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## A Remark on the Stabilization of Partially Linear Composite Systems

Abdelhak Ferfera, Abderrahmane Iggidr

*Abstract*—In this paper, we study the global stabilization, by means of smooth state feedback, of partially linear composite systems. We show how to compute the stabilizing feedback thanks to a weak Lyapunov function for a nonlinear subsystem instead of a strict one.

*Keywords*—Nonlinear systems, feedback, global stabilization, Lyapunov function.

### I. INTRODUCTION

Many recent papers (see [1], [2], [6] and references therein) addressed the problem of The global stabilization, by means of state feedback, of nonlinear control systems of the form:

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = Ay + Bu \end{cases} \quad (1)$$

where  $x \in \mathbb{R}^n, y \in \mathbb{R}^p, u \in \mathbb{R}^k, A \in \mathcal{M}_{p,p}(\mathbb{R}), B \in \mathcal{M}_{p,k}(\mathbb{R})$  and  $f$  is a smooth vector field such that:

(h1) The pair  $(A, B)$  is stabilizable.

(h2) The equilibrium  $x = 0$  of  $\dot{x} = f(x, 0)$  is globally asymptotically stable (G.A.S).

In [6], the authors assumed that the dependence of  $f(x, y)$  on  $y$  is of the form:

(h3)  $f(x, y) = f(x, 0) + G(x, y)Cy$ , with  $C \in \mathcal{M}_{k,p}(\mathbb{R})$  and that both  $C$  and  $B$  are of full rank.

They proved that there exist a matrix  $K \in \mathcal{M}_{k,p}(\mathbb{R})$  and a symmetric positive definite matrix  $P \in \mathcal{M}_{p,p}(\mathbb{R})$  satisfying the following three conditions:

(H1)  $P(A + BK) + (A + BK)^T P = -Q$ , with  $Q$  symmetric positive ( $T$ =transpose),

(H2)  $(Q^{1/2}, A + BK)$  detectable,

(H3)  $B^T P = C$ ,

if and only if the linear subsystem

$$\begin{cases} \dot{y} = Ay + Bu \\ \tilde{y} = Cy, \quad \tilde{y} \in \mathbb{R}^k \end{cases} \quad (2)$$

is invertible, weakly minimum phase and with  $CB$  symmetric positive definite.

Using these conditions, they showed that the system (1) is globally asymptotically stabilizable and they gave the stabilizing feedback:

$$u(x, y) = Ky - \frac{1}{2}G(x, y)^T \nabla V(x)$$

where  $V$  is a smooth Lyapunov function satisfying:

$$\langle \nabla V, f(x, 0) \rangle < 0 \quad \forall x \in \mathbb{R}^n, x \neq 0 \quad (3)$$

Notice that the existence of such a strict Lyapunov function  $V$  is assured by the condition (h1) and the inverse

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Lyapunov theorem (see [3], [5]). Unfortunately, there is no systematic method to compute a strict Lyapunov function for a given G.A.S system and it is often easier to construct a weak Lyapunov function for which the hypotheses of LaSalle's invariance principle (see [4]) are satisfied. As an example one can consider the following system which evolves in  $\mathbb{R}^2$  (Liénard's equation):

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g(x_1) - h(x_1)x_2 \end{cases} \quad (4)$$

where it is assumed that for all  $x \neq 0$ :

$$xg(x) > 0, \quad h(x) > 0$$

and

$$G(x) = \int_0^x g(s) ds \rightarrow \infty \text{ as } |x| \rightarrow \infty$$

For this system, it seems difficult to construct a strict Lyapunov function. However LaSalle's theorem can be applied in an obvious way by taking:

$$V(x_1, x_2) = \frac{1}{2}x_2^2 + G(x_1)$$

In this paper we show that to compute a stabilizing feedback for the system (1), we do not need to have a strict Lyapunov function for:

$$\dot{x} = f(x, 0) \quad (5)$$

We also state that the stabilization procedure is still valid when, in the decomposition (h3) of  $f$ , the matrix  $C$  is of rank  $m < k$ , provided that  $CB$  is of full rank.

### II. NOTATIONS AND DEFINITIONS

Before stating the main theorem let us introduce the following notations and definitions.

*Definition 1:* A  $C^1$  scalar function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is a weak Lyapunov function for the system on  $\mathbb{R}^n$ :

$$\dot{x} = X(x) \quad (6)$$

if  $V$  is positive definite proper and satisfies:

$$X.V(x) \leq 0, \quad \forall x \in \mathbb{R}^n$$

where  $X.V$  is the Lie-derivative of  $V$  along the trajectories of the vector field  $X$  ( $X.V(x) = \langle \nabla V(x), X(x) \rangle$  where  $\langle \cdot, \cdot \rangle$  is the inner product in  $\mathbb{R}^n$ ).

By a *proper function* we mean a function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\{x \in \mathbb{R}^n | V(x) \leq \xi\}$  is compact for each  $\xi > 0$ . Notice that if the vector field  $X$  satisfies the definition 1 then all the trajectories of the system (6) are bounded because of  $V$  is proper and its derivative is non positive. For such a vector field,  $X_t(\cdot)$  will denote the flow of  $X$  defined on  $\mathbb{R}^n$ . A subset  $E \in \mathbb{R}^n$  is said to be  $X$ -invariant if for any  $x \in E$  on has  $X_t(x) \in E, \forall t \geq 0$ .

*Definition 2:* We shall say that the system (6) is of LaSalle-type (L-T) if there exist a weak Lyapunov function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  for (6) such that the largest  $X$ -invariant set contained in  $E = \{x \in \mathbb{R}^n \mid X.V(x) = 0\}$  is reduced to the origin of  $\mathbb{R}^n$ .

*Remark 1:* The system (6) is of (L-T) if and only if it is globally asymptotically stable about the origin of  $\mathbb{R}^n$  (see [4]).

*Remark 2:* It is often easier to find a function  $V$  satisfying the definition 2 than one satisfying (3). This is typically the case for mechanical systems for example.

### III. STABILIZATION BY LASALLE'S INVARIANCE PRINCIPLE

*Theorem 1:* Assume that the pair (A,B) is stabilizable and **(H1)**, **(H2)** and **(H3)** hold. Then if the system (5) is of L-T, so is the closed-loop system (1) with the (stabilizing) feedback:

$$u(x, y) = Ky - G(x, y)^T \nabla V(x) \quad (7)$$

where  $V$  is a weak Lyapunov function for (5) as in the definition 2.

*Proof:* First of all, if the linear subsystem (2) satisfies **(H1)**, **(H2)** and **(H3)**, then it is invertible, weakly minimum phase and with  $CB$  symmetric positive definite, and it is possible to choose the matrix  $K \in \mathcal{M}_{k,p}(\mathbb{R})$  and the symmetric positive definite matrix  $P \in \mathcal{M}_{p,p}(\mathbb{R})$  such that:

$$y^T Q y = 0 \Rightarrow C y = 0 \quad (8)$$

Indeed, as done in [6], one can assume, without loss of generality, that (2) is in the special coordinate basis (see [7]):

$$\begin{cases} \dot{y}_{01} = A_{01}y_{01} + A_{11}y_1 \\ \dot{y}_{02} = A_{02}y_{02} + A_{12}y_1 \\ \dot{y}_1 = D_{01}y_{01} + D_{02}y_{02} + D_1y_1 + CBu \\ \tilde{y} = y_1 \end{cases}$$

with  $A_{01}$  Hurwitz,  $A_{02} + A_{02}^T = 0$ , and take:

$$K = (K_{01}, K_{02}, K_1)$$

with:

$$\begin{aligned} K_{01} &= -(CB)^{-1}D_{01} + A_{11}^T P_{01} \\ K_{02} &= (CB)^{-1}D_{02} + A_{12}^T P_{01} \\ K_1 &= (CB)^{-1}D_1 + \frac{1}{2}I \end{aligned}$$

and:

$$P = \begin{pmatrix} P_{01} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (CB)^{-1} \end{pmatrix}$$

with  $P_{01}$  symmetric positive definite such that:

$$P_{01}A_{01} + A_{01}^T P_{01} = -I$$

This particular choice of  $K$  and  $P$  leads to:

$$Q = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}$$

Hence,  $y^T Q y = \|y_{01}\|^2 + \|y_1\|^2$  and so:

$$y^T Q y = 0 \Rightarrow C y = y_1 = 0$$

Assume now that the system (5) is of L-T, and set  $X(x) = f(x, 0)$ ,  $x \in \mathbb{R}^n$ . Let  $V$  be a weak Lyapunov function for (5) as in the definition 2 and denote by  $\Omega$  the largest invariant set by  $X$  contained in the locus  $E = \{x \in \mathbb{R}^n \mid X.V(x) = 0\}$ . By hypotheses **(H1)** – **(H3)**, for  $x \in \mathbb{R}^n$  and  $y \in \mathbb{R}^p$ , set:

$$Z(x, y) = \begin{pmatrix} f(x, y) \\ h(x, y) \end{pmatrix}$$

where:

$$\begin{aligned} h(x, y) &= Ay + Bu(x, y) \\ &= (A + BK)y - BG(x, y)^T \nabla V(x) \end{aligned}$$

and define (see [6]):

$$W(x, y) = V(x) + \frac{1}{2}y^T P y$$

$W$  is of class  $C^1$ , definite positive and proper, and its derivative along the trajectories of the vector field  $Z$  is given by:

$$\begin{aligned} \dot{W}(x, y) &= Z.W(x, y) \\ &= \langle Z(x, y), \nabla W(x, y) \rangle \\ &= X.V(x) + \langle \nabla V(x), G(x, y)Cy \rangle \\ &\quad + \frac{1}{2}y^T Q y + \langle y, -PBG(x, y)^T \nabla V(x) \rangle \\ &= X.V(x) + \frac{1}{2}y^T Q y \\ &\quad + \langle y, C^T G(x, y)^T \nabla V(x) \\ &\quad - PBG(x, y)^T \nabla V(x) \rangle \end{aligned}$$

So, by use of **(H3)** one has:

$$\dot{W}(x, y) = X.V(x) + \frac{1}{2}y^T Q y \leq 0$$

Notice that all the trajectories of the closed-loop system are bounded because of  $W$  is proper and its derivative is non positive. Set:

$$\begin{aligned} \tilde{E} &= \{(x, y) \in \mathbb{R}^{n+p} \mid Z.W(x, y) = 0\} \\ &= \{(x, y) \in \mathbb{R}^{n+p} \mid X.V(x) = 0, \text{ and } y^T Q y = 0\} \end{aligned}$$

According to LaSalle's theorem (see [4] pp 66-67) all the solutions of the closed-loop system tend to  $\tilde{\Omega}$  the largest invariant set by  $Z$  contained in  $\tilde{E}$ . In order to prove the theorem 1 let us show that  $\tilde{\Omega}$  is the origin of  $\mathbb{R}^{n+p}$ . By (8), on  $\tilde{E}$  the vector field  $Z$  is given by:

$$Z(x, y) = \begin{pmatrix} X(x) \\ Y(x, y) \end{pmatrix}$$

where:

$$\begin{aligned} X(x) &= f(x, 0) \\ Y(x, y) &= (A + BK)y - BG(x, y)^T \nabla V(x) \end{aligned}$$

so that on  $\tilde{E}$  the closed-loop system becomes:

$$\begin{cases} \dot{x} = f(x, 0) = X(x) \\ \dot{y} = (A + BK)y - BG(x, y)^T \nabla V(x) \end{cases}$$

Let  $(x(t), y(t))$  be a solution of the above system with  $(x(0), y(0)) = (x, y) \in \tilde{\Omega}$ . Since  $\tilde{\Omega}$  is  $Z$ -invariant we have  $(x(t), y(t)) \in \tilde{\Omega}$  for all  $t \geq 0$ . But one has:

$$\frac{d}{dt}(x(t)) = X(x(t))$$

so that  $x(t) = X_t(x)$ . Consider now the following set:

$$M = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^p, \text{ such that } (x, y) \in \tilde{\Omega}\}$$

If  $x \in M$  then  $(x, y) \in \tilde{\Omega}$  for some  $y \in \mathbb{R}^p$ , and for all  $t \geq 0$ ,  $(x(t), y(t)) = (X_t(x), y(t)) \in \tilde{\Omega}$  since  $\tilde{\Omega}$  is  $Z$ -invariant, that implies  $X_t(x) \in M$ . Then  $M$  is  $X$ -invariant which implies that  $M \subset \Omega = \{0\}$ . So we have shown that:

$$(x, y) \in \tilde{\Omega} \Rightarrow x = 0$$

Since  $x(t) = 0$  for all  $t \geq 0$ ,  $y(t)$  becomes a solution of  $\dot{y} = (A + BK)y$  and  $y(t)^T Q y(t) = 0$  for all  $t \geq 0$ . Hence, from **(H2)** one deduce that  $y(t) = 0$  for all  $t \geq 0$  and so  $\tilde{\Omega} = \{(0, 0)\}$  which completes the proof of theorem 1. ■

*Example:* Consider the following system evolving in  $\mathbb{R}^4$ :

$$\begin{cases} \dot{x}_1 = x_2 + (x_1 y_1)^{4/3} y_2 \\ \dot{x}_2 = -x_1^{5/3} - x_1^{4/3} x_2 + (x_1 y_2)^{4/3} y_2 \\ \dot{y}_1 = y_2 \\ \dot{y}_2 = u \end{cases} \quad (9)$$

The subsystem:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^{5/3} - x_1^{4/3} x_2 \end{cases}$$

is of the form (4) and so it is of L-T thanks to the weak Lyapunov function:

$$V(x_1, x_2) = \frac{1}{2} x_2^2 + \frac{3}{8} x_1^{8/3}$$

Besides, the assumptions **H1** – **H3** hold for the linear subsystem:

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = u \\ \tilde{y} = y_2 \end{cases}$$

with  $K = (-1, -\frac{1}{2})$  and  $P = I$ . Hence the system (9) satisfies the conditions of the theorem 1 and so it is stabilizable thanks to the feedback:

$$u(x_1, x_2, y_1, y_2) = -y_1 - \frac{1}{2} y_2 - x_1^3 y_1^{4/3} - x_2 (x_1 y_2)^{4/3}$$

*Remark 3:* Throughout all this work it is supposed that in **(h3)** one has  $C \in \mathcal{M}_{k,p}(\mathbb{R})$ , so that the linear subsystem (2) has the same number of inputs and outputs. This restriction can be relaxed by assuming that  $C \in \mathcal{M}_{m,p}(\mathbb{R})$ ,  $m \leq k$ , and (2) is right invertible, weakly minimum phase and with  $CB$  of full rank. To make this, notice that, as mentioned in [6], if  $m = k$  the assumption  $CB$  symmetric positive definite can be replaced by  $CB$  nonsingular thanks to the use of a static precompensator  $u = (CB)^{-1} \tilde{u}$ . Then the remark is deduced from the following proposition.

*Proposition 1:* Assume that  $m < k \leq p$  and that both  $B$  and  $CB$  are of full rank. Then there exists a matrix function  $G'(x, y) \in \mathcal{M}_{n,k}(\mathbb{R})$ ,  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^p$ , and a constant matrix  $C' \in \mathcal{M}_{k,p}(\mathbb{R})$  such that  $C'B$  is nonsingular and:

$$G(x, y)C = G'(x, y)C', \quad \forall (x, y) \in \mathbb{R}^{n+p}$$

Furthermore, if (2) is weakly minimum phase, so is the linear system:

$$\begin{cases} \dot{y} = Ay + Bu \\ \tilde{y}' = C'y, \quad \tilde{y}' \in \mathbb{R}^k \end{cases} \quad (10)$$

*Proof:* From the full rank property of  $B$  and  $CB$ , it is always possible to choose  $\tilde{C} \in \mathcal{M}_{k-m,p}(\mathbb{R})$  in such a way that the block-matrix:

$$C' = \begin{pmatrix} C \\ \tilde{C} \end{pmatrix} \quad (11)$$

satisfies  $C'B$  nonsingular. For such a choice, and taking  $G'(x, y) = (G(x, y), 0)$ , one has:

$$G(x, y)C = G'(x, y)C'$$

Furthermore, from (11) one can deduce that the zero dynamics of (10) are included in those of (2) which completes the proof of the proposition. ■

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