of M at infinity equipped with the nilpotent endomorphism 0.

By gluing the expressions obtained for the different values of λ in each of the three charts, we get the announced result of the lemma. \square

By construction, the complex $K^\bullet = \tilde{s}_+ e_+ T$ has cohomology in degree zero only. More precisely, as $\tilde{s} \circ e : \mathbb{P}_{\text{exc}}^1 \cup \mathbb{P}_x^1 \rightarrow \{\text{pt}\}$, that amounts to saying that $\mathbf{R}\Gamma(\mathbb{P}_{\text{exc}}^1 \cup \mathbb{P}_x^1, \mathbf{DR}^{\text{an}} T)$ is a two terms complex with a kernel reduced to zero. If we take into account the monodromy filtration M_\bullet , we have the following more precise result:

LEMMA 2.4. — $H^j(\text{gr}_\ell^M K^\bullet) = 0$ for $j \neq 0$ and $\ell \in \mathbb{Z}$.

Proof. — (Case 1+2) If $\lambda \neq \overline{\lambda_0}$ then T is supported on $\mathbb{P}_{\text{exc}}^1$ and localized at infinity. Consequently, we can see T as a $\mathbb{C}[x']\langle \partial_{x'} \rangle$ -module. The question is to show that the two terms complex

$$\text{gr}_\ell^M T \xrightarrow{\nabla_{\partial_{x'}}} \text{gr}_\ell^M T$$

has a kernel reduced to zero. With N the nilpotent endomorphism, let us set

$$\tilde{T} = \left(M(*\infty)^{\lambda\lambda_0}, \tilde{\nabla}_{\partial_{x'}} = \frac{\partial}{\partial x'} + \frac{\gamma}{x'} \text{Id} + \frac{N}{x'} \right)$$

optionally minimally extended at 0 if $\lambda = 1$, and

$$1_{\gamma_0} = \left(\mathbb{C}[x', (x' - 1)^{-1}], \frac{\partial}{\partial x'} + \frac{\gamma_0}{x' - 1} \text{Id} \right)$$

so that $T = \tilde{T} \otimes 1_{\gamma_0}$. For $m \in \tilde{T}$ and $m' \in \mathbb{C}[x', (x' - 1)^{-1}]$, we have

$$\nabla_{\partial_{x'}}(m \otimes m') = \tilde{\nabla}_{\partial_{x'}}(m) \otimes m' + m \otimes \left(\frac{\partial}{\partial x'} + \frac{\gamma_0}{x' - 1} \text{Id} \right) m'.$$

Now, we have $\text{gr}_\ell^M T = \text{gr}_\ell^M \tilde{T} \otimes 1_{\gamma_0}$ and we want to show that

$$\ker \left(\text{gr}_\ell^M \tilde{T} \otimes 1_{\gamma_0} \xrightarrow{\nabla_{\partial_{x'}}} \text{gr}_\ell^M \tilde{T} \otimes 1_{\gamma_0} \right) = 0.$$

For $m \in \text{gr}_\ell^M \tilde{T}$, let us remark that

$$\nabla_{\partial_{x'}} \left(m \otimes \frac{1}{(x' - 1)^k} \right) = \tilde{\nabla}_{\partial_{x'}}(m) \otimes \frac{1}{(x' - 1)^k} - \frac{km}{(x' - 1)^{k+1}} + \frac{\gamma_0 m}{(x' - 1)^{k+1}},$$

from which we deduce, as $\gamma_0 \notin \mathbb{Z}$, that $m \otimes (x' - 1)^{-k}$ has a pole at 1 of order $k + 1$ if $m \neq 0$. Therefore, if an element $m \otimes m'$ is such that $\nabla_{\partial_{x'}}(m \otimes m') = 0$, then $m = 0$.

(Case 3) If $\lambda = \overline{\lambda_0}$, we have the exact sequence $0 \rightarrow T_0 \rightarrow T \rightarrow T_1 \rightarrow 0$ where $T_0 = (T_0^{\overline{\lambda_0}})'$ is supported on $\mathbb{P}_{\text{exc}}^1$ and T_1 is supported on \mathbb{P}_x^1 . In an equivalent way, we can think in terms of primitive parts instead of graded parts, that we are going to do below.

The reasoning of the previous case applies in the same way for $K_0^\bullet = \tilde{s}_+ e_+ T_0$, that gives us that the complexes $P_\ell K_0^\bullet$ are concentrated in degree 0 for all $\ell \in \mathbb{N}$. As N is zero on T_1 , we have $N(T) \subset T_0$. Let us show we have equality and, for that, let us go back to the description of T at the neighbourhood of ∞ in terms of biquivers:

$$\begin{array}{ccc} (M(*\infty)_0)^2 & \begin{array}{c} \xrightarrow{p_v} \\ \xleftarrow{(x'\partial_{x'} - \gamma_0 \text{Id}, -\text{Id})} \end{array} & M(*\infty)_0 \\ \begin{array}{c} \uparrow \\ (-\gamma_0 \text{Id}, -\text{Id}) \\ \downarrow p_u \end{array} & & \\ M(*\infty)_0 & & \end{array}$$

The action of $E + \gamma_0$ on $(M(*\infty)_0)^2$ is given by

$$E + \gamma_0 = \left(\begin{array}{c|c} \gamma_0 \text{Id} & \gamma_0(x'\partial_{x'} - \gamma_0 \text{Id}) \\ \hline \text{Id} & x'\partial_{x'} - \gamma_0 \text{Id} \end{array} \right),$$

whose rank is equal to the dimension of $M(*\infty)_0$, and thus

$$(E + \gamma_0)(M(*\infty)_0)^2 = \{(\gamma_0 m, m) \in (M(*\infty)_0)^2 \mid m \in M(*\infty)_0\} \simeq M(*\infty)_0.$$

Now, a calculation shows that the following diagram

$$\begin{array}{ccc} (M(*\infty)_0)^2 & \xrightarrow{p_2 \circ (E + \gamma_0)} & M(*\infty)_0 \\ \begin{array}{c} \uparrow \\ (-\gamma_0 \text{Id}, -\text{Id}) \\ \downarrow p_u \end{array} & & \begin{array}{c} \uparrow \\ \text{Id} \\ \downarrow -x'\partial_{x'} \end{array} \\ M(*\infty)_0 & \xrightarrow{-x'\partial_{x'}} & M(*\infty)_0 \end{array}$$

is commutative, so the image by N of the biquiver representing T gives the biquiver representing T_0 , in other words $N(T) = T_0$. Consequently, we are in a situation of a minimal extension quiver:

$$T \begin{array}{c} \xrightarrow{N} \\ \xleftarrow{\quad} \end{array} T_0.$$

We deduce from [KK87, Prop. 2.1.1(iii)] that $P_\ell T \simeq P_{\ell-1} T_0$ for $\ell \geq 1$ and then the complexes $P_\ell K^\bullet$ are concentrated in degree 0 for all $\ell \geq 1$. Moreover,

as the total complex K^\bullet is concentrated in degree 0, then the complex $P_0 K^\bullet$ is also concentrated in degree 0. We have the same property for graded parts instead of primitive parts. \square

Now, with these two lemmas, we are able to show the main theorem.

Proof of Theorem 1. — (Case 1) Let us begin with the case $\lambda \notin \{1, \overline{\lambda_0}\}$ and, as a first step, without taking into account the monodromy filtration. As $\tilde{s} \circ e : \mathbb{P}_{\text{exc}}^1 \rightarrow \{\text{pt}\}$, we have $\psi_{\infty, \lambda}(\text{MC}_{\lambda_0}(M)) = \tilde{s}_+ e_+ T = \mathbf{R}\Gamma(\mathbb{P}_{\text{exc}}^1, \text{DR } T)$. Moreover, we have $\text{DR } T = j_* \mathcal{V}$ where \mathcal{V} is an irreducible non-constant local system, then $\mathbf{H}^m(\mathbb{P}_{\text{exc}}^1, \text{DR } T) = H^m(\mathbb{P}_{\text{exc}}^1, j_* \mathcal{V}) = 0$ for $m \neq 1$ and

$$\nu_{\infty, \lambda}^p(\text{MC}_{\lambda_0}(M)) = \dim \text{gr}_F^p \psi_{\infty, \lambda}(\text{MC}_{\lambda_0}(M)) = \dim \text{gr}_F^p \mathbf{H}^1(\mathbb{P}^1, \text{DR } T).$$

Let us set $\mathbf{x} = \{0, 1, \infty\}$. According to [DS13, Prop. 2.3.3], we have

$$(2.5) \quad \nu_{\infty, \lambda}^p(\text{MC}_{\lambda_0}(M)) = \delta^{p-1}(T) - \delta^p(T) - h^p(T) - h^{p-1}(T) + \sum_{x \in \mathbf{x}} \left(\sum_{\mu \neq 1} \nu_{x, \mu}^{p-1}(T) + \mu_{x, 1}^p(T) \right).$$

As $\lambda \lambda_0 \neq 1$, we have $\mathbf{H}^1(\mathbb{P}^1, \text{DR } M(*\infty)^{\lambda \lambda_0}) = 0$. If we do the same reasoning with $M(*\infty)^{\lambda \lambda_0}$ instead of T , we get

$$(2.6) \quad \delta^{p-1}(M(*\infty)^{\lambda \lambda_0}) - \delta^p(M(*\infty)^{\lambda \lambda_0}) - \nu_{\infty, \lambda \lambda_0}^p(M) = 0.$$

According to Lemma 2.3, we have

$$T = M(*\infty)^{\lambda \lambda_0} \otimes \left(\mathbb{C}[x', x'^{-1}, (x' - 1)^{-1}], d + \gamma_0 \left(-\frac{dx'}{x'} + \frac{dx'}{x' - 1} \right) \right),$$

so it is possible to apply [DS13, Prop. 2.3.2]:

$$(2.7) \quad \delta^p(T) = \delta^p(M(*\infty)^{\lambda \lambda_0}) - \nu_{\infty, \lambda \lambda_0}^p(M) + \sum_{\alpha \in]\gamma_0, 1[} \nu_{\infty, e^{-2i\pi\alpha}}^p(M(*\infty)^{\lambda \lambda_0}) + \underbrace{\sum_{\alpha \in]1-\gamma_0, 1[} \nu_{1, e^{-2i\pi\alpha}}^p(M(*\infty)^{\lambda \lambda_0})}_{=0 \text{ because } \alpha \neq 0}.$$

We now deduce from

$$\sum_{\alpha \in]\gamma_0, 1[} \nu_{\infty, e^{-2i\pi\alpha}}^p(M(*\infty)^{\lambda \lambda_0}) = \begin{cases} \nu_{\infty, \lambda \lambda_0}^p(M) & \text{if } \gamma \in (0, 1 - \gamma_0) \\ 0 & \text{if } \gamma \in (1 - \gamma_0, 1), \end{cases}$$

that

$$\delta^p(T) = \begin{cases} \delta^p(M(*\infty)^{\lambda \lambda_0}) & \text{if } \gamma \in (0, 1 - \gamma_0) \\ \delta^p(M(*\infty)^{\lambda \lambda_0}) - \nu_{\infty, \lambda \lambda_0}^p(M) & \text{if } \gamma \in (1 - \gamma_0, 1). \end{cases}$$

Moreover, using [DS13, 2.2.13, 2.2.14], we get

$$\begin{aligned} \sum_{x \in \mathfrak{X}} \sum_{\mu \neq 1} \nu_{x, \mu}^{p-1}(T) &= \sum_{\mu \neq 1} \left(\nu_{\infty, \mu \lambda_0}^{p-1}(M(*\infty)^{\lambda \lambda_0}) + \nu_{1, \mu \lambda_0}^{p-1}(M(*\infty)^{\lambda \lambda_0}) \right) \\ &= \nu_{\infty, \lambda \lambda_0}^{p-1}(M(*\infty)^{\lambda \lambda_0}) + \nu_{1, 1}^{p-1}(M(*\infty)^{\lambda \lambda_0}) \\ &= 2\nu_{\infty, \lambda \lambda_0}^{p-1}(M) \end{aligned}$$

and

$$\sum_{x \in \mathfrak{X}} \mu_{x, 1}^p(T) = \sum_{\ell \geq 0} \sum_{k=0}^{\ell} \left(\underbrace{\mu_{\infty, \lambda_0, \ell+1}^{p+k}(M(*\infty)^{\lambda \lambda_0})}_{=0} + \underbrace{\mu_{1, \lambda_0, \ell+1}^p(M(*\infty)^{\lambda \lambda_0})}_{=0} \right) = 0.$$

We are now able to apply the formula (2.5):

(i) For $\gamma \in (0, 1 - \gamma_0)$, we have:

$$\begin{aligned} \nu_{\infty, \lambda}^p(\text{MC}_{\lambda_0}(M)) &= \underbrace{\delta^{p-1}(M(*\infty)^{\lambda \lambda_0}) - \delta^p(M(*\infty)^{\lambda \lambda_0}) - \nu_{\infty, \lambda \lambda_0}^p(M)}_{=0 \text{ according to (2.6)}} \\ &\quad - \nu_{\infty, \lambda \lambda_0}^{p-1}(M) + 2\nu_{\infty, \lambda \lambda_0}^{p-1}(M) \\ &= \nu_{\infty, \lambda \lambda_0}^{p-1}(M). \end{aligned}$$

(ii) For $\gamma \in (1 - \gamma_0, 1)$, we have:

$$\begin{aligned} \nu_{\infty, \lambda}^p(\text{MC}_{\lambda_0}(M)) &= \left(\delta^{p-1}(M(*\infty)^{\lambda \lambda_0}) - \nu_{\infty, \lambda \lambda_0}^{p-1}(M) \right) - \left(\delta^p(M(*\infty)^{\lambda \lambda_0}) - \nu_{\infty, \lambda \lambda_0}^p(M) \right) \\ &\quad - \nu_{\infty, \lambda \lambda_0}^p(M) - \nu_{\infty, \lambda \lambda_0}^{p-1}(M) + 2\nu_{\infty, \lambda \lambda_0}^{p-1}(M) \\ &= \delta^{p-1}(M(*\infty)^{\lambda \lambda_0}) - \delta^p(M(*\infty)^{\lambda \lambda_0}) \\ &= \nu_{\infty, \lambda \lambda_0}^p(M). \end{aligned}$$

To resume, we have

$$\nu_{\infty, \lambda}^p(\text{MC}_{\lambda_0}(M)) = \begin{cases} \nu_{\infty, \lambda \lambda_0}^{p-1}(M) & \text{if } \gamma \in (0, 1 - \gamma_0) \\ \nu_{\infty, \lambda \lambda_0}^p(M) & \text{if } \gamma \in (1 - \gamma_0, 1). \end{cases}$$

Let us now take into account the monodromy filtration. By an argument of degeneration of spectral sequence detailed in [Mar18a, Lemma 4.2.6], we have

$$\dim P_{\ell} \psi_{\infty, \lambda}(\text{MC}_{\lambda_0}(M)) = \dim \mathbf{H}^1(\mathbb{P}^1, \text{DR } P_{\ell} T).$$

Let us fix $\ell \geq 0$ and consider the Hodge filtration of the complex $P_\ell K^\bullet$. As the connexion sends the filtered space of order p in the filtered space of order $p-1$, we have $H^j(\text{gr}_F^p P_\ell K^\bullet) = 0$ for $j \neq 0$ and $p \in \mathbb{Z}$ similarly to Lemma 2.4. Then

$$\nu_{\infty, \lambda, \ell}^p(\text{MC}_{\lambda_0}(M)) = \dim \text{gr}_F^p P_\ell \psi_{\infty, \lambda}(\text{MC}_{\lambda_0}(M)) = \dim \text{gr}_F^p \mathbf{H}^1(\mathbb{P}^1, \text{DR } P_\ell T).$$

As $\text{DR } P_\ell T$ is again of the form $j_* \mathcal{V}$, we can apply the same reasoning as for T , and get a formula similar to (2.5) for $P_\ell T$:

$$(2.8) \quad \nu_{\infty, \lambda, \ell}^p(\text{MC}_{\lambda_0}(M)) = \delta^{p-1}(P_\ell T) - \delta^p(P_\ell T) - h^p(P_\ell T) - h^{p-1}(P_\ell T) \\ + \sum_{x \in \mathbf{x}} \left(\sum_{\mu \neq 1} \nu_{x, \mu}^{p-1}(P_\ell T) + \mu_{x, 1}^p(P_\ell T) \right).$$

On the one hand we have

$$P_\ell T = P_\ell M(*\infty)^{\lambda \lambda_0} \otimes \left(\mathbb{C}[x', x'^{-1}, (x' - 1)^{-1}], d + \gamma_0 \left(-\frac{dx'}{x'} + \frac{dx'}{x' - 1} \right) \right),$$

and on the other hand we have a formula similar to (2.6) with $P_\ell M(*\infty)^{\lambda \lambda_0}$. So we can repeat the reasoning as without the monodromy filtration and get

$$\nu_{\infty, \lambda, \ell}^p(\text{MC}_{\lambda_0}(M)) = \begin{cases} \nu_{\infty, \lambda \lambda_0, \ell}^{p-1}(M) & \text{if } \gamma \in (0, 1 - \gamma_0) \\ \nu_{\infty, \lambda \lambda_0, \ell}^p(M) & \text{if } \gamma \in (1 - \gamma_0, 1). \end{cases}$$

(Case 2) Let us look at the case $\lambda = 1$, which differs from the previous one locally at the neighbourhood of 0 where we have a minimal extension. The data of \tilde{T} , defined in the proof of Lemma 2.4, is equivalent to the quiver

$$M(*\infty)_{\gamma_0}[x'] \quad \rightleftarrows \quad N(M(*\infty)_{\gamma_0})[\partial_{x'}],$$

where the action of $\partial_{x'}$ on the left term is given for $m \in M(*\infty)_{\gamma_0}$, $k \geq 1$ by

$$\partial_{x'}(mx'^k) = \partial_{x'} x'(mx'^{k-1}) = (N + \text{Id})(mx'^{k-1}),$$

and similarly for the action of x' on the right term. If we set $H = M(*\infty)_{\gamma_0}$ et $G = N(H)$, we have $N : M_\ell H \rightarrow M_{\ell-1} G$ and we deduce that $P_\ell \tilde{T}$ is locally given by the quiver

$$(P_\ell H)[x'] \quad \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{0} \end{array} \quad (P_\ell G)[\partial_{x'}].$$

In other words, $P_\ell T$ is locally given by the direct sum of $P_\ell^1 T$ defined by the quiver

$$(P_\ell H)[x'] \quad \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{0} \end{array} \quad 0$$

and $P_\ell^2 T$ defined by the quiver

$$0 \quad \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{0} \end{array} \quad (P_\ell G)[\partial_{x'}].$$

In fact, we get a global decomposition into a direct sum $P_\ell T = P_\ell^1 T \oplus P_\ell^2 T$, where $P_\ell^1 T$ is given by the same term as in Case 1, and $P_\ell^2 T$ is supported in 0 and given by $(P_\ell G)[\partial_{x'}]$. We have

$$\begin{aligned} \nu_{\infty,1,\ell}^p(\text{MC}_{\lambda_0}(M)) &= \dim \text{gr}_F^p \mathbf{H}^1(\mathbb{P}^1, \text{DR}(P_\ell^1 T \oplus P_\ell^2 T)) \\ &= \dim \text{gr}_F^p \mathbf{H}^1(\mathbb{P}^1, \text{DR } P_\ell^1 T) + \dim \text{gr}_F^p \mathbf{H}^1(\mathbb{P}^1, \text{DR } P_\ell^2 T), \end{aligned}$$

and, therefore, two dimensions to calculate. The first one can be got in repeating the argument of Case 1 because $\text{DR } P_\ell^1 T$ is again of the form $j_* \mathcal{V}$. Firstly, we have

$$\sum_{\alpha \in [\gamma_0, 1[} \nu_{\infty, e^{-2i\pi\alpha}}^p(P_\ell M(*\infty)^{\lambda_0}) = \nu_{\infty, \lambda_0, \ell}^p(M),$$

and

$$\delta^p(P_\ell^1 T) = \delta^p(P_\ell M(*\infty)^{\lambda_0}).$$

Secondly, we have

$$\begin{aligned} \sum_{x \in \mathfrak{X}} \sum_{\mu \neq 1} \nu_{x,\mu}^{p-1}(P_\ell^1 T) &= \sum_{\mu \neq 1} \left(\underbrace{\nu_{\infty, \mu\lambda_0}^{p-1}(P_\ell M(*\infty)^{\lambda_0})}_{=0 \text{ because } \mu \neq 1} + \nu_{1, \mu\lambda_0}^{p-1}(P_\ell M(*\infty)^{\lambda_0}) \right) \\ &= \nu_{1,1}^{p-1}(P_\ell M(*\infty)^{\lambda_0}) \\ &= \nu_{\infty, \lambda_0, \ell}^{p-1}(M), \end{aligned}$$

and

$$\begin{aligned} \sum_{x \in \mathfrak{X}} \mu_{x,1}^p(P_\ell^1 T) &= \sum_{\ell \geq 0} \sum_{k=0}^{\ell} \left(\underbrace{\mu_{\infty, \lambda_0, \ell+1}^{p+k}(P_\ell M(*\infty)^{\lambda_0})}_{=0} + \underbrace{\mu_{1, \overline{\lambda_0}, \ell+1}^p(P_\ell M(*\infty)^{\lambda_0})}_{=0} \right) \\ &= 0. \end{aligned}$$

Finally, according to 2.6, we get $\dim \text{gr}_F^p \mathbf{H}^1(\mathbb{P}^1, \text{DR } P_\ell^1 T) = 0$.

Let us now try to determine $\dim \text{gr}_F^p \mathbf{H}^1(\mathbb{P}^1, \text{DR } P_\ell^2 T)$. We know that $P_\ell^2 T$ is supported in 0 and given by $(P_\ell G)[\partial_{x'}]$. The Hodge filtration is given by

$$F^p((P_\ell G)[\partial_{x'}]) = \sum_{k \geq 0} \partial_{x'}^k \cdot F^{p+1+k}(P_\ell G).$$

We deduce that $\mathbf{H}^1(\mathbb{P}^1, \text{DR } P_\ell^2 T)$ is given by the cokernel of

$$(P_\ell G)[\partial_{x'}] \xrightarrow{\partial_{x'}} (P_\ell G)[\partial_{x'}]$$

which can be identified to $P_\ell G$ equipped of the filtration

$$F^p \mathbf{H}^1(\mathbb{P}^1, \text{DR } P_\ell^2 T) = F^p(P_\ell G).$$

Finally, we have

$$\begin{aligned} \dim \operatorname{gr}_F^p \mathbf{H}^1(\mathbb{P}^1, \operatorname{DR} P_\ell^2 T) &= \dim \operatorname{gr}_F^p(P_\ell G) \\ &= \dim \operatorname{gr}_F^p(P_{\ell+1} H) \\ &= \nu_{\infty, \lambda_0, \ell+1}^p(M). \end{aligned}$$

Summing the two dimensions, we get

$$\nu_{\infty, 1, \ell}^p(\operatorname{MC}_{\lambda_0}(M)) = 0 + \nu_{\infty, \lambda_0, \ell+1}^p(M) = \nu_{\infty, \lambda_0, \ell+1}^p(M).$$

(Case 3) If $\lambda = \overline{\lambda_0}$, we take again the exact sequence $0 \rightarrow T_0 \rightarrow T \rightarrow T_1 \rightarrow 0$ and we have

$$\nu_{\infty, \overline{\lambda_0}}^p(\operatorname{MC}_{\lambda_0}(M)) = \dim \operatorname{gr}_F^p \mathbf{H}^1(\mathbb{P}_{\text{exc}}^1, \operatorname{DR} T_0) + \dim \operatorname{gr}_F^p \mathbf{H}^1(\mathbb{P}_x^1, \operatorname{DR} T_1).$$

The case of the left term can be treated in the same way that for $\gamma \in (1 - \gamma_0, 1)$ in Case 1, that gives $\dim \operatorname{gr}_F^p \mathbf{H}^1(\mathbb{P}_{\text{exc}}^1, \operatorname{DR} T_0) = \nu_{\infty, 1}^p(M)$. Concerning the right term, we have

$$\dim \operatorname{gr}_F^p \mathbf{H}^1(\mathbb{P}_x^1, \operatorname{DR} T_1) = \dim \operatorname{gr}_F^p \mathbf{H}^1(\mathbb{P}^1, \operatorname{DR} \mathcal{M}) = h^p H^1(\mathbb{A}^1, \operatorname{DR} M),$$

where \mathcal{M} is the meromorphic extension of M at infinity. By [DS13, 2.2.8 & 2.3.5], we have

$$(2.9) \quad h^p H^1(\mathbb{A}^1, \operatorname{DR} M) = h^p H^1(\mathbb{P}^1, \operatorname{DR} \mathcal{M}^{\min}) + \nu_{\infty, 1, \text{prim}}^{p-1}(M),$$

and we get

$$(2.10) \quad \nu_{\infty, \overline{\lambda_0}}^p(\operatorname{MC}_{\lambda_0}(M)) = \nu_{\infty, 1}^p(M) + h^p H^1(\mathbb{P}^1, \operatorname{DR} \mathcal{M}^{\min}) + \nu_{\infty, 1, \text{prim}}^{p-1}(M).$$

Now, we have seen in the proof of Lemma 2.4 that we are in a situation of a minimal extension quiver

$$T \begin{array}{c} \xrightarrow{\text{N}} \\ \xleftarrow{\quad} \end{array} T_0.$$

with $P_\ell T \simeq P_{\ell-1} T_0$ for $\ell \geq 1$. As N is strictly compatible to the Hodge filtration, with a shift $F^\bullet \rightarrow F^{\bullet-1}$, we deduce that $\operatorname{gr}_F^p P_\ell T \simeq \operatorname{gr}_F^{p-1} P_{\ell-1} T_0$ for $\ell \geq 1$, and so

$$\nu_{\infty, \overline{\lambda_0}, \ell}^p(\operatorname{MC}_{\lambda_0}(M)) = \dim \operatorname{gr}_F^{p-1} \mathbf{H}^1(\mathbb{P}_{\text{exc}}^1, \operatorname{DR} P_{\ell-1} T_0) = \nu_{\infty, 1, \ell-1}^{p-1}(M).$$

It remains to treat the case $\ell = 0$ for which we have

$$\nu_{\infty, \overline{\lambda_0}}^p(\operatorname{MC}_{\lambda_0}(M)) = \nu_{\infty, \overline{\lambda_0}, 0}^p(\operatorname{MC}_{\lambda_0}(M)) + \sum_{\ell \geq 1} \sum_{k=0}^{\ell} \nu_{\infty, \overline{\lambda_0}, \ell}^{p+k}(\operatorname{MC}_{\lambda_0}(M))$$

$$\begin{aligned}
 \text{and } \sum_{\ell \geq 1} \sum_{k=0}^{\ell} \nu_{\infty, \overline{\lambda_0}, \ell}^{p+k}(\text{MC}_{\lambda_0}(M)) &= \sum_{\ell \geq 1} \sum_{k=0}^{\ell} \nu_{\infty, 1, \ell-1}^{p-1+k}(M) \\
 &= \sum_{\ell \geq 0} \sum_{k=0}^{\ell} \nu_{\infty, 1, \ell}^{p-1+k}(M) + \sum_{\ell \geq 0} \nu_{\infty, 1, \ell}^p(M) \\
 &= \nu_{\infty, 1}^{p-1}(M) + \nu_{\infty, 1, \text{coprim}}^p(M),
 \end{aligned}$$

so we deduce that

$$\begin{aligned}
 \nu_{\infty, \overline{\lambda_0}, 0}^p(\text{MC}_{\lambda_0}(M)) &= h^p H^1(\mathbb{P}^1, \text{DR } \mathcal{M}^{\min}) + \nu_{\infty, 1, \text{prim}}^{p-1}(M) - \nu_{\infty, 1, \text{coprim}}^p(M) \\
 &\quad + \nu_{\infty, 1}^p(M) - \nu_{\infty, 1}^{p-1}(M).
 \end{aligned}$$

Yet, a general calculation immediately shows that

$$\nu_{\infty, 1, \text{coprim}}^p(M) - \nu_{\infty, 1, \text{prim}}^{p-1}(M) = \nu_{\infty, 1}^p(M) - \nu_{\infty, 1}^{p-1}(M),$$

and we conclude that $\nu_{\infty, \overline{\lambda_0}, 0}^p(\text{MC}_{\lambda_0}(M)) = h^p H^1(\mathbb{P}^1, \text{DR } \mathcal{M}^{\min})$, which ends the proof of the theorem. \square

3. Proof of Theorems 2 and 3

Proof of Theorem 2. — Applying identity (2.2.2**) of [DS13], we have

$$h^p(\text{MC}_{\lambda_0}(M)) = \sum_{\lambda \in S^1} \nu_{\infty, \lambda}^p(\text{MC}_{\lambda_0}(M)),$$

so it suffices to sum expressions got in Theorem 1:

$$(3.1) \quad \sum_{\lambda \neq 1, \overline{\lambda_0}} \nu_{\infty, \lambda}^p(\text{MC}_{\lambda_0}(M)) = \sum_{\gamma \in (0, \gamma_0)} \nu_{\infty, \lambda}^p(M) + \sum_{\gamma \in (\gamma_0, 1)} \nu_{\infty, \lambda}^{p-1}(M)$$

$$\begin{aligned}
 (3.2) \quad \nu_{\infty, 1}^p(\text{MC}_{\lambda_0}(M)) &= \sum_{\ell \geq 0} \sum_{k=0}^{\ell} \nu_{\infty, \lambda_0, \ell+1}^{p+k}(M) \\
 &= \nu_{\infty, \lambda_0}^p(M) - \nu_{\infty, \lambda_0, \text{coprim}}^p(M) \\
 &= \nu_{\infty, \lambda_0}^{p-1}(M) - \nu_{\infty, \lambda_0, \text{prim}}^{p-1}(M)
 \end{aligned}$$

$$(3.3) \quad \nu_{\infty, \overline{\lambda_0}}^p(\text{MC}_{\lambda_0}(M)) = \nu_{\infty, 1}^p(M) + h^p H^1(\mathbb{A}^1, \text{DR } M)$$

This last equality has already been proved in §2, formulas (2.9) and (2.10). \square

Proof of Theorem 3. — We set $\gamma^p = \delta^p - \delta^{p-1}$. According to identity (2.3.5*) of [DS13], we have

$$h^p H^1(\mathbb{A}^1, \text{DR } M) = -\gamma^p(M) - h^p(M) + \sum_{i=1}^r \left(\sum_{\mu \neq 1} \mu_{x_i, \mu}^{p-1}(M) + \mu_{x_i, 1}^p(M) \right).$$

It follows from Theorem 2 that

$$(3.4) \quad h^p(\text{MC}_{\lambda_0}(M)) + h^p(M) = -\gamma^p(M) - \nu_{\infty, \lambda_0, \text{prim}}^{p-1}(M) + \sum_{\gamma \in [0, \gamma_0)} \nu_{\infty, \lambda}^p(M) \\ + \sum_{\gamma \in [\gamma_0, 1)} \nu_{\infty, \lambda}^{p-1}(M) + \sum_{i=1}^r \left(\sum_{\mu \neq 1} \mu_{x_i, \mu}^{p-1}(M) + \mu_{x_i, 1}^p(M) \right).$$

According to [DS13, Prop. 3.1.1], we have $h^p(\text{MC}_{\overline{\lambda_0}}(\text{MC}_{\lambda_0}(M))) = h^{p-1}(M)$. We can write the same formula than the one above with $\overline{\lambda_0}$ instead of λ_0 , then we apply it with $\text{MC}_{\lambda_0}(M)$ instead of M :

$$h^p(\text{MC}_{\lambda_0}(M)) + h^{p-1}(M) = -\gamma^p(\text{MC}_{\lambda_0}(M)) - \nu_{\infty, \overline{\lambda_0}, \text{prim}}^{p-1}(\text{MC}_{\lambda_0}(M)) \\ + \sum_{\gamma \in [0, 1-\gamma_0)} \nu_{\infty, \lambda}^p(\text{MC}_{\lambda_0}(M)) + \sum_{\gamma \in [1-\gamma_0, 1)} \nu_{\infty, \lambda}^{p-1}(\text{MC}_{\lambda_0}(M)) \\ + \sum_{i=1}^r \left(\sum_{\mu \neq 1} \mu_{x_i, \mu}^{p-1}(\text{MC}_{\lambda_0}(M)) + \mu_{x_i, 1}^p(\text{MC}_{\lambda_0}(M)) \right).$$

It follows from Theorem 1 that

$$(3.5) \quad \sum_{\gamma \in [0, 1-\gamma_0)} \nu_{\infty, \lambda}^p(\text{MC}_{\lambda_0}(M)) = \sum_{\gamma \in (\gamma_0, 1)} \nu_{\infty, \lambda}^{p-1}(M) + \nu_{\infty, 1}^p(\text{MC}_{\lambda_0}(M))$$

$$(3.6) \quad \sum_{\gamma \in [1-\gamma_0, 1)} \nu_{\infty, \lambda}^{p-1}(\text{MC}_{\lambda_0}(M)) = \sum_{\gamma \in (0, \gamma_0)} \nu_{\infty, \lambda}^{p-1}(M) + \nu_{\infty, \overline{\lambda_0}}^{p-1}(\text{MC}_{\lambda_0}(M))$$

We already made explicit $\nu_{\infty, 1}^p(\text{MC}_{\lambda_0}(M))$ and $\nu_{\infty, \overline{\lambda_0}}^{p-1}(\text{MC}_{\lambda_0}(M))$ in the proof of Theorem 2, but we can remark for the second that

$$\begin{aligned}
 \nu_{\infty, \lambda_0}^{p-1}(\mathrm{MC}_{\lambda_0}(M)) - \nu_{\infty, \lambda_0, \mathrm{prim}}^{p-1}(\mathrm{MC}_{\lambda_0}(M)) &= \sum_{\ell \geq 1} \sum_{k=1}^{\ell} \nu_{\infty, \lambda_0, \ell}^{p-1+k}(\mathrm{MC}_{\lambda_0}(M)) \\
 &= \sum_{\ell \geq 0} \sum_{k=0}^{\ell} \nu_{\infty, 1, \ell}^{p-1+k}(M) = \nu_{\infty, 1}^{p-1}(M).
 \end{aligned}$$

Moreover, it follows from [DS13, 3.1.2(2)] that

$$\begin{aligned}
 \sum_{i=1}^r \left(\sum_{\mu \neq 1} \mu_{x_i, \mu}^{p-1}(\mathrm{MC}_{\lambda_0}(M)) + \mu_{x_i, 1}^p(\mathrm{MC}_{\lambda_0}(M)) \right) &= \\
 \sum_{i=1}^r \left(\sum_{\gamma \in (0, 1-\gamma_0)} \mu_{x_i, \lambda}^{p-2}(M) + \sum_{\gamma \in [1-\gamma_0, 1]} \mu_{x_i, \lambda}^{p-1}(M) \right). &
 \end{aligned}$$

Finally, we get

$$\begin{aligned}
 h^p(\mathrm{MC}_{\lambda_0}(M)) &= -\gamma^p(\mathrm{MC}_{\lambda_0}(M)) - \nu_{\infty, \lambda_0, \mathrm{coprim}}^p(M) + \nu_{\infty, \lambda_0}^p(M) - \nu_{\infty, \lambda_0}^{p-1}(M) \\
 &\quad + \sum_{i=1}^r \left(\sum_{\gamma \in (0, 1-\gamma_0)} \mu_{x_i, \lambda}^{p-2}(M) + \sum_{\gamma \in [1-\gamma_0, 1]} \mu_{x_i, \lambda}^{p-1}(M) \right).
 \end{aligned}$$

If we substitute (3.4) in the previous formula, we have

$$\begin{aligned}
 \gamma^p(\mathrm{MC}_{\lambda_0}(M)) &= \gamma^p(M) + \sum_{\gamma \in [\gamma_0, 1]} (\nu_{\infty, \lambda}^p(M) - \nu_{\infty, \lambda}^{p-1}(M)) \\
 &\quad - \sum_{i=1}^r \left((\mu_{x_i, 1}^p(M) - \mu_{x_i, 1}^{p-1}(M)) + \sum_{\gamma \in (0, 1-\gamma_0)} (\mu_{x_i, \lambda}^{p-1}(M) - \mu_{x_i, \lambda}^{p-2}(M)) \right).
 \end{aligned}$$

Summing these equalities for $p' \leq p$ gives the expected formula. \square

REMARK 3.7. — If we add the assumption of scalar monodromy at infinity equal to $\lambda_0 \mathrm{Id}$ as in [DS13], we have $\nu_{\infty, \lambda, \ell}^p(M) = 0$ except if $\lambda = \lambda_0$ and $\ell = 0$. Thus we have

$$\sum_{\gamma \in [0, \gamma_0)} \nu_{\infty, \lambda}^p(M) + \sum_{\gamma \in [\gamma_0, 1]} \nu_{\infty, \lambda}^{p-1}(M) = \nu_{\infty, \lambda_0}^{p-1}(M) = \nu_{\infty, \lambda_0, \mathrm{prim}}^{p-1}(M) = h^{p-1}(M)$$

and $\sum_{\gamma \in [\gamma_0, 1]} \nu_{\infty, \lambda}^p(M) = h^p(M)$, consequently we retrieve the results 3.1.2(1) and 3.1.2(3) of [DS13].

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