



HAL
open science

An identity for the infinite sum $\sum_{n=0}^{\infty} \frac{1}{(n!)^3}$
Hassan Jolany

► **To cite this version:**

| Hassan Jolany. An identity for the infinite sum $\sum_{n=0}^{\infty} \frac{1}{(n!)^3}$. 2018. hal-01858817

HAL Id: hal-01858817

<https://hal.science/hal-01858817v1>

Preprint submitted on 21 Aug 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

An identity for the infinite sum

$$\sum_{n=0}^{\infty} \frac{1}{(n!)^3}$$

Hassan Jolany

Abstract. In this short note, we give an identity for the alpha function

$$\alpha(x, s) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^s},$$

where $s \in \mathbb{N}$, $x \in \mathbb{R}$, in the case $s = 3$.

1. INTRODUCTION We know that when $s = 1$, then the alpha function is the exponential function $\alpha(x, 1) = e^x$. The question is that what about when $s \geq 2$?. We have the following nice differential equation for the alpha function [2]. In fact, it is known that the alpha function is the solution of the following differential equation.

$$\sum_{k=1}^s \sigma_s^k x^{k-1} y^{(k)} - y = 0$$

where σ_s^k is the Stirling numbers of the second kind which satisfies in the following generating function,

$$\frac{1}{k!} (e^x - 1)^k = \sum_{s=k}^{\infty} \sigma_s^k \frac{x^n}{n!}$$

Hence by using the language of generalized hypergeometric function, we get [3],

$$\sum_{n=0}^{\infty} \frac{1}{(n!)^3} = {}_0F_2(; 1, 1; 1) \approx 2.1297$$

To deal with this question, we first give the following theorem which is the direct consequence of Bessel-Parseval identity[1].

Theorem 1. *Let we have the two following entire series, with real the coefficients,*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \text{ and } g(x) = \sum_{n=0}^{\infty} b_n x^n$$

with respectively non-zero radius R and R' . Take

$$h(x) = \sum_{n=0}^{\infty} a_n b_n x^n$$

, then if $-R < u < R$ and $-R' < v < R'$, then we have

$$h(uv) = \frac{1}{2\pi} \int_0^{2\pi} f(ue^{it})g(ve^{-it})dt$$

In the Theorem 1, if we take $f(x) = g(x) = e^x$, and $u = x, v = 1$, then we get the following amazing identity,

$$\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} = \frac{1}{2\pi} \int_0^{2\pi} e^{(x+1)\cos t} \cos((x-1)\sin t)dt$$

Hence by taking $x = 1$ we get

$$\sum_{n=0}^{\infty} \frac{1}{(n!)^2} = \frac{1}{2\pi} \int_0^{2\pi} e^{2\cos t} dt$$

But we know,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a \cos x + b \sin x)} dx = I_0(\sqrt{a^2 + b^2}), \quad (1)$$

where I_0 is the modified Bessel function of the first kind.

2. THE MAIN RESULT In this section we give an integral identity for the alpha function when $s = 3$. We take $f(x) = e^x, g(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{(x+1)\cos t} \cos((x-1)\sin t)dt$, and $u = x, v = 1$. Hence, from Theorem 1, we have

$$A = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^3} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{xe^{is}} e^{(e^{-is}+1)\cos t} \cos((e^{-is}-1)\sin t) ds dt$$

But, we have

$$e^{xe^{is}} e^{(e^{-is}+1)\cos t} = (e^{x \cos s + \cos s \cos t + \cos t}) (\cos(x \sin s - \sin s \cos t) + i \sin(x \sin s - \sin s \cos t))$$

and also,

$$\cos((e^{-is}-1)\sin t) = \cos(\cos s \sin t - \sin t) \cosh(\sin s \sin t) + i \sin(\cos s \sin t - \sin t) \sinh(\sin s \sin t)$$

Hence

$$\begin{aligned}
A &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left((e^{x \cos s + \cos s \cos t + \cos t}) (\cos(x \sin s - \sin s \cos t)) \right) \cos(\cos s \sin t - \sin t) \cosh(\sin s \sin t) ds dt \\
&\quad - \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left(e^{x \cos s + \cos s \cos t + \cos t} \right) \sin(x \sin s - \sin s \cos t) \sin(\cos s \sin t - \sin t) \sinh(\sin s \sin t) ds dt
\end{aligned}$$

and this gives an identity for the alpha function in the case $s = 3$.

REFERENCES

1. Knopp, Konrad, *Theory and Application of Infinite Series*. Dover Publications.(1990)
2. Gerard Eguether, *Nombres de Stirling et Nombres de Bell*.
<http://www.iecl.univ-lorraine.fr/Gerard.Eguether/zARTICLE/BF.pdf>
3. Wolfram Alpha,
<http://www.wolframalpha.com/>