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Contribution of kinetic electrons to GAM damping

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Abstract. This paper presents a calculation of the contribution of electrons to GAM damping. It appears that this contribution is usually not negligible due to a resonance between barely trapped/passing electrons bounce/transit frequency and the mode pulsation.

1. Introduction

Gyrokinetic codes are often verified by setting an initial flow zonal perturbation, characterizing its subsequent time evolution, and comparing it with existing analytical calculations (see e.g. [1]). This test provides the residual value of the flow, predicted by Rosenbluth and Hinton [2], and also the pulsation and damping rate of the Geodesic Acoustic Mode (GAM). The GAM frequency is well documented in the particular case of a single ion species with adiabatic electrons [3, 4, 5, 6, 7]. It usually agrees well with simulations. However adding kinetic electrons change somewhat the picture. While the residual flow and the GAM frequency are mostly unchanged [8], the GAM damping rate increases significantly [9]. This is attributed to a resonance between GAM and barely trapped/passing electron bounce/transit pulsations. The resonance amplifies the exchange of energy between this class of particles and the mode, and thus enhances GAM damping.

The objective of this work is to provide an analytical estimate of the kinetic electron contribution to GAM damping, which does not seem to be available in the literature, except one attempt for passing electrons [10]. A variational formulation is used, which is close to the one used previously to study EGAMs [11]. A generalized dispersion relation is proposed, that is shown to depend on a single dimensionless number, that measures the ratio of the GAM to the bounce frequencies. An explicit expression, though simplified, is also derived. It appears that indeed barely trapped/passing electrons enhance GAM damping, while affecting weakly the real part of the GAM pulsation.

The paper is organised as follows. The derivation of a lagrangian form is given in section 2, while the calculation of GAM electron damping is calculated in section 3. A discussion and conclusion follow.

2. Basic equations

2.1. Lagrangian formulation of electro-neutrality

We anticipate that diamagnetic effects are negligible for GAMs, i.e. we consider a plasma of electrons and hydrogenoid ions with constant density $N_e = N_i$, and constant electron and ion

temperatures T_e and T_i . The electric potential is noted Φ . It is most conveniently normalized to the ion temperature, i.e. $\phi = \frac{e\Phi}{T_i}$, where T_i is the ion temperature and e the proton charge. We consider a potential perturbation that oscillates in time with a given complex pulsation ω , i.e. $\phi(\mathbf{x}, t) = \phi_\omega(\mathbf{x}) \exp(-i\omega t) + c.c.$, where "c.c." means "complex conjugate". The perturbed distribution function of each species is normalized to the corresponding density N , and noted $f(\mathbf{x}, \mathbf{p}, t)$, where \mathbf{x} and \mathbf{p} are the position and momentum variables. To simplify the notations we omit to indicate explicitly species labels in this section. The charge number is noted Z (i.e. $Z = 1$ for ions and $Z = -1$ for electrons) and $\tau = \frac{T_i}{T}$ denotes the ratio of a species temperature to the ion temperature. The perturbed distribution function $f(\mathbf{x}, \mathbf{p}, t)$ is written in the same way as the potential. The electro-neutrality condition can be written in a variational form, which states that the functional

$$\mathcal{L}_\omega = -N_i T_i \sum_{species} Z \int d\zeta f_\omega(\mathbf{x}, \mathbf{p}) \phi_\omega^*(\mathbf{x}) \quad (1)$$

is extremum with respect to any variation of the electric potential ϕ_ω^* . Here $d\zeta = d^3\mathbf{x}d^3\mathbf{p}$ is the volume element in the phase space. The perturbed distribution function f_ω can be separated in adiabatic and non adiabatic responses, i.e. $f_\omega = -F_M Z/\tau \phi_\omega + g_\omega$. The unperturbed distribution function F_M is a Maxwellian normalized to the density $F_M = (2\pi mT)^{-3/2} \exp\{-H_{eq}/T\}$, where H_{eq} is the unperturbed Hamiltonian. The total distribution function $F = F_M + f$ is solution of the Vlasov equation $\frac{\partial F}{\partial t} - [H, F] = 0$ where $H = H_{eq} + e\Phi$ is the total Hamiltonian. In the following we ignore the mean electric potential, which is irrelevant for the GAM dynamics. A set of action-angle variables (α, \mathbf{J}) can be constructed to describe the non perturbed trajectories of particles. This is a consequence of the existence of 3 motion invariants of the unperturbed system, namely the Hamiltonian H_{eq} , the magnetic moment μ , and the canonical toroidal momentum $P_\varphi = -e\psi + mv_\parallel \frac{B_\varphi}{B} R$, where ψ is the poloidal flux normalized to 2π . The first angle is the cyclotron angle, and the corresponding action is proportional to the magnetic moment $J_1 = \frac{m}{e} \mu$. The second and third angles are related to the guiding center motion. More precisely the third angle is equal to the toroidal angle up to an offset that is a periodic function of the second angular variable α_2 . The corresponding action is the canonical toroidal momentum P_φ . The second angle describe the bounce (resp. transit) motion of trapped (resp. passing) particles. The corresponding action can be derived explicitly, but is of little use here as it can be replaced by the energy H_{eq} at given μ and P_φ . Thanks to its periodicity with respect to the angle variables, the perturbed hamiltonian can be developed as a Fourier series $\phi_\omega(\mathbf{x}) = \sum_{\mathbf{n}=(n_1, n_2, n_3)} h_{\mathbf{n}\omega}(\mathbf{J}) \exp(i\mathbf{n} \cdot \alpha)$. Using the hamiltonian character of the dynamics, a linear solution of the Vlasov equation is easily derived

$$g_{\mathbf{n}\omega} = F_M \frac{Z}{\tau} \frac{\omega}{\omega - \mathbf{n} \cdot \boldsymbol{\Omega} + i0^+} h_{\mathbf{n}\omega} \quad (2)$$

This yields the following expression of the Lagrangian

$$\mathcal{L}_\omega = N_i T_i \sum_{species} \frac{1}{\tau} \int d^3\mathbf{x} \phi_\omega \phi_\omega^* - N_i T_i \sum_{species} \frac{1}{\tau} \sum_{\mathbf{n}} \int d\zeta F_M \frac{\omega}{\omega - \mathbf{n} \cdot \boldsymbol{\Omega} + i0^+} h_{\mathbf{n}\omega} h_{\mathbf{n}\omega}^* \quad (3)$$

The imaginary part of the resonant integral is most easily calculated in the action/angle space, using the volume element $d\zeta = d^3\alpha d^3\mathbf{J}$. Also the unperturbed Hamiltonian $H_{eq}(\mathbf{J})$ is a function of the actions only and $\boldsymbol{\Omega}$ is a set of resonant frequencies $\Omega_i = \frac{\partial H_{eq}}{\partial J_i}$. Since the GAM frequency is much lower than the cyclotron frequency, only $n_1 = 0$ components are kept. Moreover, a GAM has a toroidal wave number that is null, which implies $n_3 = 0$ since $\alpha_3 = \varphi$ up to a periodic function of α_2 . Therefore the summation should be run on n_2 indices only.

2.2. Trajectories and hamiltonian components

We use a simplified geometry of circular concentric magnetic surfaces, labeled by their minor radius r . The angles (θ, φ) are the poloidal and toroidal angles. The radius and poloidal angle of the guiding center position read $r_G = r + \hat{r}$ and $\theta_G = \epsilon_c \alpha_2 + \hat{\theta}$ where $\epsilon_c = 1$ (resp. 0) for passing particles. The functions \hat{r} and $\hat{\theta}$ are functions of the actions \mathbf{J} , or equivalently to the motion invariants (H_{eq}, μ, P_φ) , and periodic functions of α_2 . We will omit to mention explicitly the dependencies on the motion invariants, to simplify the notations. The GAM electric potential can be expanded in Fourier series with respect to the poloidal angle and minor radius

$$\phi_\omega(r, \theta) = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} \frac{dK}{2\pi} \tilde{\phi}_{m\omega}(K) e^{iKr + im\theta} \quad (4)$$

The Fourier components $\tilde{\phi}_{m\omega}(K)$ are determined by initial conditions. In most theoretical calculations, the radial structure is the same for all poloidal harmonics, and in fact just one radial wave vector is chosen. Using the trajectory equations, the correspondence between Fourier and Hamiltonian components is found to be

$$h_{\mathbf{n}\omega}(\mathbf{J}) = e^{iKr} J_0 \sum_{m=-\infty}^{+\infty} \oint \frac{d\alpha_2}{2\pi} \tilde{\phi}_{m\omega}(K) e^{i[K\hat{r} + m\hat{\theta} + (\epsilon_c m - n_2)\alpha_2]} \quad (5)$$

where $\mathbf{n} = (0, n_2, 0)$ and \oint means an integral over a period in α_2 .

3. GAM damping

The exchange of energy between particles and waves per volume and time unit is given by the relation $W = 2\omega \Im(L)$, where $\mathcal{L} = \int d^3\mathbf{x} L$. A positive value of $W > 0$ means particle heating, i.e. damping. If $\Im(L) \ll \Re(L)$, the real part of the GAM pulsation is given by the dispersion relation $\Re(L) = 0$, and the damping rate is then calculated perturbatively. The above methodology provides an expression of $\Re(L)$ [6] that coincides with previous expressions [3, 4], thus providing the classical expression of the GAM pulsation at second order in $K\rho_i$. The additional damping due to electrons can be estimated by computing the particle Lagrangian of electrons.

3.1. Electron Lagrangian

Deeply passing electrons contribute weakly to GAM damping because of their large velocities. This is due to the mismatch between transit and GAM frequency. However barely passing/trapped electrons do contribute to GAM damping because their bounce/transit frequency becomes small near the passing/trapped domain and meets a resonance with the GAM pulsation [8]. Using Eq.(3), the electron resonant Lagrangian (both trapped and passing) reads

$$\mathcal{L}_{res,e} = -N_i T_i \frac{1}{\tau_e} \sum_{n_2=-\infty}^{+\infty} \int d\zeta F_M \frac{\omega}{\omega - n_2 \Omega_2 + i0^+} |h_{\mathbf{n}\omega}|^2 \quad (6)$$

where $\tau_e = T_e/T_i$. We introduce a normalized transit pulsation $\Omega_b = \Omega_2 q R_0 / v_{Te}$ where $v_{Te} = \sqrt{T_e/m_e}$ is the electron thermal velocity. We will also make use of the scaling parameter $\sigma = q\tau_e^{-1/2} (m_e/m_i)^{1/2}$, which is proportional to the square root of the electron to ion mass ratio, hence a number that is small compared to one. The electron transit frequency normalized to the ion transit frequency can then be written $\Omega_2 \frac{R_0}{v_{Ti}} = \frac{\Omega_b}{\sigma}$. The resonance condition requires $\Omega_b \simeq \sigma \ll 1$.

The volume integrated energy transfer from a GAM to electrons is $\mathcal{W}_e = 2\omega \Im(\mathcal{L}_{res,e})$. We

Notations	$\lambda = \frac{\mu B_0}{H}$	$\epsilon_b(\lambda)$	$\theta_0(\lambda)$	κ
Passing	$0 \leq \lambda \leq \lambda_{min} = \frac{1}{1+\epsilon}$	0	π	$0 \leq \kappa \leq 1$
Trapped	$\lambda_{min} \leq \lambda \leq \lambda_{max} = \frac{1}{1-\epsilon}$	1	$-\pi \leq \theta_0 \leq \pi$	$1 \leq \kappa \leq +\infty$

Table 1. Notations and conventions for passing and trapped particles.

introduce a normalized Lagrangian $\bar{\mathcal{L}}_{res,e} = \mathcal{L}_{res,e} \tau_e / [N_i T_i K^2 \rho_i^2 |\phi_0|^2]$ which is readily written as

$$\bar{\mathcal{L}}_{res,e} = - \sum_{n_2=-\infty}^{+\infty} \int d\zeta F_M \frac{\Omega}{\Omega - n_2 \frac{\Omega_b}{\sigma} + i0^+} |\tilde{h}_{n_2\omega}|^2 \quad (7)$$

where $\tilde{h}_\omega(\theta) = h_{n\omega}(\theta / (K\rho_i\phi_0))$. To simplify the notations, we omit explicit dependencies of the Hamiltonian on the actions and the first and third angle variables. Also from now on, the second angle variable is noted α instead of α_2 . The perturbed Hamiltonian reads

$$\tilde{h}_\omega(\alpha) = \sum_{n_2=-\infty}^{+\infty} \tilde{h}_{n_2\omega} e^{in_2\alpha} \quad \leftrightarrow \quad \tilde{h}_{n_2\omega} = \oint \frac{d\alpha}{2\pi} \tilde{h}_\omega(\alpha) e^{-in_2\alpha} \quad (8)$$

An equivalent form of the Lagrangian Eq.(7) is

$$\bar{\mathcal{L}}_{res,e} = 2i\pi\sigma\Omega \int d\zeta \frac{F_M}{|\Omega_b|} \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} \int_{-\infty}^{\infty} \frac{d\alpha'}{2\pi} \Theta[\epsilon_{\parallel}(\alpha' - \alpha)] \tilde{h}_\omega(\alpha) \tilde{h}_\omega^*(\alpha') \exp\left\{i\sigma \frac{\Omega}{\Omega_b} (\alpha' - \alpha)\right\} \quad (9)$$

where α' is an extended integration variable that spans the whole real axis $[-\infty, +\infty]$, Θ is the Heaviside function ($\Theta(x) = 1$ for $x > 0$, 0 otherwise), $\epsilon_{\parallel} = \text{sign}(\Omega_b)$ is the sign of the parallel velocity for passing particles, and we choose $\epsilon_{\parallel} = 1$, $\Omega_b > 0$ for trapped particles. This expression is obtained by solving the Vlasov equation in the angular variable α . The main advantage of Eq.(9) is to better identify the parametric dependencies.

3.2. Phase space volume element

The phase space density $F_M d\zeta$ in the Lagrangian Eq.(9) reads

$$F_M d\zeta = d\mathcal{V} \sum_{\epsilon_{\parallel}=\pm 1} d\lambda \sqrt{\frac{2}{\pi}} dv v^3 \frac{1}{|\Omega_b|} e^{-v^2} \quad (10)$$

where $d\mathcal{V} = 4\pi^2 R_0 r dr$ is the volume element, $\lambda = \mu B_0 / H_{eq}$ is a pitch angle variable, ϵ_{\parallel} is the sign of the parallel velocity (for passing particles only), and v is a normalized velocity $v^2 = [\frac{1}{2}mv_{\parallel}^2 + \mu B] / T$. The bounce/transit pulsation reads

$$\frac{1}{|\Omega_b|} = \frac{\sqrt{2}}{v} \frac{1 + \epsilon_b(\lambda)}{2} \int_{-\theta_0}^{\theta_0} \frac{d\theta}{2\pi} \frac{1}{(1 - \lambda + \epsilon\lambda \cos \theta)^{1/2}} \quad (11)$$

Here θ_0 is the poloidal angle at the turning point, i.e. the positive solution of $v_{\parallel}(E, \lambda, \theta) = 0$, while $\theta_0 = \pi$ for passing particles. The meaning of the notations for trapped and passing particles is given in Table 1. The pitch-angle variable λ does not allow an easy handling of the singularity at the passing/trapped boundary. It is therefore useful to introduce an alternative variable κ defined as

$$\kappa^2 = \frac{2\epsilon\lambda}{1 - \lambda(1 - \epsilon)} \quad \rightarrow \quad d\lambda = \frac{4\epsilon}{(1 + \epsilon)^2} \frac{\kappa d\kappa}{\Lambda^2(\kappa)} \quad (12)$$

It appears that

$$\frac{v_{\parallel}}{v_{Te}} = \epsilon_{\parallel} \left(\frac{\epsilon}{1+\epsilon} \right)^{1/2} \frac{2v}{\Lambda^{1/2}(\kappa)} \left[1 - \kappa^2 \sin^2 \left(\frac{\theta}{2} \right) \right]^{1/2} \quad ; \quad \left| \frac{\partial \theta}{\partial \alpha} \right| = 2\tau(\kappa) \left[1 - \kappa^2 \sin^2 \left(\frac{\theta}{2} \right) \right]^{1/2} \quad (13)$$

where $\Lambda(\kappa) = \frac{2\epsilon+(1-\epsilon)\kappa^2}{1+\epsilon}$ is smooth near $\kappa = 1$, and $\Lambda(1) = 1$. The bounce/transit frequency can be explicitied in terms of the complete elliptical function of the first kind K , namely

$$\frac{1}{\Omega_b} = \epsilon_{\parallel} \left(\frac{1+\epsilon}{\epsilon} \right)^{1/2} \Lambda^{1/2}(\kappa) \frac{\tau(\kappa)}{v} \quad ; \quad \tau(\kappa) = \frac{1}{\pi} \begin{cases} K(\kappa^2) & 0 \leq \kappa < 1 \\ \frac{2}{\kappa} K\left(\frac{1}{\kappa^2}\right) & 1 \leq \kappa \leq +\infty \end{cases} \quad (14)$$

The relation between the angles θ and α involves the Jacobi elliptic function sn , i.e.

$$\sin \left(\frac{\theta}{2} \right) = \begin{cases} sn(\tau\alpha, \kappa^2) & 0 \leq \kappa < 1 \\ \frac{1}{\kappa} sn\left(\kappa\tau\alpha, \frac{1}{\kappa^2}\right) & 1 < \kappa < +\infty \end{cases} \quad (15)$$

Using this set of variables, the integrand of the Lagrangian Eq.(9) is recast as

$$\frac{F_M}{|\Omega_b|} d\zeta = 4\sqrt{\frac{2}{\pi}} \frac{1}{1+\epsilon} d\mathcal{V} \sum_{\epsilon_{\parallel}=\pm 1} \frac{\kappa d\kappa}{\Lambda(\kappa)} \tau^2(\kappa) v dv e^{-v^2} \quad (16)$$

3.3. Electron contribution to the GAM dispersion relation

An important quantity that appears in the Lagrangian Eq.(9) is the ratio $\sigma\Omega/|\Omega_b|$, which reduces to $\sigma\Omega/|\Omega_b| = \sigma^* \tau(\kappa) \Lambda^{1/2}(\kappa)/v$ in the limit of small inverse aspect ratio $\epsilon \rightarrow 0$. The parameter σ^* , defined as

$$\sigma^* = \frac{\sigma}{\epsilon^{1/2}} \Omega = \left(\frac{m_e}{m_i \tau_e} \right)^{1/2} \frac{q}{\epsilon^{1/2}} \Omega \quad (17)$$

is the key dimensionless parameter that characterises the resonance of a GAM with barely trapped/passing electrons. Indeed the condition $\sigma^* \simeq 1$ corresponds to the condition $\omega \simeq \omega_b$, where the bounce frequency ω_b scales as $v_{Te} \epsilon^{1/2}/(qR_0)$. Introducing $\bar{L}_{res,e}$ the Lagrangian per volume unit defined as $\bar{L}_{res,e} = \partial \bar{\mathcal{L}}_{res,e} / \partial \mathcal{V}$, the imaginary part of the Lagrangian Eq.(9) reads

$$\Im(\bar{L}_{res,e}) = 8\sqrt{2\pi} \epsilon^{1/2} \sigma^* \sum_{\epsilon_{\parallel}=\pm 1} \int_0^{+\infty} \frac{\kappa d\kappa}{\Lambda(\kappa)} \tau^2(\kappa) \int_0^{+\infty} dv v \exp(-v^2) \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} \int_{-\infty}^{\infty} \frac{d\alpha'}{2\pi} \Theta[\epsilon_{\parallel}(\alpha' - \alpha)] \tilde{h}_{\omega}(\alpha) \tilde{h}_{\omega}^*(\alpha') \cos \left\{ \sigma^* \frac{\tau}{v} \Lambda^{1/2} \epsilon_{\parallel} (\alpha' - \alpha) \right\} \quad (18)$$

or equivalently

$$\Im(\bar{L}_{res,e}) = 4\sqrt{2\pi} \epsilon^{1/2} \sigma^* \sum_{\epsilon_{\parallel}=\pm 1} \sum_{n_2=1}^{+\infty} \int_0^{+\infty} \frac{\kappa d\kappa}{\Lambda(\kappa)} \tau^2(\kappa) \int_0^{+\infty} dv v \exp(-v^2) \delta \left(n_2 - \sigma^* \frac{\tau}{v} \Lambda^{1/2} \right) \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} \int_{-\pi}^{\pi} \frac{d\alpha'}{2\pi} \tilde{h}_{\omega}(\alpha) \tilde{h}_{\omega}^*(\alpha') \exp \{ i n_2 (\alpha' - \alpha) \} \quad (19)$$

when using Eq.(6) where resonances are explicit. The resonance match parameter σ^* is smaller than 1 for usual plasma parameters. An estimate of the integral Eq.(18) appears to be quite difficult. Nevertheless some exact results can be derived. It is reminded that the main GAM poloidal dependence is of the form $\tilde{h}_{\omega}(\alpha) \sim \sin[\theta(\alpha)]$. Using the relation

$\partial^2\theta/\partial\alpha^2 = -\tau^2(\kappa)\kappa^2 \sin\theta$, a Taylor expansion in powers of σ^* of the cosine function in Eq.(18) shows that the $o([\sigma^*])$ term cancels exactly so that the next order is $o([\sigma^*]^3)$. Hence it appears that most particles contribute to an imaginary part of the Lagrangian Eq.(18) that scales as $o([\sigma^*]^3)$ when $\sigma^* \rightarrow 0$. This yields a very small contribution to damping and can be neglected against ion damping in most conditions. However this expansion breaks down whenever $\sigma^*\tau(\kappa)\Lambda^{1/2}(\kappa) > v$, i.e. near resonant curves $\sigma^*\tau(\kappa)\Lambda^{1/2}(\kappa) = n_2v$ in the phase space (v, κ) . Since σ^* is small, this requires large values of the period $\tau(\kappa)$ or small values of the velocity modulus v . This situation occurs near the trapped/passing boundary $\kappa \sim 1$, where $\tau \sim \ln|\kappa - 1|$, or at low velocities $v \simeq \sigma^*$. However because of the integrand in velocity that behaves as v , the contribution from low velocities is quite small (typically $[\sigma^*]^3$). This means that most of the integral comes from a boundary layer near the rapped/passing boundary $\kappa \sim 1$, as expected. The exact calculation of Eq.(19) (or equivalently Eq.(18)) in the region $\kappa \sim 1$ is difficult. Hence we have to resort to some approximations. One expect the bounce integrals in (α, α') to be dominated by locations where particles slow down or bounce back (turning points). Turning point are hidden when using angle variables, but appear more clearly when noting that for any function $h[\theta(\alpha)]$, even in α , one has the identity

$$\int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} h[\theta(\alpha)] = \frac{\int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{1-\kappa^2 \sin^2 \frac{\theta}{2}}} h(\theta)}{\int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{1-\kappa^2 \sin^2 \frac{\theta}{2}}}} \quad (20)$$

If $h(\theta)$ is a smooth function, a proxy of $h(\alpha)$ is $\pi \{h(\theta_0)\delta(\alpha - \frac{\pi}{2}) + h(-\theta_0)\delta(\alpha + \frac{\pi}{2})\}$. Because of the oscillating function $\exp\{in_2(\alpha' - \alpha)\}$, most of the contribution comes from the neighborhood of $\alpha = \alpha' = \frac{\pi}{2}$ and $\alpha = \alpha' = -\frac{\pi}{2}$. If a large number of trapped particle resonances $n_2 = 1, 2, \dots$ is involved, the summation over n_2 can be replaced by a continuous integral that is trivial to execute since it applies on δ functions (integral is 1/2 as it spans half the real axis - also we use $2 \int_0^{+\infty} dv v \exp(-v^2) = 1$). Note that for passing particles, most of the weight comes from $\theta = \pi$, which is a location where the mode vanishes - hence their contribution is negligible in this simplified calculation. This procedure provides an estimate of the form function

$$\Im(\bar{L}_{res,e}) \simeq \sqrt{\frac{\pi}{2}} \epsilon^{1/2} \sigma^* \int_1^{+\infty} \frac{\kappa d\kappa}{\Lambda(\kappa)} \tau^2(\kappa) \left| \tilde{h}_\omega(\theta_0(\kappa)) \right|^2 \quad (21)$$

where $\tilde{h}_\omega(\theta_0)$ is the value of the perturbed Hamiltonian at the bounce point when expressed in the θ variable. The normalised amplitude of the Hamiltonian perturbation $\tilde{h}_\omega(\theta) = h_{n\omega}(\theta)/(K\rho_i\phi_0)$ appears to be of the form $\tilde{h}_\omega(\theta) = \eta_1 \sin(\theta)$, where [6]

$$\eta_1 = - \left\{ 1 + \frac{2 + \tau_e}{q^2\Omega^2} + o\left(\frac{1}{q^4\Omega^4}\right) \right\} \frac{\tau_e}{\Omega} + o(K\rho_i) \quad (22)$$

Combining the exact expression of the Lagrangian Eq.(19) (or equivalently Eq.(18)) with the estimate Eq.(21) one finds

$$\Im(\bar{L}_{res,e}) = G(q, \tau_e) \epsilon^{1/2} \sigma^* \mathcal{D}(\sigma^*) \quad (23)$$

where G is a function of q and τ_e only (in the limit of large aspect ratio)

$$G(q, \tau_e) = \eta_1^2 \sqrt{\frac{\pi}{2}} \int_0^{+\infty} d\kappa \kappa \frac{\tau^2(\kappa)}{\Lambda(\kappa)} \sin^2(\theta_0(\kappa)) \quad (24)$$

and $\sin\left(\frac{\theta_0}{2}\right) = \frac{1}{\kappa}$. The function $\mathcal{D}(\sigma^*)$ measures the weight of the region in the phase space where $\sigma^*\tau(\kappa)\Lambda^{1/2}(\kappa)/v > 1$, i.e. the number of near resonant particles. The special case $\mathcal{D}(\sigma^*) = 1$

corresponds to the estimate based on a strong weight of bounce points, i.e. when Eq.(21) is exact. Since only a fraction of particles fulfill this criterion, one expects that $\mathcal{D}(\sigma^*) \leq 1$. Strictly speaking the form factor $\mathcal{D}(\sigma^*)$ depends also on ϵ , but this dependence is weak in the limit of large aspect ratio. One important consequence of Eq.(23) is the dimensionless form $\mathfrak{S}(\bar{L}_{res,e})$, which offers an efficient way to probe the parametric dependencies. Since the frequency Ω and potential amplitude are functions of (q, τ_e) only, this means that the mass scaling provides a strong constraint on the dependence on safety factor q and electron to ion ratio temperature $\tau_e = T_e/T_i$.

4. Estimate of the GAM damping rate due to electrons

Using Eqs.(23), and $W_e = 2\Omega\mathfrak{S}(L_{res,e})\frac{v_{Ti}}{R_0}$, the following expression of the power exchange between electrons and GAM is found

$$W_e = 2G(q, \tau_e)\Omega^2\mathcal{D}(\sigma^*)\left(\frac{m_e}{m_i}\right)^{1/2}\frac{q}{\tau_e^{3/2}}N_iT_i\frac{v_{Ti}}{R_0}K^2\rho_i^2|\phi_0|^2 \quad (25)$$

The total dispersion relation with electrons is $\bar{L} = 0$, where

$$\begin{aligned} \bar{L} = & K^2\rho_i^2|\phi_0|^2\left\{\Lambda_1 - \Lambda_2K^2\rho_i^2 + i\frac{1}{2}\sqrt{\frac{\pi}{2}}q^5\Omega^3e^{-\frac{q^2\Omega^2}{2}}\left[1 + 2\frac{1+2\tau_e}{q^2\Omega^2}\right]\right. \\ & \left.+ i\frac{1}{1024}\sqrt{\frac{\pi}{2}}K^2\rho_i^2q^9\Omega^5e^{-\frac{q^2\Omega^2}{8}}\left[1 + 16\frac{1+\tau_e}{q^2\Omega^2}\right] + iG(q, \tau_e)\mathcal{D}(\sigma^*)\left(\frac{m_e}{m_i}\right)^{1/2}\frac{q}{\tau_e^{3/2}}\Omega\right\} \end{aligned}$$

where

$$\Lambda_1 = 1 - \left(\frac{7}{2} + 2\tau_e\right)\frac{1}{\Omega^2} - \left(\frac{23}{2} + 8\tau_e + 2\tau_e^2\right)\frac{1}{q^2\Omega^4} \quad (26)$$

and

$$\Lambda_2 = \frac{3}{4} - \left(\frac{13}{2} + 6\tau_e + 2\tau_e^2\right)\frac{1}{\Omega^2} + \left(\frac{747}{8} + \frac{481}{8}\tau_e + \frac{35}{2}\tau_e^2 + 2\tau_e^3\right)\frac{1}{\Omega^4} \quad (27)$$

A rough estimate of the damping rate due to electrons is obtained by fitting the parenthesis in the real part of the dispersion relation by $1 - \frac{\Omega_0^2}{\Omega^2}$, where Ω_0 is solution of $\Re[\bar{L}(\Omega_0)] = 0$. A perturbative calculation then provides the normalised damping rate

$$\frac{R_0\gamma}{v_{Ti}} \simeq \frac{1}{2}G(q, \tau_e)\mathcal{D}(\sigma^*)\left(\frac{m_e}{m_i}\right)^{1/2}\frac{q}{\tau_e^{3/2}}\Omega_0^2 \quad (28)$$

Using the value of η_1 given by Eq.(22), the following results are found

$$G(q, \tau_e) = 0.69\left(1 + \frac{2 + \tau_e}{q^2\Omega_0^2}\right)^2\frac{\tau_e^2}{\Omega_0^2} \quad (29)$$

and

$$\frac{R_0\gamma}{v_{Ti}} \simeq 0.34\left(1 + \frac{2 + \tau_e}{q^2\Omega_0^2}\right)^2q\tau_e^{1/2}\mathcal{D}(\sigma^*)\left(\frac{m_e}{m_i}\right)^{1/2} \quad (30)$$

where Ω_0 is a function of q and τ_e . It is stressed here that the expression Eq.(30) is exact, though the weight function $\mathcal{D}(\sigma^*) \leq 1$ is unknown at this stage. Nevertheless this formulation greatly constrains the dependencies on q , τ_e and m_e/m_i , as anticipated. Hence a way to check this expression is to perform first a scan on one parameter to determine the function $\mathcal{D}(\sigma^*)$, and then check the variation with respect to the other parameters. Previous simulations indicate that the damping rate due to trapped electrons scales as $(m_e/m_i)^{1/2}$ [1]. This suggests that $\mathcal{D}(\sigma^*)$ is constant and close to $\mathcal{D}(\sigma^*) \simeq 1.0$. If so, Eq.(30) can then be used to test other dependencies, in particular on τ_e and q .

5. Conclusion

The contribution of electrons to GAM damping has been derived. It appears that this contribution is usually not negligible thanks to a resonance between barely trapped/passing electrons bounce/transit frequency and the mode pulsation. Damping is estimated via the computation of the exchange of energy between the mode and electrons. This expression appears to be quite intricate as it involves a quadruple integral over the phase space and poloidal angles. A more tractable expression is obtained by assuming a strong weight of trapped particle bounce points in this integral. The ratio between the exact result and the approximate is a weight function that depends only one dimensionless number, which characterises the ratio of the mode pulsation to the thermal bounce frequency. This weight function thus measures the number of resonant trapped electrons that participate in mode damping. Comparison with available data in the literature suggests that this weight function is nearly constant and close to 1, thus confirming a prominent role of barely trapped particles. Upcoming numerical simulations should be able to test this conjecture.

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