



HAL
open science

The generic limit set of cellular automata

Saliha Djenaoui, Pierre Guillon

► **To cite this version:**

| Saliha Djenaoui, Pierre Guillon. The generic limit set of cellular automata. 2018. hal-01861590v1

HAL Id: hal-01861590

<https://hal.science/hal-01861590v1>

Preprint submitted on 24 Aug 2018 (v1), last revised 8 Nov 2023 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The generic limit set of cellular automata

Saliha Djenaoui* and Pierre Guillon†

†*Université d'Aix-Marseille CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France

*Département de Mathématiques, Université Badji Mokhtar-Annaba, B.P. 12, Sidi Amar, 23220 Annaba, Algérie, sdjenaoui1@yahoo.fr

†Interdisciplinary Scientific Center J.-V. Poncelet (ISCP), Independent University of Moscow, CNRS, UMI 2615, Russia, pierre.guillon@math.cnrs.fr

August 24, 2018

Abstract

In this article, we consider a topological dynamical system. The generic limit set is the smallest closed subset which has a comeager realm of attraction. We study some of its topological properties, and the links with equicontinuity and sensitivity. We emphasize the case of cellular automata, for which the generic limit set is included in all subshift attractors, and discuss directional dynamics, as well as the link with measure-theoretical similar notions.

Keywords : cellular automata, basin of attraction, limit set, attractor, directional dynamics, Baire category, symbolic dynamics.

1 Introduction

In topological dynamics, the limit set is the set of points that appear arbitrarily late during the evolution of the dynamical system (see [4]). But the latter may include points which look transient, because they do not appear arbitrarily late around any orbit. To capture only the points that can be approached by iteration of the dynamical system, one can study the asymptotic set, which consists of all limit points of orbits (see [8, 12]). Nevertheless, this set still includes some points that are, in some sense, very unlikely to be seen in the dynamics.

J. Milnor, interested in the dynamics on the space of measures, introduced in [22] the notion of likely limit set, that provides a useful tool for studying asymptotic behavior for almost all orbits. Following the former reference, the likely limit set is the smallest closed set that includes the asymptotic set of a set of full measure. More precisely, it is the smallest closed set that has a realm of attraction of measure one. Equivalently, the likely limit set is the unique maximal μ -attractor (see [22, 15, 26]).

J. Milnor also implicitly defines a topological version of the same intuitive idea, that he calls the *generic limit set*. The goal of our article is to formalize this concept. In other words, we focus on the asymptotic behavior for almost all orbits in the sense of Baire category theory. We study some topological properties of the generic limit set, which is the smallest closed set that has a comeager realm of attraction.

For cellular automata, we know that all subshift attractors have a dense open realm of attraction (see [14, 9, 17, 16]), thus the generic limit set is included in all subshift attractors.

M. Delacourt *et al.* studied directional dynamics of cellular automata (see [7]); qualitative behaviour (equicontinuity, sensitivity, expansiveness) appear along arbitrary curves in space-time. A section of the present article is devoted to studying the generic limit set of cellular automata which have two directions of almost equicontinuity.

The paper is structured as follows : In Section 2, we provide the basic background on the subject of topological dynamical systems and cellular automata. In Section 3, we show some preliminary results about attractors, limit sets, realms. In Section 4, we define the generic limit set and prove the main results about it. In Section 5, we show some consequences on directional dynamics of cellular automata. In Section 6, we compare the generic limit set with similar measure-theoretical concepts. In Section 7, we give some simple examples.

2 Preliminaries

2.1 Topology

In this article, (X, d) is a compact metric space. We put $B_\delta(x) = \{y \in X \mid d(x, y) < \delta\}$ and call it the **open ball** with center $x \in X$ and radius $\delta > 0$. For $U \subseteq X$, we also write $B_\delta(U) = \{y \in X \mid d(U, y) < \delta\}$ and $\overline{B}_\delta(U)$ its closure.

A subset $U \subseteq X$ is called **comeager** in X if it includes a countable intersection of dense open sets. A subset U is **meager** if $X \setminus U$ is comeager in X . By the Baire category theorem, the intersection of countably many dense open sets in X is dense in X . Hence, a comeager set is dense in X .

A subset A of X is said to have the **Baire property** (in short BP) if there is an open set U such that the symmetric difference $A \Delta U = (A \setminus U) \cup (U \setminus A)$ is meager in X . Every Borel subset has the Baire property (for more details, see for instance [24]).

We shall use the following remark to show Lemma 2.4.

Remark 2.1. *If $W \subseteq X$ is not comeager and has the Baire property, then there exists a nonempty open set U in which $W \cap U$ is meager.*

Proof. Indeed, $W^C \Delta U$ is meager for some open set U , and $W \cap U \subseteq W^C \Delta U$. □

2.2 Topological dynamics

Now we introduce some key concepts of the topological dynamics. A (time-nonuniform) **dynamical system** (DS) is any sequence $\mathfrak{F} = (F_t)_{t \in \mathbb{N}}$ of continuous self-maps of some compact metric space X . This general formalism will be useful when studying directional dynamics of cellular automata, but the reader should keep in mind the following specific, more classical, case. \mathfrak{F} is **uniform** if $F_t = G^t$, for all $t \in \mathbb{N}$ and some continuous self-map G of X . We may then write the uniform DS simply as G .

We are interested in the **orbits** $\mathcal{O}_\mathfrak{F}(x) = \{F_t(x) \mid t \in \mathbb{N}\}$ of points $x \in X$. We say that a set U is \mathfrak{F} -invariant if $F_{t+1}(U) \subseteq F_t(U)$, for every $t \in \mathbb{N}$.

Equicontinuity. For $\varepsilon > 0$, we define the set $\mathcal{E}_{\mathfrak{F}, \varepsilon}$ of ε -**stable** points for \mathfrak{F} as the set of $x \in X$ for which there exists $\delta > 0$ such that $\forall y \in B_\delta(x), \forall t \in \mathbb{N}, d(F_t(x), F_t(y)) < \varepsilon$. The set $\mathcal{E}_\mathfrak{F} \subseteq X$ of **equicontinuous** points for \mathfrak{F} is the set of points which are ε -stable for every $\varepsilon > 0$. If $\mathcal{E}_\mathfrak{F}$ is comeager, then we say that \mathfrak{F} is **almost equicontinuous**. If $\mathcal{E}_\mathfrak{F} = X$, then we say that \mathfrak{F} is **equicontinuous**. Equivalently by compactness, for every $\varepsilon > 0$, there is a uniform $\delta > 0$ such that $\forall x \in X, \forall y \in B_\delta(x), \forall t \in \mathbb{N}, d(F_t(x), F_t(y)) < \varepsilon$.

\mathfrak{F} is **sensitive** if there exists $\varepsilon > 0$ such that $\forall x \in X, \forall \delta > 0, \exists y \in B_\delta(x), \exists t \in \mathbb{N}, d(F_t(x), F_t(y)) \geq \varepsilon$. This implies that $\mathcal{E}_\mathfrak{F} = \emptyset$.

Remark 2.2. $\mathcal{E}_\mathfrak{F}$ is comeager in some nonempty open set $U \subseteq X$ if and only for every $n \geq 1$, $\mathcal{E}_{\mathfrak{F}, 1/n}$ is.

Proof. $\mathcal{E}_\mathfrak{F}$ can be written as the decreasing countable intersection $\bigcap_{n \geq 1} \mathcal{E}_{\mathfrak{F}, 1/n}$ of open subsets. □

A class which is between nonuniform and uniform DS is the following: \mathfrak{F} is **semi-uniform** if $F_t = G_t \cdots G_1$, for all $t \in \mathbb{N}$ and some equicontinuous sequence $(G_t)_{t \geq 1}$ of self-maps of X . This is trivially satisfied when $\{G_t \mid t \geq 1\}$ is finite; in particular, in the case of any uniform DS: $G_t = G_1$ for every $t \geq 1$. We will sometimes denote $G_{T+[1,t]}$ the composition $G_{T+t} \cdots G_{T+2} G_{T+1}$, so that $F_{t+T} = G_{T+[1,t]} F_T$.

2.3 Symbolic dynamics

Configurations. Let A be a finite set called the **alphabet**. A word over A is any finite sequence of elements of A . Denote $A^* = \bigcup_{n \in \mathbb{N}} A^n$ the set of all finite words $u = u_0 \dots u_{n-1}$; $|u| = n$ is the length of u . We say that v is a **subword** of u and write $v \sqsubset u$, if there are $k, l < |u|$ such that $v = u_{[k, l]}$. $A^{\mathbb{Z}}$ is the **space of configurations**, equipped with the following metric:

$$d(x, y) := 2^{-n}, \text{ where } n = \min \{ i \in \mathbb{N} \mid x_i \neq y_i \text{ or } x_{-i} \neq y_{-i} \}.$$

Any element of $A^{\mathbb{Z}}$ is called **configuration**. $A^{\mathbb{Z}}$ is a Cantor space. In $A^{\mathbb{Z}}$ the **cylinder** of $u \in A^*$ in position i is $[u]_i = \{ x \in A^{\mathbb{Z}} \mid x_{[i, i+|u|]} = u \}$. Cylinders are clopen (closed and open set). The **shift map** $\sigma : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ defined by $\sigma(x)_i = x_{i+1}$ is continuous for $i \in \mathbb{Z}$. The dynamical system $(A^{\mathbb{Z}}, \sigma)$ is called the **full shift**.

The (spatially) **periodic** configuration ${}^\infty u^\infty$ is defined by $({}^\infty u^\infty)_{k|u|+i} = u_i$ for $k \in \mathbb{Z}, 0 \leq i < |u|$ and $u \in A^*$. A **monochrome** configuration is one with only one symbol: ${}^\infty 0^\infty$, for some $0 \in A$.

Subshifts. A **subshift** is any subsystem (Σ, σ) of a full shift; we usually simply write Σ , which is then simply a closed σ -invariant subset of $A^{\mathbb{Z}}$. Equivalently, there exists a **forbidden language** $\mathcal{F} \subseteq A^*$ such that $\Sigma = \Sigma_{\mathcal{F}} = \{ x \in A^{\mathbb{Z}} \mid \forall u \sqsubset x, u \notin \mathcal{F} \}$. If \mathcal{F} can be taken finite, then one says that $\Sigma_{\mathcal{F}}$ is a **subshift of finite type** (SFT); in that case \mathcal{F} can be taken included in A^k for some $k \in \mathbb{N}$, which is an **order** for the SFT. We write that $\Sigma_{\mathcal{F}}$ is a k -SFT.

Let $\Sigma \subseteq A^{\mathbb{Z}}$ be a subshift. Then $\mathcal{L}(\Sigma) = \{ u \in A^* \mid \exists x \in \Sigma, u \sqsubset x \}$ is the **language** of Σ .

We shall use the following lemma to show results concerning the dynamics of cellular automata.

Lemma 2.3. *Let $\varepsilon > 0$ and $V \subseteq X$.*

1. $\sigma^j(B_\varepsilon(V)) \subseteq B_{2^j\varepsilon}(\sigma^j(V))$ for all $j \in \mathbb{Z}$. If, moreover, V is σ -invariant, and there exists integer $p > 0$ such that for all $n \in \mathbb{N}$, $\sigma^{pn}(V) \in B_\varepsilon(V)$, then $\forall i \in \mathbb{Z}, \sigma^i(V) \in B_{2^p\varepsilon}(V)$.
2. If $\forall i \in \mathbb{Z}, \sigma^i(x) \in B_{2^{-k}}(V)$ and V is a k -SFT, then $x \in V$.

Proof.

1. Let $x \in \sigma^j(B_\varepsilon(V))$. Then $\sigma^{-j}(x) \in B_\varepsilon(V) \iff d(\sigma^{-j}(x), V) < \varepsilon$.
Moreover, $d(x, \sigma^j(V)) = d(\sigma^j \sigma^{-j}(x), \sigma^j(V)) < 2^j d(\sigma^{-j}(x), V) < 2^j \varepsilon$. So, $x \in B_{2^j\varepsilon}(\sigma^j(V))$.
Hence, $\forall i \in \mathbb{Z}, \sigma^i(V) = \sigma^{i \bmod p} \sigma^{\lfloor i/p \rfloor p}(V) \in \sigma^{i \bmod p}(B_\varepsilon(V)) \subseteq B_{2^{i \bmod p}\varepsilon}(\sigma^{i \bmod p}(V))$. Since V is σ -invariant, we also have that $B_{2^{i \bmod p}\varepsilon}(\sigma^{i \bmod p}(V)) \subseteq B_{2^p\varepsilon}(V)$. Hence, $\forall i \in \mathbb{Z}, \sigma^i(V) \in B_{2^p\varepsilon}(V)$.
2. Let $i \in \mathbb{Z}$.

$$\begin{aligned} \sigma^i(x) \in B_{2^{-k}}(V) &\iff d(\sigma^i(x), V) < 2^{-k} \\ &\iff \exists y \in V, \sigma^i(x)_{[-k, k]} = y_{[-k, k]} \\ &\iff \exists y \in V, x_{[i-k, i+k]} = y_{[-k, k]} \\ &\iff x_{[i-k, i+k]} \in \mathcal{L}(V). \end{aligned}$$

If this is true for every $i \in \mathbb{Z}$ and V is k -SFT (or even $(2k+1)$ -SFT), then $x \in V$. □

Transitivity. A DS \mathfrak{F} is said to be **transitive** if for any nonempty open sets U, V in X , there exists $t \in \mathbb{N}$ such that $F_t^{-1}(U) \cap V \neq \emptyset$.

It is clear that $(A^{\mathbb{Z}}, \sigma)$ is transitive, and the following lemma will be used later for this example: it states for example that subshifts are all either meager or full.

Lemma 2.4. *Let G be a transitive uniform DS, and let $W \subseteq X$ be G -invariant.*

1. If W is dense (resp. comeager) in some nonempty open set U , then W is dense (resp. comeager).

2. If W has nonempty interior (resp. has the Baire property and is not meager), then W includes a dense open set (resp. is comeager).
3. If $\bigcup_{i \in \mathbb{N}} W_i$ has nonempty interior (resp. is not meager), where each W_i is G -invariant (resp. and has the Baire property), then there exists $i \in \mathbb{N}$ such that W_i includes a dense open set (resp. is comeager).

Proof.

1.
 - Since G is transitive, for every nonempty open set V , there exists t such that $G^{-t}(V) \cap U$ is a nonempty open set. Moreover, W is dense in U , so that $G^{-t}(V) \cap U \cap W \neq \emptyset$. Since $G^t(W) \subseteq W$, $V \cap G^t(U) \cap W \supseteq G^t(G^{-t}(V) \cap U \cap W) \neq \emptyset$. So, W is dense in X .
 - One has $W \cap U \supseteq \bigcap_{n \in \mathbb{N}} U_n$ where U_n is open and such that $\overline{U_n} = U$. So, $W \supseteq \bigcap_{t \in \mathbb{N}} G^t(W) \supseteq \bigcap_{t \in \mathbb{N}} G^t(W \cap U) \supseteq \bigcap_{t \in \mathbb{N}} G^t(\bigcap_{n \in \mathbb{N}} U_n)$. Note that each $\bigcap_{t \in \mathbb{N}} G^t(U_n)$ is G -invariant, and dense in U . It is open, and by the previous point, dense, so that W indeed includes an intersection of dense open sets.
2.
 - Since W includes a nonempty open set and is G -invariant, W includes the (nonempty open) orbit of this open set. According to the previous point, W includes a dense open set.
 - By Remark 2.1, W is comeager in some nonempty open set U . By the previous point, W is comeager.
3. One of the W_i has to have nonempty interior (resp. to not be meager). We conclude by the previous point. \square

2.4 Cellular automata

A map $F : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ is a **cellular automaton** (CA) if there exist integers $r_- \leq r_+$ (**memory** and **anticipation**) and a local rule $f : A^{r_+ - r_- + 1} \rightarrow A$ such that for any $x \in A^{\mathbb{Z}}$ and any $i \in \mathbb{Z}$, $F(x)_i = f(x_{i+r_-}, \dots, x_{i+r_+})$. $d = r_+ - r_- \in \mathbb{N}$ is sometimes called the **diameter** of F . Sometimes we assume that $-r_- = r_+$, which is then called the **radius** of F (it is always possible to obtain this, by taking $r = \max\{|r_-|, |r_+|\} \in \mathbb{N}$). By Curtis, Hedlund and Lyndon [13], a map $F : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ is a CA if and only if it is continuous and commutes with the shift. In particular, CA induce uniform DS over $A^{\mathbb{Z}}$.

Directional dynamics. We call **curve** a map $h : \mathbb{N} \rightarrow \mathbb{Z}$ with bounded variation, that is: $M_h = \sup_{t \in \mathbb{N}} |h(t+1) - h(t)|$ is finite. The map is meant to give a position in space for each time step. Following [7], the **CA F in direction h** will refer to the sequence $(F^t \sigma^{h(t)})_{t \in \mathbb{N}}$. In a first reading, the reader can understand the next definitions and results by considering the classical case: h constantly 0. In general, the directional dynamics of a CA can be read on its space-time diagram, by following h as a curve when going in the time direction. The formalism includes that of [23]: a **linear** curve is $h : t \mapsto \lfloor \alpha t \rfloor$, for some real number α .

Equicontinuity. A word $u \in A^*$ is (strongly) **blocking** for a CA F along curve h if there exists an offset $s \in \mathbb{Z}$ such that for every $x, y \in [u]_s$, $\forall t \in \mathbb{N}$, $F^t(x)_{\llbracket 0, M \rrbracket} = F^t(y)_{\llbracket 0, M \rrbracket}$, where $M = \max(-r_- + \max_t(h(t+1) - h(t)), r_+ + \max_t(h(t) - h(t+1)))$, and r_- and r_+ are the (minimal) memory and anticipation for F . The terminology comes from the fact that in that case, u is both left- and right-blocking (with the same offset), which is taken as a definition in [7]: A word $u \in A^*$ is **right-blocking** for a CA F in direction h if there exists an offset $s \in \mathbb{Z}$ such that:

$$\forall x, y \in [u]_s, x_{\llbracket -\infty, s \rrbracket} = y_{\llbracket -\infty, s \rrbracket} \implies \forall t \in \mathbb{N}, F^t(x)_{\llbracket -\infty, h(t) \rrbracket} = F^t(y)_{\llbracket -\infty, h(t) \rrbracket}.$$

We define **left-blocking** words similarly.

The following proposition explains how equicontinuity in cellular automata can be rephrased in terms of blocking words. The vertical case dates back from [17, 18], the linear directions from [23], the directions with bounded variations from [7], and the general case can be found in the proofs of [6, Prop 3.1.3, Cor 3.1.4].

Proposition 2.5 ([7]). *Let F be a CA and h a curve.*

1. *If there is a left- and right-blocking word u for F in direction h , then $\mathcal{E}_{F,h}$ includes the comeager set of configurations where u appears infinitely many times on both sides.*
2. *Otherwise, F is sensitive in direction h .*

In other words, in the bounded-variation case, $\mathcal{E}_{F,h}$ is either empty or comeager. The question is open whether this remains true in the unbounded-variation case (see [6, Remark 3.1.1]).

3 Attractors and realms

We will define notions that deal with asymptotic behavior of a DS $\mathfrak{F} = (F_t)_t$.

3.1 Limit sets

The (Ω -) **limit set** of $U \subseteq X$ is the set $\Omega_{\mathfrak{F}}(U) = \bigcap_{T \in \mathbb{N}} \overline{\bigcup_{t \geq T} F_t(U)}$, and the **asymptotic set** of $U \subseteq X$ is the set $\omega_{\mathfrak{F}}(U) = \bigcup_{x \in U} \Omega_{\mathfrak{F}}(\{x\})$. By compactness, these sets are nonempty. $\Omega_{\mathfrak{F}}(U)$ is compact, but $\omega_{\mathfrak{F}}(U)$ may not be, even for $U = X$ (see Example 7.7). Remark that $\Omega_{\mathfrak{F}}(U) \supseteq \bigcap_{t \in \mathbb{N}} F_t(U)$, and this is an equality if U is a closed \mathfrak{F} -invariant set.

We note $\Omega_{\mathfrak{F}} = \Omega_{\mathfrak{F}}(X)$ and $\omega_{\mathfrak{F}} = \omega_{\mathfrak{F}}(X)$. For more about the asymptotic set of dynamical systems, one can refer to [12]. Note that it was called *accessible* set in [8], and *ultimate* set in [11].

Remark 3.1. *By compactness of X , for every U and $\varepsilon > 0$, there exists $T \in \mathbb{N}$ such that:*

$$\forall t \geq T, d(F_t(U), \Omega_{\mathfrak{F}}(U)) < \varepsilon .$$

In the uniform case, it is clear that asymptotic sets are invariant. Here is a generalization of this fact.

Proposition 3.2. *Let $\mathfrak{F} = (G_{\llbracket 1, t \rrbracket})_t$ be a semi-uniform DS, $U \subseteq X$, and $j \in \mathbb{N}$.*

1. *If $y \in \Omega_{\mathfrak{F}}(U)$, then $(G_{t+\llbracket 1, j \rrbracket}(y))_t$ admits a limit point in $\Omega_{\mathfrak{F}}(U)$.*
2. *Conversely, if $z \in \Omega_{\mathfrak{F}}(U)$, then it is a limit point of $(G_{t+\llbracket 1, j \rrbracket}(y))_t$ for some $y \in \Omega_{\mathfrak{F}}(U)$.*

Of course, this remains true for the ω , which is defined as a union of Ω .

Proof.

1. By assumption, there are increasing times $(t_k)_{k \in \mathbb{N}}$ and points $(x_k)_{k \in \mathbb{N}} \in U^{\mathbb{N}}$ such that $\lim_{k \rightarrow \infty} F_{t_k}(x_k) = y$. Let $\varepsilon > 0$. By equicontinuity of the $(G_{t+\llbracket 1, j \rrbracket})_t$, there exists $\delta > 0$ such that for all z, z' with $d(z, z') < \delta$, we have $\forall t \in \mathbb{N}, d(G_{t+\llbracket 1, j \rrbracket}(z), G_{t+\llbracket 1, j \rrbracket}(z')) < \varepsilon/2$. Then there is $K \in \mathbb{N}$ such that for all $k \geq K$, $d(F_{t_k}(x_k), y) < \delta$, so that $d(G_{t_k+\llbracket 1, j \rrbracket}(x_k), G_{t_k+\llbracket 1, j \rrbracket}(y)) < \varepsilon/2$. If z is a limit point for $(G_{t_k+\llbracket 1, j \rrbracket}(y))_{k \in \mathbb{N}}$, one sees that there exists $K' \in \mathbb{N}$ such that for all $k \geq K'$, $d(F_{t_k+\llbracket 1, j \rrbracket}(x_k), z) < \varepsilon$, so that z is also in $\Omega_{\mathfrak{F}}(\{x_k \mid k \in \mathbb{N}\}) \subseteq \Omega_{\mathfrak{F}}(U)$.
2. Now let $z \in \Omega_{\mathfrak{F}}(U)$, so that it is the limit point of $(F_{t_k}(x_k))_{k \in \mathbb{N}}$ for some $(x_k)_{k \in \mathbb{N}} \in U^{\mathbb{N}}$ and (t_k) an increasing, say nonzero, sequence. By compactness, $(F_{t_k-j}(x_k))_{k \in \mathbb{N}}$ admits a limit point $y \in \Omega_{\mathfrak{F}}(U)$. By triangular inequality, we have $d(G_{t_k+\llbracket 1, j \rrbracket}(y), z) \leq d(G_{t_k+\llbracket 1, j \rrbracket}(y), F_{t_k}(x_k)) + d(F_{t_k}(x_k), z)$. When k goes to ∞ , the second term of the sum converges to 0, and a subsequence of the first term converges to 0, thanks to equicontinuity of $(G_{t_k+\llbracket 1, j \rrbracket})_{k \in \mathbb{N}}$. \square

The following corollary is useless for the purpose of the present paper, but may help the reader to connect with the know, uniform case.

Corollary 3.3. *If \mathfrak{F} is a uniform DS, $U \subseteq X$ and $j \in \mathbb{N}$, then $\mathfrak{F}(\Omega_{\mathfrak{F}}(U)) = \Omega_{\mathfrak{F}}(U)$.*

3.2 Realms of attraction

The **realm of attraction** of V is:

$$\mathcal{D}_{\mathfrak{F}}(V) = \{x \in X \mid \omega_{\mathfrak{F}}(x) \subseteq V\} = \left\{ x \in X \mid \lim_{t \rightarrow \infty} d(F_t(x), V) = 0 \right\}.$$

It is nonempty if and only if V intersects $\omega_{\mathfrak{F}}$.

The **direct realm** of V is the set $\mathfrak{d}_{\mathfrak{F}}(V) = \bigcup_{T \in \mathbb{N}} \bigcap_{t \geq T} F_t^{-1}(V)$ of configurations whose orbits lie ultimately in V . It is included in the realm of V .

Just from the definition, the reader can be convinced of the following.

Remark 3.4. Let $V_i \subseteq X$ be subsets, with i indexed within any set I .

1. $\mathcal{D}_{\mathfrak{F}}(\bigcup_{i \in I} V_i) \supseteq \bigcup_{i \in I} \mathcal{D}_{\mathfrak{F}}(V_i)$.
2. $\mathcal{D}_{\mathfrak{F}}(\bigcap_{i \in I} V_i) = \bigcap_{i \in I} \mathcal{D}_{\mathfrak{F}}(V_i)$.

3.3 Linked concepts

Recurrence. A point $x \in X$ is **recurrent** if $x \in \omega_{\mathfrak{F}}(x)$. We say that \mathfrak{F} is **nonwandering** if the set of recurrent points is comeager (see for instance [5] for equivalent definitions, for uniform DS).

Nilpotence. We say that \mathfrak{F} is **nilpotent** if there is a point $z \in X$ such that $\exists T \in \mathbb{N}, \forall x \in X, \forall t > T, F_t(x) = z$. \mathfrak{F} is **asymptotically nilpotent** if $\omega_{\mathfrak{F}}$ is a singleton.

It is known that CA are nilpotent if and only if their limit set is finite (see for instance [4]). Also, it has been shown [12] that asymptotically nilpotent CA are actually nilpotent. In that case (see for instance [4, 12]), $z = \sigma(z)$, so that the CA is actually nilpotent in every direction.

Asymptotic pairs. Two points $x, y \in X$ are said to be **asymptotic** to each other whenever

$$\lim_{t \rightarrow \infty} d(F_t(x), F_t(y)) = 0.$$

Let us generalize the realm notations to every sequence $(V_t)_{t \in \mathbb{N}}$ of subsets of X , by defining: $\mathcal{D}_{\mathfrak{F}}((V_t)_t) = \{x \in X \mid \lim_{t \rightarrow \infty} d(F_t(x), V_t) = 0\}$ and $\mathfrak{d}_{\mathfrak{F}}((V_t)_t) = \{x \in X \mid \exists T \in \mathbb{N}, \forall t \geq T, F_t(x) \in V_t\}$. We may also note $\mathcal{D}_{\mathfrak{F}}((y_t)_t)$ and $\mathfrak{d}_{\mathfrak{F}}((y_t)_t)$ if V_t is a singleton $\{y_t\}$. With this notation, $\mathcal{A}_{\mathfrak{F}}(y) = \mathcal{D}_{\mathfrak{F}}(\mathcal{O}_{\mathfrak{F}}(y))$.

One can observe from the definition that $U \subseteq \mathcal{A}_{\mathfrak{F}}(U) \subseteq \mathcal{D}_{\mathfrak{F}}(\omega_{\mathfrak{F}}(U))$.

Remark 3.5. Let G be a uniform DS over a finite space X , and $x, y \in X$. If x and y are asymptotic, then $\exists t \in \mathbb{N}, G^t(x) = G^t(y)$. In particular, if G is injective (or surjective), then $x = y$.

Proof. The first statement comes from X being discrete. The second statement is clear because if X is finite, then injectivity or surjectivity of G are equivalent to bijectivity of any G^t . \square

The following remark states that when an asymptotic class is big, then it should contain many equicontinuous points.

Remark 3.6. If $\mathcal{A}_{\mathfrak{F}}(y)$ is comeager in some nonempty open subset $U \subseteq X$, for some $y \in X$, then $\mathcal{E}_{\mathfrak{F}}$ is comeager in U .

In particular, note that $\mathcal{E}_{\mathfrak{F}} \cap \mathcal{A}_{\mathfrak{F}}(y)$ is also comeager in U .

Proof. The assumption gives that for every $n \geq 1$, the union $\bigcup_{T \in \mathbb{N}} \bigcap_{t \geq T} F_t^{-1}(\overline{B}_{1/n}(F_t(y)))$ of closed sets is comeager in U , as a superset of $\mathcal{A}_{\mathfrak{F}}(y)$. This means that there is $T \in \mathbb{N}$ such that $\bigcap_{t \geq T} F_t^{-1}(\overline{B}_{1/n}(F_t(y)))$ includes a nonempty open subset $W \subseteq U$. For every $x \in W$, there exists $\delta > 0$ such that for every $z \in B_{\delta}(x)$, $z \in W$, which means that $\forall t \geq T, F_t(z) \in \overline{B}_{1/n}(F_t(y)) \subseteq B_{3/n}(F_t(x))$. We deduce that x is $3/n$ -stable. In other words, the set of $3/n$ -stable points includes nonempty open subsets of every nonempty open subset of U , for every $n \in \mathbb{N}$. We conclude by Remark 2.2. \square

3.4 Decomposition of realms.

This proposition can be compared partly to [22, Lemma 3].

Proposition 3.7. *Let $\mathfrak{F} = (G_{\llbracket 1, t \rrbracket})_{t \in \mathbb{N}}$ be a semi-uniform DS. Suppose $(V_i)_i$ is a finite collection of closed pairwise disjoint sets which are invariant by every G_t . Then $\mathcal{D}_{\mathfrak{F}}(\bigsqcup_i V_i) = \bigsqcup_i \mathcal{D}_{\mathfrak{F}}(V_i)$.*

Proof. Since the V_i are closed and pairwise disjoint, there are at positive pairwise distance. Let $\varepsilon = \min_{i \neq j} d(V_i, V_j)/2 > 0$. By equicontinuity of (G_t) , there exists $\delta > 0$ such that $\forall t \in \mathbb{N}, \forall x, y \in X, d(x, y) < \delta \implies d(G_t(x), G_t(y)) < \varepsilon$. Let $x \in \mathcal{D}_{\mathfrak{F}}(\bigsqcup_i V_i)$, so that there exists $T \in \mathbb{N}$ such that $\forall t \geq T, d(F_t(x), \bigsqcup_i V_i) < \min(\delta, \varepsilon)$. In particular, there exists i such that $d(F_T(x), V_i) < \min(\delta, \varepsilon)$. Let us show by recurrence on $t \geq T$ that $d(F_t(x), V_i) < \min(\delta, \varepsilon)$. Since $G_{t+1}(V_i) \subseteq V_i$, we have $d(F_{t+1}(x), V_i) \leq d(F_{t+1}(x), G_{t+1}(V_i))$. This is less than ε by equicontinuity of (G_t) , using the recurrence hypothesis. By definition of ε , we have $\min_{j \neq i} d(F_{t+1}(x), V_j) \geq \min_{j \neq i} (d(V_j, V_i) - d(F_{t+1}(x), V_i)) \geq \varepsilon$. So $\min(\delta, \varepsilon) \geq d(F_{t+1}(x), \bigsqcup_j V_j) = \min_j (d(F_{t+1}(x), V_j)) \geq \min(\varepsilon, d(F_{t+1}(x), V_i))$. It results that $d(F_{t+1}(x), V_i) \leq \min(\delta, \varepsilon)$, as wanted. Since for every $t \geq T$ and $j \neq i$, $d(F_t(x), V_j) \geq \varepsilon$, we deduce $d(F_t(x), V_i) = \min_i d(F_t(x), V_i) = d(F_t(x), \bigsqcup_i V_i)$ converges to 0.

The other inclusion comes from Point 1 of Remark 3.4. \square

Realms of finite sets.

Lemma 3.8. *Let $\mathfrak{F} = (G_{\llbracket 1, t \rrbracket})_{t \in \mathbb{N}}$ be a semi-uniform DS and V be finite. Then there exists $\delta > 0$ such that for all $x, x' \in \mathcal{D}_{\mathfrak{F}}(V)$ and $T \in \mathbb{N}$ such that $d(F_T(x), F_T(x')) \leq \delta$, and $\forall t \geq T, d(F_t(x), V) \leq \delta$ and $d(F_t(x'), V) \leq \delta$, (x, x') is an asymptotic pair.*

Proposition 3.9. *Let $\mathfrak{F} = (G_{\llbracket 1, t \rrbracket})_{t \in \mathbb{N}}$ be a semi-uniform DS and V be finite. Then there are at most $|V|$ asymptotic classes in $\mathcal{D}_{\mathfrak{F}}(V)$.*

Proof. For $0 \leq i \leq |V|$, let $x_i \in \mathcal{D}_{\mathfrak{F}}(V)$, and δ be as in Lemma 3.8. There exists $T \in \mathbb{N}$ such that for all $i, \forall t \geq T, d(F_t(x_i), V) < \delta/2$, and in particular, $\exists y_i \in V, d(F_T(x_i), y_i) < \delta/2$. By the pigeon-hole principle, there are distinct i, j such that $y_i = y_j$, so that $d(F_T(x_i), F_T(x_j)) < \delta$ by the triangular inequality. By Lemma 3.8, we then know that (x_i, x_j) is an asymptotic pair. Hence we can partition $\mathcal{D}_{\mathfrak{F}}(V)$ into at most $|V|$ asymptotic classes. \square

Proof of Lemma 3.8. Let $\varepsilon = \frac{1}{3} \min \{d(y, y') \mid y, y' \in V, y \neq y'\} > 0$. By equicontinuity of (G_t) , there exists $\delta > 0$ such that $\forall t \in \mathbb{N}, \forall x, x' \in X, d(x, x') \leq \delta \implies d(G_t(x), G_t(x')) < \varepsilon/2$. Without loss of generality, we can assume $\delta \leq \varepsilon$. Let x, x' be as in the statement of the lemma, and for $t \in \mathbb{N}$, let $y(t) \in V$ be such that $d(F_t(x), y(t)) = d(F_t(x), V)$, and $y'(t)$ be defined similarly. Let us show by induction on $t \geq T$ that $y(t) = y'(t)$, which by definition of ε , is equivalent to $d(y(t), y'(t)) < 3\varepsilon$. First, by the triangular inequality, $d(y(T), y'(T)) \leq d(y(T), F_T(x)) + d(F_T(x), F_T(x')) + d(F_T(x'), y'(T)) < 3\delta \leq 3\varepsilon$.

Now suppose this is true for $t \geq T$, and let us prove it for $t + 1$. By the triangular inequality, we also have $d(y(t+1), y'(t+1)) \leq d(y(t+1), F_{t+1}(x)) + d(G_{t+1}F_t(x), G_{t+1}F_t(x')) + d(F_{t+1}(x'), y'(t+1))$. The first and third terms are at most δ by hypothesis, while the central one is at most ε by definition of δ . All in all, we get that $y(t+1) = y'(t+1)$. We can conclude with, once again, the triangular inequality: $d(F_t(x), F_t(x')) \leq d(F_t(x), y(t)) + d(y(t), y'(t)) + d(y'(t), F_t(x'))$. If $t \geq T$, this is $d(F_t(x), V) + 0 + d(V, F_t(x')) \rightarrow_{t \rightarrow \infty} 0$. \square

3.5 Realms for cellular automata

The following proposition is very important to show Proposition 5.10.

Proposition 3.10. *Let $\mathfrak{F} = (F_t)$ be a sequence of CA over $X = A^{\mathbb{Z}}$, and $V \subseteq A^{\mathbb{Z}}$.*

1. $\mathcal{D}_{\mathfrak{F}}(\sigma(V)) = \sigma(\mathcal{D}_{\mathfrak{F}}(V))$.
2. If V is σ -invariant and $\mathcal{D}_{\mathfrak{F}}(V)$ has nonempty interior, then $\mathcal{D}_{\mathfrak{F}}(V)$ includes a dense open set.

3. If, moreover, V is a k -SFT, then $\mathfrak{d}_{\mathfrak{F}}(V)$ is dense.
4. If V is a subshift, then $\mathfrak{d}_{\mathfrak{F}}(V)$ is meager, unless $F_T^{-1}(V)$ is full, for some $T \in \mathbb{N}$.

Proof.

1. This is clear by definition, noting that $\omega_{\mathfrak{F}}(\sigma(x)) = \sigma(\omega_{\mathfrak{F}}(x))$.
2. $\mathcal{D}_{\mathfrak{F}}(V)$ has nonempty interior and, from the previous point, is σ -invariant. By Point 2 of Lemma 2.4 (applied to $G = \sigma$), $\mathcal{D}_{\mathfrak{F}}(V)$ then includes a dense open set.
3. Let us show that, for an arbitrary $w \in A^*$, $[w] \cap \mathfrak{d}_{\mathfrak{F}}(V)$ is nonempty. Since $\mathcal{D}_{\mathfrak{F}}(V)$ has nonempty interior, there exists $u \in A^*$ such that $[u] \subseteq \mathcal{D}_{\mathfrak{F}}(V)$.

Let $x = {}^\infty(uw)^\infty \in [u]$ be the periodic configuration of period $p = |uw|$ and such that $x_{[0,p[} = uw$.

Since $x \in [u] \subseteq \mathcal{D}_{\mathfrak{F}}(V)$, there exists $T \in \mathbb{N}$ such that $\forall t > T, d(F_t(x), V) < 2^{-k-p}$. Since $\forall n \in \mathbb{Z}, x = \sigma^{np}(x)$, we even have:

$$\forall t > T, \forall n \in \mathbb{Z}, d(F_t \sigma^{np}(x), V) < 2^{-k-p}.$$

By Point 1 of Lemma 2.3, for all such $t > T, \forall i \in \mathbb{Z}, \sigma^i F_t(x) \in B_{2^k(V)}$. Since V is a k -SFT, $F_t(x) \in V$, by Point 2 of Lemma 2.3, that is, $x \in \mathfrak{d}_{\mathfrak{F}}(V)$. By shift-invariance, we also have that $\sigma^{|u|}(x) \in [w] \cap \mathfrak{d}_{\mathfrak{F}}(V)$.

4. By definition, $\mathfrak{d}_{\mathfrak{F}}(V) \subseteq \bigcup_{T \in \mathbb{N}} F_T^{-1}(V)$. If for every $T \in \mathbb{N}, F_T^{-1}(V)$ is not full, Point 2 of Lemma 2.4 gives that it has empty interior (because it is closed and σ -invariant). In the end, $\mathfrak{d}_{\mathfrak{F}}(V)$ is meager. \square

Proposition 3.11. *Let F be a CA, h a curve, and U be σ -invariant. Then $F(\omega_{F,h}(U)) = \sigma(\omega_{F,h}(U)) = \omega_{F,h}(U)$.*

Proof. By Point 1 of Proposition 3.10, we know that $\omega_{F,h}(U)$ is σ -invariant, and since F commutes with σ , we have that $\forall t \geq 1, F\sigma^{h(t)-h(t-1)}(\omega_{F,h}(U)) = F(\omega_{F,h}(U))$. By Proposition 3.2 (applied to $j = 1$ and $G_t = F\sigma^{h(t)-h(t-1)}$, so that $\{G_t | t \in \mathbb{N}\}$ is finite), we obtain that for any $y, z \in \omega_{F,h}(U)$, $F\sigma^k(y) \in \omega_{F,h}(U)$ for some k , and $z = F\sigma^l(x)$ for some l and some $x \in \omega_{F,h}(U)$. \square

Corollary 3.12. *Let F be a CA, h a curve and U such that $V = \omega_{F,h}(U)$ is finite and σ -invariant, and $U = \mathcal{D}_{F,h}(V)$. Then F induces a bijection of V , and $U = \bigsqcup_{y \in V} \mathcal{A}_{F,h}(y)$.*

Proof. By Proposition 3.11, we see that $F(V) = V$, so that F induces a surjection, hence a bijection of V . By Proposition 3.9, there are at most $|V|$ asymptotic classes in U . By the first point, $V \subseteq U$, so that each $y \in V$ should be in one of these classes. By Remark 3.5, they are all in distinct classes, so that we obtain the wanted result (the converse inclusion being trivial). \square

3.6 Attractors

We say that V is an **attractor** for \mathfrak{F} if it is closed and equal to $\Omega(U)$ for some open \mathfrak{F} -invariant set U (in particular $V \subseteq U$). One can see that this notion is equivalent to some classical definitions, such as the one used in [19, 9], which is focused on subshift attractors of CA: in that case the attractor enjoys a definition as the limit set of a so-called *spreading* cylinder. $\Omega_{\mathfrak{F}} = \Omega_{\mathfrak{F}}(X)$ is then the (unique) maximal attractor. A **quasi-attractor** is a nonempty intersection of attractors.

Proposition 3.13. *Consider some CA F in some direction h .*

1. *The realm of a subshift attractor is a dense open set.*
2. *The realm of a subshift quasi-attractor is comeager.*

Point 1 was already stated in [21].

Proof.

1. This is direct from Point 2 of Proposition 3.10.
2. Suppose $V = \bigcap_{n \in \mathbb{N}} V_n$, where V_n is an attractor. Then $\mathcal{O}_\sigma(V) = \bigcap_{n \in \mathbb{N}} \mathcal{O}_\sigma(V_n)$, so that $\mathcal{D}_{F,h}(\mathcal{O}_\sigma(V)) = \bigcap_{n \in \mathbb{N}} \mathcal{D}_{F,h}(\mathcal{O}_\sigma(V_n))$. For every $n \in \mathbb{N}$, $\mathcal{D}_{F,h}(\mathcal{O}_\sigma(V_n))$ is shift-invariant, and includes the nonempty open subset $\mathcal{D}_{F,h}(V_n)$. From Point 2 of Proposition 3.10, it includes a dense open set. Globally, the realm of $\mathcal{O}_\sigma(V)$ is thus comeager. \square

This allows to answer a conjecture stated in [17].

Corollary 3.14. *If a CA admits a minimal quasi-attractor in some direction, then:*

1. *it is a subshift;*
2. *its realm is comeager.*

In particular, in that case, all (possibly not subshifts) attractors have dense open realms (see Example 7.3 in direction 0).

Proof.

1. Just note that if V is a quasi-attractor, then it is closed, and, since σ is a conjugacy, any $\sigma^n(V)$, with $n \in \mathbb{Z}$, must also be a quasi-attractor. We deduce that a minimal quasi-attractor should also be included in $\bigcap_{n \in \mathbb{N}} \sigma^n(V)$, which is a subshift. Applied to itself, we get that this is equal to V , which is thus a subshift.
2. This now corresponds to Point 2 of Proposition 3.13. \square

This property of having a comeager realm motivates the next definition.

4 The generic limit set

Milnor [22] suggests the following definition, which is the purpose of the present section.

Definition 4.1 ([22]). *Being given a DS \mathfrak{F} , the **generic limit set** $\tilde{\omega}_\mathfrak{F}$ is the intersection of all the closed subsets of X which have a comeager realm of attraction.*

The generic limit set $\tilde{\omega}_\mathfrak{F}$ can actually be defined as the smallest closed subset of X with a comeager realm, thanks to the following proposition.

Proposition 4.2. *Let \mathfrak{F} be a DS. The realm of the generic limit set is comeager.*

In other words, it is the smallest closed set which includes all limit points of all *generic* orbits.

Proof. Any compact metric space admits a countable basis: there exists a countable set $\{U_i \mid i \in \mathbb{N}\}$ of closed subsets such that every closed set U can be written as $\bigcap_{i \in I_U} U_i$ for some $I_U \subseteq \mathbb{N}$. In particular, $\tilde{\omega}_\mathfrak{F}$ is the intersection $\bigcap_U \bigcap_{i \in I_U} U_i$, where U ranges over closed sets with comeager realm; that is, $\tilde{\omega}_\mathfrak{F} = \bigcap_{i \in I} U_i$, where I is the union of I_U , for every closed U with comeager realm. If i is in I , then it is in some I_U , so that $U \subseteq U_i$, where U has comeager realm, so that U_i has comeager realm, too.

By Point 2 of Remark 3.4, $\mathcal{D}_\mathfrak{F}(\tilde{\omega}_\mathfrak{F}) = \mathcal{D}_\mathfrak{F}(\bigcap_{i \in I} U_i) = \bigcap_{i \in I} \mathcal{D}_\mathfrak{F}(U_i)$. We know that an intersection of countably many comeager sets is comeager. Then $\mathcal{D}_\mathfrak{F}(\tilde{\omega}_\mathfrak{F})$ is comeager. \square

Remark 4.3. *For every DS \mathfrak{F} and subset V , $\mathcal{D}_\mathfrak{F}(V^C)^C = \bigcap_{T \in \mathbb{N}} \bigcup_{t \geq T} F_t^{-1}(V)$ is the set of points whose orbits visit V infinitely many times, which includes $\mathcal{D}_\mathfrak{F}(V)$.*

We deduce that if, for every $\varepsilon > 0$ and every $T \in \mathbb{N}$, $\bigcup_{t \geq T} F_t^{-1}(B_\varepsilon(V))$ is dense, then \bar{V} intersects $\omega_\mathfrak{F}(x)$ for generic x ; in particular it intersects $\tilde{\omega}_\mathfrak{F}$.

Proof. The first statement is direct.

Now suppose that for every $\varepsilon > 0$ and every $T \in \mathbb{N}$, $\bigcup_{t \geq T} F_t^{-1}(B_\varepsilon(V))$ is dense; by continuity it is also open, so that $\mathcal{D}_{\mathfrak{F}}(B_\varepsilon(V)^C)^C$ is comeager, and so is $\mathcal{D}_{\mathfrak{F}}(\overline{V}^C)^C = \bigcap_{n \in \mathbb{N}} \mathcal{D}_{\mathfrak{F}}(B_{1/n}(V)^C)^C$. The orbits from this set visit $B_{1/n}(V)$ infinitely many times for every $n \in \mathbb{N}$, so that they all have a limit point in \overline{V} . Since $\mathcal{D}_{\mathfrak{F}}(\overline{V}^C)$ is meager, \overline{V}^C cannot include the generic limit set, which give us the last statement. \square

4.1 First properties for CA

Proposition 4.4. *Let $\mathfrak{F} = (F_t)_t$ be a sequence of CA. Then $\tilde{\omega}_{\mathfrak{F}}$ is a subshift. Its realm is σ -invariant.*

Proof. By Definition, $\tilde{\omega}_{\mathfrak{F}}$ is closed. Let $\tilde{\omega}_{\mathfrak{F}} = \overline{\omega_{\mathfrak{F}}(U)}$ where U is comeager. Since σ is a homeomorphism, $\sigma^k(U)$ is also comeager for all $k \in \mathbb{Z}$. Then $W = \bigcap_{k \in \mathbb{Z}} \sigma^k(U)$ is still comeager, as an intersection of countably many comeager sets. One has $W \subseteq U$, so that $\overline{\omega_{\mathfrak{F}}(W)} \subseteq \tilde{\omega}_{\mathfrak{F}}$. Conversely, the definition of $\tilde{\omega}_{\mathfrak{F}}$ gives that it is included in $\overline{\omega_{\mathfrak{F}}(W)}$. Overall, $\overline{\omega_{\mathfrak{F}}(W)} = \tilde{\omega}_{\mathfrak{F}}$. Since W is σ -invariant, $\tilde{\omega}_{\mathfrak{F}} = \overline{\omega_{\mathfrak{F}}(W)}$ is also σ -invariant. \square

Moreover, Proposition 3.13 directly gives that $\tilde{\omega}_{\mathfrak{F}}$ is included in all subshift attractors.

Proposition 4.5. *Consider the CA F in some direction h . Then $\tilde{\omega}_{F,h}$ is an F -invariant subshift.*

Proof. We just apply Proposition 3.11 to $\tilde{\omega}_{F,h} = \overline{\omega_{F,h}(\mathcal{D}(\tilde{\omega}_{F,h}))}$. \square

4.2 Nonwandering systems

For nonwandering dynamical systems, the generic limit set is the full space.

Proposition 4.6. *Let \mathfrak{F} be a nonwandering DS over space X . Then $\tilde{\omega}_{\mathfrak{F}} = X$.*

Proof. Consider the comeager set \mathcal{R} of recurrent points of \mathfrak{F} . The set $U = \mathcal{D}_{\mathfrak{F}}(\tilde{\omega}_{\mathfrak{F}}) \cap \mathcal{R}$ is comeager, and, by inclusion, $\tilde{\omega}_{\mathfrak{F}} \supseteq \overline{\omega_{\mathfrak{F}}(U)}$. Since $U \subseteq \mathcal{R}$, by definition, $U \subseteq \omega_{\mathfrak{F}}(U) \subseteq \overline{\omega_{\mathfrak{F}}(U)} \subseteq \tilde{\omega}_{\mathfrak{F}}$. By the Baire category theorem, U is dense. So the closed superset $\tilde{\omega}_{\mathfrak{F}}$ is X . \square

Since it is known that surjective CA are all nonwandering (see for instance [18]), we get the following.

Corollary 4.7. *A CA F over $A^{\mathbb{Z}}$ is surjective if and only if $\tilde{\omega}_F = A^{\mathbb{Z}}$.*

4.3 Indecomposability

Now we prove that the generic limit set of a cellular automaton is indecomposable in some sense.

Proposition 4.8. *Let $V = \bigsqcup_{i=0}^{n-1} V_i$, where $n \in \mathbb{N}$ and the V_i are closed subsets which are invariant by some CA F in some direction h , and by σ^p , for some $p > 0$. If $\mathcal{D}_{F,h}(V)$ has nonempty interior (resp. is not meager), then there exists $i \in \llbracket 0, n \llbracket$ such that $\mathcal{D}_{F,h}(V_i)$ is dense (resp. comeager).*

Proof. One has $\mathcal{D}_{F,h}(V) = \bigsqcup_{i=0}^{n-1} \mathcal{D}_{F,h}(V_i)$ by Proposition 3.7.

By Point 3 of Lemma 2.4, there exists $i \in \llbracket 0, n \llbracket$ such that $\mathcal{D}_{F,h}(V_i)$ is dense (resp. comeager). \square

Corollary 4.9. *Let F be a CA and h a curve. $\tilde{\omega}_{F,h}$ cannot be decomposed as a disjoint union of non-trivial subshift subsystems (or even non-trivial σ^p -invariant subsystems, for some $p > 0$).*

In other words, we can say that $\tilde{\omega}_{F,h}$ is connected, when considering the dynamical pseudo-metric related to the action of (F, σ) : $\tilde{d}(x, y) = \inf_{i,j \in \mathbb{Z}, s,t \in \mathbb{N}} d(F^s \sigma^i(x), F^t \sigma^j(y))$.

Proof. We assume that $\tilde{\omega}_{F,h} = \bigsqcup_{i=0}^{n-1} V_i$, where the V_i are closed, invariant, σ^p -invariant sets. By Proposition 4.8, there exists $i \in \llbracket 0, n \llbracket$ such that $\mathcal{D}_{F,h}(V_i)$ is comeager. By definition of $\tilde{\omega}_{F,h}$, it is then included in V_i , and hence equal. \square

4.4 Finite generic limit set

Proposition 4.10. *Let \mathfrak{F} be a semi-uniform DS and $V = \omega(U)$ be finite, for some set U which is comeager in some nonempty open subset $W \subseteq X$. Then $\mathcal{E}_{\mathfrak{F}}$ is comeager in W .*

Of course this means that $\mathcal{E}_{\mathfrak{F}} \cap U$ is comeager in W .

We immediately deduce the following.

Corollary 4.11.

- If $\tilde{\omega}_{\mathfrak{F}}$ is finite, then \mathfrak{F} is almost equicontinuous.
- If \mathfrak{F} is sensitive (or has not equicontinuous point), then the asymptotic and limit sets of all non-meager sets are infinite.

Proof of Proposition 4.10. Let $W' \subseteq W$ be a nonempty open subset. According to Proposition 3.9, $\bigsqcup_{y \in I} \mathcal{A}_{\mathfrak{F}}(y) \supseteq U$, where I is finite, must also be comeager in W . One such $\mathcal{A}_{\mathfrak{F}}(y)$ should not be meager in W' . By Remark 2.1, it is then comeager in some $W'' \subseteq W'$. Remark 3.6 says that $\mathcal{E}_{\mathfrak{F}}$ is then comeager in W'' . It results that $\mathcal{E}_{\mathfrak{F}}$ is not meager, in any nonempty open subset $W' \subseteq W$, and again Remark 2.1 tells us that it is the comeager in W . \square

In the case of cellular automata with finite generic limit set, we have the following.

Proposition 4.12. *Let F be a CA and h a curve, such that $\tilde{\omega}_{F,h}$ is finite. Then $\tilde{\omega}_{F,h}$ contains one single (periodic) orbit, of a monochrome configuration y , and $\mathcal{A}_{F,h}(y)$ is comeager.*

Note that the orbit of this monochrome configuration may be nontrivial (see Example 7.9), but still generic configurations are all asymptotic to the same configuration of that orbit. This could seem paradoxical, since it contrasts with the usual, uniform and synchronous aspect of dynamics of CA over the full set $A^{\mathbb{Z}}$, but here the genericity notion is not at all F -invariant.

Proof. By Proposition 4.5, $\tilde{\omega}_{F,h}$ is a finite F -invariant subshift, so that all configurations are periodic (for the shift). Let $p > 0$ be a common period: $\tilde{\omega}_{F,h}$ can be decomposed as $\bigsqcup_{y \in V} \mathcal{O}_{F,h}(y)$, where $V \subseteq \tilde{\omega}_{F,h}$ is a set of orbit representatives.

We can apply Corollary 3.12: $\mathcal{D}_F(\tilde{\omega}_{F,h}) = \bigsqcup_{y \in \tilde{\omega}_{F,h}} \mathcal{A}_{F,h}(y)$. Since every $y \in \tilde{\omega}_{F,h}$ is σ^p -invariant, we can apply Point 3 of Lemma 2.4 (to $G = \sigma^p$), and get that there is $y \in \tilde{\omega}_{F,h}$ such that $\mathcal{A}_{F,h}(y)$ is comeager. It results that $\tilde{\omega}_{F,h} = \mathcal{O}_{F,h}(y)$. Moreover, since σ is an automorphism of F , $\sigma(\mathcal{A}_{F,h}(y)) = \mathcal{A}_{F,h}(\sigma(y))$ is also comeager. Then $\mathcal{A}_{F,h}(y) \cap \mathcal{A}_{F,h}(\sigma(y))$ is also comeager, and in particular nonempty. By transitivity of the asymptoticity relation, y is asymptotic to $\sigma(y)$. Since they both lie in the bijective subsystem of F induced over the finite $\tilde{\omega}_{F,h}$, Remark 3.5 gives that $y = \sigma(y)$, which means that y is monochrome. \square

4.5 Asymptotic set of equicontinuous points

We shall show that if the system is almost equicontinuous, then its generic limit set is exactly the closure of the asymptotic set of its set of equicontinuous points. Also, if the system is equicontinuous, then its generic limit set is its limit set.

Proposition 4.13. *Let \mathfrak{F} be a DS and $(V_t)_{t \in \mathbb{N}}$ a sequence of subsets of X . Then $\mathcal{E}_{\mathfrak{F}} \cap \overline{\mathcal{D}_{\mathfrak{F}}((V_t)_t)} \subseteq \mathcal{D}_{\mathfrak{F}}((V_t)_t)$.*

Proof. Let $x \in \mathcal{E}_{\mathfrak{F}} \cap \overline{\mathcal{D}_{\mathfrak{F}}((V_t)_t)}$, and $\varepsilon > 0$. Since $x \in \mathcal{E}_{\mathfrak{F}}$, there exists $\delta > 0$ such that

$$\forall y \in B_{\delta}(x), \forall t \in \mathbb{N}, d(F_t(x), F_t(y)) < \varepsilon/2 .$$

Since $x \in \overline{\mathcal{D}_{\mathfrak{F}}((V_t)_t)}$, there exists $y \in B_{\delta}(x) \cap \mathcal{D}_{\mathfrak{F}}((V_t)_t)$. Hence,

$$\exists T \in \mathbb{N}, \forall t > T, d(F_t(y), V_t) < \varepsilon/2 .$$

For this T , $\forall t > T, d(F_t(x), V_t) < \varepsilon$. Since this is true for every $\varepsilon > 0$, we get that $x \in \mathcal{D}_{\mathfrak{F}}((V_t)_t)$. \square

In particular, for $z \in X$, we have $\mathcal{A}_{\mathfrak{F}}(z) \supseteq \overline{\mathcal{A}_{\mathfrak{F}}(z)} \cap \mathcal{E}_{\mathfrak{F}}$. For almost equicontinuous DS, Proposition 4.13 means that it is enough to prove that some realm is dense to prove that it is comeager.

Corollary 4.14. *If \mathfrak{F} is an almost equicontinuous DS, then $\tilde{\omega}_{\mathfrak{F}} = \overline{\omega_{\mathfrak{F}}(\mathcal{E}_{\mathfrak{F}})}$.*

Proof. Since $\mathcal{D}_{\mathfrak{F}}(\tilde{\omega}_{\mathfrak{F}})$ is dense, $\mathcal{E}_{\mathfrak{F}} \subseteq \mathcal{D}_{\mathfrak{F}}(\tilde{\omega}_{\mathfrak{F}})$ by Proposition 4.13. Hence, $\omega_{\mathfrak{F}}(\mathcal{E}_{\mathfrak{F}}) \subseteq \tilde{\omega}_{\mathfrak{F}}$. Since $\tilde{\omega}_{\mathfrak{F}}$ is closed, $\overline{\omega_{\mathfrak{F}}(\mathcal{E}_{\mathfrak{F}})} \subseteq \tilde{\omega}_{\mathfrak{F}}$.

Conversely, $\tilde{\omega}_{\mathfrak{F}}$ is the intersection of all closed subsets with comeager realms, among which $\overline{\omega_{\mathfrak{F}}(\mathcal{E}_{\mathfrak{F}})}$ (whose realm includes the comeager $\mathcal{E}_{\mathfrak{F}}$). So, $\tilde{\omega}_{\mathfrak{F}} = \overline{\omega_{\mathfrak{F}}(\mathcal{E}_{\mathfrak{F}})}$. \square

Proposition 4.15. *If \mathfrak{F} is an equicontinuous DS over space X , then $\tilde{\omega}_{\mathfrak{F}} = \overline{\omega_{\mathfrak{F}}} = \Omega_{\mathfrak{F}}$.*

Proof. Let $y \in \Omega_{\mathfrak{F}}$ and $\varepsilon > 0$. We will show that $B_{\varepsilon}(y)$ intersects $\omega_{\mathfrak{F}}$. There exists δ such that for every $x \in \mathcal{E}_{\mathfrak{F}} = X$ and every $t \in \mathbb{N}$, $F_t(B_{\delta}(x)) \subseteq B_{\varepsilon/2}(F_t(x))$. By compactness of X , there exists a finite $I \subseteq X$ such that $X = \bigcup_{x \in I} B_{\delta}(x)$. Since $y \in \Omega_{\mathfrak{F}}$, there is an infinite $J \subseteq \mathbb{N}$, and for all $t \in J$, some $x_t \in X$ such that $F_t(x_t) \in B_{\varepsilon/2}(y)$. By the pigeon-hole principle, there exists $x \in I$ such that $B_{\delta}(x)$ contains infinitely many x_t , $t \in J$. This means that for infinitely many t , $d(F_t(x), y) \leq d(F_t(x), F_t(x_t)) + d(F_t(x_t), y) \leq \varepsilon/2 + \varepsilon/2$. We conclude that the orbit of x has a limit point $z \in \omega_{\mathfrak{F}}(x) \cap B_{\varepsilon}(y)$.

This proves that $\omega_{\mathfrak{F}}(X)$ is dense in $\Omega_{\mathfrak{F}}$; by Corollary 4.14, we obtain $\tilde{\omega}_{\mathfrak{F}} = \overline{\omega_{\mathfrak{F}}(X)} = \Omega_{\mathfrak{F}}$. \square

Another remark: it is known that a cellular automaton is nilpotent if and only if its limit set is finite. Hence, it is nilpotent if and only if it is equicontinuous and its generic limit set is finite.

5 Directional dynamics

In this section, we study cellular automata while varying the directions.

Definition 5.1 ([7, Def 2.5]). *The set of curves (and, abusively, of directions) is noted \mathcal{B} .*

For $h, h' \in \mathcal{B}$, we put $h \preceq h'$ if there exists $M > 0$ such that $h(t) \leq h'(t) + M$ for all $t \in \mathbb{N}$. We put $h \prec h'$ if, besides $h \preceq h'$, \preceq is a preorder relation on \mathcal{B} , and we note \sim the corresponding equivalence relation.

We also note $h \ll h'$ if $\lim_{t \rightarrow \infty} h'(t) - h(t) = +\infty$. \ll is a transitive relation which is finer than \prec .

*The preorder \preceq induces a notion of (closed, open, semi-open) curve **interval**, with some bounds $h' \preceq h''$, noted $[h', h'']$, $]h', h''[$, $[h', h''[$, $]h', h'']$. We say that the interval is **nondegenerate** if $h' \prec h''$. For an interval $S \subseteq \mathcal{B}$ with bounds h' and h'' , we also note $\mathcal{I}(S) = \{h \in \mathcal{B} \mid h' \ll h \ll h''\} \subset]h', h''[$.*

A **direction** will implicitly refer to an equivalence class for \sim (sometimes abusively confused with one representative). The reader will note that all dynamical properties (equicontinuous points, finiteness of the generic limit set, etc.) are invariant through this equivalence class.

An example of curve is given by the (possibly irrational) lines: $\alpha \in \mathbb{R}$ will stand for the direction $t \mapsto [t\alpha]$. The dynamics *along* α then corresponds to that studied in [23, 1].

Here are simple remarks about direction-invariance.

Remark 5.2. *Let F be a CA and $U, V_t \subseteq A^{\mathbb{Z}}$ be shift-invariant, for $t \in \mathbb{N}$.*

1. $\mathfrak{D}_{F,h}((V_t)_t)$ does not depend on $h \in \mathcal{B}$.

2. $\Omega_{F,h}(U)$ does not depend on $h \in \mathcal{B}$.

Proof.

1. Let $x \in \mathfrak{D}_{F,h}((V_t)_t)$, that is, there exists a time $T \in \mathbb{N}$ above which for all times $t > T$, $F^t \sigma^{h(t)}(x) \in V_t$.

By assumption, we get that $F^t \sigma^{h'(t)}(x) \in V_t$ for any other $h' : \mathbb{N} \rightarrow \mathbb{Z}$.

2. $\Omega_{F,h}(U) = \bigcap_{T \in \mathbb{N}} \overline{\bigcup_{t \geq T} F^t \sigma^{h(t)}(U)} = \bigcap_{T \in \mathbb{N}} \overline{\bigcup_{t \geq T} F^t(U)} = \Omega_F(U)$. \square

5.1 Oblique directions

Let F be a CA with memory $r_- \in \mathbb{Z}$ and anticipation $r_+ \in \mathbb{Z}$. Then for every $t \in \mathbb{N}$, F^t can be defined by a rule of memory r_-t and anticipation r_+t . But it could be that smaller parameters also fit. This motivates the following definition.

For a sequence $(F_t)_{t \in \mathbb{N}}$ of CA, let us denote $r_-(t)$ and $r_+(t)$ the minimum possible memory and anticipation for F_t . Formally, $r_-(t) = \sup \{ i \in \mathbb{Z} \mid \forall x, y \in A^{\mathbb{Z}}, x_{\llbracket i, +\infty \llbracket} = y_{\llbracket i, +\infty \llbracket} \implies F_t(x)_0 = F_t(y)_0 \}$ and $r_+(t) = \inf \{ i \in \mathbb{Z} \mid \forall x, y \in A^{\mathbb{Z}}, x_{\llbracket -\infty, i \rrbracket} = y_{\llbracket -\infty, i \rrbracket} \implies F_t(x)_0 = F_t(y)_0 \}$.

Remark 5.3.

1. For every $t \in \mathbb{N}$, $-\infty < r_-(t) \leq r_+(t) < +\infty$, if and only if F_t is not a constant function.
2. $(F_t \sigma^{h(t)})_{t \in \mathbb{N}}$ is equicontinuous if and only if $r_+ \leq -h \leq r_-$ (in particular, if F_t is never constant, $h \sim -r_- \sim -r_+$).

In the uniform case, some F_t is constant if and only if the CA is nilpotent.

Proof. The first statement is direct from continuity of F_t .

If $(F_t \sigma^{h(t)})_t$ is equicontinuous, then there exists $r \in \mathbb{N}$ such that for all $t \in \mathbb{N}$, $x, y \in A^{\mathbb{Z}}$, $x_{\llbracket -r, r \rrbracket} = y_{\llbracket -r, r \rrbracket} \implies F_t \sigma^{h(t)}(x)_0 = F_t \sigma^{h(t)}(y)_0$.

Since F commutes with σ , we get $x_{\llbracket -r-h(t), r-h(t) \rrbracket} = y_{\llbracket -r-h(t), r-h(t) \rrbracket} \implies F_t(x)_0 = F_t(y)_0$. We get that $-r - h(t) \leq r_-(t)$ and $r_+(t) \leq r - h(t)$.

Conversely, assume that there exists $M \in \mathbb{N}$ such that $\forall t \in \mathbb{N}$, $r_+(t) \leq M - h(t)$ and $-h(t) \leq M + r_-(t)$, and let $l \in \mathbb{N}$. Then for every $t \in \mathbb{N}$ and $x, y \in A^{\mathbb{Z}}$ such that $x_{\llbracket -l-M, l+M \rrbracket} = y_{\llbracket -l-M, l+M \rrbracket}$, consider the configuration $z \in A^{\mathbb{Z}}$ such that $z_i = x_i$ for every $i \in \llbracket -\infty, l+M \rrbracket$ and $z_i = y_i$ for every $i \in \llbracket -l-M, +\infty \llbracket$. By the assumed inequalities, $x_{\llbracket -\infty, l+r_+(t)+h(t) \rrbracket} = z_{\llbracket -\infty, l+r_+(t)+h(t) \rrbracket}$ and $z_{\llbracket -l+r_-(t)+h(t), +\infty \llbracket} = y_{\llbracket -l+r_-(t)+h(t), +\infty \llbracket}$, so that $F_t \sigma^{h(t)}(x)_{\llbracket -l, l \rrbracket} = F_t \sigma^{h(t)}(z)_{\llbracket -l, l \rrbracket} = F_t \sigma^{h(t)}(y)_{\llbracket -l, l \rrbracket}$. This proves equicontinuity of $(F_t \sigma^{h(t)})_t$.

The consequence between parentheses comes from the first point. \square

In the case of a single CA, we have seen that $r_-(t) \geq r_-t$ and $r_+(t) \leq r_+t$. It is known that these sequences will be asymptotically linear, the slopes being called the Lyapunov exponents (see for instance [25]).

We say that a direction h is **oblique** for CA sequence $(F_t)_t$ if $h \notin [-r_+, -r_-]$. We will show that the generic limit set in an oblique direction is equal to the limit set.

We say that a DS \mathfrak{F} over space X is **semi-transitive** if for every open sets U, V such that $V \cap \Omega_{\mathfrak{F}} \neq \emptyset$, there exists $T \in \mathbb{N}$ such that $U \cap F_T^{-1}(V) \neq \emptyset$. Equivalently, $\bigcup_{t \geq T} F_t^{-1}(V)$ is dense. \mathfrak{F} is **weakly semi-mixing** if for every open sets U, V, U', V' such that V and V' intersect $\Omega_{\mathfrak{F}}$, there exists (a common) $T \in \mathbb{N}$ such that $U \cap F_T^{-1}(V) \neq \emptyset$ and $U' \cap F_T^{-1}(V') \neq \emptyset$. Clearly if a system is weakly semi-mixing or transitive, then it is semi-transitive. But weak semi-mixing has a stronger consequence.

Remark 5.4. Any weakly semi-mixing DS \mathfrak{F} is sensitive or admits a trivial limit set.

Proof. If $\Omega_{\mathfrak{F}}$ is not trivial, then there are two open subsets V and V' which are at positive distance $\varepsilon > 0$. Then for every $x \in X$ and $\delta > 0$, $B_\delta(x)$ intersects both $F_T^{-1}(V)$ and $F_T^{-1}(V')$, for some $T \in \mathbb{N}$, so that there are points y and y' in it, for which $d(F_T(y), F_T(y')) > \varepsilon$. By the triangular inequality, $F_T(x)$ should be at distance at least $\varepsilon/2$ of one of the two, which means that x is not $\varepsilon/2$ -stable. \square

Proposition 5.5. For every semi-transitive DS \mathfrak{F} , $\tilde{\omega}_{\mathfrak{F}} = \Omega_{\mathfrak{F}}$.

Proof. Let us consider $x \in \Omega_{\mathfrak{F}}$. By assumption, for every $\varepsilon > 0$ and $T \in \mathbb{N}$, $\bigcup_{t \geq T} F_t^{-1}(B_\varepsilon(x))$ is dense so that we can apply Remark 4.3: $\{x\} = \overline{\{x\}}$ intersects all generic orbits, hence is in $\tilde{\omega}_{\mathfrak{F}}$.

The other inclusion is always true. \square

Proposition 5.6. If $(F_t)_t$ is a CA sequence and h an oblique curve, then the DS $(F_t \sigma^{h(t)})_t$ is weakly semi-mixing.

Proof. It is enough to prove the property for $U = [u]_n$ a cylinder, $V = [v]_m$ and $V' = [v']_{m'}$ two cylinders of patterns appearing in $\Omega_{F,h}$, with $m, m', n \in \mathbb{Z}$. By extending v and/or v' (into a pattern which still appears in the limit set), we can suppose that they have the same length and that $m = m'$. Suppose, without loss of generality, that h is left-oblique: $-r_- < h$; in particular, there exists $T \in \mathbb{N}$ such that $r_-(T) + h(T) + m > n + |u|$. $F_T \sigma^{h(T)}$ is a CA of memory $r_-(T) + h(T)$ and anticipation $r_+(T) + h(T)$: there exists $w_T \in A^{|v|+(r_+(T)-r_-(T))}$ such that

$$F_T^{-1}([v]_m) \supseteq [w_T]_{r_-(T)+h(T)+m}.$$

Hence, there exist $T \in \mathbb{N}$ and $y \in A^{\mathbb{Z}}$ such that $y \in [u]_n \cap [w_T]_{r_-(T)+h(T)+m}$. The same is true for v' (for the same T). \square

Corollary 5.7. *Let F be a CA over $A^{\mathbb{Z}}$ and h an oblique curve. Then $\tilde{\omega}_{F,h} = \Omega_{F,h}$; it is either sensitive or nilpotent.*

Proof. The equality is direct from Propositions 5.6 and 5.5. Sensitivity comes from Proposition 5.6, Remark 5.4 and the known fact that the limit set of a CA is trivial if and only if it is nilpotent. \square

5.2 Almost equicontinuity in two directions

The purpose of this subsection is to show that if the CA is almost equicontinuous in two distinct directions, then its generic limit set is finite. We essentially reprove [6, Prop 3.2.3, Prop 3.3.4] (or the corresponding result for linear directions from [23]), but additionally discuss the generic limit set.

Here is the main lemma for understanding directional dynamics. It is based on the fact that if a word u is blocking along $h' \in \mathcal{B}$, and s' is the minimal corresponding offset, then in particular for every $t \in \mathbb{N}$ there exists $a_{u,h'}(t) \in A$ such that $\forall z \in [u]_{s'}, a_{u,h'}(t) = F^t(z)_{h'(t)}$.

Lemma 5.8. *Let F be a CA over $A^{\mathbb{Z}}$ with blocking words u along $h' \in \mathcal{B}$ with offset $s' \in \mathbb{Z}$ and v along $h'' \in \mathcal{B}$ with offset $s'' \in \mathbb{Z}$, and $q' = h' + |v| - s'$ and $q'' = h'' - s''$. Then for every $z \in [v]_0$ and $j \in \llbracket q'(t), q''(t) \rrbracket$, $F^t(z)_j = a_{u,h'}(t)$.*

Of course the same is true for $[u]$. The following statement is direct from the definition. Of course the symmetric statement holds for right-blocking words.

Remark 5.9.

1. From the definition, one can see that if u is blocking for CA F along curve h , then any word containing u also.
2. If two directions are almost equicontinuous, Proposition 2.5 states that they admit blocking words u and v . From the previous point, they admit a common blocking word uv .
3. From the definition, one can see that: if u is a right-blocking word for CA F along directions h' and h'' , then also along any direction $h \succ \min(h', h'')$.
4. From the previous point and the symmetric statement, if u is right- and left-blocking along directions h' and h'' , then also along any direction $h \in [\min(h', h''), \max(h', h'')]$.
5. In particular, right- and left-blockingness are preserved by \sim (which is not the case for strong blockingness).

Proof of Lemma 5.8. Since they are left-blocking and right-blocking, respectively, for every $t \in \mathbb{N}$:

$$\begin{cases} \forall x, y \in [u]_{s'}, x_{\llbracket s', +\infty \rrbracket} = y_{\llbracket s', +\infty \rrbracket} \implies F^t(x)_{\llbracket h'(t), +\infty \rrbracket} = F^t(y)_{\llbracket h'(t), +\infty \rrbracket} \\ \forall x, y \in [v]_{s''}, x_{\llbracket -\infty, s'' \rrbracket} = y_{\llbracket -\infty, s'' \rrbracket} \implies F^t(x)_{\llbracket -\infty, h''(t) \rrbracket} = F^t(y)_{\llbracket -\infty, h''(t) \rrbracket} \end{cases}.$$

By definition, $\forall z \in [u]_{s'}, a_{u,h'}(t) = F^t(z)_{h'(t)}$.

Now let $j \in \llbracket q'(t), q''(t) \rrbracket$, so that there is a configuration $y \in [v]_0 \cap [u]_{j-h'(t)+s'}$ such that $y_{\llbracket -\infty, 0 \rrbracket} = z_{\llbracket -\infty, 0 \rrbracket}$ and $\sigma^{j-h'(t)}(y)_{\llbracket s', +\infty \rrbracket} = x_{\llbracket s', +\infty \rrbracket}$. Since $\sigma^{j-h'(t)}(y)$ is in $[u]_{s'}$ and u is left-blocking along h' , we get: $F^t(y)_j = a_{u,h'}(t)$. On the other hand, since v is right-blocking along h'' , $\forall t \in \mathbb{N}$, $F^t(z)_{\llbracket -\infty, h''(t)-s'' \rrbracket} = F^t(y)_{\llbracket -\infty, h''(t)-s'' \rrbracket}$. In particular, we get that $F^t(z)_j = F^t(y)_j = a_{u,h'}(t)$. \square

Proposition 5.10. *Let F be a CA with a blocking word u along two distinct directions h' and h'' , with $h'' \not\preceq h'$. Along any direction $h \in \mathcal{B}$, the direct realm $\mathfrak{D}_{F,h}((\infty a_{u,h'}(t)^\infty)_t)$ includes all σ -periodic configurations where u appears; Moreover, the realm $\mathcal{D}_{F,h}((\infty a_{u,h'}(t)^\infty)_t)$ includes:*

1. All configurations where u appears infinitely many times on the left and on the right, if $h' \preceq h \preceq h''$;
2. All configurations where u appears infinitely many times on the right, if $h' \prec h \preceq h''$;
3. All configurations where u appears infinitely many times on the left, if $h' \preceq h \prec h''$;
4. All configurations where u appears, if $h' \prec h \prec h''$.

Proof. Consider a configuration $z \in [u]_i$, for some $i \in \mathbb{Z}$, with some σ -period $p \geq 1$. Let q' and q'' be as in Lemma 5.8. Since $q'' \sim h'' \not\preceq h' \sim q'$, there exists $T \in \mathbb{N}$ such that $q'(T) \leq q''(T) + p$. Lemma 5.8 says that then $F^T(z)_j = a_{u,h'}(T)$ for all $j \in \llbracket i + q'(T), i + q''(T) \rrbracket$, and, by periodicity, for all $j \in \mathbb{Z}$. This means that $F^T(z)$ is monochrome, and it is clear that it stays monochrome for $t \geq T$. Since, by definition of $a_{u,h'}(t)$, it appears in $F^t(z)$, we deduce that the latter is equal to $\infty a_{u,h'}(t)^\infty$.

1. From Point 4 of Remark 5.9, u is right- and left-blocking along every curve $h \in [\min(h', h''), \max(h', h'')]$. So from Proposition 2.5, configurations with infinitely many occurrences of u on the left and on the right are equicontinuous. From Proposition 4.13 and the previous point that $\mathcal{D}_{F,h}((\infty a_{u,h'}(t)^\infty)_t) \supseteq \mathfrak{D}_{F,h}((\infty a_{u,h'}(t)^\infty)_t)$ is dense, these equicontinuous configurations must also be in $\mathfrak{D}_{F,h}((\infty a_{u,h'}(t)^\infty)_t)$.
2. Let $z \in \bigcap_{i \in \mathbb{Z}} \bigcup_{j \geq i} [u]_j$ and $n \in \mathbb{N}$. If $h \preceq h'' \sim q''$, then there exists $j \geq \max_{t \in \mathbb{N}} h(t) - q''(t) + n$ such that $z \in [u]_j$. If $q' \sim h' \prec h$, then there exists $T \in \mathbb{N}$ such that $\forall t \geq T, q'(t) + j \leq h(t) - n$. From Lemma 5.8 and these inequalities, for all $t \geq T$, the pattern $F^t(z)_{\llbracket h(t)-n, h(t)+n \rrbracket}$ is monochrome. Since this is true for every n , every limit point of the orbit must be monochrome.
3. This case is symmetric to the previous one.
4. Let $z \in [u]_j$ for some $j \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $q' \sim h' \prec h \prec h'' \sim q''$, then there exists $T \in \mathbb{N}$ such that for all $t \geq T, h(t) + n \leq q''(t) + j$ and $q'(t) + j \leq h(t) - n$. From Lemma 5.8 and these inequalities, for all $t \geq T$, the pattern $F^t(z)_{\llbracket h(t)-n, h(t)+n \rrbracket}$ is monochrome. \square

5.3 Classification of generic limit sets up to directions

We recall the classifications of CA up to shift from [23, 7] and emphasize the properties of each class in terms of generic limit set. By *strictly almost equicontinuous*, we mean almost equicontinuous but not equicontinuous.

Theorem 5.11. *Every CA F with memory r_- and anticipation r_+ , satisfies exactly one of the following statements:*

1. F is nilpotent; there is a symbol $a \in A$ such that for all $h \in \mathcal{B}$, $\tilde{\omega}_{F,h} = \{\infty a^\infty\}$.
2. F is equicontinuous along a single direction $h' \in [-r_+, -r_-]$, and sensitive along other directions; for every $h \in \mathcal{B}$, $\tilde{\omega}_{F,h} = \Omega_F$ is infinite.
3. F is strictly almost equicontinuous along a nondegenerate interval $S \subseteq [-r_+, -r_-]$ and sensitive along other directions; there exists $a \in A$ such that $\tilde{\omega}_{F,h} = \mathcal{O}_F(\infty a^\infty)$ for every $h \in S$, and $\tilde{\omega}_{F,h}$ is infinite for every $h \notin S$; moreover, $\mathcal{E}_{F,h} \subseteq \mathcal{D}_{F,h}(\tilde{\omega}_{F,h}) = \mathcal{A}_{F,h}(\infty a^\infty)$, and if $h \in \mathcal{I}(S)$, it includes a dense open set.

4. F is strictly almost equicontinuous along a single direction $h' \in [-r_+, -r_-]$ and sensitive along other directions; for every $h \in \mathcal{B}$, $\tilde{\omega}_{F,h}$ is infinite.
- 4'. F is strictly almost equicontinuous along a single direction $h' \in [-r_+, -r_-]$ and sensitive along other directions; $\tilde{\omega}_{F,h}$ is finite if and only if $h = h'$.
5. F is sensitive in every direction; $\tilde{\omega}_{F,h}$ is infinite along all $h \in \mathcal{B}$.

Compared to [7, Theorem 2.9], we have merged the last two classes, because expansiveness is not relevant in terms of generic limit set, except that it implies surjectivity. Surjective CA have their generic limit set equal to the full shift of configurations in every direction: they are either in Class 2, Class 4 or Class 5. In Class 3, the bounds of the interval of almost equicontinuity can be closed or not; actually all four cases can happen: see [6] for some examples (the bound would be included if one allows directions with unbounded variation).

See Section 7 for examples of all classes (in particular, both subclasses of Class 4 can occur).

Proof. 1. Nilpotent CA have a trivial limit set (in every direction), which includes the generic limit set.

2. Now suppose that F is not nilpotent, which is equivalent to Ω_F being infinite. Assume also that there is at least one direction of equicontinuity. By Remark 5.3, all other directions are oblique, hence sensitive by Corollary 5.7. By Proposition 4.15 and Corollary 5.7, $\tilde{\omega}_{F,h} = \Omega_F$, for all $h \in \mathcal{B}$.

3. Now suppose that F admits no direction of equicontinuity, but two distinct directions h' and h'' of almost equicontinuity. By Point 2 of Remark 5.9, these two directions have a common blocking word u , so that we can apply Proposition 5.10: u is also blocking for directions in the interval $[\min(h', h''), \max(h', h'')]$. Since this is true for every h', h'' , we deduce that the set S of almost equicontinuous directions is convex: it is a nondegenerate interval. By Corollary 5.7, it is included in $[-r_+, -r_-]$.

Proposition 5.10 also states that $\mathcal{D}_{F,h}((^\infty a_{u,h'}(t)^\infty)_t)$ is dense for every $h \in \mathcal{B}$, so that it includes all equicontinuous points, by Proposition 4.13. If $h \in S$, Corollary 4.14 gives that the generic limit set is the closure of the asymptotic set of $\mathcal{E}_{F,h}$, that is then the asymptotic set of $\{^\infty a_{u,h'}(t)^\infty | t \in \mathbb{N}\}$, which is a set of monochrome configurations; in particular, it does not depend on h . By Proposition 4.12, it is the orbit by F of a monochrome configuration.

For other directions, the generic limit set of a sensitive DS is infinite, by Corollary 4.11.

Finally, if $h \in \mathcal{I}(S)$, it is easy to find $q', q'' \in S$ such that $q' \ll h \ll q''$. Point 4 of Proposition 5.10 gives that $\mathcal{D}_{F,h}((^\infty a_{u,q'}(t)^\infty)_t)$ contains a dense open set. The same argument as above gives that this is in the realm of the generic limit set.

4. The cases remain when there is at most one direction of almost equicontinuity; it cannot be oblique, and other directions have to all have infinite generic limit by Corollary 4.11. This settles the last three classes. \square

6 Links with the measure-theoretical approach

By a **measure**, we mean a Borel probability measure on X . The topological **support** \mathcal{S}_μ of a measure μ is the smallest closed subset of measure 1. If $\mathcal{S}_\mu = X$, we say that μ has **full support**.

We say that $\mathfrak{F} = (F_t)_{t \in \mathbb{N}}$ is μ -**equicontinuous** if $\mu(\mathcal{E}_{\mathfrak{F}}) = 1$. Proposition 2.5 and Corollary 6.4 give that, if μ is σ -ergodic, a CA F along some direction h is μ -equicontinuous unless $\mathcal{E}_{F,h} \cap \mathcal{S}_\mu = \emptyset$. In the case of Bernoulli measure, this corresponds to [10, Prop 3.5].

6.1 μ -likely limit set and μ -limit set

The generic limit set is the topological variant of the μ -likely limit set $\Lambda_{\mathfrak{F},\mu}$, which is the smallest closed subset of X which has a realm of attraction of measure one. [22, Examples 5, 6] point that there are no general inclusion relations between the two sets, but that they intersect. Here is a formalization of this argument.

Proposition 6.1. *For every DS \mathfrak{F} and full-support measure μ , $\Lambda_{\mathfrak{F},\mu} \cap \tilde{\omega}_{\mathfrak{F}} \neq \emptyset$.*

Proof. Remark that for every $\varepsilon > 0$ and every $T \in \mathbb{N}$, $\bigcup_{t \geq T} F_t^{-1}(B_\varepsilon(\Lambda_{\mathfrak{F},\mu})) \supseteq \mathcal{D}_{\mathfrak{F}}(\Lambda_{\mathfrak{F},\mu})$ has measure 1, so it is dense. We apply Remark 4.3 and get that $\Lambda_{\mathfrak{F},\mu}$ intersects every generic orbit, hence $\tilde{\omega}_{\mathfrak{F}}$. \square

The μ -likely limit set should not be confused with the μ -limit set $\Omega_{\mathfrak{F},\mu}$, from [26, 7], which is the intersection of all closed subsets U such that $\lim_{t \rightarrow \infty} \mu(F_t^{-1}(U)) = 1$. We prove one general inclusion, though; it is a generalization of [26, Prop 1].

Proposition 6.2. *For every DS \mathfrak{F} and Borel probability measure μ , $\Omega_{\mathfrak{F},\mu} \subseteq \Lambda_{\mathfrak{F},\mu}$.*

Proof. It is enough to prove that, for every $\varepsilon > 0$, $\lim_{t \rightarrow \infty} \mu(F_t^{-1}(\overline{B_\varepsilon(V)})) = 1$, where $V = \Lambda_{\mathfrak{F},\mu}$. Suppose, for the sake of contradiction, that it is not the case: there is $\alpha > 0$ and an infinite set $I \subseteq \mathbb{N}$ such that for all $t \in I$, $\mu(A_t) \geq \alpha$, where $A_t = F_t^{-1}(\overline{B_\varepsilon(V)})^C$. For every $T \in \mathbb{N}$, consider the set $B_T = A_T \setminus \bigcup_{t > T} A_t$ of points which are *for the last time* in some A_T . Note that the set of those which are *still expecting one visit*, $C_T = \bigcup_{t \in \mathbb{N}} A_t \setminus \bigcup_{t \leq T} B_t = \bigcup_{t > T} A_t$, includes A_t for some $t \in I$. It results that $\mu(C_T) \geq \mu(A_t) \geq \alpha$. Since $(C_T)_T$ is a decreasing sequence of subsets, we get that $\mu(C) \geq \alpha$, where $C = \bigcap_{T \in \mathbb{N}} C_T$. Now if $x \in C$, then for all $T \in \mathbb{N}$, there exists $t > T$ such that $F_t(x) \in \overline{B_\varepsilon(V)}^C$. This means that $x \notin \mathcal{D}_F(V)$, so that $\mathcal{D}_F(V)$ has measure at most $1 - \alpha$. \square

The converse is in general false: Example 7.6 is a counter-example, as proved in [26]. However, we prove the converse in a specific case.

Proposition 6.3. *Let \mathfrak{F} be a μ -equicontinuous DS for some measure μ . Then $\Omega_{\mathfrak{F},\mu} = \Lambda_{\mathfrak{F},\mu} = \overline{\omega_{\mathfrak{F}}(\mathcal{E}_{\mathfrak{F}} \cap \mathcal{S}_\mu)}$.*

In particular, if μ has full support, then these sets are equal to $\overline{\omega_{\mathfrak{F}}(\mathcal{E}_{\mathfrak{F}})}$.

Proof. Since $\mathcal{D}_F(\Lambda_{\mathfrak{F},\mu})$ is dense in \mathcal{S}_μ , it includes the full-measure set $\mathcal{E}_{\mathfrak{F}} \cap \mathcal{S}_\mu$, thanks to Proposition 4.13. In the same way as in the proof of Corollary 4.14, we can quickly deduce the second equality.

Now let $y \in \overline{\omega_{\mathfrak{F}}(\mathcal{E}_{\mathfrak{F}} \cap \mathcal{S}_\mu)}$ and $\varepsilon > 0$. There is a point $x \in \mathcal{E}_{\mathfrak{F}} \cap \mathcal{S}_\mu$, and a subsequence $(t_n)_n$ such that $\forall n \in \mathbb{N}$, $F_{t_n}(x) \in B_\varepsilon(y)$. By equicontinuity of x , there exists $\delta > 0$ such that $\forall t \in \mathbb{N}$, $F_t(B_\delta(x)) \subseteq B_{\varepsilon/2}(F_t(x))$. In particular, for $n \in \mathbb{N}$, we get $F_{t_n}(B_\delta(x)) \subseteq B_\varepsilon(y)$, so that $\mu(F_{t_n}^{-1}(B_\varepsilon(y))) \geq \mu(B_\delta(x)) > 0$.

Let U be a closed subset of $B_\varepsilon(y)^C$. Then for every $n \in \mathbb{N}$, $\mu(F_{t_n}^{-1}(U)) \leq \mu(F_{t_n}^{-1}(B_\varepsilon(y)^C)) \leq 1 - \mu(B_\delta(x))$. Since this is positive and independent of n , we get that $\mu(F_t^{-1}(U))$ does not converge to 1.

By contrapositive, we get that every closed U such that $\mu(F_t^{-1}(U))$ converges to 1 should intersect $B_\varepsilon(y)$, for every $\varepsilon > 0$. Hence it contains y , and we can conclude that $y \in \Omega_{\mathfrak{F},\mu}$.

The converse inclusion comes from Proposition 6.2. \square

6.2 μ -likely limit set of cellular automata

If G is a uniform DS, μ is G -invariant if $\mu(G^{-1}(U)) = \mu(U)$ for all measurable subsets U of X . A G -invariant measure μ is G -ergodic if every G -invariant set U has either zero or full measure. This is the measure-theoretical counterpart of the topological notion of transitivity, as emphasized by Lemma 2.4, which gives the following.

Corollary 6.4. *Let G be a transitive uniform DS, μ a full-support ergodic measure, and let $W \subseteq X$ be G -invariant. Then the following are equivalent.*

1. W is not meager.
2. W is comeager.
3. W has measure 1.
4. W has positive measure.

Proof. The first two points are equivalent thanks to Lemma 2.4, and the last two by definition of ergodicity. Now if W is comeager, it can be written as a countable intersection $\bigcap_{n \in \mathbb{N}} W_n$ of dense open sets W_n , and hence as a countable intersection $\bigcap_{n \in \mathbb{N}} \mathcal{O}_G(W_n)$ of dense open G -invariant sets. By full support, their measure should be positive, and by ergodicity, they should be 1, so that of W also. Otherwise W is meager, so that the previous argument holds for its complement. \square

In [22], J. Milnor asks for a *good* criterion for equality between the likely and generic limit sets. Here is at least a criterion, in the case of cellular automata.

Corollary 6.5. *If \mathfrak{F} is a sequence of CA, and μ is a full-support σ -ergodic measure, then $\tilde{\omega}_{\mathfrak{F}} = \Lambda_{\mathfrak{F}, \mu}$.*

Proof. Proposition 4.4 gives that the generic limit set is the intersection of all closed subsets with σ -invariant comeager realms. The same simple argument shows that the likely limit set is the intersection of all closed subsets with σ -invariant realms of measure 1. Thanks to Corollary 6.4, these realms are actually the same, so that the two sets are equal. \square

It results that all results from the previous sections hold for the μ -likely limit set in that case. Actually, even when they are not equal, most results on the generic limit have a parallel result on the likely limit set, which can be proven with the same proof tools.

7 Examples

Example 7.1 (Shift). *Let σ be the CA over $A^{\mathbb{Z}}$ defined by $\sigma(x)_i = x_{i+1}$. This CA is reversible, hence surjective. By Corollary 4.7, $\tilde{\omega}_{\sigma, h} = A^{\mathbb{Z}}$ along all $h \in \mathcal{B}$. Along direction -1 , it corresponds to the identity CA. This CA has only one equicontinuous direction: it is in Class 2 of Theorem 5.11.*

Let F be a CA with alphabet A , memory $r_- \in \mathbb{Z}$, anticipation $r_+ > r_-$ and local rule f . A state $0 \in A$ is **spreading** if for all $u \in A^{r_+ - r_- + 1}$ such that $0 \sqsubset u$, one has $f(u) = 0$. Note that any CA is simulated, in a strong natural sense, by a CA with a spreading state (simply by artificially adding it to the alphabet). In particular, unlike the asymptotic set (which includes the nonwandering set), the generic limit set does not support the topological entropy (in the sense of [2]).

Remark 7.2. *Let F be a CA over $A^{\mathbb{Z}}$ with memory $r_- \in \mathbb{Z}$, anticipation $r_+ > r_-$, and spreading state $0 \in A$. Then it is in Class 4: $\tilde{\omega}_{F, h} = \{\infty 0^\infty\}$ along all $h \in [-r_+, -r_-]$ and $\tilde{\omega}_{F, h} = \Omega_F$ along all $h \notin [-r_+, -r_-]$.*

Proof. We suppose that F has a spreading state $0 \in A$. By definition, it is a (left- and right-) blocking word along all $h \in [-r_+, -r_-]$. By Proposition 5.10, there exists $a \in A$ such that $\tilde{\omega}_{F, h} = \mathcal{O}_F(\infty a^\infty) = \overline{\omega(\mathcal{E}_{F, h})}$ along all $h \in [-r_+, -r_-]$. In this case, a is nothing else than 0. Moreover, since any $h \notin [-r_+, -r_-]$ is oblique, $\tilde{\omega}_{F, h} = \Omega_F$ by Corollary 5.7. \square

The simplest example of spreading state is the following.

Example 7.3 (Min). *Let Min be defined over $\{0, 1\}^{\mathbb{Z}}$ by $\text{Min}(x)_i = \min(x_i, x_{i+1})$. By Remark 7.2, we know that this CA is in Class 3: $\tilde{\omega}_{\text{Min}, h} = \{\infty 0^\infty\}$ along all $h \in [-1, 0]$ and $\tilde{\omega}_{\text{Min}, h} = \Omega_{\text{Min}} = \{x \in \{0, 1\}^{\mathbb{Z}} \mid \forall k > 0, 10^k 1 \not\sqsubset x\}$ along all $h \notin [-1, 0]$.*

1. *Along direction 0 (or, symmetrically, direction -1): the realm of $\{\infty 0^\infty\}$ is $\bigcap_{k \in \mathbb{Z}} \{x \in \{0, 1\}^{\mathbb{Z}} \mid \exists i \geq k, x_i = 0\}$, it is a comeager set. $\{\infty 0^\infty\} = \bigcap_{k \geq 0} V_k$, where $V_k = \Omega_{\text{Min}}([0]_k) = \{x \in \Omega_{\text{Min}} \mid \forall i \leq k, x_i = 0\}$ is an attractor but not a subshift, for every $k \in \mathbb{Z}$ (see [18]).*

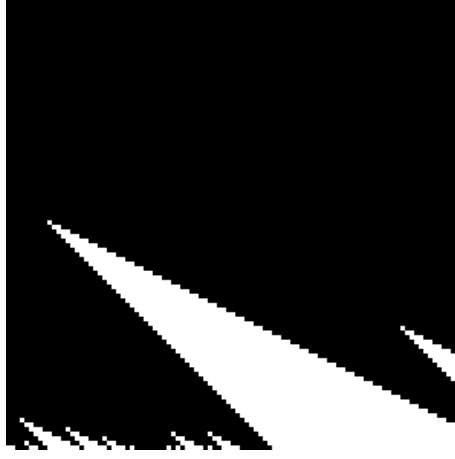


Figure 1: Oblique Min CA (0 (resp. 1) are represented by black squares (resp. white squares)).

2. Along direction $-1/2$: *Min* corresponds, up to a power, to the three-neighbor Min CA, defined by $\sigma^{-1}\text{Min}^2(x)_i = \min(x_{i-1}, x_i, x_{i+1})$. Like for all directions in $] -1, 0[$, the realm of $\{\infty 0^\infty\}$ is the dense open set $\{0, 1\}^{\mathbb{Z}} \setminus \{\infty 1^\infty\}$.
3. Direction 1 is oblique: the realm of the generic limit set is $\{0, 1\}^{\mathbb{Z}}$. A typical space-time diagram is represented in Figure 1.

Example 7.4 (Finite generic limit set). Consider the CA $\text{Min} \times \sigma^{-1}\text{Min}$ defined over $(\{0, 1\}^{\mathbb{Z}})^2$ by $(\text{Min} \times \sigma^{-1}\text{Min})(x, y)_i = (\min(x_i, x_{i+1}), \min(y_{i-1}, y_i))$. According to Example 7.3, $\tilde{\omega}_{\text{Min}, h} = \{\infty 0^\infty\}$ along all $h \in [-1, 0]$ and $\tilde{\omega}_{\text{Min}, h} = \Omega_{\text{Min}}$ along all $h \notin [-1, 0]$; of course $\tilde{\omega}_{\sigma^{-1}\text{Min}, h} = \{\infty 0^\infty\}$ along all $h \in [0, 1]$ and $\tilde{\omega}_{\sigma^{-1}\text{Min}, h} = \Omega_{\text{Min}}$ along all $h \notin [0, 1]$. Hence, $\tilde{\omega}_{\text{Min} \times \sigma^{-1}\text{Min}, 0} = \{\infty 0^\infty\}^2$, and $\tilde{\omega}_{\text{Min} \times \sigma^{-1}\text{Min}, h} = \{\infty 0^\infty\} \times \Omega_{\text{Min}}$ (resp. $\tilde{\omega}_{\text{Min} \times \sigma^{-1}\text{Min}, h} = \Omega_{\text{Min}} \times \{\infty 0^\infty\}$) along all $h \in [-1, 0[$ (resp. $h \in]0, 1]$), and $\tilde{\omega}_{\text{Min} \times \sigma^{-1}\text{Min}, h} = \Omega_{\text{Min}}^2$ along all $h \notin [-1, 1]$. In particular, $\text{Min} \times \sigma^{-1}\text{Min}$ has only one almost equicontinuous direction: $\{0\}$. Hence, it is in Class 4.

Example 7.5 (Sensitivity in every direction). Consider the CA $\sigma^{-1}\text{Min} \times \sigma$ defined over $(\{0, 1\}^{\mathbb{Z}})^2$ by $(\sigma^{-1}\text{Min} \times \sigma)(x, y)_i = (\min(x_{i-1}, x_i), y_{i+1})$. According to Example 7.1, $\tilde{\omega}_{\sigma, h} = \{0, 1\}^{\mathbb{Z}}$ along all $h \in \mathcal{B}$. According to Example 7.3, $\tilde{\omega}_{\sigma^{-1}\text{Min}, h} = \{\infty 0^\infty\}$ along all $h \in [0, 1]$ and $\tilde{\omega}_{\sigma^{-1}\text{Min}, h} = \Omega_{\text{Min}}$ along all $h \notin [0, 1]$. Hence, $\tilde{\omega}_{\sigma^{-1}\text{Min} \times \sigma, h} = \{\infty 0^\infty\} \times \{0, 1\}^{\mathbb{Z}}$ along all $h \in [0, 1]$, and $\tilde{\omega}_{\sigma^{-1}\text{Min} \times \sigma, h} = \Omega_{\text{Min}} \times \{0, 1\}^{\mathbb{Z}}$ along all $h \notin [0, 1]$. Since there is no almost equicontinuous direction, this CA is in Class 5.

Example 7.6 (Just Gliders). Let $A = \{\leftarrow, 0, \rightarrow\}$ and F the CA defined by the following local rule:

$$f(x_{i-1}, x_i, x_{i+1})_i = \begin{cases} \rightarrow & \text{if } x_{i-1} = \rightarrow \text{ and } x_i \neq \leftarrow \text{ and } (x_{i+1} \neq \leftarrow \text{ or } x_i = \rightarrow) \\ \leftarrow & \text{if } x_{i+1} = \leftarrow \text{ and } x_i \neq \rightarrow \text{ and } (x_{i-1} \neq \rightarrow \text{ or } x_i = \leftarrow) \\ 0 & \text{otherwise .} \end{cases}$$

A typical space-time diagram of this CA is shown in Figure 2. It is possible to interpret it as a background of 0's where particles \rightarrow and \leftarrow go to the right and to the left, respectively. When two opposite particles meet they disappear.

This example is known to have a μ -limit set which is strictly included in the μ -likely limit set, when μ is the uniform Bernoulli measure (see [26, Ex 3] and [20, Ex 4]). One can see (or read in these references) that the limit set is $\Omega_F = \{x \in A^{\mathbb{Z}} \mid \forall k \in \mathbb{N}, \leftarrow 0^k \rightarrow \not\subset x\}$. We prove here that F is weakly semi-mixing in every direction $h \notin \{-1, +1\}$. Hence $\tilde{\omega}_{F, h} = \Omega_{F, h}$. Moreover, $\tilde{\omega}_{F, -1} = \{0, \rightarrow\}^{\mathbb{Z}}$, $\tilde{\omega}_{F, +1} = \{0, \leftarrow\}^{\mathbb{Z}}$, and F is sensitive in every direction; it is in Class 5.

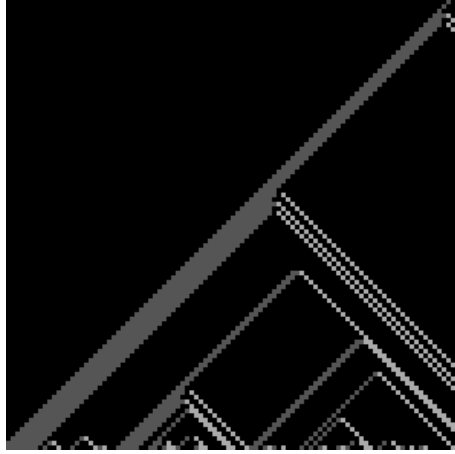


Figure 2: Just Gliders (\leftarrow are represented by light grey squares and \rightarrow are represented by dark grey squares).

Proof. By induction on $t \in \mathbb{N}$, one can see that $F^t(x)_k = \rightarrow$ if and only if $x_{-t+k} = \rightarrow$ and $u = x_{[-t+k+1, t+k]}$ is a **right-balanced** pattern, that is it does not *send* any particle to the left, or more formally:

$$\forall j \in \llbracket 0, |u| \rrbracket, \sum_{i=0}^j u_i \geq 0, \text{ where } \rightarrow = +1 \text{ and } \leftarrow = -1.$$

Generalizing this induction, we can see that if $k \in \mathbb{Z}$, $t \in \mathbb{N}$, $\leftarrow \not\sqsubset w$ and $u \in A^{2t}$ is right-balanced, then $F^t([wu]_k) \subseteq [w]_{k+t}$. We define **left-balanced** patterns symmetrically, and get that if $\rightarrow \not\sqsubset z$ and u is left-balanced, then $F^t([uz]_k) \subseteq [z]_{k+t}$.

- Let $[u]_m$, $[v]_n$ and $[v']_{n'}$ be three cylinders, the last two intersecting Ω_F . By the expression of Ω_F , note that we can decompose them as $v = wz$, $v' = w'z'$, with $\leftarrow \not\sqsubset w, w'$ and $\rightarrow \not\sqsubset z, z'$. We prove that there is common time step $t \in \mathbb{N}$ such that $F^t \sigma^{h(t)}([u]_m) \cap [v]_n \neq \emptyset$ and $F^t \sigma^{h(t)}([u]_m) \cap [v']_{n'} \neq \emptyset$. If h is oblique, the result follows from Proposition 5.6, so we can assume that $h \in]-1, +1[$.

We can assume that u is a left- and right-balanced pattern: just extend it with the suitable number of \rightarrow on the left, and the suitable number of \leftarrow on the right (the obtained cylinder is included in the original one).

Since $h > -1$, there exists $t \in \mathbb{N}$ such that $h(t) > -t + \max(n + |w|, n' + |w'|) - m$. Since $h < 1$, there exists $t \in \mathbb{N}$ such that $h(t) < t + \min(n + |w|, n' + |w'|) - m - |u|$. These t could be distinct, but it is not difficult to be convinced that, since h has bounded variation, there is a common $t \in \mathbb{N}$ which satisfies both. In that case we can define $\tilde{u} = 0^{t-n+h(t)-|w|+m} u 0^{t-m-|u|+n-h(t)+|w|}$. Clearly, it is still left- and right-balanced, so that $F^t \sigma^{h(t)}([w\tilde{u}]_{n-h(t)-t}) \subseteq [w]_n$ and $F^t \sigma^{h(t)}([\tilde{u}z]_{n-h(t)+|w|-t}) \subseteq [z]_{n+|w|}$; taking the intersection we get $F^t \sigma^{h(t)}([w\tilde{u}z]_{n-h(t)-t}) \subseteq [v]_n$. Moreover, $[w\tilde{u}z]_{n-h(t)-t} \subseteq [\tilde{u}]_{n-h(t)-t+|w|} \subseteq [u]_m$; we get the wanted nonempty intersection. The exact same can be achieved for $[v']_{n'}$ for the same t .

- Proposition 5.5 then gives that for every $h \notin \{-1, +1\}$, $\tilde{\omega}_{F,h} = \Omega_F$ and F is sensitive. Since the set of almost equicontinuous directions is an interval, then at least one direction in $\{-1, +1\}$ should also be sensitive. Since the definition of the local rule is exactly symmetric, we get that both directions are also sensitive.
- Now consider direction $h = +1$. For $i \in \mathbb{N}$, let W_i be the set of configurations $x \in A^{\mathbb{Z}}$ such that $x_{[1,i]}$ is not right-balanced. If $x \in W_i$, then by definition $x_{[1,t]}$ is not right-balanced, for $t \geq i$, so that, by the first claim of the proof, $F^t \sigma^t(x)_0 \neq \rightarrow$. Since every pattern can be extended to the right

in a pattern which is not right-balanced, we see that $W = \bigcup_{i \in \mathbb{N}} W_i$ is a dense open set. We get that $\omega_{F,1}(W) \cap [\rightarrow] = \emptyset$. Hence $\omega_{F,1}(\bigcap_{n \in \mathbb{Z}} \sigma^n(W)) \subseteq \{0, \leftarrow\}^{\mathbb{Z}}$. $\bigcap_{n \in \mathbb{Z}} \sigma^n(W)$ being comeager, we get that $\tilde{\omega}_{F,1} \subseteq \{0, \leftarrow\}^{\mathbb{Z}}$. Conversely, for every cylinders $[u]_m$ and $[v]_n$, the latter intersecting $\{0, \leftarrow\}^{\mathbb{Z}}$, the same argument above allows to find $t \in \mathbb{N}$ such that $F^t([u]_m)$ intersect $[v]_n$, so that Remark 4.3 states that $[v]_n$ intersects the generic limit set.

- The exact symmetric argument settles the case of $h = -1$. □

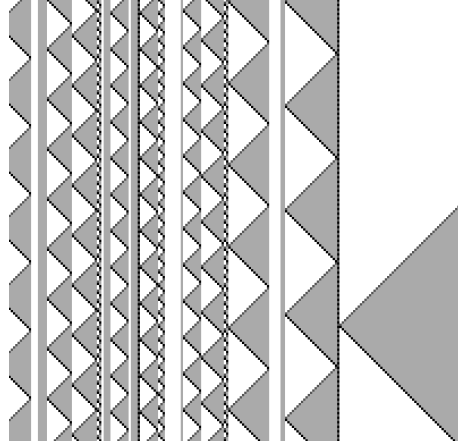


Figure 3: Lonely Gliders (\leftarrow (resp. \rightarrow) are represented by black squares (resp. white squares) and \rightarrow (resp. \leftarrow) are represented by dark grey squares (resp. light grey squares)).

Example 7.7 (Lonely gliders). Let $A = \{>, <, \rightarrow, \leftarrow\}$, and F the CA, defined by the following local rule:

$$f : (x_{-1}, x_0, x_1) \mapsto \begin{cases} \rightarrow & \text{if } x_{-1} = \rightarrow \text{ and } x_0 = < \\ \rightarrow & \text{if } x_{-1} \neq > \text{ and } x_0 = \leftarrow \\ < & \text{if } x_{-1} = \rightarrow \text{ and } x_0 = \leftarrow \\ > & \text{if } x_0 = \rightarrow \text{ and } x_1 = < \\ \leftarrow & \text{if } x_0 = \rightarrow \text{ and } x_1 \neq < \\ \leftarrow & \text{if } x_0 = > \text{ and } x_1 = \leftarrow \\ x_0 & \text{otherwise .} \end{cases}$$

A typical space-time diagram of this CA is shown in Figure 3.

Intuitively, each configuration can be decomposed into valid zones, which contain at most one arrow, towards with chevrons $<$ and $>$ are supposed to point. The arrow moves in the direction to which it points, until it reaches the end of the zone, in which case it turns back. With this in mind, it is not difficult to understand that F is reversible (hence surjective), and that any invalid pattern ab , where $a \neq >$ and $b \in \{>, \rightarrow, \leftarrow\}$, or symmetric, is a blocking word. Hence this CA is in Class 4'.

The reason to introduce this CA rather any simpler surjective almost equicontinuous one is the following feature, which answers a question left open in [11]: $\omega_F \subsetneq A^{\mathbb{Z}} = \tilde{\omega}_F$, is comeager (like for every surjective CA) but not full.

Proof. Let us prove that some configuration x with an infinite valid zone which contains one arrow cannot be the limit point of an orbit. Indeed, any configuration whose orbit comes arbitrarily close to x should also have an infinite valid zone (because the zones are invariant), and hence at most one arrow in it. Any limit point of such an orbit has no arrow in its infinite valid zone (the arrow goes to infinity). □

The last two examples are only cited, and not defined formally, because they can fit a whole article by themselves. The first one shows that it is relevant to study arbitrary curves rather than just linear directions.

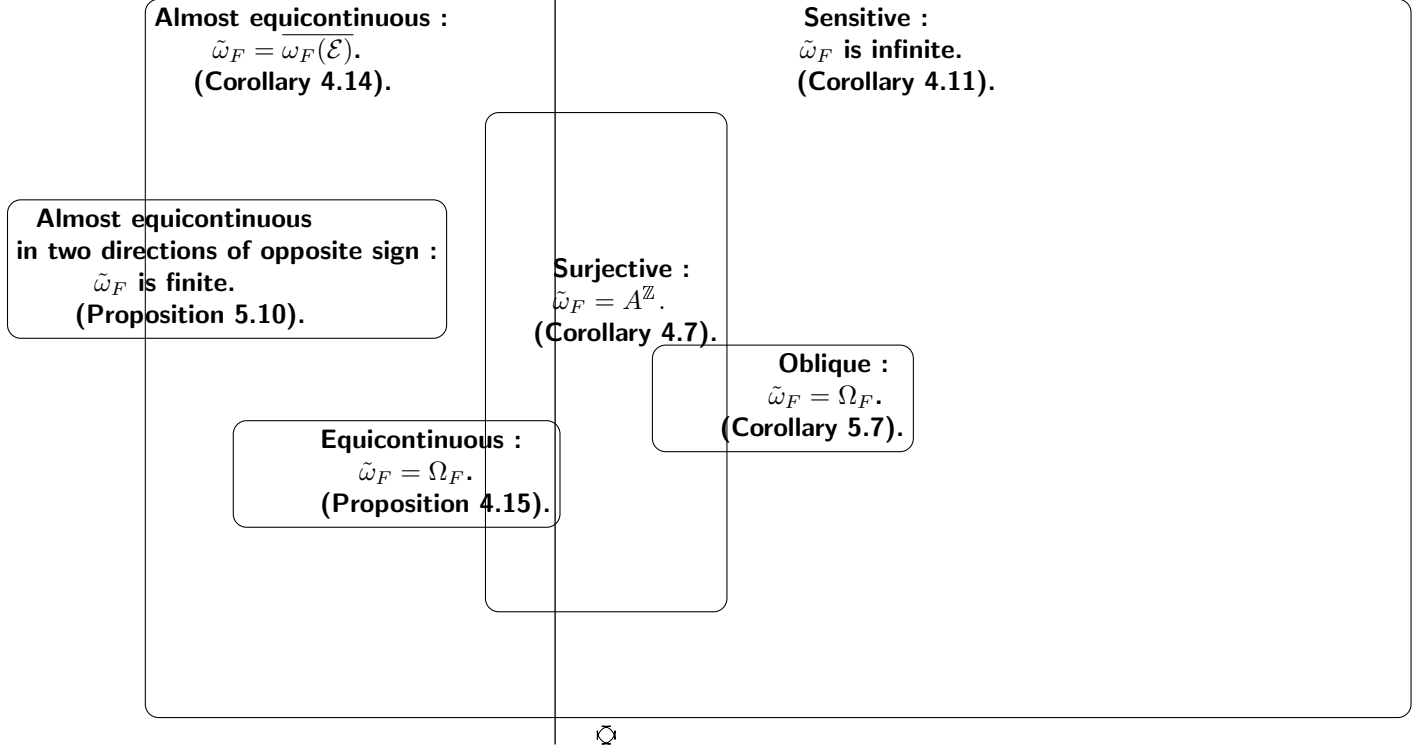


Figure 4: Summary of results. Nilpotent CA are not included, for a better readability.

Example 7.8. In [7, Prop 3.3], a CA is built, which is almost equicontinuous along h if and only if $0 \prec h \prec p$, where p is an explicit function close to the square root function. This CA is in Class 3: there is a nondegenerate interval of almost equicontinuous directions, but only one of them is linear.

With some horizontal bulking operation and a product with the CA built in [7, Prop 3.1] which deals with the other side of a parabola, one can even obtain a CA which is still in Class 3, but which is sensitive along all linear directions.

Example 7.9. In [3, Thm 6.1], a CA F is built with a word u which is blocking in a nondegenerate interval of directions, such that $\omega_F([u]_0)$ is the nontrivial orbit of a monochrome configuration (in particular, $\tilde{\omega}_{F^2} = \Omega_{F^2, \mu} \subsetneq \Omega_{F, \mu} = \tilde{\omega}_F$). This shows that CA of Class 3 are not always generically nilpotent: they can converge to a nontrivial orbit, provided that the blocking words are Gardens of Eden.

8 Conclusion

We studied the generic limit set of dynamical systems, and we emphasized the example of cellular automata. Our main results are:

- The generic limit set of a nonwandering system (in particular, of a surjective CA) is full.
- The generic limit set of an almost equicontinuous system is exactly the closure of the asymptotic set of its set of equicontinuity points. In particular, the generic limit set of an equicontinuous dynamical system is its limit set.
- The generic limit set of a cellular automaton which has two distinct directions of almost equicontinuity is finite; it is the periodic orbit of a monochrome configuration.

- The generic limit set of a sensitive system is infinite.
- The generic limit set of a semi-transitive system (in particular, of an oblique CA) is its limit set.

A summary of these results, for non-nilpotent CA, is represented in Figure 4.

Among the interesting questions that the directional classification brings, one can wonder whether, fixing one CA and making the directions vary, we obtain only finitely many generic limit sets, or whether they should intersect, at least as the orbit of a monochrome configuration (which is not clear for the last two classes).

Of course, another natural question is about what happens for two-dimensional CA: in that case almost equicontinuity does not correspond to existence of blocking words, and neither to non-sensitivity, so that everything becomes much more complex.

Acknowledgements. The authors wish to thank Martin Delacourt and Guillaume Theyssier for fruitful discussions about their results on μ -limit sets and on directional dynamics, and the latter also for his software ACML which helped easily draw our space-time diagrams.

This research has been held while the first author was welcomed in the Institut de Mathématiques de Marseille thanks to a grant from the Algerian Ministry of High Education and Scientific Research.

References

- [1] Luigi Acerbi, Alberto Dennunzio, and Enrico Formenti. Conservation of some dynamical properties for operations on cellular automata. *Theoretical Computer Science*, 410(38-40):3685–3693, 2009.
- [2] Rufus Bowen. Topological entropy and axiom A. In *Proc. Sympos. Pure Math.*, volume 14, pages 23–41, 1970.
- [3] L. Boyer, M. Delacourt, V. Poupet, M. Sablik, and G. Theyssier. μ -Limit sets of cellular automata from a computational complexity perspective. *Journal of Computer and System Sciences*, 81(8):1623–1647, December 2015.
- [4] Karel Čulik II, Jan Pachl, and Sheng Yu. On the limit set of cellular automata. *SIAM Journal on computing*, 18(4):831–842, 1989.
- [5] Jan de Vries. *Topological dynamical systems*. De Gruyter Studies in Mathematics, 2014.
- [6] Martin Delacourt. *Automates cellulaires : dynamique directionnelle et asymptotique typique*. France, 2011.
- [7] Martin Delacourt, Victor Poupet, Mathieu Sablik, and Guillaume Theyssier. Directional dynamics along arbitrary curves in cellular automata. *Theoretical Computer Science*, 412:3800–3821, 2011.
- [8] Bruno Durand and Jacques Mazoyer. Growing patterns in one dimensional cellular automata. *Complex Systems*, 8:419–434, 1994.
- [9] Enrico Formenti and Petr Kůrka. Subshift attractors of cellular automata. *Nonlinearity*, 20(1):1–15, 2007.
- [10] Robert H. Gilman. Classes of linear automata. *Ergodic Theory and Dynamical Systems*, 7:105–118, 1987.
- [11] Pierre Guillon and Gaétan Richard. Nilpotency and limit sets of cellular automata. In Edward Ochmański and Jerzy Tyszkiewicz, editors, *33rd International Symposium on the Mathematical Foundations of Computer Science (MFCS’08)*, volume 5162 of *Lecture Notes in Computer Science*, pages 375–386, Toruń, Poland, August 2008. Springer-Verlag.

- [12] Pierre Guillon and Gaétan Richard. Asymptotic behavior of dynamical systems. arXiv: 1004.4743, 2010.
- [13] Gustav A. Hedlund. Endomorphisms and automorphisms of the shift dynamical system. *Mathematical systems theory*, 3(4):320–375, 1969.
- [14] Mike Hurley. Attractors in cellular automata. *Ergodic Theory and Dynamical Systems*, 10(1):131–140, 1990.
- [15] Mike Hurley. Ergodic aspects of cellular automata. *Ergodic Theory and Dynamical Systems*, 10:671–685, 1990.
- [16] Mike Hurley. Varieties of periodic attractor in cellular automata. *American Mathematical Society*, 326(2):701–726, 1991.
- [17] Petr Kůrka. Languages, equicontinuity and attractors in cellular automata. *Ergodic Theory and Dynamical Systems*, 17(2):417–433, 1997.
- [18] Petr Kůrka. *Topological and symbolic dynamics*. Société Mathématique de France, 2003.
- [19] Petr Kůrka. Cellular automata with an infinite number of subshift attractors. *Complex Systems*, 17(3):219–230, 2007.
- [20] Petr Kůrka and Alejandro Maass. Stability of subshifts in cellular automata. *Fundamenta Informaticæ*, 52(1-3):143–155, 2002.
- [21] Pietro Di Lena and Luciano Margara. Computational complexity of dynamical systems: the case of cellular automata. *Information and Computation*, 206:104–1116, 2008.
- [22] John Milnor. On the concept of attractor. *Commun. Math. Phys.*, 99(2):177–195, 1985.
- [23] Mathieu Sablik. Directional dynamics for cellular automata: A sensitivity to initial condition approach. *Theoretical Computer Science*, 400:1–18, 2008.
- [24] S. M. Srivastava. *A course on Borel sets*. Springer, 1998.
- [25] Pierre Tisseur. Cellular automata and Lyapunov exponents. *Nonlinearity*, 13(5):1547–1560, 2000.
- [26] Petr Kůrka and Alejandro Maass. Limit sets of cellular automata associated to probability measures. *Journal of Statistical Physics*, 100:1031–1047, 2000.