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# Simultaneous estimation of multiple sensor and process faults for non-linear discrete-time systems

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**Abstract:** The paper deals with the problem of simultaneous estimation of sensor and process faults. For that purpose, a novel scheme is proposed and its complete design procedure is described. The approach results in a robust estimation strategy with guaranteed convergence. In particular, apart from simultaneous estimation ability the proposed approach makes it possible to attenuate the exogenous disturbances up to the predefined level. Finally, the design procedure boils down to solving a set of linear matrix inequalities. The last part of the paper shows an illustrative example with the application dedicated to the laboratory twin-rotor aero-dynamical MIMO system.

*Keywords:* Fault diagnosis, multiple fault estimation, sensor fault, process fault, robust estimator

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## 1. INTRODUCTION

The growing requirements for reliability and safety of contemporary complex processes and systems are the inspiration for researchers to develop newer and newer approaches concerning the Fault Diagnosis (FD) (Blanke et al., 2006; Chen and Patton, 1999; Isermann, 2011; Korbicz et al., 2004; Patton et al., 2013). Significant attention for the latest few decades is dedicated to robust Fault-Tolerant Control (FTC) approaches which can be divided into active and passive ones. Contrarily to the passive FTC, which is designed in such a way to be robust against some predefined faults treated in the same fashion as disturbances, the active one is especially attractive. It allows changing the control action according to occurring faults to compensate their impact effect on the controlled system and as a consequence to guide the system correctly and safely even in a faulty case. In order to achieve such a goal, it is necessary to obtain an information about faults in every instances of time. In the literature, there are plenty methods of identification and estimation to acquire the knowledge about faults. Among them in (Foo et al., 2013; Keller and Darouach, 1999) a Kalman filter-based schemes are considered. A sliding mode observer and an high-gain observer are presented in (Brahim et al., 2015; Khalil and Praly, 2014). In the works (Zhang et al., 2010; López-Estrada et al., 2015) adaptive estimators are proposed. Moreover, a minimum-variance estimator is considered in (Gillijns and De Moor, 2007) while in (Chen et al., 2016; Nobrega et al., 2008; Zhang et al., 2014) an  $\mathcal{H}_\infty$  approach is used.

Having in mind the fact that modern control systems can be split into actuators, sensors and process dynamics (Chen and Patton, 1999; Ducard, 2009), the three classes of faults may be taken into consideration: actuator faults, sensor faults and process (or component) faults. However, most of the listed strategies are dedicated to the one of them only. In the other words, they can be used to either actuator or sensor fault estimation, exclusively.

The main objective of the paper is to deal with the problem of simultaneous estimation of the process and sensor faults. The proposed strategy is robust against the process and measurement uncertainties and it is based on the  $\mathcal{H}_\infty$  approach. The proposed estimator is able to estimate the process and sensor faults and as a consequence it gives a state estimate free from sensor faults. Indeed, it is very important to take into account simultaneous estimation of the process and sensor faults because in the real environmental conditions both of them might appear at the same time.

The paper is organised as follows: Firstly, section 2 presents preliminaries regarding the class of considered systems. Subsequently, section 3 shows the robust design procedure for the estimator able to estimate the state as well as sensor and process faults. Section 4 concerns an illustrative example with application to the twin-rotor aero-dynamical MIMO system and finally, Section 5 concludes the paper.

## 2. PRELIMINARIES

Let us consider a discrete-time system:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \sum_{i=1}^n \mathbf{D}_i f_{i,k} \mathbf{x}_{i,k} \quad (1)$$

$$+ \mathbf{g}(\mathbf{x}_k) + \mathbf{W}_1 \mathbf{w}_k, \quad (2)$$

$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{f}_{s,k} + \mathbf{W}_2 \mathbf{w}_k,$$

where  $\mathbf{x}_k \in \mathbb{X} \subset \mathbb{R}^n$  is the state vector,  $\mathbf{u}_k \in \mathbb{R}^r$  stands for the control input,  $\mathbf{y}_k \in \mathbb{R}^m$  is the output and  $\mathbf{g}(\mathbf{x}_k)$  describes the non-linear behaviour of the system. It is assumed that  $\mathbf{g}(\mathbf{x}_k)$  is differentiable and has bounded derivatives. Moreover  $\mathbf{f}_{s,k} \in \mathbb{F}_s \subset \mathbb{R}^m$  is the sensor fault vector and  $\mathbf{w}_k$  express the exogenous disturbance vector obeying

$$l_2 = \{\mathbf{w} \in \mathbb{R}^n \mid \|\mathbf{w}\|_{l_2} < +\infty\}, \quad (3)$$

$$\|\mathbf{w}\|_{l_2} = \left( \sum_{k=0}^{\infty} \|\mathbf{w}_k\|^2 \right)^{\frac{1}{2}}. \quad (4)$$

Note that, it can be easily shown that the disturbance vector can be split in such a way  $\mathbf{w}_k = [\mathbf{w}_{1,k}^T, \mathbf{w}_{2,k}^T]^T$  which represent the process and measurement uncertainties, respectively. Furthermore,  $\mathbf{W}_1$  and  $\mathbf{W}_2$  denote the noise distribution matrices. Moreover,  $\mathbf{D}$  stands for the process fault distribution matrix, in other words, it describes the way in which the fault influences the system. It can be observed that the problem of estimating  $\mathbf{f}_{i,k}$  directly is non-linear because of multiplication, even for the linear system. To tackle this problem, throughout the paper, the system (1)–(2) is transformed into the following form:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{D}\mathbf{z}_k + \mathbf{g}(\mathbf{x}_k) + \mathbf{W}_1 \mathbf{w}_k, \quad (5)$$

$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{f}_{s,k} + \mathbf{W}_2 \mathbf{w}_k, \quad (6)$$

where the vector  $\mathbf{z}_k \in \mathbb{R}^n$  is composed with a multiplication of the fault  $\mathbf{f}_k$  and state  $\mathbf{x}_k$  in such a way as

$$\mathbf{z}_k = \begin{bmatrix} f_{1,k} \cdot x_{1,k} \\ f_{2,k} \cdot x_{2,k} \\ \vdots \\ f_{n,k} \cdot x_{n,k} \end{bmatrix}. \quad (7)$$

For further deliberations, let us call the vector  $\mathbf{z}_k$  a *process fault*. The idea is to estimate  $\mathbf{z}_k$  instead of estimating  $\mathbf{f}_{i,k}$  directly.

### 3. JOINT SENSOR AND PROCESS FAULTS ESTIMATOR DESIGN

The main goal of the section is to design the estimator, which will be able to estimate the state as well as the process and all the sensor faults, simultaneously. To cope with such requirements, the estimator of the following form is proposed:

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}\hat{\mathbf{x}}_k + \mathbf{B}\mathbf{u}_k + \mathbf{D}\hat{\mathbf{z}}_k + \mathbf{g}(\hat{\mathbf{x}}_k) + \mathbf{K}_x (\mathbf{y}_k - \mathbf{C}\hat{\mathbf{x}}_k - \hat{\mathbf{f}}_{s,k}), \quad (8)$$

$$\hat{\mathbf{z}}_{k+1} = \hat{\mathbf{z}}_k + \mathbf{K}_z (\mathbf{y}_k - \mathbf{C}\hat{\mathbf{x}}_k - \hat{\mathbf{f}}_{s,k}), \quad (9)$$

$$\hat{\mathbf{f}}_{s,k+1} = \hat{\mathbf{f}}_{s,k} + \mathbf{K}_s (\mathbf{y}_k - \mathbf{C}\hat{\mathbf{x}}_k - \hat{\mathbf{f}}_{s,k}), \quad (10)$$

where  $\mathbf{K}_x$ ,  $\mathbf{K}_z$  and  $\mathbf{K}_s$  are the estimator gain matrices for the state, process fault and sensor fault, respectively. Having regard (5)–(6) as well as (8)–(10) the state estimation error is given

$$\begin{aligned} \mathbf{e}_{k+1} &= \mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{D}\mathbf{z}_k + \mathbf{g}(\mathbf{x}_k) \\ &\quad + \mathbf{W}_1 \mathbf{w}_k - \mathbf{A}\hat{\mathbf{x}}_k - \mathbf{B}\mathbf{u}_k - \mathbf{D}\hat{\mathbf{z}}_k - \mathbf{g}(\hat{\mathbf{x}}_k) \\ &\quad - \mathbf{K}_x \mathbf{C}\mathbf{x}_k - \mathbf{K}_x \mathbf{f}_{s,k} - \mathbf{K}_x \mathbf{W}_2 \mathbf{w}_k + \mathbf{K}_x \mathbf{C}\hat{\mathbf{x}}_k \\ &\quad + \mathbf{K}_x \hat{\mathbf{f}}_{s,k} = [\mathbf{A} - \mathbf{K}_x \mathbf{C}] \mathbf{e}_k + \mathbf{g}(\mathbf{x}_k) - \mathbf{g}(\hat{\mathbf{x}}_k) \\ &\quad - \mathbf{D}\mathbf{e}_{z,k} - \mathbf{K}_x \mathbf{e}_{s,k} - \mathbf{K}_x \mathbf{W}_2 \mathbf{w}_k, \end{aligned} \quad (11)$$

with  $\mathbf{e}_{z,k} = \mathbf{z}_k - \hat{\mathbf{z}}_k$  and  $\mathbf{e}_{s,k} = \mathbf{f}_{s,k} - \hat{\mathbf{f}}_{s,k}$ . In a similar way, the process fault estimation error is defined

$$\begin{aligned} \mathbf{e}_{z,k+1} &= \mathbf{z}_{k+1} - \hat{\mathbf{z}}_{k+1} = \mathbf{z}_{k+1} + \mathbf{z}_k - \mathbf{z}_k - \hat{\mathbf{z}}_k \\ &\quad - \mathbf{K}_z \mathbf{C}\mathbf{x}_k - \mathbf{K}_z \mathbf{f}_{s,k} - \mathbf{K}_z \mathbf{W}_2 \mathbf{w}_k + \mathbf{K}_z \mathbf{C}\hat{\mathbf{x}}_k \\ &\quad + \mathbf{K}_z \hat{\mathbf{f}}_{s,k} = \mathbf{e}_{z,k} + \mathbf{e}_{z,k} - \mathbf{K}_z \mathbf{C}\mathbf{e}_k - \mathbf{K}_z \mathbf{e}_{s,k} \\ &\quad - \mathbf{K}_z \mathbf{W}_2 \mathbf{w}_k, \end{aligned} \quad (12)$$

while the sensor fault estimation error is governed by

$$\begin{aligned} \mathbf{e}_{s,k+1} &= \mathbf{f}_{s,k+1} - \hat{\mathbf{f}}_{s,k+1} = \mathbf{f}_{s,k+1} + \mathbf{f}_{s,k} - \mathbf{f}_{s,k} \\ &\quad - \hat{\mathbf{f}}_{s,k} - \mathbf{K}_s \mathbf{C}\mathbf{x}_k - \mathbf{K}_s \mathbf{f}_{s,k} - \mathbf{K}_s \mathbf{W}_2 \mathbf{w}_k \\ &\quad + \mathbf{K}_s \mathbf{C}\hat{\mathbf{x}}_k + \mathbf{K}_s \hat{\mathbf{f}}_{s,k} = \mathbf{e}_{s,k} + [\mathbf{I} - \mathbf{K}_s \mathbf{C}] \mathbf{e}_{s,k} \\ &\quad - \mathbf{K}_s \mathbf{C}\mathbf{e}_k - \mathbf{K}_s \mathbf{W}_2 \mathbf{w}_k. \end{aligned} \quad (13)$$

Subsequently, using Differential Mean Value Theorem (DMVT) (Zemouche and Boutayeb, 2006), it can be shown that

$$\mathbf{g}(\mathbf{x}_k, \mathbf{u}_k) - \mathbf{g}(\hat{\mathbf{x}}_k, \mathbf{u}_k) = \mathbf{M}_k (\mathbf{x}_k - \hat{\mathbf{x}}_k), \quad (14)$$

with

$$\mathbf{M}_k = \begin{bmatrix} \frac{\partial g_1}{\partial x}(\mathbf{c}_1, \mathbf{u}_k) \\ \vdots \\ \frac{\partial g_n}{\partial x}(\mathbf{c}_n, \mathbf{u}_k) \end{bmatrix}, \quad (15)$$

where  $\mathbf{c}_1, \dots, \mathbf{c}_n \in \text{Co}(\mathbf{x}_k, \hat{\mathbf{x}}_k)$ ,  $\mathbf{c}_i \neq \mathbf{x}_k$ ,  $\mathbf{c}_i \neq \hat{\mathbf{x}}_k$ ,  $i = 1, \dots, n$ . Having regard the fact that all states are bounded in real system,  $\mathbf{x}_k \in \mathbb{X}$  let

$$\underline{x}_{i,j} \leq \frac{\partial g_i(\mathbf{x})}{\partial x_j} \leq \bar{x}_{i,j}, \quad i = 1, \dots, n, \quad j = 1, \dots, n, \quad (16)$$

it is clear that there exist  $\mathbf{M}_k \in \mathbb{M}$  such that

$$\mathbb{M} = \{\mathbf{M}_k \in \mathbb{R}^{n \times n} \mid \underline{x}_{i,j} \leq m_{k,i,j} \leq \bar{x}_{i,j}, \quad i, j = 1, \dots, n\}. \quad (17)$$

Indeed, now the state estimation error can be rewritten without the non-linear part into the following form

$$\begin{aligned} \mathbf{e}_{k+1} &= \mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1} [\mathbf{A} + \mathbf{M}_k - \mathbf{K}_x \mathbf{C}] \mathbf{e}_k \\ &\quad - \mathbf{D}\mathbf{e}_{z,k} - \mathbf{K}_x \mathbf{e}_{s,k} - \mathbf{K}_x \mathbf{W}_2 \mathbf{w}_k. \end{aligned} \quad (18)$$

By constructing the following super-vectors:

$$\bar{\mathbf{e}}_{k+1} = \begin{bmatrix} \mathbf{e}_{k+1} \\ \mathbf{e}_{z,k+1} \\ \mathbf{e}_{s,k+1} \end{bmatrix}, \quad (19)$$

$$\mathbf{v}_k = \begin{bmatrix} \mathbf{w}_k \\ \mathbf{e}_{z,k} \\ \mathbf{e}_{s,k} \end{bmatrix}, \quad (20)$$

let us to present the state as well as process and sensor faults estimation error in a compact form

$$\bar{\mathbf{e}}_{k+1} = \mathbf{X}_k \bar{\mathbf{e}}_k + \mathbf{Z} \mathbf{v}_k, \quad (21)$$

with:

$$\mathbf{X}_k = \bar{\mathbf{A}}_k - \bar{\mathbf{K}} \bar{\mathbf{C}}, \quad (22)$$

$$\mathbf{Z} = \bar{\mathbf{W}} - \bar{\mathbf{K}} \bar{\mathbf{V}}, \quad (23)$$

where:

$$\begin{aligned}\bar{\mathbf{A}}_k &= \begin{bmatrix} \mathbf{A} + \mathbf{M}_k & \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \bar{\mathbf{C}} = [\mathbf{C} \ \mathbf{0} \ \mathbf{I}], \\ \bar{\mathbf{W}} &= \begin{bmatrix} \mathbf{W}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \bar{\mathbf{V}} = [\mathbf{W}_2 \ \mathbf{0} \ \mathbf{0}], \\ \bar{\mathbf{K}} &= \begin{bmatrix} \mathbf{K}_x \\ \mathbf{K}_z \\ \mathbf{K}_s \end{bmatrix}.\end{aligned}\quad (24)$$

Based on the aforementioned estimation error for the state and faults, the following theorem can be defined:

*Theorem 1.* For a prescribed disturbance attenuation level  $\mu > 0$  for the state and fault estimation error (21), the estimator (8)–(10) design problem for the system (5)–(6) and estimator (8)–(10) is solvable if there exist matrices  $\mathbf{P}$ ,  $\mathbf{U}$  and  $\mathbf{N}$  such that the following inequality is satisfied

$$\begin{bmatrix} \mathbf{I} - \mathbf{P} & \mathbf{0} & \bar{\mathbf{A}}_k^T \mathbf{U} - \bar{\mathbf{C}}^T \mathbf{N}^T \\ \mathbf{0} & -\mu^2 \mathbf{I} & \bar{\mathbf{W}} \mathbf{U} - \bar{\mathbf{V}}^T \mathbf{N}^T \\ \mathbf{U} \bar{\mathbf{A}}_k - \mathbf{N} \bar{\mathbf{C}} & \mathbf{U} \bar{\mathbf{W}} - \mathbf{N} \bar{\mathbf{V}} & \mathbf{P} - \mathbf{U} - \mathbf{U}^T \end{bmatrix} \prec 0. \quad (25)$$

**Proof.** The problem of the designing the  $\mathcal{H}_\infty$  observer (Li and Fu, 1997; Zemouche et al., 2008) is to obtain matrices  $\mathbf{N}$ ,  $\mathbf{U}$  and  $\mathbf{P}$  such that

$$\lim_{k \rightarrow \infty} \bar{\mathbf{e}}_k = \mathbf{0} \quad \text{for} \quad \mathbf{v}_k = \mathbf{0}, \quad (26)$$

$$\|\bar{\mathbf{e}}_k\|_{l_2} < \mu \|\mathbf{v}_k\|_{l_2} \quad \text{for} \quad \mathbf{v}_k \neq \mathbf{0}, \bar{\mathbf{e}}_0 = \mathbf{0}. \quad (27)$$

To solve such a problem, it is satisfactory to find a Lyapunov function of the following form

$$\Delta V_k + \bar{\mathbf{e}}_k^T \bar{\mathbf{e}}_k - \mu^2 \mathbf{v}_k^T \mathbf{v}_k < 0, \quad (28)$$

where:  $\Delta V_k = V_{k+1} - V_k$ ,  $V_k = \bar{\mathbf{e}}_k^T \mathbf{P} \bar{\mathbf{e}}_k$  and  $\mathbf{P} \succ 0$ . If  $\mathbf{v}_k = \mathbf{0}$ , then the Lyapunov function (28) takes the following simplified form

$$\Delta V_k + \bar{\mathbf{e}}_k^T \bar{\mathbf{e}}_k < 0 \quad (29)$$

and hence  $\Delta V_k < 0$ , which leads to (26). If  $\mathbf{v}_k \neq \mathbf{0}$  and taking into account the fact that

$$\sum_{k=0}^{\infty} (\Delta V_k) = V_{s,\infty} - V_{s,0}, \quad V_{s,\infty} = V_{s,0} = 0,$$

then (28) yields:

$$\sum_{k=0}^{\infty} (\Delta V_k) + \sum_{k=0}^{\infty} (\bar{\mathbf{e}}_k^T \bar{\mathbf{e}}_k) - \mu^2 \sum_{k=0}^{\infty} (\mathbf{v}_k^T \mathbf{v}_k) < 0 \implies$$

$$-V_0 + \sum_{k=0}^{\infty} (\bar{\mathbf{e}}_k^T \bar{\mathbf{e}}_k) - \mu^2 \sum_{k=0}^{\infty} (\mathbf{v}_k^T \mathbf{v}_k) < 0 \implies$$

$$\sum_{k=0}^{\infty} (\bar{\mathbf{e}}_k^T \bar{\mathbf{e}}_k) - \mu^2 \sum_{k=0}^{\infty} (\mathbf{v}_k^T \mathbf{v}_k) < 0 \implies$$

$$\sum_{k=0}^{\infty} (\bar{\mathbf{e}}_k^T \bar{\mathbf{e}}_k) < \mu^2 \sum_{k=0}^{\infty} (\mathbf{v}_k^T \mathbf{v}_k) \implies$$

$$\|\bar{\mathbf{e}}_k\|_{l_2} < \mu \|\mathbf{v}_k\|_{l_2},$$

which leads to (27). As a consequence by using (21) it is easy to show that

$$\begin{aligned}\Delta V_k + \bar{\mathbf{e}}_k^T \bar{\mathbf{e}}_k - \mu^2 \mathbf{v}_k^T \mathbf{v}_k &= \\ \bar{\mathbf{e}}_k^T (\mathbf{X}_k^T \mathbf{P} \mathbf{X}_k + \mathbf{I} - \mathbf{P}) \bar{\mathbf{e}}_k & \\ + \bar{\mathbf{e}}_k^T (\mathbf{X}_k^T \mathbf{P} \mathbf{Z}) \mathbf{v}_k + \mathbf{v}_k^T (\mathbf{Z}^T \mathbf{P} \mathbf{X}_k) \bar{\mathbf{e}}_k & \\ + \mathbf{v}_k^T (\mathbf{Z}^T \mathbf{P} \mathbf{Z} - \mu^2 \mathbf{I}) \mathbf{v}_k &< 0,\end{aligned}\quad (30)$$

and by introducing

$$\bar{\mathbf{v}}_k = \begin{bmatrix} \bar{\mathbf{e}}_k \\ \mathbf{v}_k \end{bmatrix}, \quad (31)$$

it can be shown that (30) can be rewritten into the following form

$$\bar{\mathbf{v}}_k^T \begin{bmatrix} \mathbf{X}_k^T \mathbf{P} \mathbf{X}_k + \mathbf{I} - \mathbf{P} & \mathbf{X}_k^T \mathbf{P} \mathbf{Z} \\ \mathbf{Z}^T \mathbf{P} \mathbf{X}_k & \mathbf{Z}^T \mathbf{P} \mathbf{Z} - \mu^2 \mathbf{I} \end{bmatrix} \bar{\mathbf{v}}_k \prec 0, \quad (32)$$

which is equivalent to

$$\begin{bmatrix} \mathbf{X}_k^T \\ \mathbf{Z}^T \end{bmatrix} \mathbf{P} [\mathbf{X}_k \ \mathbf{Z}] + \begin{bmatrix} \mathbf{I} - \mathbf{P} & \mathbf{0} \\ \mathbf{0} & -\mu^2 \mathbf{I} \end{bmatrix} \prec 0. \quad (33)$$

Now, let us recall the following Lemma:

*Lemma 1.* The following statements are equivalent:

(1) There exists  $\mathbf{X}_k \succ 0$  such that

$$\mathbf{V}^T \mathbf{X}_k \mathbf{V} - \mathbf{W} \prec 0. \quad (34)$$

(2) There exist  $\mathbf{X}_k \succ 0$  such that

$$\begin{bmatrix} -\mathbf{W} & \mathbf{V}^T \mathbf{U}^T \\ \mathbf{U} \mathbf{V} & \mathbf{X}_k - \mathbf{U} - \mathbf{U}^T \end{bmatrix} \prec 0. \quad (35)$$

Applying lemma 1 to (33) gives

$$\begin{bmatrix} \mathbf{I} - \mathbf{P} & \mathbf{0} & \mathbf{X}_k^T \mathbf{U}^T \\ \mathbf{0} & -\mu^2 \mathbf{I} & \mathbf{Z}^T \mathbf{U}^T \\ \mathbf{U} \mathbf{X}_k & \mathbf{U} \mathbf{Z} & \mathbf{P} - \mathbf{U} - \mathbf{U}^T \end{bmatrix} \prec 0, \quad (36)$$

and then substituting:

$$\mathbf{U} \mathbf{X}_k = \mathbf{U} \bar{\mathbf{A}}_k - \mathbf{U} \bar{\mathbf{K}} \bar{\mathbf{C}} = \mathbf{U} \bar{\mathbf{A}}_k - \mathbf{N} \bar{\mathbf{C}}, \quad (37)$$

$$\mathbf{U} \mathbf{Z} = \mathbf{U} \bar{\mathbf{W}} - \mathbf{U} \bar{\mathbf{K}} \bar{\mathbf{V}} = \mathbf{U} \bar{\mathbf{W}} - \mathbf{N} \bar{\mathbf{V}}, \quad (38)$$

completes the proof.  $\blacksquare$

Note that,  $\mathbb{M}$  specified by (17) can be equivalently expressed by

$$\mathbb{M} = \left\{ \mathbf{M}(\alpha) : \mathbf{M}(\alpha) = \sum_{i=1}^N \alpha_i \mathbf{M}_k, \sum_{i=1}^N \alpha_i = 1, \alpha_i \geq 0 \right\}, \quad (39)$$

where  $N = 2^{n^2}$ . Note that, this is a general description, which does not take into account that some elements of  $\mathbf{M}$  may be constant. In such cases,  $\mathbf{N}$  is given by  $\mathbf{N} = 2^{(n-c)^2}$ , where  $c$  stands for the number of constant elements of  $\mathbf{M}$ . Thus, the system can be described in a Linear Parameter Varying (LPV) form. Solving (25) is equivalent to (for  $i = 1, \dots, N$ )

$$\begin{bmatrix} \mathbf{I} - \mathbf{P} & \mathbf{0} & \bar{\mathbf{A}}_i^T \mathbf{U}^T - \bar{\mathbf{C}}^T \mathbf{N}^T \\ \mathbf{0} & -\mu^2 \mathbf{I} & \bar{\mathbf{W}}^T \mathbf{U}^T - \bar{\mathbf{V}}^T \mathbf{N}^T \\ \mathbf{U} \bar{\mathbf{A}}_i - \mathbf{N} \bar{\mathbf{C}} & \mathbf{U} \bar{\mathbf{W}} - \mathbf{N} \bar{\mathbf{V}} & \mathbf{P} - \mathbf{U} - \mathbf{U}^T \end{bmatrix} \preceq 0. \quad (40)$$

As a result, the gain matrices are obtained as follows

$$\bar{\mathbf{K}} = \begin{bmatrix} \mathbf{K}_x \\ \mathbf{K}_z \\ \mathbf{K}_s \end{bmatrix} = \mathbf{U}^{-1} \mathbf{N}. \quad (41)$$

The purpose of the subsequent section is to show the correctness of the proposed approach by its application to the twin-rotor MIMO system.

#### 4. ILLUSTRATIVE EXAMPLE

##### 4.1 Results

The performance of the proposed approach can be efficiently shown by its implementation to the non-linear twin-rotor MIMO system. The considered system is depicted in Fig. 1. Such a system has been designed to simulate the flight object in laboratory conditions. The system can be described by a highly non-linear model with cross coupled axes using the following equations:

$$\frac{d\omega_v}{dt} = \frac{k_b k_2}{J_{mr} R_b} u_v - \left( \frac{B_{mr}}{J_{mr}} + \frac{k_b^2}{J_{mr} R_b} \right) \omega_v - \frac{g_4(\omega_v)}{J_{mr}}, \quad (42)$$

$$\begin{aligned} \frac{d\Omega_v}{dt} &= \frac{l_m g_5(\omega_v) + k g \Omega_h g_5(\omega_v) \cos(\theta_v) - k_{ov} \Omega_v}{J_v} \\ &+ \frac{g((K_A - K_B) \cos(\theta_v) - K_C \sin(\theta_v))}{J_v} \\ &- \frac{\Omega_h^2 K_H \sin(\theta_v) \cos(\theta_v)}{J_v} \\ &+ \frac{k_t \left( \frac{k_a k_1}{R_a} u_h - \left( B_{tr} + \frac{k_a^2}{R_a} \right) \omega_h - g_1(\omega_h) \right)}{J_v J_{tr}}, \end{aligned} \quad (43)$$

$$\frac{d\theta_v}{dt} = \Omega_v, \quad (44)$$

$$\begin{aligned} \frac{d\omega_h}{dt} &= \frac{k_a k_1}{J_{tr} R_a} u_h - \left( \frac{B_{tr}}{J_{tr}} + \frac{k_a^2}{J_{tr} R_a} \right) \omega_h \\ &- \frac{g_1(\omega_h)}{J_{tr}}, \end{aligned} \quad (45)$$

$$\begin{aligned} \frac{d\Omega_h}{dt} &= \frac{k_{oh} g_2(\omega_h) \cos(\theta_v) - k_{oh} \Omega_h - g_3(\theta_h) + g_6(\theta_v)}{\Phi} \\ &+ \frac{k_m \omega_v \sin(\theta_v) \Omega_v (-\Phi - 2K_E \cos^2(\theta_v))}{(\Phi)^2} \\ &+ \frac{k_m \sin(\theta_v) \left( \frac{k_b k_2}{R_b} u_v - \left( B_{mr} + \frac{k_b^2}{R_b} \right) \omega_v - g_4(\omega_v) \right)}{J_{mr} \Phi}, \end{aligned} \quad (46)$$

$$\frac{d\theta_h}{dt} = \Omega_h, \quad (47)$$

where  $\Phi = K_D \cos^2(\theta_v) + K_E \sin^2(\theta_v) + K_F$ ,  $\omega_v$ ,  $\Omega_v$ ,  $\theta_v$ ,  $\omega_h$ ,  $\Omega_h$  and  $\theta_h$  are the rotational velocity of the main rotor, angular velocity around horizontal axes, pitch angle of the beam, rotational velocity of the tail rotor, angular velocity around vertical axes and yaw angle of the beam, respectively. For more information the reader is referred to (Rotondo et al., 2013). The system state vector is  $\mathbf{x} = [\omega_v^T, \Omega_v^T, \theta_v^T, \omega_h^T, \Omega_h^T, \theta_h^T]^T$  and the system input vector is  $\mathbf{u} = [u_v^T, u_h^T]^T$  where  $u_v$  and  $u_h$  are the voltages of the main and tail rotors. The rest of the parameters are inherited from (Rotondo et al., 2013).

The non-linear model, which describes the behaviour of the system has been discretized with a sampling time

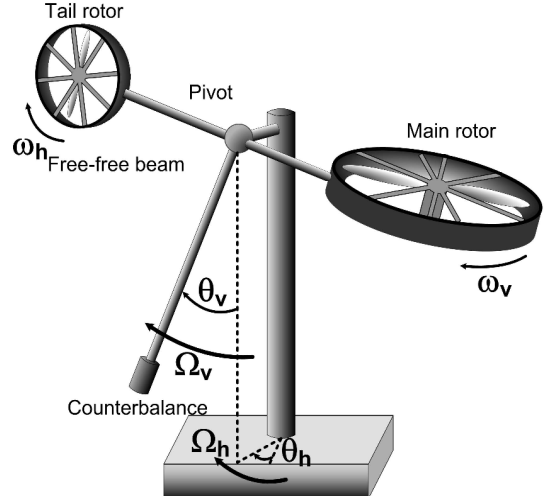


Fig. 1. Twin-rotor aero-dynamical system

$T_s = 0.01[s]$  which leads to the state-space representation (5)–(6). It is worth to emphasize that the angular velocity around both, vertical and horizontal axes, were not measured directly during the experiment, which implies the output equation expressed by

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (48)$$

Moreover, let us consider following fault scenarios for the sensor fault  $\mathbf{f}_{s,k} = [\mathbf{f}_{s,1,k}, \mathbf{f}_{s,3,k}, \mathbf{f}_{s,4,k}, \mathbf{f}_{s,6,k}]$  with:

$$\begin{aligned} \mathbf{f}_{s,1,k} &= 0 \\ \mathbf{f}_{s,3,k} &= \begin{cases} \mathbf{y}_{2,k} - 0.2, & 60[s] \leq t \leq 80[s], \\ 0, & \text{otherwise,} \end{cases} \\ \mathbf{f}_{s,4,k} &= \begin{cases} \mathbf{y}_{3,k} + 77, & 70[s] \leq t \leq 90[s], \\ 0, & \text{otherwise,} \end{cases} \\ \mathbf{f}_{s,6,k} &= 0 \end{aligned} \quad (49)$$

and for the process fault  $\mathbf{f}_k$

$$\mathbf{f}_{1,k} = \begin{cases} -0.01, & 58[s] \leq t \leq 72[s], \\ 0, & \text{otherwise,} \end{cases} \quad (50)$$

with the process fault distribution matrix

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (51)$$

The fault distribution matrix  $\mathbf{D}$  indicates that the process fault acts directly onto the rotational velocity of the main rotor  $\omega_v$ . Moreover, the rest of the process faults influencing directly onto variables  $\Omega_v, \theta_v, \omega_h, \Omega_h, \theta_h$  are equal to  $\mathbf{0}$ . The number describing the sensor fault is adequate to the state of the system. By analysing the above scenarios, it can be concluded that the intermittent faults occur in two sensors. At some time instances the sensor faults overlap each other. This means that they give wrong measurements, simultaneously. Moreover, at the same time, as it is for the sensor faults, the process fault occurs. Such a dangerous situation might happen in real industrial system and could cause its incorrect operation. As a consequence, it even might be a reason

of a total failure of the system. Summarising, in the above scenarios it is considered that two sensor faults and the process fault are occurring simultaneously.

Let us assume that the initial state for the system and for the estimator are  $\mathbf{x}_0 = [0, 0, 0.001, 0, 0, 0.001]$  and  $\hat{\mathbf{x}}_0 = [1, 0, 0.01, 1, 0, 0.01]$ , respectively. The sensor faults as well as the process fault are initialised by  $\hat{\mathbf{f}}_{s,0} = [0, 0, 0, 0]$  and  $\hat{\mathbf{f}}_{1,0} = 0$ , respectively.

By solving a set of LMIs (25) described in previous section, the following gain matrices for the state as well as for the sensor and process faults have been obtained:

$$\mathbf{K}_x = \begin{bmatrix} 0.0012 & 0.0000 & 0.0000 & -0.0000 \\ -0.0001 & 0.0003 & -0.0001 & -0.0001 \\ 0.0000 & -0.0010 & 0.0000 & 0.0001 \\ 0.0000 & -0.0001 & 0.0003 & -0.0004 \\ 0.0001 & 0.0001 & -0.0001 & 0.0002 \\ -0.0000 & 0.0001 & 0.0001 & -0.0003 \end{bmatrix}, \quad (52)$$

$$\mathbf{K}_z = \begin{bmatrix} 0.5930 & 0.0000 & 0.0000 & 0.0000 \\ -0.0000 & -0.0000 & -0.0000 & 0.0000 \\ 0.0000 & -0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.0000 & 0.0000 & -0.0000 \\ -0.0000 & -0.0000 & 0.0000 & 0.0000 \\ -0.0000 & 0.0000 & -0.0000 & -0.0000 \end{bmatrix}, \quad (53)$$

$$\mathbf{K}_s = \begin{bmatrix} 0.9907 & -0.0001 & 0.0003 & -0.0007 \\ -0.0001 & 0.9750 & -0.0014 & 0.0035 \\ 0.0002 & -0.0017 & 0.9858 & 0.0052 \\ -0.0002 & 0.0002 & -0.0055 & 0.9843 \end{bmatrix}. \quad (54)$$

#### 4.2 Discussion

Figures 2–4 present the exemplary response of the system in the open-loop without the control feedback. In those pictures, a black solid line represent the real state of the system. A red dashed line stands for the estimate and a dash-dotted blue line is the measured output. The state estimate for the  $\omega_v$  has been following the real state during the experiment in spite of the process fault occurrence. Meanwhile, the other two states have been measured incorrectly. This means that the sensor faults occurred. However, despite the sensor faults the state estimates have not been following the measured, faulty output but they have been estimating the real state with a very good accuracy. The real sensor faults (red dashed

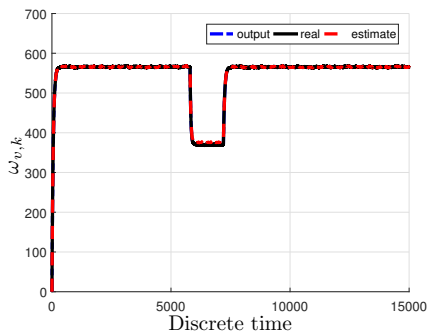


Fig. 2. Rotational velocity of the main rotor and its estimate

(red dashed line) for the pitch angle and for the rotational velocity of the tail rotor as well as their estimates (blue solid line) are

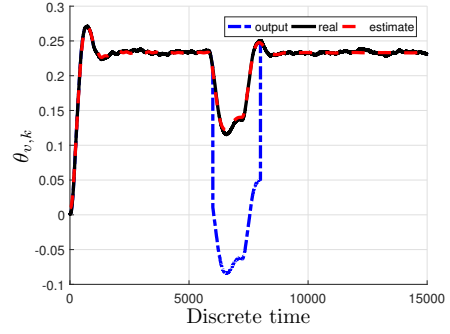


Fig. 3. Pitch angle of the beam and its estimate

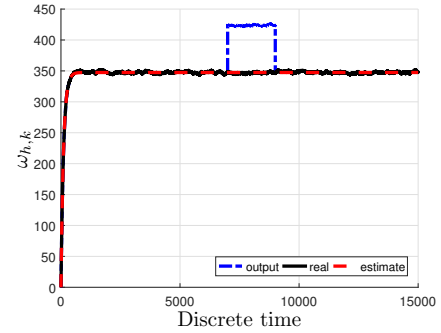


Fig. 4. Rotational velocity of the tail rotor and its estimate shown in Figs. 5–6, respectively. It can be seen that even if the sensor faults were occurring simultaneously they have been estimated very well. Finally, Fig. 7 present the real

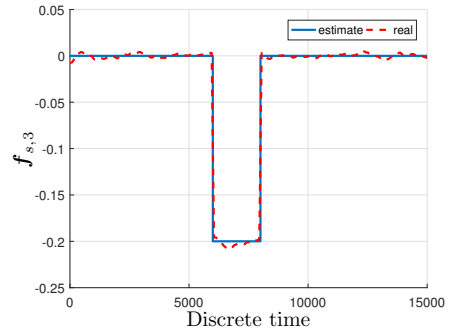


Fig. 5. The sensor fault for the pitch angle

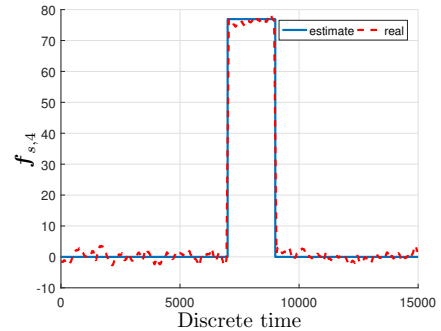


Fig. 6. The sensor fault for rotational velocity of the tail rotor

process fault (which is the multiplication of the fault and

state) depicted by blue solid line and its estimate by red dashed line. Despite the sensor and process faults the real states as well as the sensor and process faults have been estimated properly. It may be concluded that the proposed approach can be efficiently applied for the class of non-linear systems.

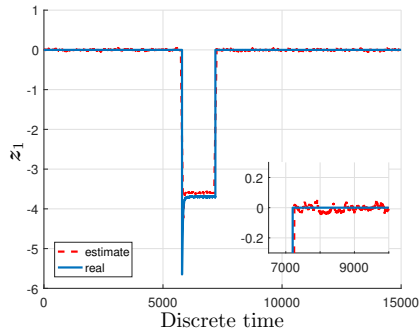


Fig. 7. The process fault and its estimate

## 5. CONCLUSIONS

The paper deals with the simultaneous estimation problem for the state as well as for the sensor and process faults. The proposed estimator was able to estimate the sensor and process faults, simultaneously. Having a knowledge concerning the sensor faults, provided by the fault estimator, it was possible to estimate the real state of the system, instead of the faulty output, which gives a sensor fault-free results. To cope with the process and measurement uncertainties the proposed strategy is based on  $\mathcal{H}_\infty$  approach. It provides a convergence analysis for the proposed estimator and guaranties a predefined disturbance attenuation level. The design procedure brings down to solve a set of LMIs. The final part of the paper showed an illustrative example with an application to the twin-rotor aero-dynamical system. The achieved results confirm the correctness and performances of the proposed approach. The future work will be focused on the problem of compensating the simultaneous sensor and process faults. The controller design strategy will be proposed to cope with the stated problem for the considered system.

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