



**HAL**  
open science

# A pseudo hermitian operator and non-trivial Riemann zeros

Yu Li

► **To cite this version:**

| Yu Li. A pseudo hermitian operator and non-trivial Riemann zeros. 2018. hal-01860885v2

**HAL Id: hal-01860885**

**<https://hal.science/hal-01860885v2>**

Preprint submitted on 27 Aug 2018 (v2), last revised 15 Sep 2019 (v6)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A pseudo hermitian operator and non-trivial Riemann zeros

Yu Li

## Abstract

Abstract: In this paper we construct a pseudo hermitian operator and reveal a connection between its eigenvalues and the non-trivial Riemann zeros.

**Keywords.** Riemann hypothesis, non-trivial Riemann zero, Pseudo hermitian operator

## 1 Introduction

One of the most attracting problems in Mathematics is Riemann hypothesis and it states that all the non-trivial zeros of Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt$$

have the real part  $\frac{1}{2}$ .

One proposed way of proving Riemann hypothesis, known as Hilbert-Pólya conjecture, is to find an unbounded self-adjoint operator such that the imaginary part of Riemann zeros corresponds to the eigenvalues of this operator. It has been suggested [4] that the possible approach to solve Hilbert-Pólya conjecture lies in a quantization of the classical Hamiltonian  $H = XP$ , where  $P$  is the canonical momentum operator associated with position operator  $X$ . Inspired of the previous works, an operator  $\tilde{H}$  similar to quantum analogy was proposed [3]. However, the momentum operator  $P$  in this case has no self-adjoint extension [2].

In this paper we define an operator  $\tilde{H}$  on the Hilbert space  $\mathcal{H} = L^2[\frac{1}{2}, 1]$ , then we show that momentum operator in this setting has the unique self-adjoint extension. Finally, we demonstrate that the operator is pseudo hermitian and prove a relationship between its eigenvalues and the non-trivial Riemann zeros.

## 2 Hurwitz zeta functions

The Hurwitz zeta function [1, 6] is originally defined for complex arguments  $s$  with  $\mathcal{R}(s) > 1$  and  $x > 0$  by

$$\zeta(s, x) = \sum_{n \geq 0} \frac{1}{(n+x)^s}$$

Hurwitz zeta function has an analytic continuation on  $\mathbb{C}$  with the only singularity  $s = 1$  and it can be represented

$$\zeta(s, x) = -\frac{\Gamma(1-s)}{2\pi i} \int_{\mathcal{C}} \frac{\xi^{s-x} e^{x\xi}}{e^\xi - 1} d\xi, \quad s \neq 1 \quad (1)$$

where  $\mathcal{C}$  is the contour around the negative real axis and it starts at  $-\infty$ , encircles the origin once in the positive direction without enclosing any of the points  $\xi = \pm 2ni, \pm 4ni, \dots$  and returns to  $-\infty$  [6]. The famous Riemann zeta function is the special case of Hurwitz zeta function with  $x = 1$

$$\zeta(s, 1) = \zeta(s)$$

and also with  $x = \frac{1}{2}$

$$\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s) \quad (2)$$

The partial derivative of Hurwitz zeta function with respect to  $x$

$$\frac{\partial \zeta(s, x)}{\partial x} = -s\zeta(s+1, x), \quad s \neq 0, 1; \mathcal{R}(x) > 0 \quad (3)$$

## 3 Non-trivial Riemann zeros

Let  $X$  be the position operator on  $L^2[\frac{1}{2}, 1]$  and  $P_0$  the momentum operator  $P_0 = -i\frac{d}{dx}$  densely defined on  $L^2[\frac{1}{2}, 1]$  with domain

$$D(P_0) = \left\{ f \in C^\infty[\frac{1}{2}, 1] : f(\frac{1}{2}) = f(1) \right\} \quad (4)$$

and obviously  $P_0$  is essentially self-adjoint [5] and it has the unique self-adjoint extension  $\overline{P_0} = P$ . The functions  $\varphi_n(x) = e^{2\pi i n(2x-1)}$  form an orthogonal basis for  $L^2[\frac{1}{2}, 1]$  and the closure of  $D(P_0)$  is all functions of  $L^2[\frac{1}{2}, 1]$

$$\overline{D(P_0)} = L^2[\frac{1}{2}, 1]$$

From the Stone's theorem on one-parameter unitary groups,  $\{e^{itP}\}_t$  is a strongly continuous one-parameter group of unitary operators on  $L^2[\frac{1}{2}, 1]$ .

So both operators  $e^{\pm iP} - I$  are bijective and bounded, and then the bounded inverses  $(e^{\pm iP} - I)^{-1}$  exist, from the bounded inverse theorem in functional analysis.

We construct an operator  $\tilde{H}$  on  $L^2[\frac{1}{2}, 1]$

$$\tilde{H} = (e^{iP} - I)^{-1} (XP + PX) (e^{iP} - I)$$

It holds obviously for  $\frac{1}{2} \leq x \leq 1$  and  $0 < \mathcal{R}(s) < 1$

$$(XP + PX)x^{-s} = i(2s - 1)x^{-s} \quad (5)$$

Here the  $H = XP + PX$  is called Berry-Keating operator [4].

The operator  $\tilde{H}$  is pseudo hermitian, i.e. there exists an invertible operator  $\eta$  such that the adjoint operator  $\tilde{H}^\dagger$  can be represented

$$\tilde{H}^\dagger = \eta \tilde{H} \eta^{-1}$$

**Proposition 3.1.**  *$\tilde{H}$  is pseudo hermitian and all the eigenvalues are real.*

*Proof.* Let  $\eta = (e^{iP} - I)^\dagger (e^{iP} - I)$ , then

$$\tilde{H}^\dagger = \eta \tilde{H} \eta^{-1}$$

The operator  $\tilde{H}$  is similar to its adjoint  $\tilde{H}^\dagger$ , then all eigenvalues are real.  $\square$

Let  $s$  denote a point in critical strip  $0 < \mathcal{R}(s) < 1$  and we define a function  $\zeta_s$  on interval  $[\frac{1}{2}, 1]$  for Hurwitz zeta function

$$\zeta_s(x) := \zeta(s, x), \quad \frac{1}{2} \leq x \leq 1$$

If a complex number  $z$  is a non-trivial Riemann zero,

$$\zeta_z(1) = 0$$

then the function  $\zeta_z$  vanishes at  $x = \frac{1}{2}$  from (2)

$$\zeta_z(\frac{1}{2}) = 0$$

**Proposition 3.2.** *For each non-trivial Riemann zero  $z$ ,  $\zeta_z$  is an eigenfunction of  $\tilde{H}$  with eigenvalue  $i(2z - 1)$ .*

*Proof.* We consider for non-trivial Riemann zeros  $z$  the term

$$\frac{1}{iP} \left( \frac{iP}{e^{iP} - I} \right) x^{-z}, \quad \frac{1}{2} \leq x \leq 1$$

where  $\frac{1}{iP}$  is defined as an integral operator on  $L^2[\frac{1}{2}, 1]$

$$\frac{1}{iP} := \int_{\frac{1}{2}}^x f(\xi) d\xi, \quad f \in L^2[\frac{1}{2}, 1]$$

and  $\frac{iP}{e^{iP} - I}$  is interpreted as operator-valued Bernoulli generating function

$$\frac{iP}{e^{iP} - I} f(x) := \sum_{n \geq 0} \frac{B_n}{n!} (iP)^n f(x), \quad f \in L^2[\frac{1}{2}, 1]$$

This series is not always convergent but it is Borel summable.

Notice

$$(iP)^n x^{-z} = \frac{\Gamma(-z+1)}{\Gamma(-z+1-n)} x^{-z-n}$$

We use Hankel loop contour integral formula [6, 7]

$$\frac{1}{\Gamma(-z+1-n)} = \frac{1}{2\pi i} \int_{\mathcal{C}} \xi^{-(1-z-n)} e^{\xi} d\xi$$

where the integration is done along the Hankel-Bromwich contour and it begins at  $-\infty$ , circles the origin once in the positive direction, and returns to  $-\infty$

$$\begin{aligned} (iP)^n x^{-z} &= \Gamma(1-z) \left( \frac{1}{2\pi i} \int_{\mathcal{C}} \xi^{-(1-z-n)} e^{\xi} d\xi \right) x^{-z-n} \\ &= \frac{\Gamma(1-z)}{2\pi i} \int_{\mathcal{C}} \left( \frac{\xi}{x} \right)^{z+n-1} e^{\xi} d \frac{\xi}{x} \end{aligned}$$

and obtain Borel sum

$$\begin{aligned} \frac{iP}{e^{iP} - I} x^{-z} &= \sum_{n \geq 0} \frac{B_n}{n!} (iP)^n x^{-z} \\ &= \frac{\Gamma(1-z)}{2\pi i} \sum_{n \geq 0} \int_{\mathcal{C}} \frac{B_n}{n!} \left( \frac{\xi}{x} \right)^{z+n-1} e^{\xi} d \frac{\xi}{x} \\ &= \frac{\Gamma(1-z)}{2\pi i} \int_{\mathcal{C}} \left( \frac{\xi}{x} \right)^{z-1} \frac{\xi}{e^{\frac{\xi}{x}} - 1} e^{(\frac{\xi}{x})x} d \frac{\xi}{x} \end{aligned}$$

Let  $u = \frac{\xi}{x}$  and  $\mathcal{C}' = \frac{\mathcal{C}}{x}$  is still a loop around the negative real axis, then Hurwitz zeta function can be reproduced (1)

$$\frac{iP}{e^{iP} - I} x^{-z} = \frac{\Gamma(1-z)}{2\pi i} \int_{\mathcal{C}'} \frac{u^z}{e^u - 1} e^{ux} du = -z\zeta(z+1, x)$$

Then from (3)

$$\begin{aligned} \frac{1}{iP} \left( \frac{iP}{e^{iP} - I} \right) x^{-z} &= \int_{\frac{1}{2}}^x -z \zeta(z+1, \xi) d\xi \\ &= \zeta(z, x) - \zeta(z, \frac{1}{2}) = \zeta_z(x) \end{aligned}$$

The function  $\zeta_z$  vanishes at  $x = \frac{1}{2}$  for non-trivial Riemann zero  $z$ , then

$$\left( \frac{1}{e^{iP} - I} \right) x^{-z} = \frac{1}{iP} \left( \frac{iP}{e^{iP} - I} \right) x^{-z} = \zeta_z(x)$$

. Combining (5), we reach

$$\tilde{H}\zeta_z = i(2z - 1)\zeta_z$$

□

**Theorem 3.3.** *The Riemann zeta function has its non-trivial zeros only at the complex numbers with real part  $\frac{1}{2}$ .*

**Corollary.** *The Hurwitz zeta function*

$$\zeta(s, \frac{1}{2})$$

*has its non-trivial zeros only at the complex numbers with real part  $\frac{1}{2}$ .*

## References

- [1] D. H. Bailey, J. M. Borwein, N. J. Calkin, R. Girgensohn, D. R. Luke, and V. Moll. *Experimental Mathematics in Action*. A K Peters/CRC Press, May 2007.
- [2] J. Belissard. Comment on "Hamiltonian for the Zeros of the Riemann Zeta Function".
- [3] C. M. Bender, D. C. Brody, and M. P. Mller. Hamiltonian for the Zeros of the Riemann Zeta Function. *PHYSICAL REVIEW LETTERS*, 2017.
- [4] M. V. Berry and J. P. Keating.  $H = XP$  and the Riemann zeros. *Symmetry and trace formula: Chaos and disorder*, 1999.
- [5] B. C. Hall. *Quantum Theory for Mathematicians (Graduate Texts in Mathematics)*. Springer, June 2013.
- [6] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST Handbook of Mathematical Functions Paperback and CD-ROM*. Cambridge University Press, May 2010.

- [7] T. Schmelzer and L. N. Trefethen. Computing the Gamma function using contour integrals and rational approximation. *SIAM journal of numerical analysis*, 45(2):558–571, 2007.