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To cite this version:
Yu Li. A pseudo hermitian operator and non-trivial Riemann zeros. 2018. hal-01860885v2

HAL Id: hal-01860885
https://hal.science/hal-01860885v2
Preprint submitted on 27 Aug 2018 (v2), last revised 15 Sep 2019 (v6)

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A pseudo hermitian operator and non-trivial Riemann zeros

Yu Li

Abstract

Abstract: In this paper we construct a pseudo hermitian operator and reveal a connection between its eigenvalues and the non-trivial Riemann zeros.

Keywords. Riemann hypothesis, non-trivial Riemann zero, Pseudo hermitian operator

1 Introduction

One of the most attracting problems in Mathematics is Riemann hypothesis and it states that all the non-trivial zeros of Riemann zeta function

\[ \zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt \]

have the real part \(\frac{1}{2}\).

One proposed way of proving Riemann hypothesis, known as Hilbert-Pólya conjecture, is to find an unbounded self-adjoint operator such that the imaginary part of Riemann zeros corresponds to the eigenvalues of this operator. It has been suggested [4] that the possible approach to solve Hilbert-Pólya conjecture lies in a quantization of the classical Hamiltonian \(H = XP\), where \(P\) is the canonical momentum operator associated with position operator \(X\). Inspired of the previous works, an operator \(\tilde{H}\) similar to quantum analogy was proposed [3]. However, the momentum operator \(P\) in this case has no self-adjoint extension [2].

In this paper we define an operator \(\tilde{H}\) on the Hilbert space \(\mathcal{H} = L^2[1, 1]\), then we show that momentum operator in this setting has the unique self-adjoint extension. Finally, we demonstrate that the operator is pseudo hermitian and prove a relationship between its eigenvalues and the non-trivial Riemann zeros.
2 Hurwitz zeta functions

The Hurwitz zeta function \([1, 6]\) is originally defined for complex arguments \(s\) with \(\Re(s) > 1\) and \(x > 0\) by

\[
\zeta(s, x) = \sum_{n \geq 0} \frac{1}{(n + x)^s}
\]

Hurwitz zeta function has an analytic continuation on \(\mathbb{C}\) with the only singularity \(s = 1\) and it can be represented

\[
\zeta(s, x) = -\frac{\Gamma(1 - s)}{2\pi i} \int_C \frac{\xi^{s-x}e^{x\xi}}{e^\xi - 1} d\xi, \quad s \neq 1
\] (1)

where \(C\) is the contour around the negative real axis and it starts at \(-\infty\), encircles the origin once in the positive direction without enclosing any of the points \(\xi = \pm 2ni, \pm 4ni, \ldots\) and returns to \(-\infty\) \([6]\). The famous Riemann zeta function is the special case of Hurwitz zeta function with \(x = 1\)

\[
\zeta(s, 1) = \zeta(s)
\]

and also with \(x = \frac{1}{2}\)

\[
\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)
\] (2)

The partial derivative of Hurwitz zeta function with respect to \(x\)

\[
\frac{\partial \zeta(s, x)}{\partial x} = -s\zeta(s + 1, x), \quad s \neq 0, 1; \Re(x) > 0
\] (3)

3 Non-trivial Riemann zeros

Let \(X\) be the position operator on \(L^2[\frac{1}{2}, 1]\) and \(P_0\) the momentum operator \(P_0 = -i \frac{d}{dx}\) densely defined on \(L^2[\frac{1}{2}, 1]\) with domain

\[
D(P_0) = \left\{ f \in C^\infty[\frac{1}{2}, 1] : f(\frac{1}{2}) = f(1) \right\}
\] (4)

and obviously \(P_0\) is essentially self-adjoint \([5]\) and it has the unique self-adjoint extension \(\overline{P_0} = P\). The functions \(\varphi_n(x) = e^{2\pi in(2x-1)}\) form an orthogonal basis for \(L^2[\frac{1}{2}, 1]\) and the closure of \(D(P_0)\) is all functions of \(L^2[\frac{1}{2}, 1]\)

\[
\overline{D(P_0)} = L^2[\frac{1}{2}, 1]
\]

From the Stone’s theorem on one-parameter unitary groups, \(\{e^{itP}\}_t\) is a strongly continuous one-parameter group of unitary operators on \(L^2[\frac{1}{2}, 1]\).
So both operators $e^{\pm iP} - I$ are bijective and bounded, and then the bounded inverses $(e^{\pm iP} - I)^{-1}$ exist, from the bounded inverse theorem in functional analysis.

We construct an operator $\tilde{H}$ on $L^2[\frac{1}{2}, 1]$

$$\tilde{H} = (e^{iP} - I)^{-1}(XP + PX)(e^{iP} - I)$$

It holds obviously for $\frac{1}{2} \leq x \leq 1$ and $0 < R(s) < 1$

$$(XP + PX)x^{-s} = i(2s - 1)x^{-s}$$

(5)

Here the $H = XP + PX$ is called Berry-Keating operator [4].

The operator $\tilde{H}$ is pseudo hermitian, i.e. there exists an invertible operator $\eta$ such that the adjoint operator $\tilde{H}^\dagger$ can be represented

$$\tilde{H}^\dagger = \eta \tilde{H} \eta^{-1}$$

**Proposition 3.1.** $\tilde{H}$ is pseudo hermitian and all the eigenvalues are real.

**Proof.** Let $\eta = (e^{iP} - I)^\dagger(e^{iP} - I)$, then

$$\tilde{H}^\dagger = \eta \tilde{H} \eta^{-1}$$

The operator $\tilde{H}$ is similar to its adjoint $\tilde{H}^\dagger$, then all eigenvalues are real. \qed

Let $s$ denote a point in critical strip $0 < R(s) < 1$ and we define a function $\zeta_s$ on interval $[\frac{1}{2}, 1]$ for Hurwitz zeta function

$$\zeta_s(x) := \zeta(s, x), \quad \frac{1}{2} \leq x \leq 1$$

If a complex number $z$ is a non-trivial Riemann zero,

$$\zeta_z(1) = 0$$

then the function $\zeta_z$ vanishes at $x = \frac{1}{2}$ from (2)

$$\zeta_z(\frac{1}{2}) = 0$$

**Proposition 3.2.** For each non-trivial Riemann zero $z$, $\zeta_z$ is an eigenfunction of $\tilde{H}$ with eigenvalue $i(2z - 1)$.

**Proof.** We consider for non-trivial Riemann zeros $z$ the term

$$\frac{1}{iP} \left( \frac{iP}{e^{iP} - I} \right) x^{-z}, \quad \frac{1}{2} \leq x \leq 1$$

3
where $\frac{1}{iP}$ is defined as an integral operator on $L^2[\frac{1}{2}, 1]$

$$\frac{1}{iP} := \int_{\frac{1}{2}}^{x} f(\xi) d\xi, \quad f \in L^2[\frac{1}{2}, 1]$$

and $\frac{iP}{e^{iP} - I}$ is interpreted as operator-valued Bernoulli generating function

$$\frac{iP}{e^{iP} - I} f(x) := \sum_{n \geq 0} \frac{B_n}{n!} (iP)^n f(x), \quad f \in L^2[\frac{1}{2}, 1]$$

This series is not always convergent but it is Borel summable.

Notice

$$(iP)^n x^{-z} = \frac{\Gamma(-z + 1)}{\Gamma(-z + 1 - n)} x^{-z-n}$$

We use Hankel loop contour integral formula [6, 7]

$$\frac{1}{\Gamma(-z + 1 - n)} = \frac{1}{2\pi i} \int_{C} \xi^{-(1-z-n)} e^{\xi} d\xi$$

where the integration is done along the Hankel-Bromwich contour and it begins at $-\infty$, circles the origin once in the positive direction, and returns to $-\infty$

$$(iP)^n x^{-z} = \frac{\Gamma(1-z)}{2\pi i} \int_{C} \left( \frac{\xi}{x} \right)^{z+n-1} e^{\xi} d\xi$$

and obtain Borel sum

$$\frac{iP}{e^{iP} - I} x^{-z} = \sum_{n \geq 0} \frac{B_n}{n!} (iP)^n x^{-z}$$

$$= \frac{\Gamma(1-z)}{2\pi i} \sum_{n \geq 0} \int_{C} \frac{B_n}{n!} \left( \frac{\xi}{x} \right)^{z+n-1} e^{\xi} d\xi$$

$$= \frac{\Gamma(1-z)}{2\pi i} \int_{C} \left( \frac{\xi}{x} \right)^{z-1} \frac{\xi}{e^{\xi} - 1} e^{(\xi)x} d\xi$$

Let $u = \frac{\xi}{x}$ and $C' = \frac{C}{x}$ is still a loop around the negative real axis, then Hurwitz zeta function can be reproduced (1)

$$\frac{iP}{e^{iP} - I} x^{-z} = \frac{\Gamma(1-z)}{2\pi i} \int_{C'} \frac{u^z}{e^u - 1} e^{uz} du = -z \zeta(z + 1, x)$$
Then from (3)
\[
\frac{1}{iP} \left( \frac{iP}{e^{iP} - 1} \right) x^{-z} = \int_{1/2}^{x} -z\zeta(z + 1, \xi) \, d\xi
\]
\[= \zeta(z, x) - \zeta(z, \frac{1}{2}) = \zeta_z(x)
\]
The function $\zeta_z$ vanishes at $x = \frac{1}{2}$ for non-trivial Riemann zero $z$, then
\[
\left( \frac{1}{e^{iP} - 1} \right) x^{-z} = \frac{1}{iP} \left( \frac{iP}{e^{iP} - 1} \right) x^{-z} = \zeta_z(x)
\]
. Combining (5), we reach
\[
\tilde{H}\zeta_z = i(2z - 1)\zeta_z
\]

\[\blacksquare\]

**Theorem 3.3.** The Riemann zeta function has its non-trivial zeros only at the complex numbers with real part $\frac{1}{2}$.

**Corollary.** The Hurwitz zeta function
\[
\zeta(s, \frac{1}{2})
\]
has its non-trivial zeros only at the complex numbers with real part $\frac{1}{2}$.

**References**


