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# A study of the eigenvalue sensitivity by homotopy and perturbation methods

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In this work, the sensitivity to material characteristics of eigenvalues is studied. From an initial structure, some defects of material or/and geometry are introduced. A method is proposed to solve the new eigenvalue problem from the initial one without using classical techniques. This method is based on the association of a homotopy transformation and the perturbation method.

## 1. Introduction

Studying the evolution of the eigenvalues in relation to modifications of a structure's material or geometrical characteristics is interesting from an industrial point of view. Indeed, the industrialist has to choose the most efficient structure (in its field of use). All of these alterations require a solution for the free vibration problem, for each modification of the stiffness matrix ( $K_0$ ) and/or the mass matrix ( $M_0$ ) and/or the geometry of the structure. Classical methods, such as the Subspace technique or the Lanczos algorithm [1,2] are usually called upon to resolve the corresponding eigenvalue problem. Nevertheless, with these methods, the computation can lead to excessive CPU times if the number of unknowns is large. There are alternative techniques like the one presented in the references [3,4]. In these methods, the initial and final structures are linked up, which means that all the parameters vary in the same way. This is why a new method, based on the independence of the two structures, is proposed in this work. The idea is to start from an initial structure ( $K_0, M_0$ ) for which a given number of eigenvalues  $\lambda_0$  and eigenvectors  $U_0$  have already been computed. The latter are then used to calculate the new values ( $\lambda, U$ ) of the modified problem ( $K, M$ ). This should generate a gain in CPU times. The method presented here is relevant to a homotopy technique, and to the perturbation method. This paper begins by formulating the problem to be resolved, and goes on to develop the method, presenting some numerical examples in the field of vibrations of plates.

## 2. Problem to be resolved

The idea is to start from an initial structure with the following characteristics: Young's modulus  $E_0$ , volumic mass  $\rho_0$  and thickness  $h_0$ . The finite element discretization of the free vibration problem leads to a resolution of the following linear equation:

$$\begin{cases} [K_0 - \lambda_0^i M_0] U_0^i = 0 \\ {}^t U_0^i M_0 U_0^i = 1 \end{cases} \quad (1)$$

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where  $K_0$  stands for the stiffness matrix and  $M_0$  is the mass matrix. The eigenvalues  $\lambda_0^i$  and eigenvectors  $U_0^i$  of the  $i$ th mode, are computed by using either the Subspace method or the Lanczos' method [1]. The second equation is the  $M_0$ -normalization condition.

The problem to be resolved can be obtained in different ways. The first is to introduce some defects  $\Delta_E$ ,  $\Delta_\rho$ ,  $\Delta_h$ , defined as follows:

$$E = E_0(1 + \Delta_E), \quad \rho = \rho_0(1 + \Delta_\rho), \quad \text{and} \quad h = h_0(1 + \Delta_h). \quad (2)$$

The values of the three defect parameters can be different. The second way is to distort the geometric structure by deforming the initial mesh by a displacement or a curvature. These alterations can involve a part or the totality of the structure. In both cases, the new matrices of stiffness ( $K$ ) and mass ( $M$ ) are calculated, and are not linked to  $K_0$ ,  $M_0$ . Thus, the problem and the normalization condition to resolve become:

$$\begin{cases} [K - \lambda^i M]U^i = 0 \\ {}^t U^i M U^i = 1 \end{cases} \quad (3)$$

where  $\lambda^i$  and  $U^i$  represent respectively the  $i$ th eigenvalue and eigenvector of the new structure. These eigensolutions can be obtained by classical methods like Subspace or Lanczos methods, which can induce considerable CPU time when the number of unknowns is large. Some iterative algorithms can reduce this time but they require the whole equation to be solved for each change of material or geometry. Another method proposed by Chen [3,4] is to use the solution of the initial problem. For this, the actual problem is determined from a perturbation of the initial problem as follows:

$$\begin{cases} K = K_0 + \varepsilon \delta K \\ M = M_0 + \varepsilon \delta M. \end{cases} \quad (4)$$

In this method, we note that the final and initial structures are linked up and that the stiffness matrix  $K_0$  and the mass matrix  $M_0$  undergo the same perturbation. That is to say that  $\Delta_E$ ,  $\Delta_\rho$ ,  $\Delta_h$  must be the same.

In the following section, a method based on the homotopy transformation and the perturbation method is presented to determine the eigenvalues of the problem (3). This method makes it possible to obtain them, even if the defects  $\Delta_E$ ,  $\Delta_\rho$ ,  $\Delta_h$  are different, and for a final structure independent of the initial ones. It can be noted that a perturbation method is also used in the reference [3,4], the defect in this case being the perturbation parameter.

### 3. The proposed numerical method

The problem to solve is indicated in Eq. (3). Its eigenvalues could be obtained through the Subspace method [1], but, as it is assumed that an initial problem ( $K_0$ ,  $M_0$ ) has already been solved, its solutions ( $\lambda_0$ ,  $U_0$ ) will be used to compute the new unknowns ( $\lambda$ ,  $U$ ).

#### 3.1. Homotopy-Perturbation: Set of linear problems

The idea is to establish a method, leading to a set of linear problems, so the homotopy technique [5] will be applied first. This technique has already been applied in other fields such as nonlinear eigenvalue problems [6]. It consists in introducing a parameter  $\varepsilon$  in the Eq. (3) in such a way that for  $\varepsilon$  equal to one, the problem (3) turns out again. The stiffness matrix  $K_0$  and the mass matrix  $M_0$  are also introduced, and the following problem is obtained:

$$\begin{cases} [K_0 + \varepsilon(K - K_0)]v^i - \beta^i[M_0 + \varepsilon(M - M_0)]v^i = 0 \\ (1 - \varepsilon){}^t v^i M_0 v^i + \varepsilon {}^t v^i M v^i = 1. \end{cases} \quad (5)$$

The subscript 0 is relative to the initial problem (1), and the topscript  $i$  indicates the mode number, so that  $\beta^i$  and  $v^i$  stand respectively for the eigenvalue and the eigenvector at the  $i$ th mode. It is obvious that for  $\varepsilon = 0$ , the initial problem turns out (1) with  $\beta^i(\varepsilon = 0) = \lambda_0^i$  and  $v^i(\varepsilon = 0) = U_0^i$ , and as announced, for  $\varepsilon = 1$  the current problem (3) with  $\beta^i(\varepsilon = 1) = \lambda^i$  and  $v^i(\varepsilon = 1) = U^i$ . At this stage, the unknowns  $\beta^i$  and  $v^i$  are sought in form of integro-power series with respect to the homotopy parameter  $\varepsilon$ :

$$\begin{cases} \beta^i = \lambda_0^i + \sum_{p=1}^P \varepsilon^p \beta_p^i \\ v^i = U_0^i + \sum_{p=1}^P \varepsilon^p v_p^i \end{cases} \quad (6)$$

where  $P$  is the order of truncation,  $\lambda_0^i$  and  $U_0^i$  are the eigenvalue and eigenvector of the initial structure, and are chosen as the starting point ( $\beta_0^i = \lambda_0^i$  and  $v_0^i = U_0^i$ ). After insertion of the expansions (6) into (5), a set of recurrent linear problems that comes from the identification of the like powers in  $\varepsilon$  is obtained:

Order 1:

$$\begin{cases} (K_0 - \lambda_0^i M_0) v_1^i = F_1^i \\ F_1^i = \beta_1^i M_0 U_0^i - (K - K_0) U_0^i + \lambda_0^i (M - M_0) U_0^i \\ 2^t U_0^i M_0 v_1^i = 1 - {}^t U_0^i M U_0^i. \end{cases} \quad (7)$$

Order  $p \geq 2$ :

$$\begin{cases} (K_0 - \lambda_0^i M_0) v_p^i = F_p^i \\ F_p^i = \beta_p^i M_0 U_0^i - (K - K_0) v_{p-1}^i + \sum_{k=1}^{p-1} \beta_k^i M_0 v_{p-k}^i + \sum_{k=0}^{p-1} \beta_k^i (M - M_0) v_{p-1-k}^i \\ 2^t U_0^i M_0 v_p^i = F_p' \\ F_p' = \sum_{k=0}^{p-1} {}^t v_k^i (M_0 - M) v_{p-1-k}^i - \sum_{k=1}^{p-1} {}^t v_k^i M_0 v_{p-k}^i. \end{cases} \quad (8)$$

The unknowns  $v_p^i$  and  $\beta_p^i$  ( $p = 1, P$ ) appear in each previous equation. To simplify the problem to solve, they will be sought separately: first the eigenvalue, then the eigenvector.

### 3.2. Eigenvalues

Multiplying by  ${}^t U_0$  each Eq. (7) and (8) enables the expression of the eigenvalue  $\beta_p^i$  for each order, independently of the eigenvector  $v_p^i$ . In order to simplify the obtained scalar equation, the  $M$ -normalization condition of the initial problem (1) is used. Then, the eigenvalues can be written as follows:

Order 1:

$$\beta_1^i = {}^t U_0^i K U_0^i - \lambda_0^i {}^t U_0^i M U_0^i. \quad (9)$$

Order  $p \geq 2$ :

$$\beta_p^i = {}^t U_0^i (K - K_0) v_{p-1}^i - \sum_{k=1}^{p-1} \beta_k^i {}^t U_0^i M_0 v_{p-k}^i - \sum_{k=0}^{p-1} \beta_k^i {}^t U_0^i (M - M_0) v_{p-1-k}^i. \quad (10)$$

It can be noted that the eigenvalue  $\beta_p^i$  is only dependent on terms of the previous orders. Once it is determined, the corresponding  $v_p^i$  remains to be found.

### 3.3. Eigenvectors

To compute the vectors  $v_p^i$  for each order  $p$ , the linear problems (7) and (8) have to be resolved. In these equations, since the operator is singular, its triangulation is not possible and therefore the following method of projection will be used:

$$v_p^i = \alpha_p^i U_0^i + \sum_{j=1 \text{ and } j \neq i}^n \alpha_p^j U_0^j \quad (11)$$

where  $1 \leq n \leq N$  and  $N$  represent the required initial number of modes. To determine the vectors  $v_p^i$ , the computation of the scalars  $\alpha_p^i$  and  $\alpha_p^j$  are needed. The evaluation of  $\alpha_p^j$  is described first. By introducing the linear combination (11) into the Eq. (8), this latter becomes:

$$(K_0 - \lambda_0^i M_0) \left( \sum_{j=1 \text{ and } j \neq i}^n \alpha_p^j U_0^j \right) = F_p^i. \quad (12)$$

To isolate  $\alpha_p^k$  ( $k \in [1, n]$  and  $k \neq i$ ), this equation is multiplied by  ${}^t U_0^k$ :

$$\underbrace{\alpha_p^k ({}^t U_0^k K_0 U_0^k - \lambda_0^i {}^t U_0^k M_0 U_0^k)}_{=\lambda_0^k} + \sum_{j=1 \text{ and } j \neq i, k}^n \alpha_p^j {}^t U_0^k (K_0 U_0^j - \lambda_0^i M_0 U_0^j) = {}^t U_0^k F_p^i. \quad (13)$$

Finally, the scalar  $\alpha_p^k$  ( $k \in [1, n]$  and  $k \neq i$ ) is computed by using:

$$\alpha_p^k = \frac{{}^t U_0^k F_p^i}{\lambda_0^k - \lambda_0^i}. \quad (14)$$

The  $M$ -normalization conditions expressed previously will be useful to get the expression of  $\alpha_p^i$ . Then, injecting the latter gives the expression of  $\alpha_1^i$  and  $\alpha_p^i$ :

$$\alpha_1^i = \frac{1 - {}^t U_0^i M U_0^i}{2} \quad \text{and} \quad \alpha_p^i = \frac{1}{2} F_p^i. \quad (15)$$

In this way, the eigenvectors  $v_p^i$  can be determined.

#### 4. Numerical examples

The method presented above is used to compute the eigenvalues for a basic cantilever plate and for a cantilever box beam.

The method's accuracy has been evaluated with the following relation:

$$\text{error} = \log_{[10]} \frac{|\lambda_{\text{exact}} - \lambda|}{\lambda_{\text{exact}}}. \quad (16)$$

$\lambda_{\text{exact}}$  will be estimated using the Subspace method,  $\lambda$  is estimated through the proposed method or Chen's method [3].

##### 4.1. Cantilever plate

The cantilever plate under scrutiny has the following dimensions: 1 m \* 1 m \* 0.01 m, and its characteristics are  $2.110^{11}$  N m<sup>-2</sup> for Young's modulus, 7800 kg m<sup>-3</sup> for the volumic mass and a Poisson's factor of 0.3. A classical triangle DKT-element type has been used to obtain the mesh. The structure has 132 nodes and 792 degrees of freedom.

###### 4.1.1. Comparison with Chen's method

Chen's method considers three necessarily identical defects, their values having been set to 0.2 then 2, so that the variation with the initial structure is equal to 20% then 300%. In order to compare with the method presented here, the three defects in the latter have been given the same settings. Then a test was settled to prove the efficiency of the proposed algorithm when three different defects are considered. In the event, only the 4th mode was picked, as it was noted, that it was representative of the previous and the following modes in every test presented. In Fig. 1, the evolution of the error (16) is plotted versus the truncation order for the defect's two values. In Fig. 1(a), the results obtained with the proposed method are compared to those obtained with Chen's method [3], whereas only the results with the proposed method are plotted in Fig. 1(b). It can be noted that there is convergence in the series. For a large variation of the structure, both methods give the same results, but they are divergent (see series of the proposed method in Fig. 1(b)). Nevertheless, Fig. 1(b) shows that the proposed method can be improved by using Rayleigh's ratio. In this numerical example, the initial and the final modes are nearly the same, which is why the use of Rayleigh's ratio corrects the results obtained with the proposed method. As the results for small and large defects are the same with the two methods, they validate the presented method.

###### 4.1.2. Different defects for each material characteristic

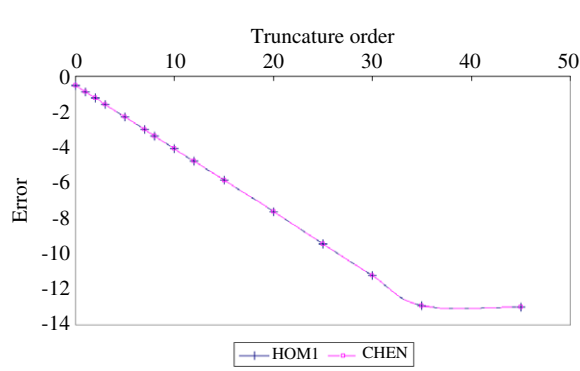
In order to illustrate cases in which the three defects are different (Fig. 2), the defect on  $E$  has been set to 0.4, the defect on  $\rho$  to 0.3, and the defect on  $h$  to 0.2. Only the series results of the proposed method are indicated as Chen's method cannot bring about a three different defect problem. The results show the efficiency of the proposed method, which is convergent even when the defects are different.

##### 4.2. Cantilever box beam

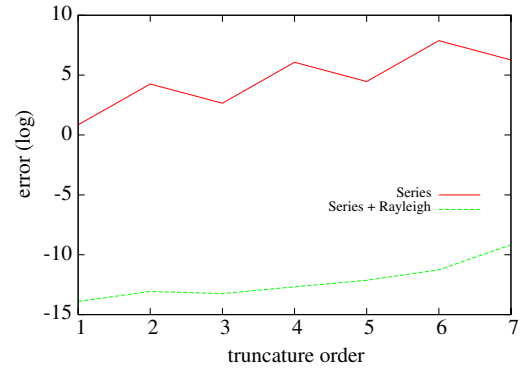
In this last example, the cantilever box beam represented in Fig. 3 is considered [7]. This structure is composed of 5 plates. Its material properties are: Young's modulus  $E = 72 \times 10^9$  N m<sup>-2</sup>, Poisson's ratio  $\nu = 0.3$  and density  $\rho = 2700$  kg m<sup>-3</sup>. The geometric characteristics are: length  $L = 3$  m, width  $l = 0.2$  m, height  $h = 0.1$  m and the thickness of the plates  $t$  is equal to 2 mm. This example is interesting because when the thickness of plate 3 is modified, some modes are crossing. In Table 1, the eigenvalues obtained by the Subspace method and by the proposed method, are compared when the thickness of plate number 3 is modified ( $\Delta t = 0.2$  and  $\Delta t = 0.4$ ). A good accuracy between the eigenvalues can be noted. To check the pairing of the eigenmodes for the initial and modified structural dynamic system, the number of Modal Assurance Criterium (MAC) is used [8]:

$$\text{MAC}_{(U_0^i, U^j)} = \frac{\| {}^t U_0^i U^j \|^2}{({}^t U_0^i U_0^i)({}^t U^j U^j)}. \quad (17)$$

$U_0^i$  represents the  $i$ th initial mode and  $U^j$  the  $j$ th current mode. With this number, we have been able to verify the following physical phenomenon: the inversion of the 6th and 7th mode (see the Table 2) when the thickness of the plates increases.

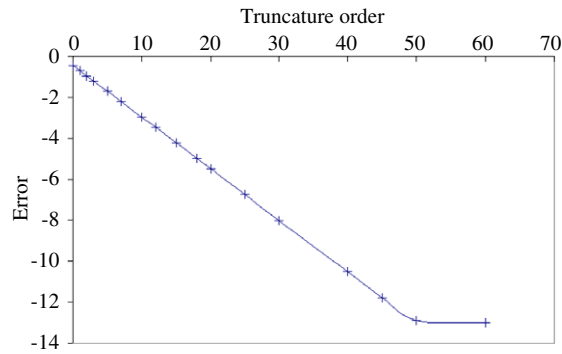


(a) Variation = 20%.

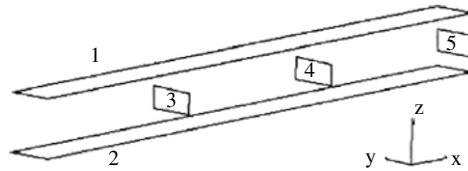


(b) Variation = 300%.

**Fig. 1.** Effect of the truncature order for (a) small variation: 20% (b) large variation: 300% (cantilever plate).



**Fig. 2.** Error obtained when all the defects are different (cantilever plate).



**Fig. 3.** The cantilever box beam [7].

**Table 1**

Comparison between eigenvalues obtained by the Subspace method and the proposed method (the cantilever box beam).

Defect Mode	$\Delta t = 0.2$		$\Delta t = 0.4$	
	Subs.	Hom.	Subs.	Hom.
1	26.08	26.40	26.57	27.51
2	240.75	240.77	238.98	239.02
3	669.43	670.68	669.37	673.07
4	3182.21	3197.77	3231.04	3913.49
5	3988.03	4029.65	4140.78	4548.93
6	4604.15	4606.98	4616.75	4624.77
7	4617.69	4681.22	4723.92	5010.16

**Table 2**

MAC number computed with the proposed method (the cantilever box beam).

Mode	Initial structure	Current structure $\Delta t = 0.4$
(6, 6)	1	$2.559 \cdot 10^{-4}$
(6, 7)	$9.75102 \cdot 10^{-23}$	<b>0.997</b>
(7, 6)	$9.75103 \cdot 10^{-23}$	<b>0.998</b>
(7, 7)	1	$1.937 \cdot 10^{-4}$

## 5. Conclusion

The method presented here is based on the perturbation method and on a homotopy technique. It makes it possible to resolve a series of linear problems aiming at computing the eigenvalues and eigenvectors of a modified structure from an initial structure's already-known elements (mass and stiffness matrices, eigenvalues and eigenvectors). A set of numerical results validates this method. Unlike Chen's method, it is possible with this method to use different defects for Young's modulus, volumic mass and thickness. It also makes it possible to obtain physical phenomenon like mode inversion. For large defects however, the quality of the solutions could be improved by the use of the Padé approximants [3,5] for example or a continuation method or iterative methods such as those defined in [6].

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