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# A class of cutoff functions for non-trivial Riemann zeros 

Yu Li


#### Abstract

Abstract: In this paper we construct a pseudo ermitian operator $\tilde{H}$ on the Hilbert space $L^{2}\left[\frac{1}{2}, 1\right]$ and show that the Hurwitz zeta functions over the interval $\left[\frac{1}{2}, 1\right]$ are eigenfunctions of $\tilde{H}$ for all complex numbers.


Keywords. Riemann hypothesis, non-trivial Riemann zero, Pseudo hermitian operator, Berry-Keating operator

## 1 Introduction

One of the most attracting problems in Mathematics is Riemann hypothesis and it states that all the non-trivial zeros of Riemann zeta function

$$
\zeta(s)=\sum_{n \geq 1} \frac{1}{n^{s}}=\frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{t^{s-1}}{e^{t}-1} d t
$$

have the real part $\frac{1}{2}$.
One proposed way of proving Riemann hypothesis, known as Hilbert-Pólya conjecture, is to find an unbounded self-adjoint operator such that the imaginary part of Riemann zeros corresponds to the eigenvalues of this operator. It has been suggested [5] that the possible approach to solve Hilbert-Póplya conjecture lies in a quantization of the classical Hamiltonian $H=X P$, where $P$ is the canonical momentum operator associated with position operator $X$. Inspired of the previous works, an operator $\tilde{H}$ similar to quantum analogy was proposed [4] and asymptotic analysis was performed [3]. However, the momentum operator $P$ in this case has no self-adjoint extension [2].

## 2 Hurwitz Zeta functions

The Hurwitz zeta function is originally defined for complex arguments $s$ with $\mathcal{R}(s)>$ 1 and $x>0$ by

$$
\zeta(s, x)=\sum_{n \geq 0} \frac{1}{(n+x)^{s}}
$$

Hurwitz zeta function has an analytic continuation on $\mathbb{C}$ with the only singularity $s=1$ and it can be represented $[1,10]$

$$
\begin{equation*}
\zeta(s, x)=-\frac{\Gamma(1-s)}{2 \pi i} \int_{\mathcal{C}} \frac{\xi^{s-1} e^{x \xi}}{e^{\xi}-1} d \xi, \quad s \neq 1 \tag{1}
\end{equation*}
$$

where $\mathcal{C}$ is the contour around the negative real axis and it starts at $-\infty$, encircles the origin once in the positive direction without enclosing any of the points $\xi= \pm 2 n i, \pm 4 n i, \ldots$ and returns to $-\infty[10]$. The famous Riemann zeta function is the special case of Hurwitz zeta function with $x=1$

$$
\zeta(s, 1)=\zeta(s)
$$

and also with $x=\frac{1}{2}$

$$
\begin{equation*}
\zeta\left(s, \frac{1}{2}\right)=\left(2^{s}-1\right) \zeta(s) \tag{2}
\end{equation*}
$$

The partial derivative of Hurwitz zeta function with respect to $x$

$$
\begin{equation*}
\frac{\partial \zeta(s, x)}{\partial x}=-s \zeta(s+1, x), \quad s \neq 0,1 ; x>0 \tag{3}
\end{equation*}
$$

Let $s$ denote a point in critical strip $0<\mathcal{R}(s)<1$ and we define a cutoff function $\zeta_{s}$ of Hurwitz function on interval $\left[\frac{1}{2}, 1\right]$

$$
\begin{equation*}
\zeta_{s}(x):=\zeta(s, x), \quad \frac{1}{2} \leq x \leq 1 \tag{4}
\end{equation*}
$$

The functions $\zeta_{s}$ is continuous and even smooth on the closed interval $\left[\frac{1}{2}, 1\right]$ and have finite norm and belong to the Hilbert space, $\zeta_{s} \in L^{2}\left[\frac{1}{2}, 1\right]$. If a complex number $z$ is a non-trivial Riemann zero,

$$
\zeta_{z}(1)=0
$$

then the function $\zeta_{z}$ vanishes at $x=\frac{1}{2}$ from (2)

$$
\zeta_{z}\left(\frac{1}{2}\right)=0
$$

## 3 Operators in Hilbert space $L^{2}\left[\frac{1}{2}, 1\right]$

We confine our discussion over the region $\left[\frac{1}{2}, 1\right]$. Let $X$ be the position operator on $L^{2}\left[\frac{1}{2}, 1\right]$ and $P_{0}$ the momentum operator $P_{0}=-i \frac{d}{d x}$ densely defined on $L^{2}\left[\frac{1}{2}, 1\right]$ with domain

$$
\begin{equation*}
D\left(P_{0}\right)=\left\{f \in C^{\infty}\left[\frac{1}{2}, 1\right]: f\left(\frac{1}{2}\right)=f(1)\right\} \tag{5}
\end{equation*}
$$

[6] and it has a unique self-adjoint extension $\overline{P_{0}}=P$. The closure of the graph of $P_{0}$ is the graph of $P$

$$
\Gamma(P)=\overline{\Gamma\left(P_{0}\right)}=L^{2}\left[\frac{1}{2}, 1\right] \times L^{2}\left[\frac{1}{2}, 1\right]
$$

From the Stone's theorem on one-parameter unitary groups, $\left\{e^{i t P}\right\}_{t}$ is a strongly continuous one-parameter group of unitary operators on $L^{2}\left[\frac{1}{2}, 1\right]$. Thus, both operators $e^{ \pm i P}-I$ are bijective and bounded, and then the bounded inverses $\left(e^{ \pm i P}-I\right)^{-1}$ exist, from the bounded inverse theorem in functional analysis.

We construct an operator $\tilde{H}$ on $L^{2}\left[\frac{1}{2}, 1\right]$

$$
\tilde{H}=\left(e^{i P}-I\right)^{-1}(X P+P X)\left(e^{i P}-I\right)
$$

It holds obviously for $\frac{1}{2} \leq x \leq 1$ and $0<\mathcal{R}(s)<1$

$$
\begin{equation*}
(X P+P X) x^{-s}=i(2 s-1) x^{-s} \tag{6}
\end{equation*}
$$

The operator $\tilde{H}$ is pseudo hermitian [7, 8, 9], i.e. there exists an invertible and selfadjoint operator $\eta$ such that the adjoint operator $\tilde{H}^{\dagger}$ can be represented

$$
\tilde{H}^{\dagger}=\eta \tilde{H} \eta^{-1}
$$

Proposition 3.1. $\tilde{H}$ is pseudo hermitian and all eigenvalues are real.
Proof. Let $\eta=\left(e^{i P}-I\right)^{\dagger}\left(e^{i P}-I\right)$ and it is obviously self-adjoint, then

$$
\tilde{H}^{\dagger}=\eta \tilde{H} \eta^{-1}
$$

The operator $\eta$ is positive definite and self-adjoint by construction, the operator $\tilde{H}$ is hermitian in the new inner product with respect to $\eta$

$$
\langle\tilde{H} \phi, \varphi\rangle_{\eta}=\langle\tilde{H} \phi, \eta \varphi\rangle=\left\langle\phi, \eta \tilde{H} \eta^{-1} \eta \varphi\right\rangle=\langle\phi, \tilde{H} \varphi\rangle_{\eta}, \quad \phi, \varphi \in L^{2}\left[\frac{1}{2}, 1\right]
$$

Then all eigenvalues of $\tilde{H}$ are real.
Proposition 3.2. For any complex number $z$, the function $\zeta_{z}(x)-\zeta_{z}\left(\frac{1}{2}\right)$ is an eigenfunction of $\tilde{H}$ with eigenvalue $i(2 z-1)$

$$
\tilde{H}\left(\zeta_{z}(x)-\zeta_{z}\left(\frac{1}{2}\right)\right)=i(2 z-1)\left(\zeta_{z}(x)-\zeta_{z}\left(\frac{1}{2}\right)\right)
$$

Proof. We consider for non-trivial Riemann zeros $z$ the term

$$
\frac{1}{i P}\left(\frac{i P}{e^{i P}-I}\right) x^{-z}, \quad \frac{1}{2} \leq x \leq 1
$$

where $\frac{1}{i P}$ is defined as an integral operator on $L^{2}\left[\frac{1}{2}, 1\right]$ with boundary $x=\frac{1}{2}$

$$
\frac{1}{i P} f(x):=\int_{\frac{1}{2}}^{x} f(\xi) d \xi, \quad f \in L^{2}\left[\frac{1}{2}, 1\right]
$$

and $\frac{i P}{e^{P P}-I} f$ is defined as a Bernoulli generating function

$$
\frac{i P}{e^{i P}-I} f(x):=\sum_{n \geq 0} \frac{B_{n}}{n!}(i P)^{n} f(x), \quad f \in L^{2}\left[\frac{1}{2}, 1\right]
$$

Notice

$$
(i P)^{n} x^{-z}=\frac{\Gamma(-z+1)}{\Gamma(-z+1-n)} x^{-z-n}
$$

We use Hankel loop contour integral formula [10, 11]

$$
\frac{1}{\Gamma(-z+1-n)}=\frac{1}{2 \pi i} \int_{\mathcal{C}} \xi^{-(1-z-n)} e^{\xi} d \xi
$$

where the integration is done along the Hankel-Bromwich contour and it begins at $-\infty$, circles the origin once in the positive direction, and returns to $-\infty$

$$
\begin{aligned}
(i P)^{n} x^{-z} & =\Gamma(1-z)\left(\frac{1}{2 \pi i} \int_{\mathcal{C}} \xi^{-(1-z-n)} e^{\xi} d \xi\right) x^{-z-n} \\
& =\frac{\Gamma(1-z)}{2 \pi i} \int_{\mathcal{C}}\left(\frac{\xi}{x}\right)^{z+n-1} e^{\xi} d \frac{\xi}{x}
\end{aligned}
$$

and then

$$
\begin{aligned}
\frac{i P}{e^{i P}-I} x^{-z} & =\sum_{n \geq 0} \frac{B_{n}}{n!}(i P)^{n} x^{-z} \\
& =\frac{\Gamma(1-z)}{2 \pi i} \sum_{n \geq 0} \int_{\mathcal{C}} \frac{B_{n}}{n!}\left(\frac{\xi}{x}\right)^{z+n-1} e^{\xi} d \frac{\xi}{x}
\end{aligned}
$$

Let $u=\frac{\xi}{x}$ and $\mathcal{C}^{\prime}=\frac{\mathcal{C}}{x}$ is still a loop around the negative real axis. Uniform convergence on closed interval justifies the interchange of integral and sum. Then Hurwitz zeta function can be reproduced (1)

$$
\frac{i P}{e^{i P}-I} x^{-z}=\frac{\Gamma(1-z)}{2 \pi i} \int_{\mathcal{C}^{\prime}} \frac{u^{z}}{e^{u}-1} e^{u x} d u=-z \zeta(z+1, x)
$$

and the inverse

$$
\frac{i P}{e^{i P}-I} x^{-z}=-z \zeta(z+1, x)
$$

Then from (3)

$$
\begin{align*}
\frac{1}{i P}\left(\frac{i P}{e^{i P}-I}\right) x^{-z} & =\int_{\frac{1}{2}}^{x}-z \zeta(z+1, \xi) d \xi \\
& =\zeta(z, x)-\zeta\left(z, \frac{1}{2}\right) \tag{7}
\end{align*}
$$

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