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A class of cutoff functions for non-trivial Riemann zeros

Yu Li

Abstract

Abstract: In this paper we construct a pseudo hermitian operator \tilde{H} and introduce a class of cutoff functions ζ_s , then we show that the cutoff functions are eigenfunction of \tilde{H} for all non-trivial Riemann zeros.

Keywords. Riemann hypothesis, non-trivial Riemann zero, Pseudo hermitian operator, Berry-Keating operator

1 Introduction

One of the most attracting problems in Mathematics is Riemann hypothesis and it states that all the non-trivial zeros of Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt$$

have the real part $\frac{1}{2}$.

One proposed way of proving Riemann hypothesis, known as Hilbert-Pólya conjecture, is to find an unbounded self-adjoint operator such that the imaginary part of Riemann zeros corresponds to the eigenvalues of this operator. It has been suggested [5] that the possible approach to solve Hilbert-Pólya conjecture lies in a quantization of the classical Hamiltonian $H = XP$, where P is the canonical momentum operator associated with position operator X . Inspired of the previous works, an operator \tilde{H} similar to quantum analogy was proposed [4] and asymptotic analysis was performed [3]. However, the momentum operator P in this case has no self-adjoint extension [2].

In the whole paper we study a cutoff function ζ_s instead of raw Hurwitz zeta function, which is the main difference from that in [3, 4] and construct a pseudo hermitian operator \tilde{H} on the Hilbert space $\mathcal{H} = L^2[\frac{1}{2}, 1]$. Then we show that the cutoff function is eigenfunction of \tilde{H} for non-trivial Riemann zeros.

2 A cutoff function of Hurwitz zeta function

The Hurwitz zeta function is originally defined for complex arguments s with $\Re(s) > 1$ and $x > 0$ by

$$\zeta(s, x) = \sum_{n \geq 0} \frac{1}{(n+x)^s}$$

Hurwitz zeta function has an analytic continuation on \mathbb{C} with the only singularity $s = 1$ and it can be represented [1, 10]

$$\zeta(s, x) = -\frac{\Gamma(1-s)}{2\pi i} \int_{\mathcal{C}} \frac{\xi^{s-1} e^{x\xi}}{e^\xi - 1} d\xi, \quad s \neq 1 \quad (1)$$

where \mathcal{C} is the contour around the negative real axis and it starts at $-\infty$, encircles the origin once in the positive direction without enclosing any of the points $\xi = \pm 2ni, \pm 4ni, \dots$ and returns to $-\infty$ [10]. The famous Riemann zeta function is the special case of Hurwitz zeta function with $x = 1$

$$\zeta(s, 1) = \zeta(s)$$

and also with $x = \frac{1}{2}$

$$\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s) \quad (2)$$

The partial derivative of Hurwitz zeta function with respect to x

$$\frac{\partial \zeta(s, x)}{\partial x} = -s\zeta(s+1, x), \quad s \neq 0, 1; x > 0 \quad (3)$$

Let s denote a point in critical strip $0 < \mathcal{R}(s) < 1$ and we define a cutoff function ζ_s of Hurwitz function on interval $[\frac{1}{2}, 1]$

$$\zeta_s(x) := \zeta(s, x), \quad \frac{1}{2} \leq x \leq 1 \quad (4)$$

The functions ζ_s is continuous and even smooth on the closed interval $[\frac{1}{2}, 1]$ and have finite norm and belong to the Hilbert space, $\zeta_s \in L^2[\frac{1}{2}, 1]$. If a complex number z is a non-trivial Riemann zero,

$$\zeta_z(1) = 0$$

then the function ζ_z vanishes at $x = \frac{1}{2}$ from (2)

$$\zeta_z(\frac{1}{2}) = 0$$

3 Non-trivial Riemann zeros

Let X be the position operator on $L^2[\frac{1}{2}, 1]$ and P_0 the momentum operator $P_0 = -i\frac{d}{dx}$ densely defined on $L^2[\frac{1}{2}, 1]$ with domain

$$D(P_0) = \left\{ f \in C^\infty[\frac{1}{2}, 1] : f(\frac{1}{2}) = f(1) \right\} \quad (5)$$

Integrating by part, the boundary terms cancels, it shows that the momentum operator P_0 is symmetric. The functions $\{e^{2\pi in(2x-1)} : \frac{1}{2} \leq x \leq 1\}_n$ form an orthogonal basis for $L^2[\frac{1}{2}, 1]$, consisting of eigenfunction for P_0 with eigenvalues $4\pi n$. Thus, the momentum

operator P_0 is essentially self-adjoint [6] and it has a unique self-adjoint extension $\overline{P_0} = P$. The closure of the graph of P_0 is the graph of P

$$\Gamma(P) = \overline{\Gamma(P_0)} = L^2[\frac{1}{2}, 1] \times L^2[\frac{1}{2}, 1]$$

From the Stone's theorem on one-parameter unitary groups, $\{e^{itP}\}_t$ is a strongly continuous one-parameter group of unitary operators on $L^2[\frac{1}{2}, 1]$. Thus, both operators $e^{\pm iP} - I$ are bijective and bounded, and then the bounded inverses $(e^{\pm iP} - I)^{-1}$ exist, from the bounded inverse theorem in functional analysis.

We construct an operator \tilde{H} on $L^2[\frac{1}{2}, 1]$

$$\tilde{H} = (e^{iP} - I)^{-1} (XP + PX) (e^{iP} - I)$$

It holds obviously for $\frac{1}{2} \leq x \leq 1$ and $0 < \mathcal{R}(s) < 1$

$$(XP + PX)x^{-s} = i(2s - 1)x^{-s} \quad (6)$$

Here the operator $H_{\text{BK}} = XP + PX$ is called Berry-Keating operator [5].

The operator \tilde{H} is pseudo hermitian [7, 8, 9], i.e. there exists an invertible and self-adjoint operator η such that the adjoint operator \tilde{H}^\dagger can be represented

$$\tilde{H}^\dagger = \eta \tilde{H} \eta^{-1}$$

Proposition 3.1. *\tilde{H} is pseudo hermitian and all eigenvalues are real.*

Proof. Let $\eta = (e^{iP} - I)^\dagger (e^{iP} - I)$ and it is obviously self-adjoint, then

$$\tilde{H}^\dagger = \eta \tilde{H} \eta^{-1}$$

The operator η is positive definite and self-adjoint by construction, the operator \tilde{H} is hermitian in the new inner product with respect to η

$$\langle \tilde{H}\phi, \varphi \rangle_\eta = \langle \tilde{H}\phi, \eta\varphi \rangle = \langle \phi, \eta \tilde{H} \eta^{-1} \eta\varphi \rangle = \langle \phi, \tilde{H}\eta\varphi \rangle_\eta, \quad \phi, \varphi \in L^2[\frac{1}{2}, 1]$$

Then all eigenvalues of \tilde{H} are real. □

Proposition 3.2. *For every non-trivial Riemann zero z , the cutoff function ζ_z is an eigenfunction of \tilde{H} with eigenvalue $i(2z - 1)$.*

Proof. We consider for non-trivial Riemann zeros z the term

$$\frac{1}{iP} \left(\frac{iP}{e^{iP} - I} \right) x^{-z}, \quad \frac{1}{2} \leq x \leq 1$$

where $\frac{1}{iP}$ is defined as an integral operator on $L^2[\frac{1}{2}, 1]$ with boundary $x = \frac{1}{2}$

$$\frac{1}{iP} f(x) := \int_{\frac{1}{2}}^x f(\xi) d\xi, \quad f \in L^2[\frac{1}{2}, 1]$$

and $\frac{iP}{e^{iP}-I}f$ is defined as a Bernoulli generating function

$$\frac{iP}{e^{iP}-I}f(x) := \sum_{n \geq 0} \frac{B_n}{n!} (iP)^n f(x), \quad f \in L^2\left[\frac{1}{2}, 1\right]$$

Notice

$$(iP)^n x^{-z} = \frac{\Gamma(-z+1)}{\Gamma(-z+1-n)} x^{-z-n}$$

We use Hankel loop contour integral formula [10, 11]

$$\frac{1}{\Gamma(-z+1-n)} = \frac{1}{2\pi i} \int_{\mathcal{C}} \xi^{-(1-z-n)} e^{\xi} d\xi$$

where the integration is done along the Hankel-Bromwich contour and it begins at $-\infty$, circles the origin once in the positive direction, and returns to $-\infty$

$$\begin{aligned} (iP)^n x^{-z} &= \Gamma(1-z) \left(\frac{1}{2\pi i} \int_{\mathcal{C}} \xi^{-(1-z-n)} e^{\xi} d\xi \right) x^{-z-n} \\ &= \frac{\Gamma(1-z)}{2\pi i} \int_{\mathcal{C}} \left(\frac{\xi}{x} \right)^{z+n-1} e^{\xi} d\frac{\xi}{x} \end{aligned}$$

and then

$$\begin{aligned} \frac{iP}{e^{iP}-I} x^{-z} &= \sum_{n \geq 0} \frac{B_n}{n!} (iP)^n x^{-z} \\ &= \frac{\Gamma(1-z)}{2\pi i} \sum_{n \geq 0} \int_{\mathcal{C}} \frac{B_n}{n!} \left(\frac{\xi}{x} \right)^{z+n-1} e^{\xi} d\frac{\xi}{x} \end{aligned}$$

Let $u = \frac{\xi}{x}$ and $\mathcal{C}' = \frac{\mathcal{C}}{x}$ is still a loop around the negative real axis. Uniform convergence on closed interval justifies the interchange of integral and sum. Then Hurwitz zeta function can be reproduced (1)

$$\frac{iP}{e^{iP}-I} x^{-z} = \frac{\Gamma(1-z)}{2\pi i} \int_{\mathcal{C}'} \frac{u^z}{e^u-1} e^{ux} du = -z\zeta(z+1, x)$$

Then from (3)

$$\begin{aligned} \frac{1}{iP} \left(\frac{iP}{e^{iP}-I} \right) x^{-z} &= \int_{\frac{1}{2}}^x -z\zeta(z+1, \xi) d\xi \\ &= \zeta(z, x) - \zeta\left(z, \frac{1}{2}\right) = \zeta_z(x) \end{aligned}$$

In general $\frac{1}{iP} \left(\frac{iP}{e^{iP}-I} \right) \neq \frac{1}{e^{iP}-I}$, but the function ζ_z takes the value 0 at $x = \frac{1}{2}$ for non-trivial Riemann zero z , then

$$\left(\frac{1}{e^{iP}-I} \right) x^{-z} = \frac{1}{iP} \left(\frac{iP}{e^{iP}-I} \right) x^{-z} = \zeta_z(x)$$

Combining (6), we reach

$$\tilde{H}\zeta_z = i(2z - 1)\zeta_z$$

□

Remark. For a complex number s such that $\zeta_s(\frac{1}{2}) \neq 0$, it holds

$$\tilde{H}\zeta_s(x) = i(2s - 1) \left[\zeta_s(x) + \zeta_s\left(\frac{1}{2}\right) \right]$$

In this case, the second term $\zeta_s(\frac{1}{2})$ remains, which implies ζ_s is not an eigenfunction of \tilde{H} .

In summary, we identify a Hilbert space $L^2[\frac{1}{2}, 1]$ spanned by all pairwise orthogonal eigenfunctions $\{e^{2\pi i n(2x-1)} : \frac{1}{2} \leq x \leq 1\}_n$ of \tilde{H} that satisfies the boundary condition

$$f\left(\frac{1}{2}\right) = f(1)$$

Moreover, we establish the essential self-adjointness of \tilde{H} on this space and demonstrate that the cutoff functions equipped with non-trivial Riemann zeros are eigenfunctions of \tilde{H} with eigenvalues $i(2z - 1)$.

Theorem 3.3. *The Riemann zeta function has its non-trivial zeros only at the complex numbers with real part $\frac{1}{2}$.*

Corollary. *The Hurwitz zeta function*

$$\zeta\left(s, \frac{1}{2}\right)$$

has its non-trivial zeros only at the complex numbers with real part $\frac{1}{2}$.

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