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# A class of cutoff functions for non-trivial Riemann zeros

Yu Li

## Abstract

Abstract: In this paper we construct a pseudo hermitian operator  $\tilde{H}$  and introduce a class of cutoff functions  $\zeta_s$ , then we show that the cutoff functions are eigenfunction of  $\tilde{H}$  for all non-trivial Riemann zeros.

**Keywords.** Riemann hypothesis, non-trivial Riemann zero, Pseudo hermitian operator, Berry-Keating operator

## 1 Introduction

One of the most attracting problems in Mathematics is Riemann hypothesis and it states that all the non-trivial zeros of Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt$$

have the real part  $\frac{1}{2}$ .

One proposed way of proving Riemann hypothesis, known as Hilbert-Pólya conjecture, is to find an unbounded self-adjoint operator such that the imaginary part of Riemann zeros corresponds to the eigenvalues of this operator. It has been suggested [5] that the possible approach to solve Hilbert-Pólya conjecture lies in a quantization of the classical Hamiltonian  $H = XP$ , where  $P$  is the canonical momentum operator associated with position operator  $X$ . Inspired of the previous works, an operator  $\tilde{H}$  similar to quantum analogy was proposed [4] and asymptotic analysis was performed [3]. However, the momentum operator  $P$  in this case has no self-adjoint extension [2].

In the whole paper we study a cutoff function  $\zeta_s$  instead of raw Hurwitz zeta function, which is the main difference from that in [3, 4] and construct a pseudo hermitian operator  $\tilde{H}$  on the Hilbert space  $\mathcal{H} = L^2[\frac{1}{2}, 1]$ . Then we show that the cutoff function is eigenfunction of  $\tilde{H}$  for non-trivial Riemann zeros.

## 2 A cutoff function of Hurwitz zeta function

The Hurwitz zeta function is originally defined for complex arguments  $s$  with  $\Re(s) > 1$  and  $x > 0$  by

$$\zeta(s, x) = \sum_{n \geq 0} \frac{1}{(n+x)^s}$$

Hurwitz zeta function has an analytic continuation on  $\mathbb{C}$  with the only singularity  $s = 1$  and it can be represented [1, 10]

$$\zeta(s, x) = -\frac{\Gamma(1-s)}{2\pi i} \int_{\mathcal{C}} \frac{\xi^{s-1} e^{x\xi}}{e^\xi - 1} d\xi, \quad s \neq 1 \quad (1)$$

where  $\mathcal{C}$  is the contour around the negative real axis and it starts at  $-\infty$ , encircles the origin once in the positive direction without enclosing any of the points  $\xi = \pm 2ni, \pm 4ni, \dots$  and returns to  $-\infty$  [10]. The famous Riemann zeta function is the special case of Hurwitz zeta function with  $x = 1$

$$\zeta(s, 1) = \zeta(s)$$

and also with  $x = \frac{1}{2}$

$$\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s) \quad (2)$$

The partial derivative of Hurwitz zeta function with respect to  $x$

$$\frac{\partial \zeta(s, x)}{\partial x} = -s\zeta(s+1, x), \quad s \neq 0, 1; x > 0 \quad (3)$$

Let  $s$  denote a point in critical strip  $0 < \mathcal{R}(s) < 1$  and we define a cutoff function  $\zeta_s$  of Hurwitz function on interval  $[\frac{1}{2}, 1]$

$$\zeta_s(x) := \zeta(s, x), \quad \frac{1}{2} \leq x \leq 1 \quad (4)$$

The functions  $\zeta_s$  is continuous and even smooth on the closed interval  $[\frac{1}{2}, 1]$  and have finite norm and belong to the Hilbert space,  $\zeta_s \in L^2[\frac{1}{2}, 1]$ . If a complex number  $z$  is a non-trivial Riemann zero,

$$\zeta_z(1) = 0$$

then the function  $\zeta_z$  vanishes at  $x = \frac{1}{2}$  from (2)

$$\zeta_z(\frac{1}{2}) = 0$$

### 3 Non-trivial Riemann zeros

Let  $X$  be the position operator on  $L^2[\frac{1}{2}, 1]$  and  $P_0$  the momentum operator  $P_0 = -i\frac{d}{dx}$  densely defined on  $L^2[\frac{1}{2}, 1]$  with domain

$$D(P_0) = \left\{ f \in C^\infty[\frac{1}{2}, 1] : f(\frac{1}{2}) = f(1) \right\} \quad (5)$$

Integrating by part, the boundary terms cancels, it shows that the momentum operator  $P_0$  is symmetric. The functions  $\{e^{2\pi in(2x-1)} : \frac{1}{2} \leq x \leq 1\}_n$  form an orthogonal basis for  $L^2[\frac{1}{2}, 1]$ , consisting of eigenfunction for  $P_0$  with eigenvalues  $4\pi n$ . Thus, the momentum

operator  $P_0$  is essentially self-adjoint [6] and it has a unique self-adjoint extension  $\overline{P_0} = P$ . The closure of the graph of  $P_0$  is the graph of  $P$

$$\Gamma(P) = \overline{\Gamma(P_0)} = L^2[\frac{1}{2}, 1] \times L^2[\frac{1}{2}, 1]$$

From the Stone's theorem on one-parameter unitary groups,  $\{e^{itP}\}_t$  is a strongly continuous one-parameter group of unitary operators on  $L^2[\frac{1}{2}, 1]$ . Thus, both operators  $e^{\pm iP} - I$  are bijective and bounded, and then the bounded inverses  $(e^{\pm iP} - I)^{-1}$  exist, from the bounded inverse theorem in functional analysis.

We construct an operator  $\tilde{H}$  on  $L^2[\frac{1}{2}, 1]$

$$\tilde{H} = (e^{iP} - I)^{-1} (XP + PX) (e^{iP} - I)$$

It holds obviously for  $\frac{1}{2} \leq x \leq 1$  and  $0 < \mathcal{R}(s) < 1$

$$(XP + PX)x^{-s} = i(2s - 1)x^{-s} \quad (6)$$

Here the operator  $H_{\text{BK}} = XP + PX$  is called Berry-Keating operator [5].

The operator  $\tilde{H}$  is pseudo hermitian [7, 8, 9], i.e. there exists an invertible and self-adjoint operator  $\eta$  such that the adjoint operator  $\tilde{H}^\dagger$  can be represented

$$\tilde{H}^\dagger = \eta \tilde{H} \eta^{-1}$$

**Proposition 3.1.**  *$\tilde{H}$  is pseudo hermitian and all eigenvalues are real.*

*Proof.* Let  $\eta = (e^{iP} - I)^\dagger (e^{iP} - I)$  and it is obviously self-adjoint, then

$$\tilde{H}^\dagger = \eta \tilde{H} \eta^{-1}$$

The operator  $\eta$  is positive definite and self-adjoint by construction, the operator  $\tilde{H}$  is hermitian in the new inner product with respect to  $\eta$

$$\langle \tilde{H}\phi, \varphi \rangle_\eta = \langle \tilde{H}\phi, \eta\varphi \rangle = \langle \phi, \eta \tilde{H} \eta^{-1} \eta\varphi \rangle = \langle \phi, \tilde{H}\eta\varphi \rangle_\eta, \quad \phi, \varphi \in L^2[\frac{1}{2}, 1]$$

Then all eigenvalues of  $\tilde{H}$  are real. □

**Proposition 3.2.** *For every non-trivial Riemann zero  $z$ , the cutoff function  $\zeta_z$  is an eigenfunction of  $\tilde{H}$  with eigenvalue  $i(2z - 1)$ .*

*Proof.* We consider for non-trivial Riemann zeros  $z$  the term

$$\frac{1}{iP} \left( \frac{iP}{e^{iP} - I} \right) x^{-z}, \quad \frac{1}{2} \leq x \leq 1$$

where  $\frac{1}{iP}$  is defined as an integral operator on  $L^2[\frac{1}{2}, 1]$  with boundary  $x = \frac{1}{2}$

$$\frac{1}{iP} f(x) := \int_{\frac{1}{2}}^x f(\xi) d\xi, \quad f \in L^2[\frac{1}{2}, 1]$$

and  $\frac{iP}{e^{iP}-I}f$  is defined as a Bernoulli generating function

$$\frac{iP}{e^{iP}-I}f(x) := \sum_{n \geq 0} \frac{B_n}{n!} (iP)^n f(x), \quad f \in L^2[\frac{1}{2}, 1]$$

Notice

$$(iP)^n x^{-z} = \frac{\Gamma(-z+1)}{\Gamma(-z+1-n)} x^{-z-n}$$

We use Hankel loop contour integral formula [10, 11]

$$\frac{1}{\Gamma(-z+1-n)} = \frac{1}{2\pi i} \int_{\mathcal{C}} \xi^{-(1-z-n)} e^{\xi} d\xi$$

where the integration is done along the Hankel-Bromwich contour and it begins at  $-\infty$ , circles the origin once in the positive direction, and returns to  $-\infty$

$$\begin{aligned} (iP)^n x^{-z} &= \Gamma(1-z) \left( \frac{1}{2\pi i} \int_{\mathcal{C}} \xi^{-(1-z-n)} e^{\xi} d\xi \right) x^{-z-n} \\ &= \frac{\Gamma(1-z)}{2\pi i} \int_{\mathcal{C}} \left( \frac{\xi}{x} \right)^{z+n-1} e^{\xi} d\frac{\xi}{x} \end{aligned}$$

and then

$$\begin{aligned} \frac{iP}{e^{iP}-I} x^{-z} &= \sum_{n \geq 0} \frac{B_n}{n!} (iP)^n x^{-z} \\ &= \frac{\Gamma(1-z)}{2\pi i} \sum_{n \geq 0} \int_{\mathcal{C}} \frac{B_n}{n!} \left( \frac{\xi}{x} \right)^{z+n-1} e^{\xi} d\frac{\xi}{x} \end{aligned}$$

Let  $u = \frac{\xi}{x}$  and  $\mathcal{C}' = \frac{\mathcal{C}}{x}$  is still a loop around the negative real axis. Uniform convergence on closed interval justifies the interchange of integral and sum. Then Hurwitz zeta function can be reproduced (1)

$$\frac{iP}{e^{iP}-I} x^{-z} = \frac{\Gamma(1-z)}{2\pi i} \int_{\mathcal{C}'} \frac{u^z}{e^u-1} e^{ux} du = -z\zeta(z+1, x)$$

Then from (3)

$$\begin{aligned} \frac{1}{iP} \left( \frac{iP}{e^{iP}-I} \right) x^{-z} &= \int_{\frac{1}{2}}^x -z\zeta(z+1, \xi) d\xi \\ &= \zeta(z, x) - \zeta(z, \frac{1}{2}) = \zeta_z(x) \end{aligned}$$

In general  $\frac{1}{iP} \left( \frac{iP}{e^{iP}-I} \right) \neq \frac{1}{e^{iP}-I}$ , but the function  $\zeta_z$  takes the value 0 at  $x = \frac{1}{2}$  for non-trivial Riemann zero  $z$ , then

$$\left( \frac{1}{e^{iP}-I} \right) x^{-z} = \frac{1}{iP} \left( \frac{iP}{e^{iP}-I} \right) x^{-z} = \zeta_z(x)$$

Combining (6), we reach

$$\tilde{H}\zeta_z = i(2z - 1)\zeta_z$$

□

**Remark.** For a complex number  $s$  such that  $\zeta_s(\frac{1}{2}) \neq 0$ , it holds

$$\tilde{H}\zeta_s(x) = i(2s - 1) \left[ \zeta_s(x) + \zeta_s(\frac{1}{2}) \right]$$

In this case, the second term  $\zeta_s(\frac{1}{2})$  remains, which implies  $\zeta_s$  is not an eigenfunction of  $\tilde{H}$ .

In summary, we identify a Hilbert space  $L^2[\frac{1}{2}, 1]$  spanned by all pairwise orthogonal eigenfunctions  $\{e^{2\pi i n(2x-1)} : \frac{1}{2} \leq x \leq 1\}_n$  of  $\tilde{H}$  that satisfies the boundary condition

$$f(\frac{1}{2}) = f(1)$$

Moreover, we establish the essential self-adjointness of  $\tilde{H}$  on this space and demonstrate that the cutoff functions equipped with non-trivial Riemann zeros are eigenfunctions of  $\tilde{H}$  with eigenvalues  $i(2z - 1)$ .

**Theorem 3.3.** *The Riemann zeta function has its non-trivial zeros only at the complex numbers with real part  $\frac{1}{2}$ .*

**Corollary.** *The Hurwitz zeta function*

$$\zeta(s, \frac{1}{2})$$

*has its non-trivial zeros only at the complex numbers with real part  $\frac{1}{2}$ .*

## References

- [1] D. H. Bailey, J. M. Borwein, N. J. Calkin, R. Girgensohn, D. R. Luke, and V. Moll. *Experimental Mathematics in Action*. A K Peters/CRC Press, May 2007.
- [2] J. Belissard. Comment on "Hamiltonian for the Zeros of the Riemann Zeta Function".
- [3] C. M. Bender and D. C. Brody. Asymptotic analysis on a pseudo-Hermitian Riemann-zeta Hamiltonian. *Journal of Physics A: Mathematical and Theoretical*, 51(13), Mar. 2018.
- [4] C. M. Bender, D. C. Brody, and M. P. Müller. Hamiltonian for the Zeros of the Riemann Zeta Function. *PHYSICAL REVIEW LETTERS*, 2017.

- [5] M. V. Berry and J. P. Keating.  $H = XP$  and the Riemann zeros. *Symmetry and trace formula: Chaos and disorder*, 1999.
- [6] B. C. Hall. *Quantum Theory for Mathematicians (Graduate Texts in Mathematics)*. Springer, June 2013.
- [7] G. W. Mackey. *Commutative Banach Algebras*. Fasciculo Publicado Pelo Instituto de Matematica Pura e Aplicada do Conselho Nacional de Pequisas, 1959.
- [8] A. Mostafazadeh. Pseudo-Hermiticity versus PT symmetry: The necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian. *Journal of Mathematical Physics*, 43(205), 2002.
- [9] A. Mostafazadeh. Pseudo-supersymmetric quantum mechanics and isospectral pseudo-Hermitian Hamiltonians. *Nuclear Physics B*, 640(3):419–434, Sept. 2002.
- [10] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST Handbook of Mathematical Functions Paperback and CD-ROM*. Cambridge University Press, May 2010.
- [11] T. Schmelzer and L. N. Trefethen. Computing the Gamma function using contour integrals and rational approximation. *SIAM journal of numerical analysis*, 45(2):558–571, 2007.