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To cite this version:
Yu Li. A pseudo hermitian operator and non-trivial Riemann zeros. 2018. hal-01860885v3

HAL Id: hal-01860885
https://hal.science/hal-01860885v3
Preprint submitted on 27 Aug 2018 (v3), last revised 15 Sep 2019 (v6)
A pseudo hermitian operator and non-trivial Riemann zeros

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Abstract

Abstract: In this paper we construct a pseudo hermitian operator and reveal a connection between its eigenvalues and the non-trivial Riemann zeros.

Keywords. Riemann hypothesis, non-trivial Riemann zero, Pseudo hermitian operator

1 Introduction

One of the most attracting problems in Mathematics is Riemann hypothesis and it states that all the non-trivial zeros of Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt$$

have the real part $\frac{1}{2}$.

One proposed way of proving Riemann hypothesis, known as Hilbert-Pólya conjecture, is to find an unbounded self-adjoint operator such that the imaginary part of Riemann zeros corresponds to the eigenvalues of this operator. It has been suggested [4] that the possible approach to solve Hilbert-Pólya conjecture lies in a quantization of the classical Hamiltonian $H = XP$, where $P$ is the canonical momentum operator associated with position operator $X$. Inspired of the previous works, an operator $\tilde{H}$ similar to quantum analogy was proposed [3]. However, the momentum operator $P$ in this case has no self-adjoint extension [2].

In this paper we define an operator $\tilde{H}$ on the Hilbert space $\mathcal{H} = L^2[1, 1]$, then we show that momentum operator in this setting has the unique self-adjoint extension. Finally, we demonstrate that the operator is pseudo hermitian and prove a relationship between its eigenvalues and the non-trivial Riemann zeros.
2 Hurwitz zeta functions

The Hurwitz zeta function is originally defined for complex arguments $s$ with $\Re(s) > 1$ and $x > 0$ by

$$\zeta(s, x) = \sum_{n \geq 0} \frac{1}{(n + x)^s}$$

Hurwitz zeta function has an analytic continuation on $\mathbb{C}$ with the only singularity $s = 1$ and it can be represented [1, 6]

$$\zeta(s, x) = -\frac{\Gamma(1-s)}{2\pi i} \int_{C} \frac{\xi^{s-x} e^{x\xi}}{e^\xi - 1} d\xi, \quad s \neq 1$$ (1)

where $C$ is the contour around the negative real axis and it starts at $-\infty$, encircles the origin once in the positive direction without enclosing any of the points $\xi = \pm 2ni, \pm 4ni, ...$ and returns to $-\infty$ [6]. The famous Riemann zeta function is the special case of Hurwitz zeta function with $x = 1$

$$\zeta(s, 1) = \zeta(s)$$

and also with $x = \frac{1}{2}$

$$\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)$$ (2)

The partial derivative of Hurwitz zeta function with respect to $x$

$$\frac{\partial \zeta(s, x)}{\partial x} = -s\zeta(s + 1, x), \quad s \neq 0, 1; x > 0$$ (3)

3 Non-trivial Riemann zeros

Let $X$ be the position operator on $L^2[\frac{1}{2}, 1]$ and $P_0$ the momentum operator $P_0 = -i \frac{d}{dx}$ densely defined on $L^2[\frac{1}{2}, 1]$ with domain

$$D(P_0) = \left\{ f \in C^\infty[\frac{1}{2}, 1] : f(\frac{1}{2}) = f(1) \right\}$$ (4)

and obviously $P_0$ is essentially self-adjoint [5] and it has the unique self-adjoint extension $\overline{P_0} = P$. The functions $\varphi_n(x) = e^{2\pi in(x-\frac{1}{2})}$ form an orthogonal basis for $L^2[\frac{1}{2}, 1]$ and the closure of $D(P_0)$ is all functions of $L^2[\frac{1}{2}, 1]$

$$D(P) = \overline{D(P_0)} = L^2[\frac{1}{2}, 1]$$

From the Stone’s theorem on one-parameter unitary groups, $\{e^{itP}\}_{t}$ is a strongly continuous one-parameter group of unitary operators on $L^2[\frac{1}{2}, 1]$. 
So both operators $e^{\pm iP} - I$ are bijective and bounded, and then the bounded inverses $(e^{\pm iP} - I)^{-1}$ exist, from the bounded inverse theorem in functional analysis.

We construct an operator $\tilde{H}$ on $L^2[1/2, 1]$

$$\tilde{H} = (e^{iP} - I)^{-1} (XP + PX) (e^{iP} - I)$$

It holds obviously for $\frac{1}{2} \leq x \leq 1$ and $0 < \Re(s) < 1$

$$(XP + PX)x^{-s} = i(2s - 1)x^{-s}$$

(5)

Here the $H = XP + PX$ is called Berry-Keating operator [4].

The operator $\tilde{H}$ is pseudo hermitian, i.e. there exists an invertible operator $\eta$ such that the adjoint operator $\tilde{H}^\dagger$ can be represented

$$\tilde{H}^\dagger = \eta \tilde{H} \eta^{-1}$$

**Proposition 3.1.** $\tilde{H}$ is pseudo hermitian and all the eigenvalues are real.

*Proof.* Let $\eta = (e^{iP} - I)^\dagger (e^{iP} - I)$, then

$$\tilde{H}^\dagger = \eta \tilde{H} \eta^{-1}$$

The operator $\tilde{H}$ is similar to its adjoint $\tilde{H}^\dagger$, then all eigenvalues are real. $\square$

Let $s$ denote a point in critical strip $0 < \Re(s) < 1$ and we define a function $\zeta_s$ on interval $[1/2, 1]$ for Hurwitz zeta function

$$\zeta_s(x) := \zeta(s, x), \quad \frac{1}{2} \leq x \leq 1$$

If a complex number $z$ is a non-trivial Riemann zero,

$$\zeta_z(1) = 0$$

then the function $\zeta_z$ vanishes at $x = \frac{1}{2}$ from (2)

$$\zeta_z\left(\frac{1}{2}\right) = 0$$

**Proposition 3.2.** For each non-trivial Riemann zero $z$, $\zeta_z$ is an eigenfunction of $\tilde{H}$ with eigenvalue $i(2z - 1)$.

*Proof.* We consider for non-trivial Riemann zeros $z$ the term

$$\frac{1}{iP} \left( \frac{iP}{e^{iP} - I} \right) x^{-z}, \quad \frac{1}{2} \leq x \leq 1$$

3
where $\frac{1}{iP}$ is defined as an integral operator on $L^2[\frac{1}{2}, 1]$ with boundary $x = \frac{1}{2}$

$$\frac{1}{iP} := \int_{\frac{1}{2}}^{x} f(\xi) d\xi, \quad f \in L^2[\frac{1}{2}, 1]$$

and $\frac{iP}{e^{iP} - I} f$ is defined as a Bernoulli generating function

$$\frac{iP}{e^{iP} - I} f(x) := \sum_{n \geq 0} B_n \frac{n!}{n!} (iP)^n f(x), \quad f \in L^2[\frac{1}{2}, 1]$$

Notice

$$(iP)^n x^{-z} = \frac{\Gamma(-z + 1)}{\Gamma(-z + 1 - n)} x^{-z-n}$$

We use Hankel loop contour integral formula [6, 7]

$$\frac{1}{\Gamma(-z + 1 - n)} = \frac{1}{2\pi i} \int_{C} \xi^{-(1-z-n)} e^{\xi} d\xi$$

where the integration is done along the Hankel-Bromwich contour and it begins at $-\infty$, circles the origin once in the positive direction, and returns to $-\infty$

$$(iP)^n x^{-z} = \Gamma(1 - z) \left( \frac{1}{2\pi i} \int_{C} \xi^{-(1-z-n)} e^{\xi} d\xi \right) x^{-z-n}$$

$$= \frac{\Gamma(1 - z)}{2\pi i} \int_{C} \left( \frac{\xi}{x} \right)^{z+n-1} e^{\xi} d\xi$$

and then

$$\frac{iP}{e^{iP} - I} x^{-z} = \sum_{n \geq 0} B_n \frac{n!}{n!} (iP)^n x^{-z}$$

$$= \frac{\Gamma(1 - z)}{2\pi i} \sum_{n \geq 0} \int_{C} \frac{B_n}{n!} \left( \frac{\xi}{x} \right)^{z+n-1} e^{\xi} d\xi$$

Let $u = \frac{\xi}{x}$ and $C' = \frac{\xi}{x}$ is still a loop around the negative real axis. The uniform convergence on the closed interval justifies the interchange of the integral and the sum. Then Hurwitz zeta function can be reproduced (1)

$$\frac{iP}{e^{iP} - I} x^{-z} = \frac{\Gamma(1 - z)}{2\pi i} \int_{C'} \frac{u^z}{e^{u} - 1} e^{ux} du = -z\zeta(z + 1, x)$$

Then from (3)

$$\frac{1}{iP} \left( \frac{iP}{e^{iP} - I} \right) x^{-z} = \int_{\frac{1}{2}}^{x} -z\zeta(z + 1, \xi) d\xi$$

$$= \zeta(z, x) - \zeta(z, \frac{1}{2}) = \zeta(z, x)$$
The function $\zeta_z$ vanishes at $x = \frac{1}{2}$ for non-trivial Riemann zero $z$, then
\[
\left( \frac{1}{e^{iP} - 1} \right) x^{-z} = \frac{1}{iP} \left( \frac{iP}{e^{iP} - 1} \right) x^{-z} = \zeta_z(x)
\]
Combining (5), we reach
\[
\tilde{H} \zeta_z = i(2z - 1) \zeta_z
\]

**Theorem 3.3.** The Riemann zeta function has its non-trivial zeros only at the complex numbers with real part $\frac{1}{2}$.

**Corollary.** The Hurwitz zeta function
\[
\zeta(s, \frac{1}{2})
\]
has its non-trivial zeros only at the complex numbers with real part $\frac{1}{2}$.

**References**


