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# Consensus analysis of large-scale nonlinear homogeneous multi-agent formations with polynomial dynamics

Paolo Massioni\*, Gérard Scorletti

## SUMMARY

This paper concerns the consensus analysis of multi-agent systems made of the interconnection of identical, nonlinear agents interacting with one another through an undirected and connected graph topology. Drawing inspiration from the theory of linear “decomposable systems”, we provide a method for proving the convergence (or consensus) of such multi-agents systems in the case of polynomial dynamics. The method is based on a numerical test, namely a set of linear matrix inequalities (LMIs) providing sufficient conditions for the convergence. We also show that the use of a generalised version of the famous Kalman-Yakubovic-Popov lemma allows the development of an LMI test whose size does not directly depend on the number of agents. The method is validated in simulation on three examples, which also show how the numerical test can be used to properly tune a controller.

**KEY WORDS:** Multi-agent systems, nonlinear systems, consensus, polynomial dynamics, sum of squares, linear matrix inequalities.

## 1. INTRODUCTION

Large-scale systems are an emerging topic in the system and control community, which is devoting significant efforts on the development of analysis and control synthesis methods for them. This deep interest can clearly be seen from the large number of works published in the field in the last 40 years [2, 5, 6, 32, 11, 17, 25, 1, 29, 19, 24, 22, 35, 38, 39].

One of the main objectives of the studies is the development and validation of “distributed control laws” for obtaining a certain specified goal for a system of this kind. By “distributed control”, opposed to “centralized control”, we mean a control action that is computed locally according to the physical spatial extension of the system, which is seen as an interconnection of simpler subsystems. One of the main problems of large-scale systems is the “curse of dimensionality” that goes with them, i.e. the analysis and synthesis problems related to dynamical systems grow with the size, and for system of very high order, such problems becomes computationally infeasible. In the literature, if we restrict to linear systems, we can find a few solutions [25, 11, 2] that can effectively overcome the curse of dimensionality for a class of systems with a certain regularity, namely for what we call “homogeneous systems”, i.e. systems made of the interconnection of a huge number of identical subunits (also sometimes called “agents”).

In this paper we focus on formations made of a high number of identical nonlinear agents interacting with one another through a time-invariant undirected and connected graph interconnection. The goal is being able to check whether the agents converge eventually to a common trajectory. This problem, also known as consensus or synchronisation of multi-agent systems, has been largely investigated in the literature for both linear and nonlinear dynamics; a typical research problem is the determination of sufficient conditions which ensure the convergence

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for a specific class of agent dynamical equation (for example, first order dynamics [26, 27], second order dynamics [37, 36, 40], heterogeneous systems [4, 24], nonholonomic vehicles [20, 3, 44], etc.). In this case, we assume that the dynamical equation of each agent is of any order  $n$  and it is described by a generic polynomial in the state vector. We can then show that a linear matrix inequality (LMI) test can be devised in order to verify the relative stability of such a formation. We will also be able to formulate such a test in a form that is not strictly depending on the formation size, making it possible to check the stability of formations with virtually any number of agents, basically extending the analysis results of [25, 12] to the nonlinear (polynomial) case.

The generality of the proposed approach is of course paid for by the conservatism of the result: nevertheless, three examples are proposed to show that the proposed LMI tests are successful in proving the consensus of agent formations with very complex nonlinear dynamics. Two of the examples show that the analysis results can also be used for the synthesis of a controller, at the cost of losing the convexity of the optimisation problem: the LMIs become bilinear matrix inequalities (BMIs); in most practical cases, an acceptable solution can be found by executing an iterative search on the controller parameters and solving for the corresponding LMIs.

## 2. PRELIMINARIES

### 2.1. Notation

We denote by  $\mathbb{N}$  the set of positive integers, by  $\mathbb{R}$  the set of real numbers and by  $\mathbb{R}^{n \times m}$  the set of real  $n \times m$  matrices.  $A^\top$  indicates the transpose of a matrix  $A$ ,  $I_n$  is the identity matrix of size  $n$ ,  $\mathbf{0}_{n \times m}$  is a matrix of zeros of size  $n \times m$  and  $\mathbf{1}_n \in \mathbb{R}^n$  a column vector that contains 1 in all of its entries. The notation  $A \succeq 0$  (resp.  $A \preceq 0$ ) indicates that all the eigenvalues of the symmetric matrix  $A$  are positive (resp. negative) or equal to zero, whereas  $A \succ 0$  (resp.  $A \prec 0$ ) indicates that all such eigenvalues are strictly positive (resp. negative). The binomial coefficient is denoted by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The symbol  $\otimes$  indicates the Kronecker product, for which we remind the basic properties ( $A \otimes B$ )( $C \otimes D$ ) = ( $AC \otimes BD$ ), ( $A \otimes B$ ) $^\top$  = ( $A^\top \otimes B^\top$ ), ( $A \otimes B$ ) $^{-1}$  = ( $A^{-1} \otimes B^{-1}$ ) (with matrices of compatible sizes). We employ the symbol  $*$  to complete symmetric matrix expressions avoiding repetitions.

### 2.2. Agent dynamics

We consider a set of  $N \in \mathbb{N}$  identical agents or subsystems of order  $n$ , which interact with one another. Each agent, if taken alone, is supposed to be described by polynomial dynamics, of the kind

$$\dot{x}_i = f_d(x_i) = A_a \chi_i \quad (1)$$

where  $i = 1, \dots, N$ ,  $x_i = [x_{i,1}, x_{i,2}, \dots, x_{i,n}]^\top \in \mathbb{R}^n$  is the state of the  $i^{\text{th}}$  agent,  $f_d$  is a polynomial function of degree  $d \in \mathbb{N}$ ,  $A_a \in \mathbb{R}^{n \times \rho}$  and  $\chi_i \in \mathbb{R}^\rho$  is the vector containing all the monomials in  $x_i$  up to degree  $d$  (for example, if  $n = 2$ ,  $d = 2$ , then  $\chi_i = [1, x_{i,1}, x_{i,2}, x_{i,1}^2, x_{i,1}x_{i,2}, x_{i,2}^2]^\top$ ). The value of  $\rho$  is given (see [28]) by the expression

$$\rho = \binom{n+d}{n}. \quad (2)$$

This approach is based on the sum of squares (SOS) literature [28], which basically allows the relaxation of polynomial problems into linear algebra's. In this context, it is possible to express polynomials  $p$  up to degree  $2d$  as quadratic forms with respect to  $\chi_i$ , i.e.  $p(x_i) = \chi_i^\top \mathcal{X} \chi_i$ , with  $\mathcal{X} = \mathcal{X}^\top \in \mathbb{R}^{\rho \times \rho}$ . This quadratic expression is not unique, due to the fact that different products of monomials in  $\chi_i$  can yield the same result, for example  $x_{i,1}^2$  is either  $x_{i,1}^2$  times 1 or  $x_{i,1}$

times  $x_{i,1}$ . This implies that there exist linearly independent slack matrices  $Q_k = Q_k^\top \in \mathbb{R}^{\rho \times \rho}$ , with  $k = 1, \dots, \iota$  such that  $\chi_i^\top Q_k \chi_i = 0$ . The number of such matrices (see [28]) is

$$\iota = \frac{1}{2} \left( \binom{d+n}{d} + \binom{d+n}{d} \right) - \binom{n+2d}{2d}. \quad (3)$$

We also define

$$\Gamma = [\mathbf{0}_{n \times 1} \quad I_n \quad \mathbf{0}_{n \times (\rho-n-2)}], \quad (4)$$

a matrix that picks the elements of the state vector  $x_i$  from  $\chi_i$ , i.e. such that  $\Gamma \chi_i = x_i$ .

As a clarifying example, consider  $\chi_i = [1, x_{i,1}, x_{i,1}^2]$ . Then it is possible to express a polynomial up to degree 2 as a linear function of  $\chi_i$ , for example,  $1 + 2x_{i,1} + 4x_{i,1}^2 = [1 \ 2 \ 4] \chi_i$ . It is also possible to express a polynomial up to degree 4 as a quadratic form in  $\chi_i$ , for example

$$1 + 2x_{i,1} + 4x_{i,1}^2 + 6x_{i,1}^3 - x_{i,1}^4 = \chi_i^\top \begin{bmatrix} 1 & 1 & 2 - \tau \\ 1 & 2\tau & 3 \\ 2 - \tau & 3 & -1 \end{bmatrix} \chi_i \quad (5)$$

where the value of  $\tau$  is not important as it always cancels out, which means that its matrix coefficient

$$Q = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad (6)$$

is a slack matrix, for which it can be verified that  $\chi_i^\top Q \chi_i = 0$ .

### 2.3. Formations

Moving from one single agent to a whole formation, we employ a ‘‘pattern’’ matrix  $\mathcal{P} \in \mathbb{R}^{N \times N}$  to describe the interactions among the agents. Basically  $\mathcal{P}$  is a sparse matrix whose entries in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column indicate whether the  $i^{\text{th}}$  agent is influenced by the state of the  $j^{\text{th}}$ , according to the definition that follows.

#### Definition 1 (Formation)

We call a formation (of non-linear agents with polynomial dynamics) a dynamical system of order  $nN$ , with  $n, N \in \mathbb{N}$ , described by the dynamical equation

$$\dot{x} = (I_N \otimes A_a + \mathcal{P} \otimes A_b) \chi \quad (7)$$

where  $x = [x_1^\top, x_2^\top, \dots, x_N^\top]^\top \in \mathbb{R}^{nN}$ ,  $\chi = [\chi_1^\top, \chi_2^\top, \dots, \chi_N^\top]^\top \in \mathbb{R}^{\rho N}$  ( $\chi_i$  is the vector of monomials in  $x_i$  up to degree  $d$ ),  $\mathcal{P} \in \mathbb{R}^{N \times N}$  and  $A_a, A_b \in \mathbb{R}^{n \times \rho}$ .

This definition extends and adapts the definition of ‘‘decomposable systems’’ found for example in [25, 10] to polynomial dynamics. In the linear case, a formation defined above boils down to the dynamical equation

$$\dot{x} = (I_N \otimes A_a + \mathcal{P} \otimes A_b) x. \quad (8)$$

In [25] it has been shown that if  $\mathcal{P}$  is diagonalisable, then this system (of order  $nN$ ) is equivalent to a set of parameter-dependent linear systems of order  $n$ . This is obtained with the change of variables  $x = (S \otimes I_n) \hat{x}$ , where  $\hat{x} = [\hat{x}_1^\top, \hat{x}_2^\top, \dots, \hat{x}_N^\top]^\top \in \mathbb{R}^{nN}$  and  $S$  is the matrix diagonalising  $\mathcal{P}$ , i.e.  $S^{-1} \mathcal{P} S = \Lambda$ , with  $\Lambda$  diagonal. This turns (8) into  $\dot{\hat{x}} = (I_N \otimes A_a + \Lambda \otimes A_b) \hat{x}$ , which is a block-diagonal system equivalent to the set of equations  $\dot{\hat{x}}_i = (A_a + \lambda_i A_b) \hat{x}_i$  for  $i = 1, \dots, N$ , with  $\lambda_i$  the  $i^{\text{th}}$  eigenvalue of  $\mathcal{P}$ . This idea of decomposing a distributed system into a set of parameter-varying systems is very practical and it has inspired several works in the domain of consensus and distributed control [18, 41, 10, 7]. However, this idea as it is cannot work if the dynamics is polynomial, as it can be seen by means of a simple example. Consider the dynamics in (7) with only two subsystems ( $N = 2$ ) of order one ( $n = 1$ ), second-degree dynamics ( $d = 2$ ), so

$\chi = [1, x_{1,1}, x_{1,1}^2, 1, x_{2,1}, x_{2,1}^2]^\top$ ), and a pattern matrix

$$\mathcal{P} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ diagonalised by } S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \quad (9)$$

Applying the change of variables  $x = (S \otimes I_n)\hat{x}$  and  $\chi = (S \otimes I_\rho)\hat{\chi}$  leads to the apparently useful expression  $\hat{x}_i = (A_a + \lambda_i A_b)\hat{\chi}_i$ . The problem is that the vector  $\hat{\chi}_i$  does not contain the monomials of the powers of  $\hat{x}_i$ , but linear combinations of the powers of  $x_i$ , which means that the decomposition is only apparent. In fact for  $\hat{\chi} = (S^{-1} \otimes I_\rho)\chi$  we have

$$\hat{\chi}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 \\ x_{1,1} + x_{2,1} \\ x_{1,1}^2 + x_{2,1}^2 \end{bmatrix}, \quad \hat{\chi}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ x_{1,1} - x_{2,1} \\ x_{1,1}^2 - x_{2,1}^2 \end{bmatrix} \quad (10)$$

and

$$\hat{x} = \begin{bmatrix} \hat{x}_{1,1} \\ \hat{x}_{2,1} \end{bmatrix} = (S^{-1} \otimes I_n)x = \frac{1}{\sqrt{2}} \begin{bmatrix} x_{1,1} + x_{2,1} \\ x_{1,1} - x_{2,1} \end{bmatrix}. \quad (11)$$

It is clear that  $\hat{\chi}_{1,3} = \frac{1}{\sqrt{2}}(x_{1,1}^2 + x_{2,1}^2)$  is not the second power of  $\hat{x}_{1,1} = \frac{1}{\sqrt{2}}(x_{1,1} + x_{2,1})$  which means that the  $\hat{x}_{1,1}$  and  $\hat{x}_{1,2}$  have not really been decoupled. Nevertheless, it is still possible to adapt the idea of decomposition to polynomial dynamics, as it is described in the rest of this paper. To our knowledge, this is the first time that the idea is applied to nonlinear systems.

#### 2.4. Problem formulation

The topic of this paper is to find a proof of convergence of the state of the agents under given dynamics expressed as in (7). We do not require that each agent by itself converges to a point, but that they all converge eventually to the same state, which could be either an equilibrium point or a trajectory. In order to do so, we formulate first an assumption on the pattern matrix  $\mathcal{P}$ .

##### Assumption 1

The pattern matrix  $\mathcal{P} \in \mathbb{R}^{N \times N}$  in (7) is symmetric and it has one and only one eigenvalue equal to 0, associated to the eigenvector  $\mathbf{1}_N$ , i.e  $\mathcal{P}\mathbf{1}_N = 0$ .

This assumption is very common in the literature, it basically ensures that the interconnection matrix is a (generalised) graph Laplacian of an undirected connected graph [8]. Such matrices have real eigenvalues and eigenvectors. We can then formulate the problem on which this paper focuses.

##### Problem 1

We consider (7) with initial conditions  $x(0) \in \mathbb{R}^{nN}$ . Find a numerical test that is sufficient condition for  $\lim_{t \rightarrow \infty} \|x_i - x_j\| = 0, \forall i, j \in \{1, \dots, N\}$ , for any initial condition.

### 3. FORMATION LYAPUNOV FUNCTION

In order to be able to prove the convergence of all the agents to the same trajectory, we define a formation Lyapunov function, a special choice (quadratic) of a function tending to zero when the agents are converging. We summarise these notions in a definition and a lemma.

##### Definition 2 (Formation Lyapunov function candidate)

We define as ‘‘formation Lyapunov function candidate’’ a function

$$V(x) = x^\top \left( \sum_{i=1}^l \mathcal{P}^i \otimes L_i \right) x = x^\top \mathcal{L}x \quad (12)$$

with  $L_i = L_i^\top \in \mathbb{R}^{n \times n}$ ,  $l \in \mathbb{N}$ ,  $l \leq N$ . The reason for this special structure will be clear later on, in fact it allows the block-diagonalisation of the Lyapunov matrix in the same way as  $\mathcal{P}$  can be diagonalised; increasing the number  $l$  of addends also increases the number of degrees of freedom of  $V$ , making it more likely that a solution is found.

*Lemma 1*

Consider (7) and a formation Lyapunov function candidate  $V(x) = x^\top \mathcal{L}x$  as in (12). Let  $\mathbf{1}_N^\perp \in \mathbb{R}^{N \times (N-1)}$  be the orthogonal complement of  $\mathbf{1}_N$ , i.e.  $[\mathbf{1}_N \ \mathbf{1}_N^\perp]$  is full rank and  $\mathbf{1}_N^\top \mathbf{1}_N^\perp = 0$ .

If  $(\mathbf{1}_N^\perp \otimes I_n)^\top \mathcal{L} (\mathbf{1}_N^\perp \otimes I_n) > 0$ , then we have  $x_i = x_j \ \forall i, j \in \{1, \dots, N\}$  if and only if  $V(x) = 0$ .

*Proof*

Necessity is almost obvious: if  $x_i = x_j \ \forall i, j \in \{1, \dots, N\}$ , then  $x = \mathbf{1}_N \otimes x_i$ ; the fact that  $\mathcal{P}\mathbf{1}_N = 0$  implies that  $V(x) = 0$ .

We prove the sufficiency by contradiction, i.e. we suppose that there exist  $i$  and  $j$  for which  $x_i \neq x_j$  and  $V(x) = 0$ . The vector  $x$  must then have at least one orthogonal component with respect to the columns of  $(\mathbf{1}_N \otimes I_n)$ , because  $(\mathbf{1}_N \otimes I_n)$  contains columns with all the corresponding agent states equal. So, based on the fact that  $(\mathbf{1}_N^\perp \otimes I_n)^\top \mathcal{L} (\mathbf{1}_N^\perp \otimes I_n) > 0$ , then  $V(x) > 0$  contradicting the hypothesis.  $\square$

#### 4. MAIN RESULT

We are now ready for our main result. A preliminary lemma allows diagonalising the Lyapunov function in the same way as a linear system is decomposed in [25].

*Lemma 2*

If Assumption 1 holds, then 1) there exist a matrix  $S \in \mathbb{R}^{N \times N}$  such that  $S^\top S = SS^\top = I_N$ , and  $S^\top \mathcal{P}S = \Lambda$ , with  $\Lambda$  diagonal (matrix of the eigenvalues of  $\mathcal{P}$ ). Moreover, we have that 2)  $S^\top \mathbf{1}_N = T = [t_1 \ t_2 \ \dots \ t_N]$ , with  $t_i \in \mathbb{R}^N = 0$  if the eigenvalue  $\lambda_i = \Lambda_{i,i} \neq 0$ .

*Proof*

The first part is proven by the fact that all symmetric matrices are diagonalisable by an orthonormal matrix  $S$  (i.e.  $S^{-1} = S^\top$ ) [13]. For the second part, due to Assumption 1,  $\mathbf{1}_N$  is an eigenvector of  $\mathcal{P}$  with eigenvalue 0;  $S$  contains the normalised eigenvectors of  $\mathcal{P}$  in its columns, and all these eigenvectors are orthogonal to one another because  $S^\top S = I_N$ . So each  $t_i$  is the dot product between  $\mathbf{1}_N$  and the  $i^{\text{th}}$  eigenvector, and it is non zero if and only if  $\lambda_i = 0$ .  $\square$

*Theorem 1*

Consider (7) with given  $N$ ,  $A_a$ ,  $A_b$  and  $\mathcal{P}$  satisfying Assumption 1; moreover, we order the eigenvalues of  $\mathcal{P}$  so that the first eigenvalue is the one equal to zero, i.e.  $\lambda_1 = 0$ . If for a chosen  $l \in \mathbb{N}$ , there exist  $\tau_j \in \mathbb{R}$ , and matrices  $L_j = L_j^\top \in \mathbb{R}^{n \times n}$  such that

$$\sum_{j=1}^l \lambda_i^j L_j \succ 0 \quad (13)$$

$$\sum_{j=1}^l \tau_j Q_j \preceq 0 \quad (14)$$

$$\Pi \left( \sum_{j=1}^l \tau_j Q_j + \sum_{j=1}^l (\lambda_i^j (\Gamma^\top L_j A_a + A_a^\top L_j \Gamma + \Gamma^\top L_j \Gamma) + \lambda_i^{j+1} (\Gamma^\top L_j A_b + A_b^\top L_j \Gamma) + \epsilon \lambda_i^j (\Gamma^\top L_j \Gamma)) \right) \Pi^\top \preceq 0 \quad (15)$$

for  $i = 2, \dots, N$ , where  $\Pi = [\mathbf{0}_{(\rho-1) \times 1} \ I_{\rho-1}]$ ,  $\epsilon > 0$  (small), the slack matrices  $Q_j$  for  $j = 1, \dots, l$  with  $l$  defined in (3) are such that  $\chi_i^\top Q_j \chi_i = 0$ , and where  $\Gamma$  is defined in (4), then  $\lim_{t \rightarrow \infty} \|x_i - x_j\| = 0, \ \forall i, j \in \{1, \dots, N\}$  (Problem 1).

*Proof*

Assuring the convergence of the agents requires assuring the conditions stated in Lemma 1, namely that a function  $V(x) = x^\top \mathcal{L}x$  exists, with and  $(\mathbf{1}_N^\perp \otimes I_n)^\top \mathcal{L} (\mathbf{1}_N^\perp \otimes I_n) > 0$ , and that  $\dot{V}(x) < 0$  for  $V(x) > 0$ . For the condition  $(\mathbf{1}_N^\perp \otimes I_n)^\top \mathcal{L} (\mathbf{1}_N^\perp \otimes I_n) > 0$ , consider that  $S$  contains a scaled version of  $\mathbf{1}_N$  in its first column and  $\mathbf{1}_N^\perp$  in the rest of the matrix. Knowing that  $\mathcal{L} = (S \otimes I_n) (\sum_{i=1}^l \Lambda^i \otimes L_i) (S^\top \otimes I_n)$  thanks to Lemma 2, this condition is equivalent to (13).

For what concerns the condition  $\dot{V}(x) < 0$  for  $V(x) > 0$ , it is satisfied if  $\dot{V}(x) \leq -\epsilon x^\top \mathcal{L}x$ , i.e.

$$\begin{aligned} & \chi^\top (\mathcal{Q}_N + \Gamma_N^\top \mathcal{L}(I_N \otimes A_a) + \Gamma_N^\top \mathcal{L}(\mathcal{P} \otimes A_b) + \\ & (I_N \otimes A_a^\top) \mathcal{L} \Gamma_N + (\mathcal{P} \otimes A_b^\top) \mathcal{L} \Gamma_N + \epsilon \Gamma_N^\top \mathcal{L} \Gamma_N) \chi \leq 0 \end{aligned} \quad (16)$$

where  $\Gamma_N = (I_N \otimes \Gamma)$  (so  $\Gamma_N \chi = x$ ) and  $\mathcal{Q}_N = I_N \otimes \sum_{j=1}^l \tau_j Q_j$  (for which, by definition of  $Q_j$ ,  $\chi^\top \mathcal{Q}_N \chi = 0$  for all values of the  $\tau_i$ ). By the fact that  $\mathcal{P} = S \Lambda S^\top$  and  $I_N = S S^\top$  (Assumption 1 and Lemma 2), using the properties of Kronecker product (16) is equivalent to

$$\begin{aligned} & \hat{\chi}^\top (\mathcal{Q}_N + \Gamma_N^\top \hat{\mathcal{L}}(I_N \otimes A_a) + \Gamma_N^\top \hat{\mathcal{L}}(\Lambda \otimes A_b) + \\ & (I_N \otimes A_a^\top) \hat{\mathcal{L}} \Gamma_N + (\Lambda \otimes A_b^\top) \hat{\mathcal{L}} \Gamma_N + \epsilon \Gamma_N^\top \hat{\mathcal{L}} \Gamma_N) \hat{\chi} \leq 0 \end{aligned} \quad (17)$$

with  $\hat{\mathcal{L}} = \sum_{j=1}^l \Lambda^j \otimes L_j$  and  $\hat{\chi} = (S^\top \otimes I_\rho) \chi$ . Notice in this last inequality that the term between  $\hat{\chi}^\top$  and  $\hat{\chi}$  is block-diagonal, as it is the sum of terms of the kind  $I_N \otimes X$  or  $\Lambda^i \otimes X$  ( $i \in \mathbb{N}$ ). If we define  $\hat{\chi}_i \in \mathbb{R}^\rho$  such that  $\hat{\chi} = [\hat{\chi}_1^\top, \hat{\chi}_2^\top, \dots, \hat{\chi}_N^\top]^\top$ , then (17) is equivalent to

$$\begin{aligned} & \sum_{i=1}^N \hat{\chi}_i^\top (\sum_{j=1}^l \tau_j Q_j + \sum_{j=1}^l \lambda_i^j (\Gamma^\top L_j A_a + A_a^\top L_j \Gamma) + \\ & \sum_{j=1}^l \lambda_i^{j+1} (\Gamma^\top L_j A_b + A_b^\top L_j \Gamma) + \epsilon \sum_{j=1}^l \lambda_i^j (\Gamma^\top L_j \Gamma)) \hat{\chi}_i \leq 0. \end{aligned} \quad (18)$$

The term of the sum for  $i = 1$  is always  $\leq 0$  due to (14) (as we chose  $\lambda_1 = 0$ ), so it can be disregarded. Concerning the other terms, remember that the vectors  $\chi_i$  all contain 1 in their first entry, i.e.  $\chi_i = [1 \ \tilde{\chi}_i^\top]^\top$ ,  $\tilde{\chi}_i \in \mathbb{R}^{\rho-1}$ . For each  $\hat{\chi}_i$ , the first entry is by its definition the  $i^{\text{th}}$  entry of the vector  $e = S^\top \mathbf{1}$ , which contains zeros in all of its entries but the first (due to Lemma 2). So for  $i = 2, \dots, N$ , we have that  $\hat{\chi}_i = \Pi^\top \Pi \tilde{\chi}_i$ . So (18) is implied by

$$\begin{aligned} & \sum_{i=2}^N \hat{\chi}_i^\top \Pi^\top \Pi (\sum_{j=1}^l \tau_j Q_j + \sum_{j=1}^l \lambda_i^j (\Gamma^\top L_j A_a + A_a^\top L_j \Gamma) + \\ & \sum_{j=1}^l \lambda_i^{j+1} (\Gamma^\top L_j A_b + A_b^\top L_j \Gamma) + \epsilon \sum_{j=1}^l \lambda_i^j (\Gamma^\top L_j \Gamma)) \Pi^\top \Pi \tilde{\chi}_i \leq 0 \end{aligned} \quad (19)$$

The set of LMIs in (15) and (14) imply (19), concluding the proof.  $\square$

This theorem allows proving the convergence of  $N$  agents with two sets of  $N - 1$  parameter-dependent LMIs, whose matrix size is respectively  $n$  (i.e. the order of each agent taken alone) and  $\rho - 1$ . This result is already interesting as its LMIs do not scale with  $Nn$ , i.e. the global system order. If one were to study the consensus directly on the  $Nn$  order dynamics, the use of SOS would create linear matrix inequalities whose size would grow a lot more, due to the necessity of creating all the combinations of products of the state variables up to order  $d$ . For growing values of  $N$ , the computational complexity becomes quickly unmanageable. Let us consider an example with third degree dynamics ( $d = 3$ ), three agents ( $N = 3$ ) and order  $n = 1$ . The state vector is  $x = [x_{1,1}, x_{2,1}, x_{3,1}]^\top$ . The use of sum of squares (see [28]) on the  $Nn$  order system would require constructing a vector of monomials with all combinations up to third degree, i.e.  $\xi = [1, x_{1,1}, x_{2,1}, x_{3,1}, x_{1,1}^2, x_{1,1}x_{2,1}, x_{1,1}x_{3,1}, x_{2,1}^2, x_{2,1}x_{3,1}, x_{3,1}^2, x_{1,1}^3, x_{1,1}^2x_{2,1}, x_{1,1}^2x_{3,1}, x_{1,1}x_{2,1}x_{3,1}, x_{1,1}x_{3,1}^2, x_{2,1}^3, x_{2,1}^2x_{3,1}, x_{2,1}x_{3,1}^2, x_{3,1}^3]$ , which has 20 entries. Theorem 1 will rather produce  $N - 1$  LMIs of depending on the size of  $\chi_i = [1, x_{i,1}, x_{i,1}^2, x_{i,1}^3]^\top$ , i.e. only 4. If  $N$  grows the situation gets worse: for  $N = 4$ , the size of  $\xi$  becomes 35, whereas  $\chi_i$  stays the same (we just go from 2 LMIs of size 4 to 3 LMIs of size 4). For  $N = 4$ , the size of  $\xi$  becomes 56, whereas  $\chi_i$  stays the same (we go from 3 LMIs of size 4 to 4 LMIs of size 4). For  $N = 5$ , the size of  $\xi$  becomes 84, etc., which is already very bulky in terms of computational complexity. Theorem 1 instead will let one deal with much higher  $N$  without any problems on a normal computer.

Notice that Theorem 1 only provides sufficient condition, i.e. the multi-agent system might achieve consensus but the LMI test might still fail. In order to have more chances of getting a solution, one can either increase  $l$  (give more degrees of freedom to the Lyapunov candidate function), or increase  $d$  (embed the given polynomial functions into a space of polynomials of higher degree). Both options of course imply a higher computational cost.

In the next section, we explore whether it is possible to further reduce the computational complexity.

## 5. VARIATION ON THE MAIN RESULT

We now introduce a generalised version of the Kalman-Yakubovich-Popov (KYP) lemma [31]. This lemma turns a parameter-dependent LMI into a parameter-independent one.

## 5.1. The Kalman-Yakubovich-Popov lemma

The Kalman-Yakubovich-Popov lemma or KYP [31] is a widely celebrated result for dynamical systems that allows turning frequency-dependent inequalities into frequency-independent ones, by exploiting a state-space formulation. It turns out that such a result can be adapted and generalised to inequalities depending on any scalar parameter. Namely, we will use the following generalised version of the KYP.

*Lemma 3* (Generalized KYP [9])

Consider

$$M(\xi) = M_0 + \sum_{i=1}^{\eta} \xi_i M_i, \quad (20)$$

with  $\xi \in \mathbb{R}^l$  a vector of decision variables and  $M_i = M_i^\top \in \mathbb{R}^{n_M \times n_M}$ ,  $i = 1, \dots, \eta$ . The quadratic constraint

$$\phi(\theta)^\top M(\xi) \phi(\theta) \prec 0 \text{ for } \theta \in [\underline{\theta}, \bar{\theta}] \quad (21)$$

is verified if and only if there exist  $\mathcal{D} = \mathcal{D}^\top \succ 0$  and  $\mathcal{G} = -\mathcal{G}^\top$  such that

$$\begin{bmatrix} \tilde{C}^\top \\ \tilde{D}^\top \end{bmatrix} M(\xi) \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} + \begin{bmatrix} I & 0 \\ \tilde{A} & \tilde{B} \end{bmatrix}^\top \begin{bmatrix} -2\mathcal{D} & (\underline{\theta} + \bar{\theta})\mathcal{D} + \mathcal{G} \\ (\underline{\theta} + \bar{\theta})\mathcal{D} - \mathcal{G} & -2\underline{\theta}\bar{\theta}\mathcal{D} \end{bmatrix} \begin{bmatrix} I & 0 \\ \tilde{A} & \tilde{B} \end{bmatrix} \prec 0 \quad (22)$$

with  $\tilde{A}, \tilde{B}, \tilde{C}$  and  $\tilde{D}$  such that

$$\phi(\theta) = \tilde{D} + \tilde{C}\theta I(I - \tilde{A}\theta I)^{-1}\tilde{B} = \theta I \star \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}, \quad (23)$$

where the operator  $\star$  implicitly defined above is known as the Redheffer product [45]. The lemma still holds if the sign  $\prec$  in (21) is replaced by  $\preceq$ : in this case replace  $\prec$  with  $\preceq$  in (22).

## 5.2. Second main result

Let us define

$$\bar{\lambda} = \max_{2 \leq i \leq N} \{\lambda_i\}, \quad \underline{\lambda} = \min_{2 \leq i \leq N} \{\lambda_i\}. \quad (24)$$

Then, for  $\theta \in [\underline{\lambda}, \bar{\lambda}]$ , the following set of LMIs

$$\sum_{j=1}^l \theta^j L_j \succ 0 \quad (25)$$

$$\Pi(\sum_{j=1}^l \tau_j Q_j + \sum_{j=1}^l (\theta^j (\Gamma^\top L_j A_a + A_a^\top L_j \Gamma + \Gamma^\top L_j \Gamma) + \theta^{j+1} (\Gamma^\top L_j A_b + A_b^\top L_j \Gamma) + \epsilon \theta^j (\Gamma^\top L_j \Gamma))) \Pi^\top \preceq 0 \quad (26)$$

“embeds” the set of LMIs in (13) and (15) (notice that we have moved from a discrete set of values to a continuous interval which includes them all). Subsequently, Lemma 3 can be used to turn the  $\theta$ -dependent LMIs in (25) and (26) into parameter-independent ones. The dependence of the terms in (25) and (26) from  $\theta$  (which is ultimately  $\lambda_i$ ) is polynomial, so we need to define

$$\phi_\nu(\theta) = \left[ \theta^{\text{ceil}((l+1)/2)} I_\nu, \theta^{\text{ceil}((l+1)/2)-1} I_\nu, \dots, I_\nu \right]^\top \quad (27)$$

which requires

$$\begin{aligned} \tilde{A}_\nu &= U_\nu \otimes I_n, & \tilde{B}_\nu &= \begin{bmatrix} \mathbf{0}_{(\nu-1) \times 1} \\ 1 \end{bmatrix} \otimes I_n, \\ \tilde{C}_\nu &= \begin{bmatrix} I_\nu \\ \mathbf{0}_{1 \times \nu} \end{bmatrix} \otimes I_n, & \tilde{D}_\nu &= \begin{bmatrix} \mathbf{0}_{\nu \times 1} \\ 1 \end{bmatrix} \otimes I_n. \end{aligned} \quad (28)$$



where  $U_\nu \in \mathbb{R}^{i\nu \times \nu}$  is a matrix containing 1's in the first upper diagonal and 0's elsewhere, and  $\nu = n$  for (25) and  $\nu = \rho - 1$  for (26). We can now formulate the second main result.

*Corollary 1*

Consider (7) with given  $N$ ,  $A_a$ ,  $A_b$  and  $\mathcal{P}$  satisfying Assumption 1; excluding the first eigenvalue of  $\mathcal{P}$ , which is equal to 0, we have that  $\underline{\lambda} \leq \lambda_i \leq \bar{\lambda}$ , with  $i = 2, \dots, N$ . If for a chosen  $l \in \mathbb{N}$ , there exist  $\tau_i \in \mathbb{R}$ , and matrices  $L_i = L_i^\top \in \mathbb{R}^{n \times n}$ , and there exist  $\mathcal{D}_{\nu_k}, \mathcal{G}_{\nu_k} \in \mathbb{R}^{\nu_k \times \nu_k}$ ,  $\mathcal{D}_{\nu_k} = \mathcal{D}_{\nu_k}^\top \succ 0$  and  $\mathcal{G}_{\nu_k} = -\mathcal{G}_{\nu_k}^\top$  such that (14) holds and

$$[*]^\top M_k \begin{bmatrix} \tilde{C}_{\nu_k} & \tilde{D}_{\nu_k} \end{bmatrix} + [*]^\top \begin{bmatrix} -2\mathcal{D}_{\nu_k} & (\underline{\lambda} + \bar{\lambda})\mathcal{D}_{\nu_k} + \mathcal{G}_{\nu_k} \\ * & -2\underline{\lambda}\bar{\lambda}\mathcal{D}_{\nu_k} \end{bmatrix} \begin{bmatrix} I_{\nu_k} & 0 \\ \tilde{A}_{\nu_k} & \tilde{B}_{\nu_k} \end{bmatrix} \prec 0 \quad (29)$$

for  $k = 1, 2$ , with  $\nu_1 = n$ ,  $\nu_2 = \rho - 1$ , and with  $\tilde{A}_{\nu_k}, \tilde{B}_{\nu_k}, \tilde{C}_{\nu_k}, \tilde{D}_{\nu_k}, \phi_{\nu_k}$  defined in (28), (27), and  $M_1, M_2$  defined by

$$\phi_{\nu_1}(\lambda_i)^\top M_1 \phi_{\nu_1}(\lambda_i) = - \sum_{j=1}^l \lambda_i^j L_j \quad (30)$$

$$\begin{aligned} & \phi_{\nu_2}(\lambda_i)^\top M_2 \phi_{\nu_2}(\lambda_i) = \\ & \Pi \left( \sum_{j=1}^l \tau_j Q_j + \sum_{j=1}^l (\lambda_i^j (\Gamma^\top L_j A_a + A_a^\top L_j \Gamma + \Gamma^\top L_j \Gamma) + \lambda_i^{j+1} (\Gamma^\top L_j A_b + A_b^\top L_j \Gamma) + \epsilon \lambda_i^j (\Gamma^\top L_j \Gamma)) \right) \Pi^\top, \end{aligned} \quad (31)$$

where  $\epsilon > 0$ ,  $\Gamma = [\mathbf{0}_{n \times 1} \ I_n \ \mathbf{0}_{n \times (\rho-n-2)}]$  (i.e.  $\Gamma \chi_i = x_i$ ) and  $\Pi = [\mathbf{0}_{(\rho-1) \times 1} \ I_{\rho-1}]$ , then  $\lim_{t \rightarrow \infty} \|x_i - x_j\| = 0, \forall i, j \in \{1, \dots, N\}$  (Problem 1).

*Proof*

A direct application of Lemma 3 for  $M_1$  and  $M_2$  implies that the hypotheses of Theorem 1 are satisfied if the hypotheses here are.  $\square$

With this corollary, we replace the two sets of  $N - 1$  LMIs of size  $n$  and  $\rho - 1$ , with only two LMIs of matrix size  $n \text{ ceil}((l + 3)/2)$  and  $(\rho - 1) \text{ ceil}((l + 3)/2)$ . This is an interesting result because the computational complexity is no longer depending on  $N$ , i.e. the number of agents. On the other hand, the choice of a bigger  $l$  will improve the chances of solving the LMIs for high values of  $N$ .

*Remark 1*

The matrices  $M_1$  and  $M_2$  in Corollary 1 are implicitly defined by (30) and (31). In order to help the understanding of the paper, here are provided the explicit expressions of  $M_1$  for  $l = 3$ :

$$M_1 = - \begin{bmatrix} 0 & \frac{1}{2}L_3 & 0 \\ \frac{1}{2}L_3 & L_2 & \frac{1}{2}L_1 \\ 0 & \frac{1}{2}L_1 & 0 \end{bmatrix}, \phi_\nu(\lambda_i) = \begin{bmatrix} \lambda_i^2 I_\nu \\ \lambda_i I_\nu \\ I_\nu \end{bmatrix} \quad (32)$$

and  $l = 5$ :

$$M_1 = - \begin{bmatrix} 0 & \frac{1}{2}L_5 & 0 & 0 \\ \frac{1}{2}L_5 & L_4 & \frac{1}{2}L_3 & 0 \\ 0 & \frac{1}{2}L_3 & L_2 & \frac{1}{2}L_1 \\ 0 & 0 & \frac{1}{2}L_1 & 0 \end{bmatrix}, \phi_\nu(\lambda_i) = \begin{bmatrix} \lambda_i^3 I_\nu \\ \lambda_i^2 I_\nu \\ \lambda_i I_\nu \\ I_\nu \end{bmatrix}. \quad (33)$$

## 6. EXAMPLES

In order to provide a few challenging examples of application of the proposed method, we focus on a problem widely studied in the nonlinear dynamics community, namely the synchronisation of oscillators [30, 34]. The approach here is of course numerical and different (or complementary) with respect to those found in such a literature, where the objective is usually to find a control law and then prove stability. Here we just propose a control law and we test numerically whether

it will make the subsystems converge or not. We first consider two famous examples of nonlinear systems, namely the Van der Pol oscillator [14] and the Lorenz attractor [23], and we conclude with an example based on the dynamics of spintorque oscillators [16]. All the LMI problems have been coded in Matlab using Yalmip [21], and using SeDuMi [33] as solver.

### 6.1. Van der Pol oscillator

We consider a system of  $N$  agents of equation

$$\begin{cases} \dot{x}_{i,1} = x_{i,2} + \sigma r_i \\ \dot{x}_{i,2} = \mu(1 - x_{i,1}^2)x_{i,2} - x_{i,1} - cr_i \end{cases} \quad (34)$$

for which  $n = 2$ , with  $r_i = -(x_{i-1,1} + x_{i-1,2}) + 2(x_{i,1} + x_{i,2}) - (x_{i+1,1} + x_{i+1,2})$  (where the first index  $i$  is to be considered as modulo  $N$ , i.e.  $0 \rightarrow N, N + 1 \rightarrow 1$ ). The degree of the polynomial is  $d = 3$ , for which  $\chi_i = [1, x_{i,1}, x_{i,2}, x_{i,1}^2, x_{i,1}x_{i,2}, x_{i,2}^2, x_{i,1}^3, x_{i,1}^2x_{i,2}, x_{i,1}x_{i,2}^2, x_{i,2}^3]^\top$ . For

$$\begin{aligned} A_a &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & \mu & 0 & 0 & 0 & 0 & -\mu & 0 & 0 \end{bmatrix}, \\ A_b &= \begin{bmatrix} 0 & 2\sigma & 2\sigma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2c & -2c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \mathcal{P} &= \begin{bmatrix} 1 & -0.5 & 0 & 0 & \dots & 0 & -0.5 \\ -0.5 & 1 & -0.5 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ -0.5 & 0 & 0 & 0 & \dots & -0.5 & 1 \end{bmatrix} \end{aligned} \quad (35)$$

(34) fits within the framework of (7). Notice that  $\mathcal{P}$  is a symmetric graph Laplacian, whose (real) eigenvalues are guaranteed to be in the  $[0, 2]$  interval by the Perron-Frobenius theorem [11]. The numerical values are taken arbitrarily as  $\mu = 0.5$ ,  $N = 8$  and  $l = 8$ . The interconnection between each oscillator is given by two terms. The first term is  $\sigma$ , which we set arbitrarily as  $\sigma = 0.1$ ; it gives a repulsive contribution which destabilises the system. The second interconnection term is  $c$ , which can be interpreted as a proportional feedback gain that we want to tune in order to achieve consensus. The interconnection structure is depicted in Figure 1. For  $c = 0$ , it can be verified that

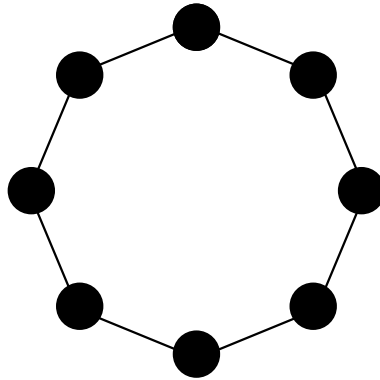


Figure 1. Interconnection graph for the coupled Van der Pol oscillators.

the system is unstable and the states diverge to infinity. Intuitively, a  $c$  high enough should make the agents converge to one another. By running a dichotomic search on the value of  $c$ , it has been found that the smallest value for which a solution for the LMIs in Corollary 1 is found, guaranteeing the convergence, is  $c = 4.8$ . Figure 2 and Figure 3 show the evolution of the system during a simulation

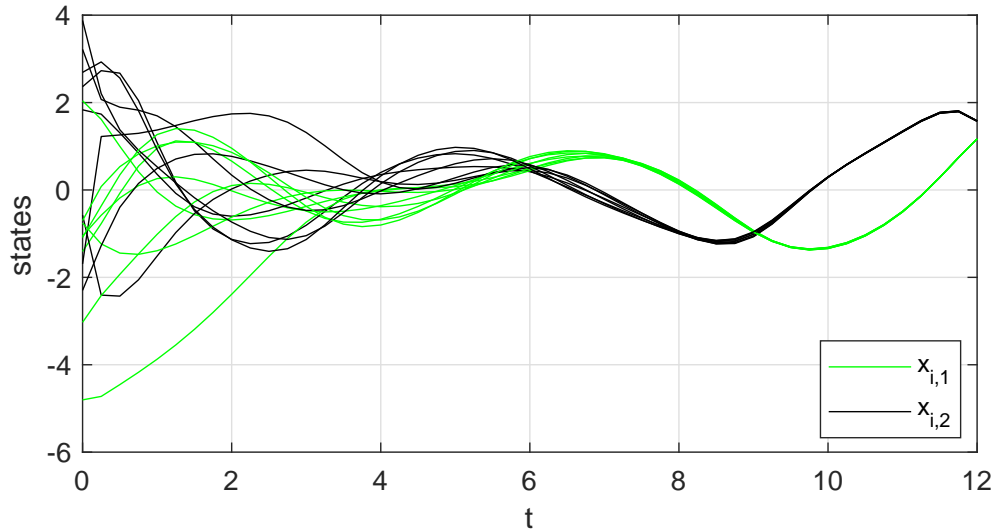


Figure 2. Evolution of the state of the 8 coupled Van der Pol oscillators, for  $c = 4.8$ .

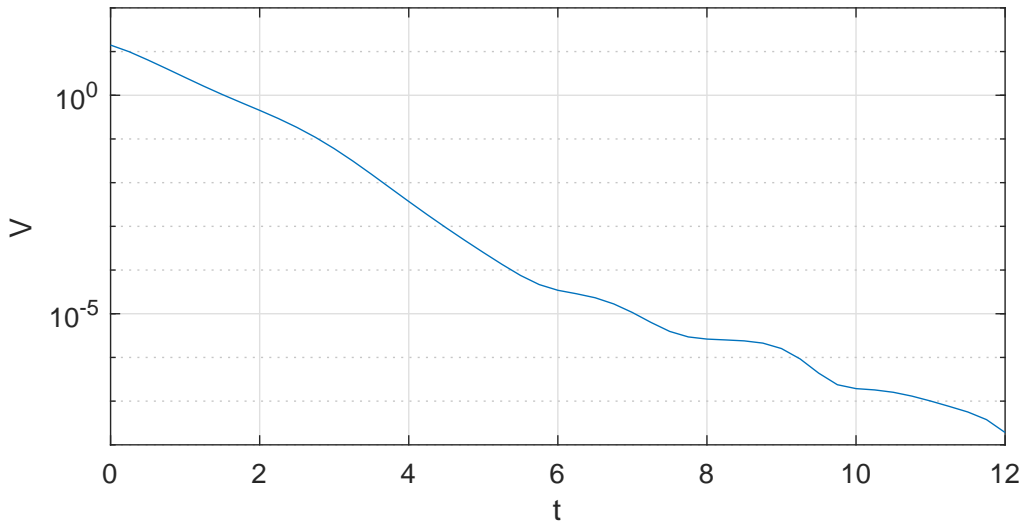


Figure 3. Value of the formation Lyapunov function  $V$  for the coupled Van der Pol oscillators ( $c = 4.8$ ).

for this value of  $c$ , the individual states are shown, together with the value of the Lyapunov function over time.

Due to the fact that Corollary 1 is conservative, the minimum value of  $c$  guaranteeing consensus is probably overestimated. In fact, the simulations show apparent convergence also for slightly smaller values: we say “apparent” because it is not possible to prove the convergence for all the possible initial conditions by means of simulation. In any case, for this example, Corollary 1 provides an efficient and reliable method to safely tune the controller.

## 6.2. Lorenz attractor

We now consider a system of  $N$  agents of equation

$$\begin{cases} \dot{x}_{i,1} = \sigma_l(x_{i,2} - x_{i,1}) - cx_{i,1} \\ \dot{x}_{i,2} = x_{i,1}(\rho_l - x_{i,3}) - x_{i,2} - cx_{i,2} \\ \dot{x}_{i,3} = x_{i,1}x_{i,2} - \beta_l x_{i,3} - cx_{i,3} \end{cases} \quad (36)$$

with  $r_{i,j} = -x_{i-1,j} + 2x_{i,j} - x_{i+1,j}$  ( $j = 1, 2, 3$ , again the index  $i$  is taken modulo  $N$ ). The equations fit in the framework of (7) for  $d = 2$ ,  $n = 3$  and with a pattern matrix  $\mathcal{P}$  that has the same structure as the one in (35). We arbitrarily set  $\rho_l = 28$ ,  $\sigma_l = 10$ ,  $\beta_l = 8/3$ ,  $N = 4$  and  $l = 4$ . This time Theorem 1 has been used; for the arbitrary value of  $c = 50$ , the LMIs in Theorem 1 are satisfied, successfully yielding a formation Lyapunov function and thus proving that a consensus is eventually reached. Figure 4 and Figure 5 again show the evolution of the system during a simulation, with individual states. The value of the Lyapunov function is also verified to be decreasing. Notice that the Lorenz oscillator does not converge to a limit cycle but to a chaotic trajectory.

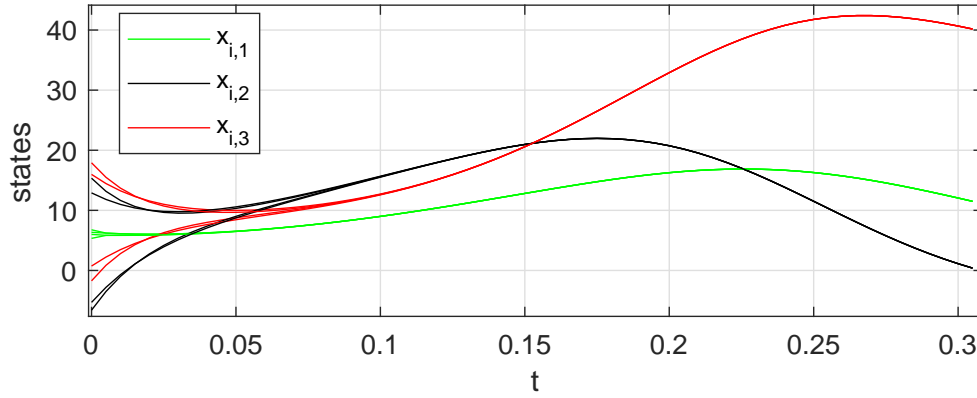


Figure 4. Evolution of the state of the coupled Lorenz systems.

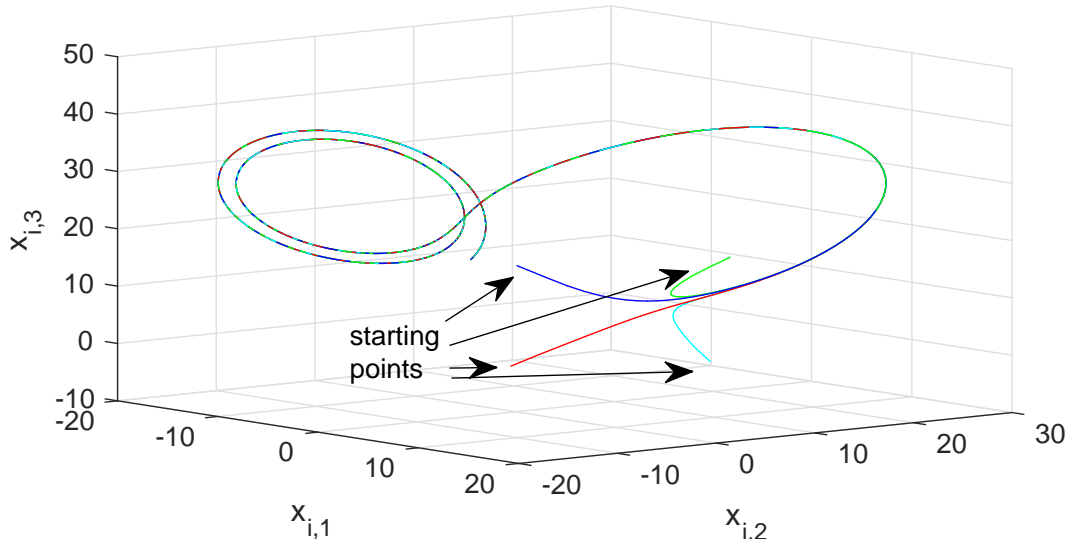


Figure 5. Tridimensional visualisation of the state of the coupled Lorenz systems of the example (the trajectories eventually converge to the consensus trajectory).

### 6.3. Spintorque oscillators

Spintorque oscillators are microscopic devices with several potential applications in modern radiofrequency electronics [16]. It is a relevant research problem to investigate how such oscillators can be synchronised [43, 42].

The simplified dynamics of a network of  $N$  coupled spintorque oscillators can be described by equations of the kind [42]

$$\begin{cases} \dot{r}_i = -g_r r_{r,i} \\ \dot{\phi}_i = 1 + r_i^2 - g_\phi \phi_{r,i} \end{cases} \quad (37)$$

with  $r_{r,i} = -\frac{1}{N-1} \sum_{j \neq i} r_j + r_i$  and  $\phi_{r,i} = -\frac{1}{N-1} \sum_{j \neq i} \phi_j + \phi_i$ . This definition implies a different interconnection topology compared to the previous examples, namely one given by the pattern

$$\mathcal{P} = \begin{bmatrix} 1 & -\frac{1}{N-1} & -\frac{1}{N-1} & \cdots & -\frac{1}{N-1} & -\frac{1}{N-1} \\ -\frac{1}{N-1} & 1 & -\frac{1}{N-1} & \cdots & -\frac{1}{N-1} & -\frac{1}{N-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ -\frac{1}{N-1} & -\frac{1}{N-1} & -\frac{1}{N-1} & \cdots & -\frac{1}{N-1} & 1 \end{bmatrix}, \quad (38)$$

which is depicted in Figure 6.

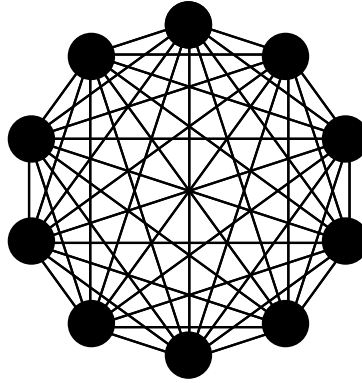


Figure 6. Interconnection graph for the spintorque oscillators example.

The values  $g_r$  and  $g_\phi$  are unknown feedback control gains, for which it is interesting to find values that guarantee the synchronisation of the oscillators. In order to do so, it is convenient to rely on Theorem 1, because the pattern  $\mathcal{P}$  in (38) has identical non-zero eigenvalues, which automatically turns its set of  $N - 1$  LMIs into just one LMI (regardless of  $N$ ).

By executing a random search on the values of  $g_r$  and  $g_\phi$ , it can be found that for  $g_r = 20$  and  $g_\phi = 0.1$  the LMIs in Theorem 1 have a solution with  $l = 2$ , proving the global convergence of the oscillators. Figure 7 shows a simulation in the case of  $N = 8$ .

Notice that in this case, we have been able to choose a value of  $l$  which is much smaller compared to  $N$  (in fact, for  $l = 2$  we are able to prove convergence for any  $N$ ). This is probably due to the highly connected topology that has been chosen, where every oscillator is connected to all of the others. This also suggests that in general the number  $l$  can be reduced if the interconnection among the agents is increased.

## 7. CONCLUSION

We have introduced a new method for proving convergence or consensus of multi-agent system with polynomial dynamics. This method is the generalisation of the analysis methods in [25] and it has proven effective in test cases featuring dynamical oscillators. Further research will investigate if the limitation given by the hypothesis of homogeneity and time invariance of the interconnection can be dropped, for example following the approach of [24, 15].

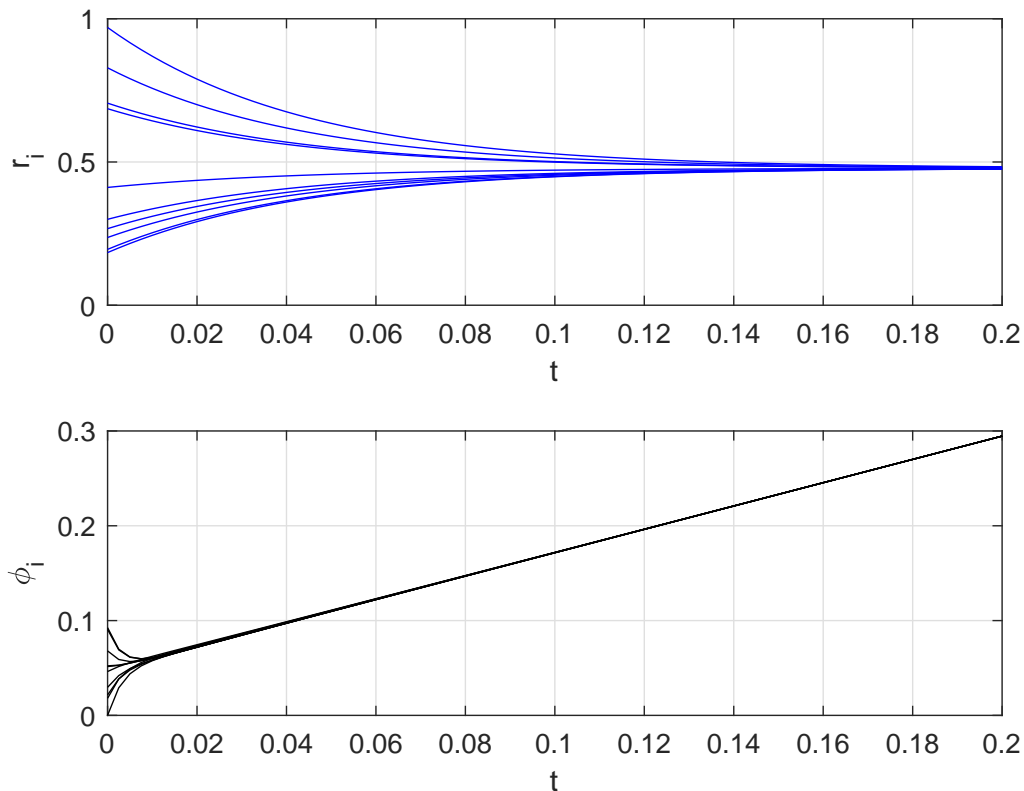


Figure 7. Evolution of the state of the coupled spintorque oscillators.

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