



HAL
open science

A class of cutoff functions for non-trivial Riemann zeros

Yu Li

► **To cite this version:**

| Yu Li. A class of cutoff functions for non-trivial Riemann zeros. 2019. hal-01860885v6

HAL Id: hal-01860885

<https://hal.science/hal-01860885v6>

Preprint submitted on 15 Sep 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A class of cutoff functions for non-trivial Riemann zeros

Yu Li

Abstract

Abstract: In this paper we construct a pseudo ermitian operator \tilde{H} on the Hilbert space $L^2[\frac{1}{2}, 1]$ and show that the Hurwitz zeta functions over the interval $[\frac{1}{2}, 1]$ are eigenfunctions of \tilde{H} for all complex numbers.

Keywords. Riemann hypothesis, non-trivial Riemann zero, Pseudo hermitian operator, Berry-Keating operator

1 Introduction

One of the most attracting problems in Mathematics is Riemann hypothesis and it states that all the non-trivial zeros of Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt$$

have the real part $\frac{1}{2}$.

One proposed way of proving Riemann hypothesis, known as Hilbert-Pólya conjecture, is to find an unbounded self-adjoint operator such that the imaginary part of Riemann zeros corresponds to the eigenvalues of this operator. It has been suggested [5] that the possible approach to solve Hilbert-Pólya conjecture lies in a quantization of the classical Hamiltonian $H = XP$, where P is the canonical momentum operator associated with position operator X . Inspired of the previous works, an operator \tilde{H} similar to quantum analogy was proposed [4] and asymptotic analysis was performed [3]. However, the momentum operator P in this case has no self-adjoint extension [2].

2 Hurwitz Zeta functions

The Hurwitz zeta function is originally defined for complex arguments s with $\Re(s) > 1$ and $x > 0$ by

$$\zeta(s, x) = \sum_{n \geq 0} \frac{1}{(n+x)^s}$$

Hurwitz zeta function has an analytic continuation on \mathbb{C} with the only singularity $s = 1$ and it can be represented [1, 10]

$$\zeta(s, x) = -\frac{\Gamma(1-s)}{2\pi i} \int_{\mathcal{C}} \frac{\xi^{s-1} e^{x\xi}}{e^\xi - 1} d\xi, \quad s \neq 1 \quad (1)$$

where \mathcal{C} is the contour around the negative real axis and it starts at $-\infty$, encircles the origin once in the positive direction without enclosing any of the points $\xi = \pm 2ni, \pm 4ni, \dots$ and returns to $-\infty$ [10]. The famous Riemann zeta function is the special case of Hurwitz zeta function with $x = 1$

$$\zeta(s, 1) = \zeta(s)$$

and also with $x = \frac{1}{2}$

$$\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s) \quad (2)$$

The partial derivative of Hurwitz zeta function with respect to x

$$\frac{\partial \zeta(s, x)}{\partial x} = -s\zeta(s+1, x), \quad s \neq 0, 1; x > 0 \quad (3)$$

Let s denote a point in critical strip $0 < \mathcal{R}(s) < 1$ and we define a cutoff function ζ_s of Hurwitz function on interval $[\frac{1}{2}, 1]$

$$\zeta_s(x) := \zeta(s, x), \quad \frac{1}{2} \leq x \leq 1 \quad (4)$$

The functions ζ_s is continuous and even smooth on the closed interval $[\frac{1}{2}, 1]$ and have finite norm and belong to the Hilbert space, $\zeta_s \in L^2[\frac{1}{2}, 1]$. If a complex number z is a non-trivial Riemann zero,

$$\zeta_z(1) = 0$$

then the function ζ_z vanishes at $x = \frac{1}{2}$ from (2)

$$\zeta_z(\frac{1}{2}) = 0$$

3 Operators in Hilbert space $L^2[\frac{1}{2}, 1]$

We confine our discussion over the region $[\frac{1}{2}, 1]$. Let X be the position operator on $L^2[\frac{1}{2}, 1]$ and P_0 the momentum operator $P_0 = -i\frac{d}{dx}$ densely defined on $L^2[\frac{1}{2}, 1]$ with domain

$$D(P_0) = \left\{ f \in C^\infty[\frac{1}{2}, 1] : f(\frac{1}{2}) = f(1) \right\} \quad (5)$$

[6] and it has a unique self-adjoint extension $\overline{P_0} = P$. The closure of the graph of P_0 is the graph of P

$$\Gamma(P) = \overline{\Gamma(P_0)} = L^2[\frac{1}{2}, 1] \times L^2[\frac{1}{2}, 1]$$

From the Stone's theorem on one-parameter unitary groups, $\{e^{itP}\}_t$ is a strongly continuous one-parameter group of unitary operators on $L^2[\frac{1}{2}, 1]$. Thus, both operators $e^{\pm iP} - I$ are bijective and bounded, and then the bounded inverses $(e^{\pm iP} - I)^{-1}$ exist, from the bounded inverse theorem in functional analysis.

We construct an operator \tilde{H} on $L^2[\frac{1}{2}, 1]$

$$\tilde{H} = (e^{iP} - I)^{-1} (XP + PX) (e^{iP} - I)$$

It holds obviously for $\frac{1}{2} \leq x \leq 1$ and $0 < \mathcal{R}(s) < 1$

$$(XP + PX)x^{-s} = i(2s - 1)x^{-s} \quad (6)$$

The operator \tilde{H} is pseudo hermitian [7, 8, 9], i.e. there exists an invertible and self-adjoint operator η such that the adjoint operator \tilde{H}^\dagger can be represented

$$\tilde{H}^\dagger = \eta \tilde{H} \eta^{-1}$$

Proposition 3.1. *\tilde{H} is pseudo hermitian and all eigenvalues are real.*

Proof. Let $\eta = (e^{iP} - I)^\dagger (e^{iP} - I)$ and it is obviously self-adjoint, then

$$\tilde{H}^\dagger = \eta \tilde{H} \eta^{-1}$$

The operator η is positive definite and self-adjoint by construction, the operator \tilde{H} is hermitian in the new inner product with respect to η

$$\langle \tilde{H}\phi, \varphi \rangle_\eta = \langle \tilde{H}\phi, \eta\varphi \rangle = \langle \phi, \eta \tilde{H} \eta^{-1} \eta\varphi \rangle = \langle \phi, \tilde{H}\eta\varphi \rangle_\eta, \quad \phi, \varphi \in L^2[\frac{1}{2}, 1]$$

Then all eigenvalues of \tilde{H} are real. □

Proposition 3.2. *For any complex number z , the function $\zeta_z(x) - \zeta_z(\frac{1}{2})$ is an eigenfunction of \tilde{H} with eigenvalue $i(2z - 1)$*

$$\tilde{H} \left(\zeta_z(x) - \zeta_z\left(\frac{1}{2}\right) \right) = i(2z - 1) \left(\zeta_z(x) - \zeta_z\left(\frac{1}{2}\right) \right)$$

Proof. We consider for non-trivial Riemann zeros z the term

$$\frac{1}{iP} \left(\frac{iP}{e^{iP} - I} \right) x^{-z}, \quad \frac{1}{2} \leq x \leq 1$$

where $\frac{1}{iP}$ is defined as an integral operator on $L^2[\frac{1}{2}, 1]$ with boundary $x = \frac{1}{2}$

$$\frac{1}{iP} f(x) := \int_{\frac{1}{2}}^x f(\xi) d\xi, \quad f \in L^2[\frac{1}{2}, 1]$$

and $\frac{iP}{e^{iP}-I}f$ is defined as a Bernoulli generating function

$$\frac{iP}{e^{iP}-I}f(x) := \sum_{n \geq 0} \frac{B_n}{n!} (iP)^n f(x), \quad f \in L^2\left[\frac{1}{2}, 1\right]$$

Notice

$$(iP)^n x^{-z} = \frac{\Gamma(-z+1)}{\Gamma(-z+1-n)} x^{-z-n}$$

We use Hankel loop contour integral formula [10, 11]

$$\frac{1}{\Gamma(-z+1-n)} = \frac{1}{2\pi i} \int_{\mathcal{C}} \xi^{-(1-z-n)} e^{\xi} d\xi$$

where the integration is done along the Hankel-Bromwich contour and it begins at $-\infty$, circles the origin once in the positive direction, and returns to $-\infty$

$$\begin{aligned} (iP)^n x^{-z} &= \Gamma(1-z) \left(\frac{1}{2\pi i} \int_{\mathcal{C}} \xi^{-(1-z-n)} e^{\xi} d\xi \right) x^{-z-n} \\ &= \frac{\Gamma(1-z)}{2\pi i} \int_{\mathcal{C}} \left(\frac{\xi}{x} \right)^{z+n-1} e^{\xi} d\frac{\xi}{x} \end{aligned}$$

and then

$$\begin{aligned} \frac{iP}{e^{iP}-I} x^{-z} &= \sum_{n \geq 0} \frac{B_n}{n!} (iP)^n x^{-z} \\ &= \frac{\Gamma(1-z)}{2\pi i} \sum_{n \geq 0} \int_{\mathcal{C}} \frac{B_n}{n!} \left(\frac{\xi}{x} \right)^{z+n-1} e^{\xi} d\frac{\xi}{x} \end{aligned}$$

Let $u = \frac{\xi}{x}$ and $\mathcal{C}' = \frac{\mathcal{C}}{x}$ is still a loop around the negative real axis. Uniform convergence on closed interval justifies the interchange of integral and sum. Then Hurwitz zeta function can be reproduced (1)

$$\frac{iP}{e^{iP}-I} x^{-z} = \frac{\Gamma(1-z)}{2\pi i} \int_{\mathcal{C}'} \frac{u^z}{e^u - 1} e^{ux} du = -z\zeta(z+1, x)$$

and the inverse

$$\frac{iP}{e^{iP}-I} x^{-z} = -z\zeta(z+1, x)$$

Then from (3)

$$\begin{aligned} \frac{1}{iP} \left(\frac{iP}{e^{iP}-I} \right) x^{-z} &= \int_{\frac{1}{2}}^x -z\zeta(z+1, \xi) d\xi \\ &= \zeta(z, x) - \zeta\left(z, \frac{1}{2}\right) \end{aligned} \tag{7}$$

□

References

- [1] D. H. Bailey, J. M. Borwein, N. J. Calkin, R. Girgensohn, D. R. Luke, and V. Moll. *Experimental Mathematics in Action*. A K Peters/CRC Press, May 2007.
- [2] J. Belissard. Comment on "Hamiltonian for the Zeros of the Riemann Zeta Function".
- [3] C. M. Bender and D. C. Brody. Asymptotic analysis on a pseudo-Hermitian Riemann-zeta Hamiltonian. *Journal of Physics A: Mathematical and Theoretical*, 51(13), Mar. 2018.
- [4] C. M. Bender, D. C. Brody, and M. P. Müller. Hamiltonian for the Zeros of the Riemann Zeta Function. *PHYSICAL REVIEW LETTERS*, 2017.
- [5] M. V. Berry and J. P. Keating. $H = XP$ and the Riemann zeros. *Symmetry and trace formula: Chaos and disorder*, 1999.
- [6] B. C. Hall. *Quantum Theory for Mathematicians (Graduate Texts in Mathematics)*. Springer, June 2013.
- [7] G. W. Mackey. *Commutative Banach Algebras*. Fasciculo Publicado Pelo Instituto de Matematica Pura e Aplicada do Conselho Nacional de Pequisas, 1959.
- [8] A. Mostafazadeh. Pseudo-Hermiticity versus PT symmetry: The necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian. *Journal of Mathematical Physics*, 43(205), 2002.
- [9] A. Mostafazadeh. Pseudo-supersymmetric quantum mechanics and isospectral pseudo-Hermitian Hamiltonians. *Nuclear Physics B*, 640(3):419–434, Sept. 2002.
- [10] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST Handbook of Mathematical Functions Paperback and CD-ROM*. Cambridge University Press, May 2010.
- [11] T. Schmelzer and L. N. Trefethen. Computing the Gamma function using contour integrals and rational approximation. *SIAM journal of numerical analysis*, 45(2):558–571, 2007.