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Optimal control of averaged state of a parabolic equation with missing boundary condition

Gisèle MOPHOU * Romario Gildas FOKO TIOMELA † Ali SEIBOU ‡

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Abstract

We consider the optimal control of general heat governed by an operator depend on an unknown parameter and with missing boundary condition. Using the notion of no-regret and low-regret control we prove that we can bring the average of the state of our model to a desired state. Then by means of Euler-Lagrange first order optimality condition, we expressed the optimal control in term of average of an appropriate adjoint state that we characterize by an optimality system. The main tools are the Lebesgue dominated convergence theorem and an appropriate Hilbert space endowed with a norm containing the average of the state.

Key-words :No-regret control, Low-regret control, Average control, Euler-Lagrange first order optimality condition.

AMS Subject Classification 35Q93,49J20, 93C05, 93C41.

1 Introduction

Let $N \in \mathbb{N}^*$ and Ω be a bounded open subset of \mathbb{R}^N with boundary Γ of class \mathcal{C}^2 . Let ω be an open non-empty subset of Ω . For $T > 0$, we set $Q = \Omega \times (0, T)$, $\omega_T = \omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. We consider the following controlled parabolic problem:

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$$\begin{cases} \frac{\partial y}{\partial t} - \operatorname{div}(\kappa(\alpha, x)\nabla y) + a_0 y &= f + v\chi_\omega & \text{in } Q, \\ y &= g & \text{on } \Sigma, \\ y(0) &= y^0 & \text{in } \Omega, \end{cases} \quad (1)$$

where diffusivity $\kappa \in C([0, 1] \times \bar{\Omega})$, the potential $a_0 \in L^\infty(Q)$, the function $f \in L^2(Q)$ is given, $v \in L^2(\omega_T)$ is the control and χ_ω is the characteristic function of the control set Ω . The initial condition y^0 is known and belongs to $L^2(\Omega)$. The function g belongs to $L^2(\Sigma)$.

Under the assumptions on the data, we know that system (1) has a unique solution $y(\alpha, v, g) = y(\alpha, x, t; v, g) \in L^2(Q)$ such that,

$$\begin{aligned} & - \int_\Omega y^0 \varphi(0) dx + \int_\Sigma \kappa(\alpha, x) g \frac{\partial \varphi}{\partial \nu} d\sigma dt + \int_Q y \left(-\frac{\partial \varphi}{\partial t} - \operatorname{div}(\kappa(\alpha, x)\nabla \varphi) + a_0 \varphi \right) dx dt \\ &= \int_Q (f + v\chi_\omega) \varphi dx dt \\ & \forall \varphi \in \mathcal{C}^\infty(\bar{Q}) \text{ such that } \varphi|_\Sigma = 0 \text{ and } \varphi(T) = 0 \text{ in } \Omega. \end{aligned}$$

System (1) is parameter dependent system with missing boundary condition. We want to act on this system in order to bring the average of the state to a desired state. More precisely, we are interested by the following problem

$$\inf_{v \in L^2(\omega_T)} J(v, g), \quad \forall g \in L^2(\Sigma), \quad (2)$$

where the cost function is given by

$$J(v, g) = \left\| \int_0^1 y(\alpha, v, g) d\alpha - z_d \right\|_{L^2(Q)}^2 + N \|v\|_{L^2(\omega_T)}^2, \quad (3)$$

with $z_d \in L^2(Q)$, $N > 0$ and $\int_0^1 y(\alpha, v, g) d\alpha \in L^2(Q) \quad \forall v \in L^2(\omega_T)$ and $\forall g \in L^2(\Sigma)$.

As (2) has no sense, we will study this problem combining the notion of no-regret and low-regret control introduce by J.L lion [2] to control problem with incomplete data with the notion of averaged control introduce by E. Zuazua [16] to control for parameter dependent system. So we first consider for any $\gamma > 0$ the low-regret problem:

$$\inf_{v \in L^2(\omega_T)} \left(\sup_{g \in L^2(\Sigma)} \left(J(v, g) - J(0, g) - \gamma \|g\|_{L^2(\Sigma)}^2 \right) \right). \quad (4)$$

Then we prove that the low-regret control converge to the no-regret control solution of the problem:

$$\inf_{v \in L^2(\omega_T)} \left(\sup_{g \in L^2(\Sigma)} (J(v, g) - J(0, g)) \right), \quad (5)$$

that we characterize by giving the corresponding optimality system.

The notion of no-regret and low regret control was introduced by J. L. Lions [2] to control a parabolic equation governed with an operator free of unknown parameter but with unknown initial condition. According to J.L. Lions, by looking for such a control, one looks for the "best possible control" v which do "at least as well" and "not much worse in the worst situation" than doing nothing. This notion was then applied to control some model with incomplete data, including model involving fractional derivative in time. We refer for instance to [4, 10, 11, 13, 12, 14, 15]. The averaged control notion was introduced by E. Zuazua [16] to analyse the problem of controlling parameter dependent systems. In this notion, one is interested by the average of the state with respect to the unknown parameter. for more literature on this topic we refer for instance to [18, 19, 17] and the reference therein. In this paper we are concerned with the control of a parameter dependent system with missing boundary condition. As far as we know this problem is new.

The rest of this paper is stated as follows. In section 2, we reformulate the low-regret and no-regret control problem (4) and (5). Section 3 is devoted to the proof of the existence of the low-regret control and its characterization. In Section 4, we prove that the low-regret control converges towards the no-regret control that we characterize with an optimality system. Concluding remarks are made in Section 5

2 Reformulation of the No regret and Low-regret control

In this section we prove using Legendre-Fenchel transform that the inf-sup problem (4) is equivalent to a classical optimal control problem.

So, let $f \in L^2(Q)$, $v \in L^2(\omega_T)$, $g \in L^2(\Sigma)$, $a_0 \in L^\infty(Q)$, $\kappa \in C([0, 1] \times \bar{\Omega})$ and $y^0 \in L^2(\Omega)$. Let also $y(\alpha, v, 0)$, $y(\alpha, 0, g)$, $y(\alpha, 0, 0)$ be respectively solutions of:

$$\begin{cases} \frac{\partial y(\alpha, v, 0)}{\partial t} - \operatorname{div}(\kappa(\alpha, x)\nabla y(\alpha, v, 0) + a_0 y(\alpha, v, 0)) = f + v\chi_\omega & \text{in } Q, \\ y(\alpha, v, 0) = 0 & \text{on } \Sigma, \\ y(0; \alpha, v, 0) = y^0 & \text{in } \Omega, \end{cases} \quad (6)$$

$$\begin{cases} \frac{\partial y(\alpha, 0, g)}{\partial t} - \operatorname{div}(\kappa(\alpha, x)\nabla y(\alpha, 0, g) + a_0 y(\alpha, 0, g)) = f & \text{in } Q, \\ y(\alpha, 0, g) = g & \text{on } \Sigma, \\ y(0; \alpha, 0, g) = y^0 & \text{in } \Omega \end{cases} \quad (7)$$

and

$$\begin{cases} \frac{\partial y(\alpha, 0, 0)}{\partial t} - \operatorname{div}(\kappa(\alpha, x)\nabla y(\alpha, 0, 0) + a_0 y(\alpha, 0, 0)) = f & \text{in } Q, \\ y(\alpha, 0, 0) = 0 & \text{on } \Sigma, \\ y(0; \alpha, 0, 0) = y^0 & \text{in } \Omega. \end{cases} \quad (8)$$

Then for almost every $\alpha \in [0, 1]$, the functions $y(\alpha, v, 0)$ and $y(\alpha, 0, 0)$ belong to $L^2((0, T); H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega))$ and we can prove by transposition that $y(\alpha, 0, g)$ belongs to $L^2(Q)$ and is such that,

$$\begin{aligned} & - \int_{\Omega} y^0 \varphi(0) dx + \int_{\Sigma} \kappa(\alpha, x) g \frac{\partial \varphi}{\partial \nu} d\sigma dt + \int_Q y \left(-\frac{\partial \varphi}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \varphi) + a_0 \varphi \right) dx dt \\ & = \int_Q f \varphi dx dt, \\ & \forall \varphi \in C^\infty(\bar{Q}) \text{ such that } \varphi|_{\Sigma} = 0 \text{ and } \varphi(T) = 0 \text{ in } \Omega. \end{aligned}$$

Remark 1 Note that $\int_0^1 y(\alpha, v, 0) d\alpha$, $\int_0^1 y(\alpha, 0, g) d\alpha$ and $\int_0^1 y(\alpha, 0, 0) d\alpha$ belong to $L^2(Q)$ because $\int_0^1 y(\alpha, v, g) d\alpha \in L^2(Q)$ for any $v \in L^2(\omega_T)$ and $g \in L^2(\Sigma)$.

Remark 2 Using Holder inequality, we have that if there exists a constant C independent of α such that $\|y\|_{L^2(Q)} \leq C$ then

$$\left\| \int_0^1 y(\alpha, \dots) d\alpha \right\|_{L^2(Q)} \leq C.$$

Lemma 2.1 For any $v \in L^2(\omega_T)$ and for any $g \in L^2(\Sigma)$, we have:

$$J(v, g) - J(0, g) = J(v, 0) - J(0, 0) + 2 \int_{\Sigma} \left(\int_0^1 \kappa(\alpha, x) \frac{\partial \zeta(\alpha, v)}{\partial \nu} d\alpha \right) g d\sigma dt, \quad (9)$$

where J is the cost function given by (3) and $\zeta(\alpha, v) := \zeta(x, t; \alpha, v) \in Z^{2,1}(Q) =: L^2((0, T); H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1((0, T); L^2(\Omega))$ is solution of

$$\begin{cases} -\frac{\partial \zeta(\alpha, v)}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \zeta(\alpha, v) + a_0 \zeta(\alpha, v)) & = - \int_0^1 \eta(\alpha, v) d\alpha & \text{in } Q, \\ \zeta(\alpha, v) & = 0 & \text{on } \Sigma, \\ \zeta(T; \alpha, v) & = 0 & \text{in } \Omega, \end{cases} \quad (10)$$

with

$$\eta(\alpha, v) = y(\alpha, v, 0) - y(\alpha, 0, 0). \quad (11)$$

Remark 3 Note that in view of Remark 2, we have that $\int_0^1 \eta(\alpha, v) d\alpha \in L^2(Q)$.

Proof. Observing that $y(\alpha, v, g) = y(\alpha, v, 0) + y(\alpha, 0, g) - y(\alpha, 0, 0)$, we have

$$\begin{aligned}
J(v, g) &= \left\| \int_0^1 (y(\alpha, v, 0) + y(\alpha, 0, g) - y(\alpha, 0, 0)) d\alpha - z_d \right\|_{L^2(Q)}^2 + N \|v\|_{L^2(\omega_T)}^2 \\
&= \left\| \int_0^1 y(\alpha, v, 0) d\alpha - z_d \right\|_{L^2(Q)}^2 + N \|v\|_{L^2(\omega_T)}^2 \\
&+ \left\| \left(\int_0^1 y(\alpha, 0, g) d\alpha - z_d \right) - \left(\int_0^1 y(\alpha, 0, 0) d\alpha - z_d \right) \right\|_{L^2(Q)}^2 \\
&+ 2 \int_Q \left[\int_0^1 y(\alpha, v, 0) d\alpha - z_d \right] \left[\int_0^1 (y(\alpha, 0, g) - y(\alpha, 0, 0)) d\alpha \right] dx dt \\
&= J(v, 0) + \left\| \int_0^1 y(\alpha, 0, g) d\alpha - z_d \right\|_{L^2(Q)}^2 + \left\| \int_0^1 y(\alpha, 0, 0) d\alpha - z_d \right\|_{L^2(Q)}^2 \\
&- 2 \int_Q \left[\int_0^1 y(\alpha, 0, g) d\alpha - z_d \right] \left[\int_0^1 y(\alpha, 0, 0) d\alpha - z_d \right] dx dt \\
&+ 2 \int_Q \left[\int_0^1 (y(\alpha, v, 0) - y(\alpha, 0, 0)) d\alpha \right] \left[\int_0^1 (y(\alpha, 0, g) - y(\alpha, 0, 0)) d\alpha \right] dx dt \\
&+ 2 \int_Q \left[\int_0^1 (y(\alpha, 0, 0) d\alpha - z_d) \right] \left[\int_0^1 (y(\alpha, 0, g) - y(\alpha, 0, 0)) d\alpha \right] dx dt \\
&= J(v, 0) + J(0, g) - J(0, 0) \\
&+ 2 \int_Q \left[\int_0^1 (y(\alpha, v, 0) - y(\alpha, 0, 0)) d\alpha \right] \left[\int_0^1 (y(\alpha, 0, g) - y(\alpha, 0, 0)) d\alpha \right] dx dt
\end{aligned}$$

Consequently,

$$\begin{aligned}
J(v, g) - J(0, g) &= J(v, 0) - J(0, 0) + \\
&2 \int_Q \left[\int_0^1 (y(\alpha, v, 0) - y(\alpha, 0, 0)) d\alpha \right] \left[\int_0^1 (y(\alpha, 0, g) - y(\alpha, 0, 0)) d\alpha \right] dx dt.
\end{aligned} \tag{12}$$

In view of Remark 3, system (10) has a unique solution $\zeta(\alpha, v) \in Z^{2,1}(Q)$. Moreover, there exists a constant $C > 0$ independent of α such that

$$\|\zeta(\alpha, v)\|_{L^2((0,T);H^2(\Omega))} \leq C \|v\|_{L^2(\omega_T)}. \tag{13}$$

Therefore, using the continuity of the trace operator, we have that

$$\left\| \frac{\partial \zeta(\alpha, v)}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C \|v\|_{L^2(\omega_T)}. \tag{14}$$

We set

$$\pi(\alpha, g) = y(\alpha, 0, g) - y(\alpha, 0, 0).$$

In view of Remark 2, we have that $\int_0^1 \pi(\alpha, g) d\alpha \in L^2(Q)$. Observing that $\pi(\alpha, g)$ is solution of

$$\begin{cases} \frac{\partial \pi}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \pi) + a_0 \pi = 0 & \text{in } Q, \\ \pi = g & \text{on } \Sigma, \\ \pi(0) = 0 & \text{in } \Omega, \end{cases} \quad (15)$$

if we multiply the first equation of (10) by $\int_0^1 \pi(\alpha, g) d\alpha$ and we integrate over Q , we have:

$$\begin{aligned} & - \int_Q \left[\int_0^1 (y(\alpha, v, 0) - y(\alpha, 0, 0)) d\alpha \right] \left[\int_0^1 (y(\alpha, 0, g) - y(\alpha, 0, 0)) d\alpha \right] dx dt \\ & = \int_0^1 \left[\int_Q \left(-\frac{\partial \zeta(\alpha, v)}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \zeta(\alpha, v) + a_0 \zeta(\alpha, v)) \pi(\alpha, g) dx dt \right) d\alpha \right] \\ & = - \int_\Sigma \left(\int_0^1 \kappa(\alpha, x) \frac{\partial \zeta(\alpha, v)}{\partial \nu} d\alpha \right) g d\sigma dt \end{aligned}$$

Hence,

$$\begin{aligned} & \int_Q \left[\int_0^1 (y(\alpha, v, 0) - y(\alpha, 0, 0)) d\alpha \right] \left[\int_0^1 (y(\alpha, 0, g) - y(\alpha, 0, 0)) d\alpha \right] dx dt \\ & = \int_\Sigma \left(\int_0^1 \kappa(\alpha, x) \frac{\partial \zeta(\alpha, v)}{\partial \nu} d\alpha \right) g d\sigma dt. \end{aligned} \quad (16)$$

If we combine (16) and (12), we obtain that

$$J(v, g) - J(0, g) = J(v, 0) - J(0, 0) + 2 \int_\Sigma \left(\int_0^1 \kappa(\alpha, x) \frac{\partial \zeta(\alpha, v)}{\partial \nu} d\alpha \right) g d\sigma dt.$$

■

Remark 4 As $\kappa \in C([0, 1] \times \overline{\Omega})$, using (14), we have that $\kappa \frac{\partial \zeta(\alpha, v)}{\partial \nu}$ is bounded in $L^2(\Sigma)$ independently of α . It then follows from Remark 1 that $\int_0^1 \kappa(\alpha, x) \frac{\partial \zeta(\alpha, v)}{\partial \nu} d\alpha \in L^2(\Sigma)$. This means that (9) has a sense.

From now on, we set

$$\frac{\partial \tilde{\zeta}(\alpha, v)}{\partial \nu} = \int_0^1 \kappa(\alpha, x) \frac{\partial \zeta(\alpha, v)}{\partial \nu} d\alpha. \quad (17)$$

Then according to (9), problem (5) is equivalent to:

$$\inf_{v \in L^2(\omega_T)} \left(\sup_{g \in L^2(\Sigma)} \left(J(v, 0) - J(0, 0) + 2 \int_\Sigma \frac{\partial \tilde{\zeta}(\alpha, v)}{\partial \nu} g d\sigma dt \right) \right). \quad (18)$$

Consequently, the no-regret control belongs to the set

$$\mathcal{O} = \left\{ v \in L^2(\omega_T) \text{ such that } \frac{\partial \tilde{\zeta}(\alpha, v)}{\partial \nu} \text{ is orthogonal to } L^2(\Sigma) \right\}. \quad (19)$$

As such a control is not easy to characterize, we consider the low-regret control problem. Using (9), problem (4) could be rewritten as

$$\begin{aligned} & \inf_{v \in L^2(\omega_T)} (J(v, 0) - J(0, 0)) \\ & + 2 \sup_{g \in L^2(\Sigma)} \left(\int_{\Sigma} \frac{\partial \tilde{\zeta}(\alpha, v)}{\partial \nu} g d\sigma dt - \frac{\gamma}{2} \|g\|_{L^2(\Sigma)}^2 \right), \end{aligned}$$

which by means of Legendre-Fenchel transform is equivalent to the following problem: For any $\gamma > 0$, find $u^\gamma \in L^2(Q)$ such that

$$J_\gamma(u^\gamma) = \inf_{v \in L^2(\omega_T)} J_\gamma(v), \quad (20)$$

where

$$J_\gamma(v) = J(v, 0) - J(0, 0) + \frac{1}{\gamma} \left\| \frac{\partial \tilde{\zeta}(\alpha, v)}{\partial \nu} \right\|_{L^2(\Sigma)}^2. \quad (21)$$

3 Existence and characterization of the low-regret control

Our main objective in this section will be to show that the low-regret problem (20) has a unique solution which converges to the no-regret control unique solution of (18). Furthermore, we will give the equations that characterize the low-regret control.

Remark 5 *From now on, we use $C(X)$ to denote a positive constant whose value varies from a line to another but depends on X .*

Proposition 3.1 *There exists a unique low-regret control $u^\gamma \in L^2(\omega_T)$ solution of (20).*

Proof. Observing that We have:

$$-J(0, 0) \leq J(v, 0) - J(0, 0) + \frac{1}{\gamma} \left\| \int_0^1 \kappa(\alpha, x) \frac{\partial \zeta(\alpha, v)}{\partial \nu} d\alpha \right\|_{L^2(\Sigma)}^2 \quad \forall v \in L^2(\omega_T),$$

we have that the set $\{J_\gamma(v), J_\gamma(v) \geq -J(0, 0), v \in L^2(\omega_T)\}$ is not empty. Consequently, there exists m_γ such that

$$m_\gamma = \inf_{v \in L^2(\omega_T)} J_\gamma(v).$$

Let $v_n \in L^2(\omega_T)$ be a minimizing sequence such that

$$J_\gamma(v_n) \longrightarrow m_\gamma. \quad (22)$$

Then, $y_n := y(x, t; \alpha, v_n, 0)$ satisfies:

$$\begin{cases} \frac{\partial y_n}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla y_n) + a_0 y_n = f + v_n \chi_\omega & \text{in } Q, \\ y_n = 0 & \text{on } \Sigma, \\ y_n(0) = y^0 & \text{in } \Omega. \end{cases} \quad (23)$$

Let us set $\tilde{\zeta}_n = \tilde{\zeta}(v_n) = \int_0^1 \kappa(\alpha, x) \frac{\partial \zeta(\alpha, v_n)}{\partial \nu} d\alpha$ where $\zeta_n = \zeta(\alpha, v_n)$ is solution of (10).

In view of (22), there exists a constant $C > 0$ independent of n and γ such that $-J(0, 0) \leq J_\gamma(v_n) \leq C$ which implies:

$$0 \leq J(v_n, 0) + \frac{1}{\gamma} \left\| \frac{\partial \tilde{\zeta}_n}{\partial \nu} \right\|_{L^2(\Sigma)}^2 \leq C + J(0, 0).$$

Hence there exists a non negative constant C independent of n and γ , such that:

$$0 \leq J(v_n, 0) + \frac{1}{\gamma} \left\| \frac{\partial \tilde{\zeta}_n}{\partial \nu} \right\|_{L^2(\Sigma)}^2 \leq C,$$

which implies that

$$\begin{aligned} J(v_n, 0) &\leq C, \\ \frac{1}{\gamma} \left\| \frac{\partial \tilde{\zeta}_n}{\partial \nu} \right\|_{L^2(\Sigma)}^2 &\leq C. \end{aligned}$$

Hence, using the fact that

$$J(v_n, 0) = \left\| \int_0^1 y(\alpha, v_n, 0) d\alpha - z_d \right\|_{L^2(Q)}^2 + N \|v_n\|_{L^2(\omega_T)}^2,$$

we deduce that

$$\|v_n\|_{L^2(\omega_T)} \leq C, \quad (24)$$

$$\left\| \frac{\partial \tilde{\zeta}_n}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C\sqrt{\gamma}, \quad (25)$$

$$\left\| \int_0^1 y_n(\alpha, \cdot, \cdot) d\alpha \right\|_{L^2(Q)} \leq C. \quad (26)$$

In view of (23) and (24), there exists $C > 0$ independent of n and γ such that

$$\|y_n\|_{L^2((0, T); H_0^1(\Omega))} \leq C. \quad (27)$$

Hence, there exist $u^\gamma \in L^2(\omega_T)$, $y^\gamma \in L^2((0, T); H_0^1(\Omega))$ and subsequences extracted from (v_n) and (y_n) (still denoted (v_n) and (y_n)) such that

$$v_n \rightharpoonup u^\gamma \quad \text{weakly in } L^2(\omega_T), \quad (28)$$

$$y_n \rightharpoonup y^\gamma \quad \text{weakly in } L^2((0, T); H_0^1(\Omega)). \quad (29)$$

We set $z_n(x, t) = \int_0^1 y_n(\alpha, x, t) d\alpha$. Then in view of (26) there exist $z^\gamma \in L^2(Q)$ such that

$$\int_Q z_n(x, t) \varphi(x, t) dx dt = \int_0^1 \left(\int_Q y_n(x, t) \varphi(x, t) dx dt \right) \rightarrow \int_Q z^\gamma(x, t) \varphi(x, t) dx dt \quad \forall \varphi \in L^2(Q). \quad (30)$$

On the other hand, using the fact that y_n is bounded independently of α and the fact that

$$\int_Q y_n(x, t) \varphi(x, t) dx dt \rightarrow \int_Q y^\gamma(x, t) \varphi(x, t) dx dt \quad \forall \varphi \in L^2(Q), \quad (31)$$

it follows from the Lebesgue dominated convergence theorem that

$$\int_0^1 \left[\int_Q y_n(\alpha, x, t) \varphi(x, t) dx dt \right] d\alpha = \int_Q \left[\int_0^1 y_n(\alpha, x, t) d\alpha \right] \varphi(x, t) dx dt$$

converges towards

$$\int_0^1 \left[\int_Q y^\gamma(\alpha, x, t) \varphi(x, t) dx dt \right] d\alpha = \int_Q \left[\int_0^1 y^\gamma(\alpha, x, t) d\alpha \right] \varphi(x, t) dx dt.$$

Therefore, from (30) and the uniqueness of the limit, we have that

$$\int_0^1 y^\gamma(\alpha, x, t) d\alpha = z^\gamma(x, t). \quad (32)$$

We thus can write

$$\int_Q \int_0^1 y(\alpha, v_n, 0) \varphi d\alpha dx dt \rightarrow \int_Q \int_0^1 y(\alpha, u^\gamma, 0) \varphi(x, t) d\alpha dx dt. \quad (33)$$

The rest of the proof will be divided into steps.

Step 1. We prove that (u^γ, y^γ) satisfies (6).

Let $\mathcal{D}(Q)$ being the set of functions of class C^∞ on Q with compact support. Multiplying the first equation in (23) by $\varphi \in \mathcal{D}(Q)$ and integrating by parts over Q ,

$$\int_Q y_n \left(-\frac{\partial \varphi}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \varphi) + a_0 \varphi \right) dx dt = \int_Q (f + v_n \chi_\omega) \varphi dx dt$$

Passing this latter identity to the limit when $n \rightarrow \infty$ while using (28) and (29), we obtain that

$$\int_Q y^\gamma \left(-\frac{\partial \varphi}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \varphi) + a_0 \varphi \right) dx dt = \int_Q (f + u^\gamma \chi_\omega) \varphi dx dt$$

which after integration by parts gives over Q ,

$$\int_Q \left(\frac{\partial y^\gamma}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla y^\gamma) + a_0 y^\gamma \right) \varphi dx dt = \int_Q (f + u^\gamma \chi_\omega) \varphi dx dt.$$

Hence, we deduce that

$$\frac{\partial y^\gamma}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla y^\gamma) + a_0 y^\gamma = f + u^\gamma \chi_\omega \text{ in } Q. \quad (34)$$

Now, as, $y^\gamma \in L^2((0, T), H_0^1(\Omega))$, using (34), we have that $\frac{\partial y^\gamma}{\partial t} \in L^2((0, T), H^{-1}(\Omega))$. This implies that $y^\gamma \in \mathcal{C}([0, T], L^2(\Omega))$. On the other hand, $y^\gamma \in L^2(Q)$ and $\operatorname{div}(\kappa(\alpha, x) \nabla y^\gamma) \in H^{-1}((0, T), L^2(\Omega))$. This implies that $y^\gamma|_\Sigma$ and $\frac{\partial y^\gamma}{\partial \nu}|_\Sigma$ exist and belong respectively to $H^{-1}((0, T); H^{-\frac{1}{2}}(\Gamma))$ and $H^{-1}((0, T); H^{-\frac{3}{2}}(\Gamma))$.

Now, let $\varphi \in \mathcal{C}^\infty(\bar{Q})$ with $\varphi|_\Sigma = 0$ and $\varphi(T) = 0$ in Ω . If we multiply the first equation in (23) by φ and we integrate over Q , we obtain:

$$\begin{aligned} \int_0^T \int_\Omega (f + v_n \chi_\omega) \varphi dx dt &= - \int_\Omega y^0 \varphi(x, 0) dx \\ &+ \int_0^T \int_\Omega y_n \left(-\frac{\partial \varphi}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \varphi) + a_0 \varphi \right) dx dt, \end{aligned}$$

which by passing to the limit when $n \rightarrow \infty$ while using (28) and (29) gives

$$\begin{aligned} \int_0^T \int_\Omega (f + u^\gamma \chi_\omega) \varphi dx dt &= - \int_\Omega y^0 \varphi(x, 0) dx \\ &+ \int_0^T \int_\Omega y^\gamma \left(-\frac{\partial \varphi}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \varphi) + a_0 \varphi \right) dx dt, \\ &\forall \varphi \in \mathcal{C}^\infty(\bar{Q}) \text{ with } \varphi|_\Sigma = 0 \text{ and } \varphi(T) = 0 \text{ in } \Omega. \end{aligned}$$

If we integrate the last integral in this latter identity, we obtain that

$$\begin{aligned} \int_0^T \int_\Omega (f + u^\gamma \chi_\omega) \varphi dx dt &= - \int_\Omega y^0 \varphi(x, 0) dx + \int_\Omega y^\gamma(x, 0) \varphi(x, 0) dx \\ &- \left\langle y^\gamma, \kappa(\alpha, x) \frac{\partial \varphi}{\partial \nu} \right\rangle_{H^{-3/2}((0, T); H^{-\frac{1}{2}}(\Gamma)), H^{3/2}((0, T); H^{\frac{1}{2}}(\Gamma))} \\ &+ \int_0^T \int_\Omega \varphi \left(\frac{\partial y^\gamma}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla y^\gamma) + a_0 y^\gamma \right) dx dt, \\ &\forall \varphi \in \mathcal{C}^\infty(\bar{Q}) \text{ with } \varphi|_\Sigma = 0 \text{ and } \varphi(T) = 0 \text{ in } \Omega. \end{aligned}$$

which in view of (34) yields

$$\begin{aligned}
0 &= \int_{\Omega} (-y^0 + y^\gamma(0))\varphi(x, 0)dx \\
&\quad - \left\langle y^\gamma, \kappa(\alpha, x) \frac{\partial \varphi}{\partial \nu} \right\rangle_{H^{-3/2}((0, T); H^{-\frac{1}{2}}(\Gamma)), H^{3/2}((0, T); H^{\frac{1}{2}}(\Gamma))}, \\
&\quad \forall \varphi \in \mathcal{C}^\infty(\bar{Q}) \text{ with } \varphi|_{\Sigma} = 0 \text{ and } \varphi(T) = 0 \text{ in } \Omega.
\end{aligned} \tag{35}$$

If we take in addition $\frac{\partial \varphi}{\partial \nu} = 0$ on Σ , we have $\int_{\Omega} (-y^0 + y^\gamma(0))\varphi(x, 0)dx = 0$, which implies:

$$y^\gamma(x, 0) = y^0(x), \quad x \in \Omega. \tag{36}$$

It then follows from (35) that

$$y^\gamma = 0 \text{ on } \Sigma. \tag{37}$$

Now, combining (34), (36) and (37), we find that $y^\gamma = y(\alpha, u^\gamma, 0)$ is solution of (6). This means that

$$\begin{cases} \frac{\partial y^\gamma}{\partial t} - \operatorname{div}(\kappa(\alpha, x)\nabla y^\gamma) + a_0 y^\gamma = f + u^\gamma \chi_\omega & \text{in } Q, \\ y^\gamma = 0 & \text{on } \Sigma, \\ y^\gamma(0) = y^0 & \text{in } \Omega. \end{cases} \tag{38}$$

Step 2. We show that $\zeta(\alpha, v_n)$ converges to $\zeta^\gamma = \zeta(\alpha, u^\gamma)$ which satisfies (10).

We know that $\zeta_n := \zeta(x, t; \alpha, v_n)$ is solution of

$$\begin{cases} -\frac{\partial \zeta_n}{\partial t} - \operatorname{div}(\kappa(\alpha, x)\nabla \zeta_n) + a_0 \zeta_n = -\int_0^1 (y_n - y(\alpha, 0, 0))d\alpha & \text{in } Q, \\ \zeta_n = 0 & \text{on } \Sigma, \\ \zeta_n(T) = 0 & \text{in } \Omega, \end{cases} \tag{39}$$

Therefore, using (26), we can say that there exists a positive constant C independent of n and γ such that

$$\|\zeta_n\|_{L^2((0, T); H^2(\Omega))} \leq C. \tag{40}$$

And it follows from the continuity of the trace that

$$\left\| \frac{\partial \zeta_n}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C. \tag{41}$$

Consequently, there exists $\zeta^\gamma \in L^2((0, T); H^2(\Omega))$ and a subsequence extracted from $(\zeta(\alpha, v_n))$ (still denoted $\zeta(\alpha, v_n)$) such that

$$\zeta_n \rightharpoonup \zeta^\gamma \text{ weakly in } L^2((0, T); H^2(\Omega)). \tag{42}$$

Multiplying the first equation in (39) by $\varphi \in \mathcal{D}(Q)$ and integrating by parts over Q ,

$$\int_Q \zeta_n \left(\frac{\partial \varphi}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \varphi) + a_0 \varphi \right) dx dt = - \int_Q \left(\int_0^1 (y_n - y(\alpha, 0, 0)) d\alpha \right) \varphi dx dt$$

Passing this latter identity to the limit when $n \rightarrow \infty$ while using (42), (30) and (33), we obtain that

$$\int_Q \zeta^\gamma \left(\frac{\partial \varphi}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \varphi) + a_0 \varphi \right) dx dt = - \int_Q \left(\int_0^1 (y^\gamma - y(\alpha, 0, 0)) d\alpha \right) \varphi dx dt$$

which after integration by parts gives over Q ,

$$\int_Q \left(-\frac{\partial \zeta^\gamma}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \zeta^\gamma) + a_0 \zeta^\gamma \right) \varphi dx dt = - \int_Q \left(\int_0^1 (y^\gamma - y(\alpha, 0, 0)) d\alpha \right) \varphi dx dt.$$

Hence, we deduce that

$$-\frac{\partial \zeta^\gamma}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \zeta^\gamma) + a_0 \zeta^\gamma = - \int_0^1 (y^\gamma - y(\alpha, 0, 0)) d\alpha \text{ in } Q. \quad (43)$$

Now, as $\zeta^\gamma \in L^2((0, T), H^2(\Omega))$, using (43), we have that $\frac{\partial y^\gamma}{\partial t} \in L^2((0, T), L^2(\Omega))$. This implies that $y^\gamma \in \mathcal{C}([0, T], L^2(\Omega))$. On the other hand, we have that $y^\gamma|_\Sigma \in L^2(\Sigma)$.

Now, let $\varphi \in \mathcal{C}^\infty(\bar{Q})$ with $\varphi|_\Sigma = 0$ and $\varphi(0) = 0$ in $\bar{\Omega}$. If we multiply the first equation in (23) by φ and we integrate over Q , we obtain:

$$- \int_Q \left(\int_0^1 (y_n - y(\alpha, 0, 0)) d\alpha \right) \varphi dx dt = \int_0^T \int_\Omega \zeta_n \left(\frac{\partial \varphi}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \varphi) + a_0 \varphi \right) dx dt,$$

which by passing to the limit when $n \rightarrow \infty$ while using (42), (30) and (33) gives

$$- \int_Q \left(\int_0^1 (y^\gamma - y(\alpha, 0, 0)) d\alpha \right) \varphi dx dt = \int_0^T \int_\Omega \zeta^\gamma \left(\frac{\partial \varphi}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \varphi) + a_0 \varphi \right) dx dt, \\ \forall \varphi \in \mathcal{C}^\infty(\bar{Q}) \text{ with } \varphi|_\Sigma = 0 \text{ and } \varphi(0) = 0 \text{ in } \Omega.$$

If we integrate the last integral in this latter identity, we obtain that

$$- \int_Q \left(\int_0^1 (y^\gamma - y(\alpha, 0, 0)) d\alpha \right) \varphi dx dt = \int_\Omega \zeta^\gamma(T) \varphi(x, T) dx \\ - \int_\Sigma \zeta^\gamma \kappa(\alpha, x) \frac{\partial \varphi}{\partial \nu} d\sigma dt \\ + \int_0^T \int_\Omega \varphi \left(-\frac{\partial \zeta^\gamma}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \zeta^\gamma) + a_0 \zeta^\gamma \right) dx dt, \\ \forall \varphi \in \mathcal{C}^\infty(\bar{Q}) \text{ with } \varphi|_\Sigma = 0 \text{ and } \varphi(0) = 0 \text{ in } \Omega.$$

which in view of (43) yields

$$0 = \int_{\Omega} \zeta^\gamma(T) \varphi(x, T) dx - \int_{\Sigma} \zeta^\gamma \kappa(\alpha, x) \frac{\partial \varphi}{\partial \nu} d\sigma dt, \quad (44)$$

$\forall \varphi \in \mathcal{C}^\infty(\bar{Q})$ with $\varphi|_{\Sigma} = 0$ and $\varphi(0) = 0$ in Ω .

If we take in addition $\frac{\partial \varphi}{\partial \nu} = 0$ on Σ , we have $\int_{\Omega} \zeta^\gamma(T) \varphi(x, T) dx = 0$, which implies:

$$\zeta^\gamma(x, T) = 0, \quad x \in \Omega. \quad (45)$$

It follows from (44) that

$$\zeta^\gamma = 0 \text{ on } \Sigma. \quad (46)$$

Now, combining (43), (45) and (46), we find that $\zeta^\gamma = \zeta(\alpha, u^\gamma)$ is solution of

$$\begin{cases} -\frac{\partial \zeta^\gamma}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \zeta^\gamma) + a_0 \zeta^\gamma = -\int_0^1 \eta(\alpha, u^\gamma) d\alpha & \text{in } Q, \\ \zeta^\gamma = 0 & \text{on } \Sigma, \\ \zeta^\gamma(T) = 0 & \text{in } \Omega, \end{cases} \quad (47)$$

where as in (11),

$$\eta(\alpha, u^\gamma) = y(\alpha, u^\gamma, 0) - y(\alpha, 0, 0).$$

Moreover, using (41), we prove that

$$\frac{\partial \zeta_n}{\partial \nu} \rightharpoonup \frac{\partial \zeta^\gamma}{\partial \nu} \text{ weakly in } L^2(\Sigma). \quad (48)$$

In view of (25), there exist $\beta \in L^2(\Sigma)$ such that

$$\frac{\partial \tilde{\zeta}_n}{\partial \nu} \rightharpoonup \beta \text{ weakly in } L^2(\Sigma). \quad (49)$$

This means that when $n \rightarrow \infty$,

$$\int_{\Sigma} \left(\int_0^1 \kappa(\alpha, x) \frac{\partial \zeta_n}{\partial \nu} d\alpha \right) \varphi d\sigma dt \rightarrow \int_{\Sigma} \beta \varphi d\sigma dt, \quad \forall \varphi \in L^2(\Sigma) \quad (50)$$

because $\frac{\partial \tilde{\zeta}_n}{\partial \nu} = \int_0^1 \kappa(\alpha, x) \frac{\partial \zeta_n}{\partial \nu} d\alpha$.

Using (41) and the fact that $\kappa \in C([0, 1] \times \bar{\Omega})$, we have that

$$\left\| \kappa(\alpha, x) \frac{\partial \zeta_n}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C \left\| \frac{\partial \zeta_n}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C. \quad (51)$$

Hence it follows from (48) that

$$\int_{\Sigma} (\kappa(\alpha, x) \frac{\partial \zeta_n}{\partial \nu}) \varphi d\sigma dt \rightarrow \int_{\Sigma} (\kappa(\alpha, x) \frac{\partial \zeta^\gamma}{\partial \nu}) \varphi d\sigma dt, \quad \forall \varphi \in L^2(\Sigma). \quad (52)$$

Using the Lebesgue dominated convergence theorem and the uniqueness of the limit, we deduce that

$$\beta = \frac{\partial \tilde{\zeta}^\gamma}{\partial \nu} = \int_0^1 \kappa(\alpha, x) \frac{\partial \zeta^\gamma}{\partial \nu} d\alpha. \quad (53)$$

Step 3. We prove that u^γ is unique.

Using the weak lower semi-continuity of the function $J_\gamma(v)$, (30), (32), (28), (49) and (53), we deduce that

$$J_\gamma(u^\gamma) \leq \liminf_{n \rightarrow \infty} J_\gamma(v_n),$$

where

$$J_\gamma(u^\gamma) = \left\| \int_0^1 y(\alpha, u^\gamma, 0) d\alpha - z_d \right\|_{L^2(Q)}^2 - J(0, 0) + \frac{1}{\gamma} \left\| \frac{\partial \tilde{\zeta}^\gamma}{\partial \nu} \right\|_{L^2(\Sigma)}^2 + N \|u^\gamma\|_{L^2(\omega_T)}^2.$$

As $\liminf_{n \rightarrow \infty} J_\gamma(v_n) = \lim_{n \rightarrow \infty} J_\gamma(v_n) = \inf_{v \in L^2(\omega_T)} J_\gamma(v)$, we obtain that u^γ is solution of (20). The strictly convexity of J_γ allows us to conclude that u^γ is unique.

■

We can now characterize the low-regret control u^γ .

Proposition 3.2 *Let u_γ be the solution of (20). Then there exist $p^\gamma \in L^2(Q)$ and $q^\gamma \in Z^{2,1}(Q)$ such that $\{y^\gamma, \zeta^\gamma, p^\gamma, q^\gamma\}$ is solution to*

$$\begin{cases} \frac{\partial y^\gamma}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla y^\gamma) + a_0 y^\gamma = f + u^\gamma \chi_\omega & \text{in } Q, \\ y^\gamma = 0 & \text{on } \Sigma, \\ y^\gamma(0) = y^0 & \text{in } \Omega, \end{cases} \quad (54)$$

$$\begin{cases} -\frac{\partial \zeta^\gamma}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \zeta^\gamma) + a_0 \zeta^\gamma = -\int_0^1 \eta(\alpha, u^\gamma) d\alpha & \text{in } Q, \\ \zeta^\gamma = 0 & \text{on } \Sigma, \\ \zeta^\gamma(T) = 0 & \text{in } \Omega. \end{cases} \quad (55)$$

$$\begin{cases} \frac{\partial p^\gamma}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla p^\gamma) + a_0 p^\gamma = 0 & \text{in } Q, \\ p^\gamma = \frac{1}{\sqrt{\gamma}} \frac{\partial \tilde{\zeta}^\gamma}{\partial \nu} & \text{on } \Sigma, \\ p^\gamma(0) = 0 & \text{in } \Omega, \end{cases} \quad (56)$$

$$\begin{cases} -\frac{\partial q^\gamma}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla q^\gamma) + a_0 q^\gamma = \int_0^1 \left(y^\gamma + \frac{p^\gamma}{\sqrt{\gamma}} \right) d\alpha - z_d & \text{in } Q, \\ q^\gamma = 0 & \text{on } \Sigma, \\ q^\gamma(T) = 0 & \text{in } \Omega \end{cases} \quad (57)$$

and

$$u^\gamma = -\frac{1}{N} \int_0^1 q^\gamma d\alpha \text{ in } \omega_T, \quad (58)$$

where $q^\gamma := q^\gamma(\alpha, u^\gamma)$ and $p^\gamma := p^\gamma(\alpha, u^\gamma)$.

Proof. We have already proved in the **Step 1** of Proposition 3.1 that the state $y^\gamma = y(\alpha, u^\gamma, 0)$ associate to optimal control u^γ satisfies (54). In **Step 2**, we obtain that $\zeta^\gamma = \zeta(\alpha, u^\gamma)$ satisfies (55). To prove (56)-(58), we write the Euler-Lagrange first order optimality condition which characterizes the optimal control u^γ :

$$\lim_{\theta \rightarrow 0} \frac{J_\gamma(u^\gamma + \theta v) - J_\gamma(u^\gamma)}{\theta} = 0 \quad \forall v \in L^2(\omega_T). \quad (59)$$

After some calculations

$$\begin{aligned} & \int_Q \left(\int_0^1 y(\alpha, u^\gamma, 0) d\alpha - z_d \right) \left(\int_0^1 \bar{y}(\alpha, v) d\alpha \right) dx dt \\ & + \frac{1}{\gamma} \int_\Sigma \left(\int_0^1 \kappa(\alpha, x) \frac{\partial \bar{\zeta}}{\partial \nu}(\alpha, v) d\alpha \right) \left(\int_0^1 \kappa(\alpha, x) \frac{\partial \zeta}{\partial \nu}(\alpha, u^\gamma) d\alpha \right) d\sigma dt \\ & + N \int_{\omega_T} u^\gamma v dx dt = 0 \quad \forall v \in L^2(\omega_T), \end{aligned} \quad (60)$$

where $\bar{y}(\alpha, v)$ is solution to

$$\begin{cases} \frac{\partial \bar{y}}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \bar{y}) + a_0 \bar{y} = v \chi_\omega & \text{in } Q, \\ \bar{y} = 0 & \text{on } \Sigma, \\ \bar{y}(0) = 0 & \text{in } \Omega, \end{cases} \quad (61)$$

and $\bar{\zeta}(\alpha, v)$ is solution to

$$\begin{cases} -\frac{\partial \bar{\zeta}}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \bar{\zeta}) + a_0 \bar{\zeta} = -\int_0^1 \bar{y}(\alpha, v) d\alpha & \text{in } Q, \\ \bar{\zeta} = 0 & \text{on } \Sigma, \\ \bar{\zeta}(T) = 0 & \text{in } \Omega. \end{cases} \quad (62)$$

Now, if we multiply the first equation in (61) and in (62) by $\int_0^1 q^\gamma(\alpha, u^\gamma) d\alpha \in L^2(Q)$ and $\frac{1}{\sqrt{\gamma}} \int_0^1 p^\gamma(\alpha, u^\gamma) d\alpha$ respectively, then integrate by parts over Q , we obtain

$$\begin{aligned} & \int_Q \left[\left(\int_0^1 \bar{y}(\alpha, v) d\alpha \right) \left(\int_0^1 \left(y(\alpha, u^\gamma, 0) + \frac{1}{\sqrt{\gamma}} p^\gamma(\alpha, u^\gamma) \right) d\alpha - z_d \right) \right] dx dt \\ & = \int_{\omega_T} v \left(\int_0^1 q^\gamma(\alpha, u^\gamma) d\alpha \right) dx dt, \end{aligned} \quad (63)$$

and

$$\begin{aligned} & \int_\Sigma \left[-\frac{1}{\gamma} \int_0^1 \left(\kappa(\alpha, x) \frac{\partial \bar{\zeta}}{\partial \nu}(\alpha, v) \right) d\alpha \right] \left[\int_0^1 \kappa(\alpha, x) \frac{\partial \zeta}{\partial \nu}(\alpha, u^\gamma) d\alpha \right] d\sigma dt \\ & = \int_Q \left[\int_0^1 \bar{y}(\alpha, v) d\alpha \right] \left[\frac{1}{\sqrt{\gamma}} \int_0^1 p^\gamma d\alpha \right] dx dt. \end{aligned} \quad (64)$$

Combining (60), (63) and (64), we have that

$$\int_{\omega_T} \left(Nu^\gamma + \int_0^1 q^\gamma(\alpha, u^\gamma) d\alpha \right) v dx dt = 0, \quad \forall v \in L^2(\omega_T), \quad (65)$$

which implies that

$$Nu^\gamma + \int_0^1 q^\gamma(\alpha, u^\gamma) = 0 \text{ in } \omega_T. \quad (66)$$

■

4 Existence and characterization of the no-regret control

Proposition 4.1 *The low-regret control u^γ converges in $L^2(Q)$ to the no-regret control \hat{u} , solution of (18).*

Proof. From (20), we have that

$$J_\gamma(u^\gamma) \leq J_\gamma(0) = 0, \quad (67)$$

which in view of the expression of J_γ given by (21) implies that

$$\left\| \int_0^1 y^\gamma d\alpha - z_d \right\|_{L^2(Q)}^2 + N \|u^\gamma\|_{L^2(\omega_T)}^2 + \frac{1}{\gamma} \left\| \frac{\partial \tilde{\zeta}^\gamma}{\partial \nu} \right\|_{L^2(\Sigma)}^2 \leq J(0, 0). \quad (68)$$

Hence, we deduce that

$$\|u^\gamma\|_{L^2(\omega_T)} \leq \frac{\sqrt{J(0, 0)}}{\sqrt{N}}, \quad (69)$$

$$\left\| \frac{\partial \tilde{\zeta}^\gamma}{\partial \nu} \right\|_{L^2(\Sigma)} \leq \sqrt{J(0, 0)} \sqrt{\gamma}, \quad (70)$$

$$\left\| \int_0^1 y^\gamma(\alpha, \cdot, \cdot) d\alpha \right\|_{L^2(Q)} \leq \sqrt{J(0, 0)} + \|z_d\|_{L^2(Q)}. \quad (71)$$

In view of (69) and (54), there exists $C > 0$ independent of γ such that

$$\|y^\gamma\|_{L^2((0, T); H_0^1(\Omega))} \leq C. \quad (72)$$

Thus, there exist $\hat{u} \in L^2(\omega_T)$, $\hat{y} \in L^2((0, T); H_0^1(\Omega))$ such that

$$u^\gamma \rightharpoonup \hat{u} \text{ weakly in } L^2(\omega_T), \quad (73)$$

$$y^\gamma \rightharpoonup \hat{y} \text{ weakly in } L^2((0, T); H_0^1(\Omega)). \quad (74)$$

Using (73), (74) and (54), we show as in pages 9-11 that $\hat{y} = y(\alpha, \hat{u}, 0)$ satisfies

$$\begin{aligned} \frac{\partial \hat{y}}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \hat{y}) + a_0 \hat{y} &= f + \hat{u} \chi_\omega & \text{in } Q, \\ \hat{y} &= 0 & \text{on } \Sigma, \\ \hat{y}(0) &= y^0 & \text{in } \Omega. \end{aligned} \quad (75)$$

Moreover, using (71), (74) and the Lebesgue dominated convergence theorem, we have that

$$\int_0^1 y(\alpha, u^\gamma, 0) d\alpha \rightharpoonup \int_0^1 y(\alpha, \hat{u}, 0) d\alpha \text{ weakly in } L^2(Q). \quad (76)$$

From (71) and (55), we prove that there exists a positive constant C such that

$$\|\zeta(\alpha, u^\gamma)\|_{L^2((0,T);H^2(\Omega))} \leq C. \quad (77)$$

Therefore the continuity of the trace allows us to write that

$$\left\| \frac{\partial \zeta^\gamma}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C. \quad (78)$$

Consequently, there exists $\hat{\zeta} \in L^2((0, T); H^2(\Omega))$ such that

$$\zeta(\alpha, u^\gamma) \rightharpoonup \hat{\zeta} \text{ weakly in } L^2((0, T); H^2(\Omega)). \quad (79)$$

Therefore proceeding as for ζ_n in pages 11-14, we prove that $\hat{\zeta} = \zeta(\alpha, \hat{u})$ is solution to

$$\begin{cases} -\frac{\partial \hat{\zeta}}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \hat{\zeta}) + a_0 \hat{\zeta} &= -\int_0^1 (\hat{y} - y(\alpha, 0, 0)) d\alpha & \text{in } Q, \\ \hat{\zeta} &= 0 & \text{on } \Sigma, \\ \hat{\zeta}(T) &= 0 & \text{in } \Omega. \end{cases} \quad (80)$$

Moreover, using (78), we have the following

$$\frac{\partial \zeta^\gamma}{\partial \nu} \rightharpoonup \frac{\partial \hat{\zeta}}{\partial \nu} \text{ weakly in } L^2(\Sigma). \quad (81)$$

In view of (70),

$$\frac{\partial \tilde{\zeta}^\gamma}{\partial \nu} \rightarrow 0 \text{ strongly in } L^2(\Sigma). \quad (82)$$

Because $\frac{\partial \tilde{\zeta}^\gamma}{\partial \nu} = \int_0^1 \kappa(\alpha, x) \frac{\partial \zeta^\gamma}{\partial \nu} d\alpha$. Using (78) and the fact that $\kappa \in C([0, 1] \times \bar{\Omega})$, we have that

$$\left\| \kappa(\alpha, x) \frac{\partial \zeta^\gamma}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C \left\| \frac{\partial \zeta_n}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C. \quad (83)$$

Hence it follows from (81) that

$$\int_{\Sigma} \left(\kappa(\alpha, x) \frac{\partial \zeta^\gamma}{\partial \nu} \right) \varphi d\sigma dt \longrightarrow \int_{\Sigma} \left(\kappa(\alpha, x) \frac{\partial \hat{\zeta}}{\partial \nu} \right) \varphi d\sigma dt, \quad \forall \varphi \in L^2(\Sigma). \quad (84)$$

Using Lebesgue dominated convergence theorem and (82), we deduce that

$$0 = \int_0^1 \kappa(\alpha, x) \frac{\partial \hat{\zeta}}{\partial \nu} d\alpha, \quad (85)$$

which in view of (19) allows to say that $\hat{u} \in \mathcal{O}$. Therefore, we conclude that the low-regret control u^γ converges to the no-regret control \hat{u} . ■

Proposition 4.2 *The no-regret control \hat{u} , solution of (5) (or equivalently (18)), is characterized by following optimality system:*

$$\begin{cases} \frac{\partial \hat{y}}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \hat{y}) + a_0 \hat{y} &= f + \hat{u} \chi_\omega & \text{in } Q, \\ \hat{y} &= 0 & \text{on } \Sigma, \\ \hat{y}(0) &= y^0 & \text{in } \Omega, \end{cases} \quad (86)$$

$$\begin{cases} -\frac{\partial \hat{\zeta}}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \hat{\zeta}) + a_0 \hat{\zeta} &= -\int_0^1 (\hat{y} - y(\alpha, 0, 0)) d\alpha & \text{in } Q, \\ \hat{\zeta} &= 0 & \text{on } \Sigma, \\ \hat{\zeta}(T) &= 0 & \text{in } \Omega, \end{cases} \quad (87)$$

$$\begin{cases} \frac{\partial \hat{p}}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \hat{p}) + a_0 \hat{p} &= 0 & \text{in } Q, \\ \hat{p} &= \lambda_1 & \text{on } \Sigma, \\ \hat{p}(0) &= 0 & \text{in } \Omega, \end{cases} \quad (88)$$

$$\begin{cases} -\frac{\partial \hat{q}}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \hat{q}) + a_0 \hat{q} &= \int_0^1 (\hat{y} + \lambda_2) d\alpha - z_d & \text{in } Q, \\ \hat{q} &= 0 & \text{on } \Sigma, \\ \hat{q}(T) &= 0 & \text{in } \Omega \end{cases} \quad (89)$$

and

$$\hat{u} = -\frac{1}{N} \int_0^1 \hat{q} d\alpha \quad \text{in } \omega_T, \quad (90)$$

where $\lambda_1 = \lim_{\gamma \rightarrow 0} \frac{1}{\sqrt{\gamma}} \frac{\partial \tilde{\zeta}^\gamma}{\partial \nu}$ and $\lambda_2 = \lim_{\gamma \rightarrow 0} \frac{1}{\sqrt{\gamma}} \int_0^1 p^\gamma(\alpha, u^\gamma) d\alpha$.

Proof. we have already relations (86) and (87) (see (75) and (80)). The proof of (88)- (90) is obtained by passing to the limit in (56) - (58). To this end we will proceed in 3 steps.

Step 1. We show that \hat{p} satisfies (88).

In view of (70), we have that there exists a positive constant C such that:

$$\left\| \frac{1}{\sqrt{\gamma}} \frac{\partial \tilde{\zeta}^\gamma}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C, \quad (91)$$

which in view of (56) allows us to say that

$$\|p^\gamma\|_{L^2(Q)} \leq C. \quad (92)$$

Hence, there exists $\lambda_1 \in L^2(\Sigma)$ and $\hat{p} \in L^2(Q)$ such that:

$$\frac{1}{\sqrt{\gamma}} \frac{\partial \tilde{\zeta}^\gamma}{\partial \nu} \rightharpoonup \lambda_1 \quad \text{weakly in } L^2(\Sigma), \quad (93)$$

$$p^\gamma \rightharpoonup \hat{p} \quad \text{weakly in } L^2(Q). \quad (94)$$

If we multiply the first equation in (56) by $\varphi \in \mathcal{D}(Q)$, we get

$$\begin{aligned} 0 &= \int_Q \left(\frac{\partial p^\gamma}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla p^\gamma) + a_0 p^\gamma \right) \varphi dx dt \\ &= \int_Q p^\gamma \left(-\frac{\partial \varphi}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \varphi) + a_0 \varphi \right) dx dt. \end{aligned}$$

Passing this latter identity to the limit when $\gamma \rightarrow 0$ while using (94), we deduce that

$$0 = \int_Q \hat{p} \left(-\frac{\partial \varphi}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \varphi) + a_0 \varphi \right) dx dt$$

which after integration by parts gives

$$0 = \int_Q \left(\frac{\partial \hat{p}}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \hat{p}) + a_0 \hat{p} \right) \varphi dx dt,$$

from which we get,

$$\frac{\partial \hat{p}}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \hat{p}) + a_0 \hat{p} = 0 \quad \text{in } Q. \quad (95)$$

As $\hat{p} \in L^2(Q)$ and $\frac{\partial \hat{p}}{\partial t} \in L^2((0, T), H^{-2}(\Omega))$ we have that $\hat{p} \in \mathcal{C}([0, T], H^{-1}(\Omega))$. On the other hand, $\hat{p} \in L^2(Q)$ and $\operatorname{div}(\kappa(\alpha, x) \nabla \hat{p}) \in H^{-1}((0, T), L^2(\Omega))$. This implies that $\hat{p}|_\Sigma$ and $\frac{\partial \hat{p}}{\partial \nu}|_\Sigma$ exist and belong respectively to $H^{-1}((0, T); H^{-\frac{1}{2}}(\Gamma))$ and $H^{-1}((0, T); H^{-\frac{3}{2}}(\Gamma))$.

Now, let $\varphi \in \mathcal{C}^\infty(\bar{Q})$ with $\varphi|_\Sigma = 0$ and $\varphi(T) = 0$ in Ω . If we multiply the first equation in (56) by φ and we integrate over Q , we obtain

$$\int_\Sigma \frac{1}{\sqrt{\gamma}} \frac{\partial \tilde{\zeta}^\gamma}{\partial \nu} \kappa(\alpha, x) \frac{\partial \varphi}{\partial \nu} d\sigma dt + \int_Q p^\gamma \left(-\frac{\partial \varphi}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \varphi) + a_0 \varphi \right) dx dt = 0$$

Passing this latter identity to the limit when $\gamma \rightarrow 0$ while using (93) and (94), we obtain

$$\int_{\Sigma} \lambda_1 \kappa(\alpha, x) \frac{\partial \varphi}{\partial \nu} + \int_Q \hat{p} \left(-\frac{\partial \varphi}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \varphi + a_0 \varphi) \right) dx dt = 0,$$

$\forall \varphi \in \mathcal{C}^\infty(\bar{Q})$ such that $\varphi|_{\Sigma} = 0$ and $\varphi(T) = 0$,

which after integration by parts yields

$$\begin{aligned} & \int_{\Sigma} \kappa(\alpha, x) \lambda_1 \frac{\partial \varphi}{\partial \nu} + \langle \hat{p}(0), \varphi(0) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)^-} \\ & \langle \hat{p}, \kappa(\alpha, x) \frac{\partial \varphi}{\partial \nu} \rangle_{H^{-3/2}((0,T); H^{-\frac{1}{2}}(\Gamma)), H^{3/2}((0,T); H^{\frac{1}{2}}(\Gamma))^+} \\ & \int_Q \left(\frac{\partial \hat{p}}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \hat{p} + a_0 \hat{p}) \right) \varphi dx dt = 0 \\ & \forall \varphi \in \mathcal{C}^\infty(\bar{Q}) \text{ such that } \varphi|_{\Sigma} = 0 \text{ and } \varphi(T) = 0. \end{aligned}$$

In view of (95), we deduce that

$$\begin{aligned} & \langle \hat{p}(0), \varphi(0) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)^-} \\ & \langle (\lambda_1 - \hat{p}), \kappa(\alpha, x) \frac{\partial \varphi}{\partial \nu} \rangle_{H^{-3/2}((0,T); H^{-\frac{1}{2}}(\Gamma)), H^{3/2}((0,T); H^{\frac{1}{2}}(\Gamma))^+} = 0, \end{aligned} \quad (96)$$

$\forall \varphi \in \mathcal{C}^\infty(\bar{Q})$ such that $\varphi|_{\Sigma} = 0$ and $\varphi(T) = 0$.

If we take in addition $\frac{\partial \varphi}{\partial \nu} = 0$ on Σ , we have

$$\int_{\Omega} \hat{p}(0) \varphi(0) dx = 0,$$

which implies that

$$\hat{p}(0) = 0 \text{ in } \Omega, \quad (97)$$

and it follows from (96) that

$$\int_{\Sigma} (\lambda_1 - \hat{p}) \kappa(\alpha, x) \frac{\partial \varphi}{\partial \nu} = 0.$$

This implies

$$\hat{p} = \lambda_1 \quad \text{on } \Sigma. \quad (98)$$

Combining (95), (97) and (98), we have that \hat{p} satisfies (88).

Step 2. We show that q^γ converge towards \hat{q} which satisfies (89).

Let q_1^γ and q_2^γ be respectively solution of

$$\begin{cases} -\frac{\partial q_1^\gamma}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla q_1^\gamma) + a_0 q_1^\gamma = \int_0^1 y^\gamma d\alpha - z_d & \text{in } Q, \\ q_1^\gamma = 0 & \text{on } \Sigma, \\ q_1^\gamma(T) = 0 & \text{in } \Omega, \end{cases} \quad (99)$$

and

$$\begin{cases} -\frac{\partial q_2^\gamma}{\partial t} - \operatorname{div}(\kappa(\alpha, x)\nabla q_2^\gamma) + a_0 q_2^\gamma &= \int_0^1 \frac{p^\gamma}{\sqrt{\gamma}} d\alpha & \text{in } Q, \\ q_2^\gamma &= 0 & \text{on } \Sigma, \\ q_2^\gamma(T) &= 0 & \text{in } \Omega. \end{cases} \quad (100)$$

Then $q_1^\gamma + q_2^\gamma = q^\gamma$, solution of (57). According to (71), there exists a constant $C > 0$ independent of γ such

$$\|q_1^\gamma\|_{L^2((0,T);H^2(\Omega))} \leq C. \quad (101)$$

Therefore there exists $\hat{q}_1 \in L^2((0, T); H^2(\Omega))$ such that

$$q_1^\gamma \rightharpoonup \hat{q}_1 \text{ weakly in } L^2((0, T); H^2(\Omega)). \quad (102)$$

Proceeding as for ζ_n in pages 11-14, we prove using (102) and (76) that \hat{q}_1 is solution to

$$\begin{cases} -\frac{\partial \hat{q}_1}{\partial t} - \operatorname{div}(\kappa(\alpha, x)\nabla \hat{q}_1) + a_0 \hat{q}_1 &= \int_0^1 \hat{y} d\alpha - z_d & \text{in } Q, \\ \hat{q}_1 &= 0 & \text{on } \Sigma, \\ \hat{q}_1(T) &= 0 & \text{in } \Omega. \end{cases} \quad (103)$$

To complete the proof of the convergence of q_θ^γ , we need to prove that prove that $\left\| \int_0^1 \frac{p^\gamma}{\sqrt{\gamma}} d\alpha \right\|_{L^2(Q)} \leq C$ for some $C > 0$ independent of γ .

We consider the following set

$$\mathcal{E} = \left\{ \Phi(v) = \int_0^1 \bar{y}(\alpha, v, 0) d\alpha \mid v \in L^2(\omega_T) \right\}. \quad (104)$$

In view of Remark 1, $\mathcal{E} \subset L^2(Q)$. We define on $\mathcal{E} \times \mathcal{E}$ the inner product:

$$\begin{aligned} \langle \Phi(v), \phi(w) \rangle_{\mathcal{E}} &= \int_Q v w \, dx dt + \int_Q \left(\int_0^1 \bar{y}(\alpha, v, 0) d\alpha \right) \left(\int_0^1 \bar{y}(\alpha, w, 0) d\alpha \right) dx dt, \\ \forall \Phi(v), \phi(w) \in \mathcal{E}. \end{aligned} \quad (105)$$

Then \mathcal{E} endowed with the norm

$$\|\Phi(v)\|_{\mathcal{E}}^2 = \int_Q |v|^2 \, dx dt + \int_Q \left(\int_0^1 \bar{y}(\alpha, v, 0) d\alpha \right)^2 \, dx dt, \forall \Phi(v) \in \mathcal{E} \quad (106)$$

is an Hilbert space. Now if we combine (60) and (64), we obtain that

$$\begin{aligned} \int_Q \left[\int_0^1 \bar{y}(\alpha, v) d\alpha \right] \left[\frac{1}{\sqrt{\gamma}} \int_0^1 p^\gamma d\alpha \right] dx dt &= N \int_{\omega_T} u^\gamma v \, dx dt + \\ \int_Q \left(\int_0^1 y(\alpha, u^\gamma, 0) d\alpha - z_d \right) \left(\int_0^1 \bar{y}(\alpha, v) d\alpha \right) dx dt &\quad \forall v \in L^2(\omega_T). \end{aligned} \quad (107)$$

We set $T_\gamma(u^\gamma) = \frac{1}{\sqrt{\gamma}} \int_0^1 p^\gamma d\alpha$. Then in view of (107),

$$\begin{aligned} \int_Q T_\gamma(u^\gamma) \Phi(v) dx dt &= N \int_Q u^\gamma v dx dt \\ &+ \int_Q \left(\int_0^1 y(\alpha, u^\gamma, 0) d\alpha - z_d \right) \left(\int_0^1 \bar{y}(\alpha, v) d\alpha \right) dx dt \\ &\quad \forall v \in L^2(Q). \end{aligned} \tag{108}$$

Observing that there exists a constant C independent of γ such that

$$\begin{aligned} \left| \int_Q \left(\int_0^1 y(\alpha, u^\gamma, 0) d\alpha - z_d \right) \left(\int_0^1 \bar{y}(\alpha, v) d\alpha \right) dx dt + N \int_Q u^\gamma v dx dt \right| \leq \\ \left(\left\| \int_0^1 y(\alpha, u^\gamma, 0) d\alpha - z_d \right\|_{L^2(Q)}^2 + N^2 \|u^\gamma\|_{L^2(Q)}^2 \right)^{1/2} \|\Phi(v)\|_\mathcal{E} \leq C \|\Phi(v)\|_\mathcal{E} \end{aligned}$$

because of (69) and (71). We then deduce from (108) that

$$\left| \int_Q T_\gamma(u^\gamma) \Phi(v) dx dt \right| \leq C \|\Phi(v)\|_\mathcal{E}.$$

This means that

$$\|T_\gamma(u^\gamma)\|_\mathcal{E} = \left\| \frac{1}{\sqrt{\gamma}} \int_0^1 p^\gamma d\alpha \right\|_\mathcal{E} \leq C.$$

In particular

$$\left\| \frac{1}{\sqrt{\gamma}} \int_0^1 p^\gamma d\alpha \right\|_{L^2(Q)} \leq C. \tag{109}$$

This implies that

$$\|q_2^\gamma\|_{L^2((0,T); H^2(\Omega))} \leq C. \tag{110}$$

Therefore there exist $\hat{q}_2 \in L^2((0,T); H^2(\Omega))$ $\lambda_2 \in L^2(Q)$ such that

$$q_2^\gamma \rightharpoonup \hat{q}_2 \text{ weakly in } L^2((0,T); H^2(\Omega)), \tag{111}$$

$$\frac{1}{\sqrt{\gamma}} \int_0^1 p^\gamma d\alpha \rightharpoonup \lambda_2 \text{ weakly in } L^2(Q). \tag{112}$$

Proceeding as for ζ_n in pages 11-14, we prove using (111) and (112) that \hat{q}_2 is solution to

$$\begin{cases} -\frac{\partial \hat{q}_2}{\partial t} - \operatorname{div}(\kappa(\alpha, x) \nabla \hat{q}_2) + a_0 \hat{q}_2 &= \lambda_2 & \text{in } Q, \\ \hat{q}_2 &= 0 & \text{on } \Sigma, \\ \hat{q}_2(T) &= 0 & \text{in } \Omega. \end{cases} \tag{113}$$

From (101) and (110), we have that $q^\gamma = q_1^\gamma + q_2^\gamma$ is such that

$$\|q^\gamma\|_{L^2(Q)} \leq C \tag{114}$$

for some $C > 0$ independent of γ . Hence,

$$q^\gamma \rightharpoonup \hat{q}_1 + \hat{q}_2 = \hat{q} \text{ weakly in } L^2(Q) \quad (115)$$

and \hat{q} satisfies (89).

Step 3. Using Remark 2 and (114), we have that

$$\left\| \int_0^1 q^\gamma d\alpha \right\|_{L^2(Q)} \leq C. \quad (116)$$

and using Lebesgue dominated convergence theorem we prove that

$$\int_0^1 q^\gamma d\alpha \rightharpoonup \int_0^1 q d\alpha \text{ weakly in } L^2(Q). \quad (117)$$

Finally passing to the limits in (66) while using (73) and (117), we deduce (90).

■

5 Concluding remarks

In this paper, we proved that averaging the cost function related to our model, the system is still controllable and gives an optimal control which does not depend of the unknown parameter. Using the Lebesgue dominated convergence theorem, we remark also that as $\gamma \rightarrow 0$, the limits of the average state is not far from the average of the limit of the corresponding original state.

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