

# A class of cutoff functions for non-trivial Riemann zeros

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## Abstract

Abstract: In this paper we construct a pseudo hermitian operator  $\tilde{H}$  on the Hilbert space  $L^2[\frac{1}{2}, 1]$  and show that the Hurwitz zeta functions over the interval  $[\frac{1}{2}, 1]$  are eigenfunctions of  $\tilde{H}$  for all complex numbers.

**Keywords.** Riemann hypothesis, non-trivial Riemann zero, Pseudo hermitian operator, Berry-Keating operator

## 1 Introduction

One of the most attracting problems in Mathematics is Riemann hypothesis and it states that all the non-trivial zeros of Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt$$

have the real part  $\frac{1}{2}$ .

One proposed way of proving Riemann hypothesis, known as Hilbert-Pólya conjecture, is to find an unbounded self-adjoint operator such that the imaginary part of Riemann zeros corresponds to the eigenvalues of this operator. It has been suggested [5] that the possible approach to solve Hilbert-Pólya conjecture lies in a quantization of the classical Hamiltonian  $H = XP$ , where  $P$  is the canonical momentum operator associated with position operator  $X$ . Inspired of the previous works, an operator  $\tilde{H}$  similar to quantum analogy was proposed [4] and asymptotic analysis was performed [3]. However, the momentum operator  $P$  in this case has no self-adjoint extension [2].

## 2 Hurwitz Zeta functions

The Hurwitz zeta function is originally defined for complex arguments  $s$  with  $\Re(s) > 1$  and  $x > 0$  by

$$\zeta(s, x) = \sum_{n \geq 0} \frac{1}{(n+x)^s}$$

Hurwitz zeta function has an analytic continuation on  $\mathbb{C}$  with the only singularity  $s = 1$  and it can be represented [1, 10]

$$\zeta(s, x) = -\frac{\Gamma(1-s)}{2\pi i} \int_{\mathcal{C}} \frac{\xi^{s-1} e^{x\xi}}{e^\xi - 1} d\xi, \quad s \neq 1 \quad (1)$$

where  $\mathcal{C}$  is the contour around the negative real axis and it starts at  $-\infty$ , encircles the origin once in the positive direction without enclosing any of the points  $\xi = \pm 2ni, \pm 4ni, \dots$  and returns to  $-\infty$  [10]. The famous Riemann zeta function is the special case of Hurwitz zeta function with  $x = 1$

$$\zeta(s, 1) = \zeta(s)$$

and also with  $x = \frac{1}{2}$

$$\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s) \quad (2)$$

The partial derivative of Hurwitz zeta function with respect to  $x$

$$\frac{\partial \zeta(s, x)}{\partial x} = -s\zeta(s+1, x), \quad s \neq 0, 1; x > 0 \quad (3)$$

Let  $s$  denote a point in critical strip  $0 < \mathcal{R}(s) < 1$  and we define a cutoff function  $\zeta_s$  of Hurwitz function on interval  $[\frac{1}{2}, 1]$

$$\zeta_s(x) := \zeta(s, x), \quad \frac{1}{2} \leq x \leq 1 \quad (4)$$

The functions  $\zeta_s$  is continuous and even smooth on the closed interval  $[\frac{1}{2}, 1]$  and have finite norm and belong to the Hilbert space,  $\zeta_s \in L^2[\frac{1}{2}, 1]$ . If a complex number  $z$  is a non-trivial Riemann zero,

$$\zeta_z(1) = 0$$

then the function  $\zeta_z$  vanishes at  $x = \frac{1}{2}$  from (2)

$$\zeta_z(\frac{1}{2}) = 0$$

### 3 Operators in Hilbert space $L^2[\frac{1}{2}, 1]$

We confine our discussion over the region  $[\frac{1}{2}, 1]$ . Let  $X$  be the position operator on  $L^2[\frac{1}{2}, 1]$  and  $P_0$  the momentum operator  $P_0 = -i\frac{d}{dx}$  densely defined on  $L^2[\frac{1}{2}, 1]$  with domain

$$D(P_0) = \left\{ f \in C^\infty[\frac{1}{2}, 1] : f(\frac{1}{2}) = f(1) \right\} \quad (5)$$

[6] and it has a unique self-adjoint extension  $\overline{P_0} = P$ . The closure of the graph of  $P_0$  is the graph of  $P$

$$\Gamma(P) = \overline{\Gamma(P_0)} = L^2[\frac{1}{2}, 1] \times L^2[\frac{1}{2}, 1]$$

From the Stone's theorem on one-parameter unitary groups,  $\{e^{itP}\}_t$  is a strongly continuous one-parameter group of unitary operators on  $L^2[\frac{1}{2}, 1]$ . Thus, both operators  $e^{\pm iP} - I$  are bijective and bounded, and then the bounded inverses  $(e^{\pm iP} - I)^{-1}$  exist, from the bounded inverse theorem in functional analysis.

We construct an operator  $\tilde{H}$  on  $L^2[\frac{1}{2}, 1]$

$$\tilde{H} = (e^{iP} - I)^{-1} (XP + PX) (e^{iP} - I)$$

It holds obviously for  $\frac{1}{2} \leq x \leq 1$  and  $0 < \mathcal{R}(s) < 1$

$$(XP + PX)x^{-s} = i(2s - 1)x^{-s} \quad (6)$$

The operator  $\tilde{H}$  is pseudo hermitian [7, 8, 9], i.e. there exists an invertible and self-adjoint operator  $\eta$  such that the adjoint operator  $\tilde{H}^\dagger$  can be represented

$$\tilde{H}^\dagger = \eta \tilde{H} \eta^{-1}$$

**Proposition 3.1.**  *$\tilde{H}$  is pseudo hermitian and all eigenvalues are real.*

*Proof.* Let  $\eta = (e^{iP} - I)^\dagger (e^{iP} - I)$  and it is obviously self-adjoint, then

$$\tilde{H}^\dagger = \eta \tilde{H} \eta^{-1}$$

The operator  $\eta$  is positive definite and self-adjoint by construction, the operator  $\tilde{H}$  is hermitian in the new inner product with respect to  $\eta$

$$\langle \tilde{H}\phi, \varphi \rangle_\eta = \langle \tilde{H}\phi, \eta\varphi \rangle = \langle \phi, \eta \tilde{H} \eta^{-1} \eta\varphi \rangle = \langle \phi, \tilde{H}\varphi \rangle_\eta, \quad \phi, \varphi \in L^2[\frac{1}{2}, 1]$$

Then all eigenvalues of  $\tilde{H}$  are real. □

**Proposition 3.2.** *For any complex number  $z$ , the function  $\zeta_z(x) - \zeta_z(\frac{1}{2})$  is an eigenfunction of  $\tilde{H}$  with eigenvalue  $i(2z - 1)$*

$$\tilde{H} \left( \zeta_z(x) - \zeta_z\left(\frac{1}{2}\right) \right) = i(2z - 1) \left( \zeta_z(x) - \zeta_z\left(\frac{1}{2}\right) \right)$$

*Proof.* We consider for non-trivial Riemann zeros  $z$  the term

$$\frac{1}{iP} \left( \frac{iP}{e^{iP} - I} \right) x^{-z}, \quad \frac{1}{2} \leq x \leq 1$$

where  $\frac{1}{iP}$  is defined as an integral operator on  $L^2[\frac{1}{2}, 1]$  with boundary  $x = \frac{1}{2}$

$$\frac{1}{iP} f(x) := \int_{\frac{1}{2}}^x f(\xi) d\xi, \quad f \in L^2[\frac{1}{2}, 1]$$

and  $\frac{iP}{e^{iP}-I}f$  is defined as a Bernoulli generating function

$$\frac{iP}{e^{iP}-I}f(x) := \sum_{n \geq 0} \frac{B_n}{n!} (iP)^n f(x), \quad f \in L^2\left[\frac{1}{2}, 1\right]$$

Notice

$$(iP)^n x^{-z} = \frac{\Gamma(-z+1)}{\Gamma(-z+1-n)} x^{-z-n}$$

We use Hankel loop contour integral formula [10, 11]

$$\frac{1}{\Gamma(-z+1-n)} = \frac{1}{2\pi i} \int_{\mathcal{C}} \xi^{-(1-z-n)} e^{\xi} d\xi$$

where the integration is done along the Hankel-Bromwich contour and it begins at  $-\infty$ , circles the origin once in the positive direction, and returns to  $-\infty$

$$\begin{aligned} (iP)^n x^{-z} &= \Gamma(1-z) \left( \frac{1}{2\pi i} \int_{\mathcal{C}} \xi^{-(1-z-n)} e^{\xi} d\xi \right) x^{-z-n} \\ &= \frac{\Gamma(1-z)}{2\pi i} \int_{\mathcal{C}} \left( \frac{\xi}{x} \right)^{z+n-1} e^{\xi} d\frac{\xi}{x} \end{aligned}$$

and then

$$\begin{aligned} \frac{iP}{e^{iP}-I} x^{-z} &= \sum_{n \geq 0} \frac{B_n}{n!} (iP)^n x^{-z} \\ &= \frac{\Gamma(1-z)}{2\pi i} \sum_{n \geq 0} \int_{\mathcal{C}} \frac{B_n}{n!} \left( \frac{\xi}{x} \right)^{z+n-1} e^{\xi} d\frac{\xi}{x} \end{aligned}$$

Let  $u = \frac{\xi}{x}$  and  $\mathcal{C}' = \frac{\mathcal{C}}{x}$  is still a loop around the negative real axis. Uniform convergence on closed interval justifies the interchange of integral and sum. Then Hurwitz zeta function can be reproduced (1)

$$\frac{iP}{e^{iP}-I} x^{-z} = \frac{\Gamma(1-z)}{2\pi i} \int_{\mathcal{C}'} \frac{u^z}{e^u - 1} e^{ux} du = -z\zeta(z+1, x)$$

and the inverse

$$\frac{iP}{e^{iP}-I} x^{-z} = -z\zeta(z+1, x)$$

Then from (3)

$$\begin{aligned} \frac{1}{iP} \left( \frac{iP}{e^{iP}-I} \right) x^{-z} &= \int_{\frac{1}{2}}^x -z\zeta(z+1, \xi) d\xi \\ &= \zeta(z, x) - \zeta\left(z, \frac{1}{2}\right) \end{aligned} \tag{7}$$

□

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