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# $Diff_+(S^1)$ –PSEUDO-DIFFERENTIAL OPERATORS AND THE KADOMTSEV-PETVIASHVILI HIERARCHY

JEAN-PIERRE MAGNOT<sup>1</sup> AND ENRIQUE G. REYES<sup>2</sup>

ABSTRACT. We establish a non-formal link between the structure of the group of Fourier integral operators  $C_{odd}^{0,*}(S^1, V) \rtimes Diff_+(S^1)$  and the solutions of the Kadomtsev-Petviashvili hierarchy, using infinite-dimensional groups of series of non-formal pseudo-differential operators.

*Keywords:* Kadomtsev-Petviashvili hierarchy, Birkhoff-Mulase factorization, infinite jets, Fourier-integral operators, odd class pseudo-differential operators.

*MSC(2010):* 35Q51; 37K10; 37K25; 37K30; 58J40 Secondary: 58B25; 47N20

## 1. INTRODUCTION

The Kadomtsev-Petviashvili hierarchy (KP hierarchy for short) is an integrable system on an infinite number of dependent variables which is related to several soliton equations [7, 30], and which also appears in quantum field theory and algebraic geometry among other fields, see for instance [18, 29, 32] and references therein. In the 1980's M. Mulase published fundamental papers on the algebraic structure and formal integrability properties of the KP hierarchy, see [31, 32, 33]. A common theme in these papers was the use of a powerful theorem on the factorization of a group of formal pseudo-differential operators of infinite order which integrates the algebra of formal pseudo-differential operators: this factorization—a delicate algebraic generalization of the Birkhoff decomposition of loop groups appearing for example in [38]—allowed him to *solve* the Cauchy problem for the KP hierarchy in an algebraic context. These results have been re-interpreted and extended in the context of (generalized) differential geometry on diffeological and Frölicher spaces and used to prove well-posedness of the KP hierarchy, see [11, 12, 25, 28]. Hereafter we make no distinction between the KP hierarchy of PDEs and the non-linear equation on (formal) pseudo-differential operators which induces it.

In these papers the operators under consideration are formal pseudo-differential operators. They are called “formal” because they cannot be understood as operators acting on smooth maps or smooth sections of vector bundles. They differ from non-formal pseudo-differential operators by an (unknown) smooth kernel operator, a so-called smoothing operator. As is well-known, any classical non-formal pseudo-differential operator  $A$  generates a formal operator (the one obtained from the asymptotic expansion of the symbol of  $A$ , see [14]), but there is no canonical way to recover a non-formal operator from a formal one.

The aim of this paper is to show how the non-linear equation on formal pseudo-differential operators which gives rise to the KP hierarchy of PDEs, see [7], can be posed on groups of (non-formal) Fourier integral operators called *groups of  $Diff(S^1)$ –pseudo-differential operators*, in which  $Diff(S^1)$  is the group of diffeomorphisms of  $S^1$ . These groups arise as central extensions of  $Diff(S^1)$  by a

group of bounded (non-formal) classical pseudo-differential operators in a specific class called the odd-class. This class was first described by Kontsevich and Vishik in [19, 20] (in order to deal with spectral functions and renormalized determinants) while, to the best of our knowledge, groups of  $Diff(S^1)$ -pseudo-differential operators were first described (with  $S^1$  replaced by a compact Riemannian manifold  $M$ ) in [26], in the context of differential geometry of non-parametrized, non-linear grassmannians.

In this paper we specialize some of the results in [26] to the group of Fourier integral operators  $FCl_{Diff_+, odd}^{0,*}(S^1, V)$ , which is described as the central extension of the group of orientation-preserving diffeomorphisms  $Diff_+(S^1)$  by the group of all odd-class, bounded and invertible classical pseudo-differential operators  $Cl_{odd}^{0,*}(S^1, V)$  which act on smooth sections of a trivial (finite rank) vector bundle  $S^1 \times V$ , see (1) below. Working with this group we can prove an analogue of the Birkhoff-Mulase decomposition. In turn, this decomposition motivates us to introduce a parameter-dependent KP hierarchy and it allows us to solve its corresponding Cauchy problem.

Our decomposition and parameter-dependent hierarchy also involves two other groups:

- The group of *integral operators*  $Cl_{odd}^{-1,*}(S^1, V)$ , which is the kernel of the principal symbol map (as a morphism of groups) defined in  $Cl_{odd}^{0,*}(S^1, V)$ ;
- The group  $Cl_{h, odd}(S^1, V)$ , defined along the lines of [25]. Its elements are formal series  $\sum A_n h^n$  in a parameter  $h$ , of classical odd class pseudo-differential operators whose constant term  $A_0$  is invertible and bounded, and such that the order of a monomial (as a classical pseudo-differential operator) is controlled by the order in  $h$ . The groups  $Cl_{odd}^{-1,*}(S^1, V)$  and  $Cl_{h, odd}(S^1, V)$  are *regular* (in a sense to be explained in Section 2) if endowed with some classical topologies, as we comment in Section 3.

The group  $FCl_{Diff_+, odd}^{0,*}(S^1, V)$  already mentioned is defined through the short exact sequence

$$(1) \quad 0 \rightarrow Cl_{odd}^{0,*}(S^1, V) \rightarrow FCl_{Diff_+, odd}^{0,*}(S^1, V) \rightarrow Diff_+(S^1) \rightarrow 0,$$

in the spirit of [26]. One of our main observations is that this sequence splits under a *Birkhoff-Mulase sequence*

$$0 \rightarrow Cl_{odd}^{-1,*}(S^1, V) \rightarrow FCl_{Diff_+, odd}^{0,*}(S^1, V) \rightarrow DO^{0,*}(S^1, V) \rtimes Diff_+(S^1) \rightarrow 0$$

in which  $DO^{0,*}(S^1, V)$  is the loop group of  $GL(V)$ , see (4) and section 4 below. In agreement with the remarks made at the beginning of this section, we choose our terminology so as to take into account Mulase's generalization [31, 32, 33] of the classical Birkhoff decomposition [38]. Motivated by Mulase's work and this splitting, we introduce our non-formal KP hierarchy and study its Cauchy problem in Section 5.

Our KP hierarchy is an  $h$ -deformed KP-hierarchy constructed with the help of the group  $Cl_{h, odd}(S^1, V)$  and a scaling introduced in [25] in order to solve the Cauchy problem for a (classical) KP hierarchy. We note that our scaling differs from the one used in [34, 41]: in these papers the authors apply scaling so as to obtain a deformation of the *dispersionless* KP hierarchy.

In Section 5 we also highlight a non-formal operator  $U_h \in Cl_{h, odd}(S^1, V)$  which depends on the initial condition of our  $h$ -deformed KP hierarchy; this operator

generates its solutions by using the Birkhoff-Mulase decomposition proved in Section 4.

Finally, in section 6 we show how to recover the operator  $U_h$  by analysing the Taylor expansion of functions in the image of the twisted operator  $A : f \in C^\infty(S^1; V) \mapsto S_0^{-1}(f) \circ g$ , in which  $g \in Diff_+(S^1)$  and  $S_0$  is our version of the dressing operator (see [7]) for the initial value of the  $h$ -deformed KP hierarchy.

## 2. PRELIMINARIES ON CATEGORIES OF REGULAR FRÖLICHER LIE GROUPS

In this section we recall briefly the formal setting which allows us to work rigorously with spaces of pseudo-differential operators and exponential mappings. We begin with the notion of a diffeological space:

**Definition 1.** *Let  $X$  be a set.*

- A  **$p$ -parametrization** of dimension  $p$  on  $X$  is a map from an open subset  $O$  of  $\mathbb{R}^p$  to  $X$ .
- A **diffeology** on  $X$  is a set  $\mathcal{P}$  of parametrizations on  $X$  such that:
  - For each  $p \in \mathbb{N}$ , any constant map  $\mathbb{R}^p \rightarrow X$  is in  $\mathcal{P}$ ;
  - For each arbitrary set of indexes  $I$  and family  $\{f_i : O_i \rightarrow X\}_{i \in I}$  of compatible maps that extend to a map  $f : \bigcup_{i \in I} O_i \rightarrow X$ , if  $\{f_i : O_i \rightarrow X\}_{i \in I} \subset \mathcal{P}$ , then  $f \in \mathcal{P}$ .
  - For each  $f \in \mathcal{P}$ ,  $f : O \subset \mathbb{R}^p \rightarrow X$ , and  $g : O' \subset \mathbb{R}^q \rightarrow O$ , in which  $g$  is a smooth map (in the usual sense) from an open set  $O' \subset \mathbb{R}^q$  to  $O$ , we have  $f \circ g \in \mathcal{P}$ .

If  $\mathcal{P}$  is a diffeology on  $X$ , then  $(X, \mathcal{P})$  is called a **diffeological space** and, if  $(X, \mathcal{P})$  and  $(X', \mathcal{P}')$  are two diffeological spaces, a map  $f : X \rightarrow X'$  is **smooth** if and only if  $f \circ \mathcal{P} \subset \mathcal{P}'$ .

The notion of a diffeological space is due to J.M. Souriau, see [40]; see also [17] and [5] for related constructions. Of particular interest to us is the following subcategory of the category of diffeological spaces.

**Definition 2.** *A **Frölicher space** is a triple  $(X, \mathcal{F}, \mathcal{C})$  such that*

- $\mathcal{C}$  is a set of paths  $\mathbb{R} \rightarrow X$ ,
- $\mathcal{F}$  is the set of functions from  $X$  to  $\mathbb{R}$ , such that a function  $f : X \rightarrow \mathbb{R}$  is in  $\mathcal{F}$  if and only if for any  $c \in \mathcal{C}$ ,  $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ ;
- A path  $c : \mathbb{R} \rightarrow X$  is in  $\mathcal{C}$  (i.e. is a **contour**) if and only if for any  $f \in \mathcal{F}$ ,  $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ .

If  $(X, \mathcal{F}, \mathcal{C})$  and  $(X', \mathcal{F}', \mathcal{C}')$  are two Frölicher spaces, a map  $f : X \rightarrow X'$  is **smooth** if and only if  $\mathcal{F}' \circ f \circ \mathcal{C} \subset C^\infty(\mathbb{R}, \mathbb{R})$ .

This definition first appeared in [13]; we use terminology appearing in Kriegl and Michor's book [21]. A short comparison of the notions of diffeological and Frölicher spaces is in [24]; the reader can also see [25, 27, 28, 42] for extended expositions.

Any family of maps  $\mathcal{F}_g$  from  $X$  to  $\mathbb{R}$  generates a Frölicher structure  $(X, \mathcal{F}, \mathcal{C})$  by setting, after [21]:

- $\mathcal{C} = \{c : \mathbb{R} \rightarrow X \text{ such that } \mathcal{F}_g \circ c \subset C^\infty(\mathbb{R}, \mathbb{R})\}$
- $\mathcal{F} = \{f : X \rightarrow \mathbb{R} \text{ such that } f \circ \mathcal{C} \subset C^\infty(\mathbb{R}, \mathbb{R})\}$ .

We call  $\mathcal{F}_g$  a **generating set of functions** for the Frölicher structure  $(X, \mathcal{F}, \mathcal{C})$ . One easily see that  $\mathcal{F}_g \subset \mathcal{F}$ . This notion will be useful for us, see for instance Proposition 3 below. A Frölicher space  $(X, \mathcal{F}, \mathcal{C})$  carries a natural topology, the pull-back topology of  $\mathbb{R}$  via  $\mathcal{F}$ . In the case of a finite dimensional differentiable manifold  $X$  we can take  $\mathcal{F}$  as the set of all smooth maps from  $X$  to  $\mathbb{R}$ , and  $\mathcal{C}$  the

set of all smooth paths from  $\mathbb{R}$  to  $X$ . Then, the underlying topology of the Frölicher structure is the same as the manifold topology [21].

We also remark that if  $(X, \mathcal{F}, \mathcal{C})$  is a Frölicher space, we can define a natural diffeology on  $X$  by using the following family of maps  $f$  defined on open domains  $D(f)$  of Euclidean spaces, see [24]:

$$\mathcal{P}_\infty(\mathcal{F}) = \prod_{p \in \mathbb{N}} \{f : D(f) \rightarrow X; \mathcal{F} \circ f \in C^\infty(D(f), \mathbb{R}) \text{ (in the usual sense)}\}.$$

If  $X$  is a finite-dimensional differentiable manifold, this diffeology has been called the *nébuleuse diffeology*. by J.-M. Souriau, see [40].

**Proposition 3.** [24] *Let  $(X, \mathcal{F}, \mathcal{C})$  and  $(X', \mathcal{F}', \mathcal{C}')$  be two Frölicher spaces. A map  $f : X \rightarrow X'$  is smooth in the sense of Frölicher if and only if it is smooth for the underlying diffeologies  $\mathcal{P}_\infty(\mathcal{F})$  and  $\mathcal{P}_\infty(\mathcal{F}')$ .*

**Proposition 4.** *Let  $(X, \mathcal{P})$  and  $(X', \mathcal{P}')$  be two diffeological spaces. There exists a diffeology  $\mathcal{P} \times \mathcal{P}'$  on  $X \times X'$  made of plots  $g : O \rightarrow X \times X'$  that decompose as  $g = f \times f'$ , where  $f : O \rightarrow X \in \mathcal{P}$  and  $f' : O \rightarrow X' \in \mathcal{P}'$ . We call it the **product diffeology**, and this construction extends to an infinite product.*

We apply this result to the case of Frölicher spaces and we derive very easily, (compare with e.g. [21]) the following:

**Proposition 5.** *Let  $(X, \mathcal{F}, \mathcal{C})$  and  $(X', \mathcal{F}', \mathcal{C}')$  be two Frölicher spaces equipped with their natural diffeologies  $\mathcal{P}$  and  $\mathcal{P}'$ . There is a natural structure of Frölicher space on  $X \times X'$  which contours  $\mathcal{C} \times \mathcal{C}'$  are the 1-plots of  $\mathcal{P} \times \mathcal{P}'$ .*

We can also state the above result for infinite products; we simply take cartesian products of the plots or of the contours.

Now we remark that given an algebraic structure, we can define a corresponding compatible diffeological structure, see for instance [22]. For example, a  $\mathbb{R}$ -vector space equipped with a diffeology is called a diffeological vector space if addition and scalar multiplication are smooth (with respect to the canonical diffeology on  $\mathbb{R}$ ). An analogous definition holds for Frölicher vector spaces. The example of Frölicher Lie groups will arise in Section 3 below.

**Remark 6.** *Frölicher and Gateaux smoothness are the same notion if we restrict to a Fréchet context, see [21, Theorem 4.11]. Indeed, for a smooth map  $f : (F, \mathcal{P}_1(F)) \rightarrow \mathbb{R}$  defined on a Fréchet space with its 1-dimensional diffeology, we have that  $\forall (x, h) \in F^2$ , the map  $t \mapsto f(x + th)$  is smooth as a classical map in  $C^\infty(\mathbb{R}, \mathbb{R})$ . And hence, it is Gateaux smooth. The converse is obvious.*

We follow [40, 17]: Let  $(X, \mathcal{P})$  be a diffeological space, and let  $X'$  be a set. Let  $f : X \rightarrow X'$  be a map. We define the **push-forward diffeology** as the coarsest (i.e. the smallest for inclusion) among the diffeologies on  $X'$ , which contains  $f \circ \mathcal{P}$ .

**Proposition 7.** *let  $(X, \mathcal{P})$  be a diffeological space and  $\mathcal{R}$  an equivalence relation on  $X$ . Then, there is a natural diffeology on  $X/\mathcal{R}$ , noted by  $\mathcal{P}/\mathcal{R}$ , defined as the push-forward diffeology on  $X/\mathcal{R}$  by the quotient projection  $X \rightarrow X/\mathcal{R}$ .*

Given a subset  $X_0 \subset X$ , where  $X$  is a Frölicher space or a diffeological space, we equip  $X_0$  with structures induced by  $X$  as follows:

- (1) If  $X$  is equipped with a diffeology  $\mathcal{P}$ , we define a diffeology  $\mathcal{P}_0$  on  $X_0$  called the **subset or trace diffeology**, see [40, 17], by setting

$$\mathcal{P}_0 = \{p \in \mathcal{P} \text{ such that the image of } p \text{ is a subset of } X_0\}.$$

- (2) If  $(X, \mathcal{F}, \mathcal{C})$  is a Frölicher space, we take as a generating set of maps  $\mathcal{F}_g$  on  $X_0$  the restrictions of the maps  $f \in \mathcal{F}$ . In this case, the contours (resp. the induced diffeology) on  $X_0$  are the contours (resp. the plots) on  $X$  whose images are a subset of  $X_0$ .

Let  $(X, \mathcal{P})$  and  $(X', \mathcal{P}')$  be diffeological spaces. Let  $M \subset C^\infty(X, X')$  be a set of smooth maps. The **functional diffeology** on  $S$  is the diffeology  $\mathcal{P}_S$  made of plots

$$\rho : D(\rho) \subset \mathbb{R}^k \rightarrow S$$

such that, for each  $p \in \mathcal{P}$ , the maps  $\Phi_{\rho,p} : (x, y) \in D(p) \times D(\rho) \mapsto \rho(y)(x) \in X'$  are plots of  $\mathcal{P}'$ . This definition allows us to prove the following classical properties, see [17]:

**Proposition 8.** *Let  $X, Y, Z$  be diffeological spaces. Then,*

$$C^\infty(X \times Y, Z) = C^\infty(X, C^\infty(Y, Z)) = C^\infty(Y, C^\infty(X, Z))$$

*as diffeological spaces equipped with functional diffeologies.*

Since we are interested in infinite-dimensional analogues of Lie groups, we need to consider tangent spaces of diffeological spaces, as we have to deal with Lie algebras and exponential maps. We state the following general definition after [9] and [6]:

- (i) the **internal tangent cone**. For each  $x \in X$ , we consider

$$C_x = \{c \in C^\infty(\mathbb{R}, X) \mid c(0) = x\}$$

and take the equivalence relation  $\mathcal{R}$  given by

$$c\mathcal{R}c' \Leftrightarrow \forall f \in C^\infty(X, \mathbb{R}), \partial_t(f \circ c)|_{t=0} = \partial_t(f \circ c')|_{t=0}.$$

Equivalence classes of  $\mathcal{R}$  are called **germs** and are denoted by  $\partial_t c(0)$  or  $\partial_t c(t)|_{t=0}$ . The **internal tangent cone** at  $x$  is the quotient  ${}^i T_x X = C_x / \mathcal{R}$ . If  $X = \partial_t c(t)|_{t=0} \in {}^i T_X$ , we define the derivation  $Df(X) = \partial_t(f \circ c)|_{t=0}$ .

- (ii) The **internal tangent space** at  $x \in X$  is the vector space generated by the internal tangent cone.

The reader may compare this definition to the one appearing in [21] for manifolds in the “convenient”  $C^\infty$ -setting. We remark that the internal tangent cone at a point  $x$  is not a vector space in many examples; this motivates item (ii) above, see [6, 9]. Fortunately, the internal tangent cone at  $x \in X$  is a vector space for the objects under consideration in this work, see Proposition 10 below; it will be called, simply, the tangent space at  $x \in X$ .

**Definition 9.** *Let  $G$  be a group equipped with a diffeology  $\mathcal{P}$ . We call it a **diffeological group** if both multiplication and inversion are smooth.*

Analogous definitions hold for Frölicher groups. Following Iglesias-Zemmour, see [17], we do not assert that arbitrary diffeological groups have associated Lie algebras; however, the following holds, see [22, Proposition 1.6.] and also [28].

**Proposition 10.** *Let  $G$  be a diffeological group. Then the tangent cone at the neutral element  $T_e G$  is a diffeological vector space.*

**Definition 11.** *The diffeological group  $G$  is a **diffeological Lie group** if and only if the derivative of the Adjoint action of  $G$  on the diffeological vector space  ${}^i T_e G$  defines a Lie bracket. In this case, we call  ${}^i T_e G$  the Lie algebra of  $G$  and we denote it by  $\mathfrak{g}$ .*

Let us concentrate on Frölicher Lie groups. If  $G$  is a Frölicher Lie group then, after (i) and (ii) above we have (see [25] and [22]):

$$\mathfrak{g} = \{\partial_t c(0); c \in \mathcal{C} \text{ and } c(0) = e_G\}$$

is the space of germs of paths at  $e_G$ . Moreover:

- Let  $(X, Y) \in \mathfrak{g}^2$ ,  $X + Y = \partial_t(c.d)(0)$  where  $c, d \in \mathcal{C}^2$ ,  $c(0) = d(0) = e_G$ ,  $X = \partial_t c(0)$  and  $Y = \partial_t d(0)$ .
- Let  $(X, g) \in \mathfrak{g} \times G$ ,  $Ad_g(X) = \partial_t(gcg^{-1})(0)$  where  $c \in \mathcal{C}$ ,  $c(0) = e_G$ , and  $X = \partial_t c(0)$ .
- Let  $(X, Y) \in \mathfrak{g}^2$ ,  $[X, Y] = \partial_t(Ad_{c(t)}Y)$  where  $c \in \mathcal{C}$ ,  $c(0) = e_G$ ,  $X = \partial_t c(0)$ .

All these operations are smooth and thus well-defined as operations on Frölicher spaces, see [22, 25, 27, 28].

The basic properties of adjoint, coadjoint actions, and of Lie brackets, remain globally the same as in the case of finite-dimensional Lie groups, and the proofs are similar: see [22] and [9] for details.

**Definition 12.** [22] *A Frölicher Lie group  $G$  with Lie algebra  $\mathfrak{g}$  is called **regular** if and only if there is a smooth map*

$$Exp : C^\infty([0; 1], \mathfrak{g}) \rightarrow C^\infty([0, 1], G)$$

such that  $g(t) = Exp(v(t))$  is the unique solution of the differential equation

$$(2) \quad \begin{cases} g(0) = e \\ \frac{dg(t)}{dt} g(t)^{-1} = v(t) \end{cases}$$

We define the exponential function as follows:

$$\begin{aligned} exp : \mathfrak{g} &\rightarrow G \\ v &\mapsto exp(v) = g(1), \end{aligned}$$

where  $g$  is the image by  $Exp$  of the constant path  $v$ .

When the Lie group  $G$  is a vector space  $V$ , the notion of regular Lie group specialize to what is called *regular vector space* in [25] and *integral vector space* in [22]; we follow the latter terminology.

**Definition 13.** [22] *Let  $(V, \mathcal{P})$  be a Frölicher vector space. The space  $(V, \mathcal{P})$  is **integral** if there is a smooth map*

$$\int_0^{(\cdot)} : C^\infty([0; 1]; V) \rightarrow C^\infty([0, 1], V)$$

such that  $\int_0^{(\cdot)} v = u$  if and only if  $u$  is the unique solution of the differential equation

$$\begin{cases} u(0) = 0 \\ u'(t) = v(t) \end{cases} .$$

This definition applies, for instance, if  $V$  is a complete locally convex topological vector space equipped with its natural Frölicher structure given by the Frölicher completion of its nébuleuse diffeology, see [17, 24, 25].

**Definition 14.** *Let  $G$  be a Frölicher Lie group with Lie algebra  $\mathfrak{g}$ . Then,  $G$  is **regular with integral Lie algebra** if  $\mathfrak{g}$  is integral and  $G$  is regular in the sense of Definitions 12 and 13.*

The properties of the specific groups which we will use in the following sections are consequences of the following two structural results which we quote for completeness:

**Theorem 15.** [25] *Let  $(A_n)_{n \in \mathbb{N}^*}$  be a sequence of integral (Frölicher) vector spaces equipped with a graded smooth multiplication operation on  $\bigoplus_{n \in \mathbb{N}^*} A_n$ , i.e. a multiplication such that for each  $n, m \in \mathbb{N}^*$ ,  $A_n \cdot A_m \subset A_{n+m}$  is smooth with respect to the corresponding Frölicher structures. Let us define the (non unital) algebra of formal series:*

$$\mathcal{A} = \left\{ \sum_{n \in \mathbb{N}^*} a_n | \forall n \in \mathbb{N}^*, a_n \in A_n \right\},$$

*equipped with the Frölicher structure of the infinite product. Then, the set*

$$1 + \mathcal{A} = \left\{ 1 + \sum_{n \in \mathbb{N}^*} a_n | \forall n \in \mathbb{N}^*, a_n \in A_n \right\}$$

*is a regular Frölicher Lie group with integral Frölicher Lie algebra  $\mathcal{A}$ . Moreover, the exponential map defines a smooth bijection  $\mathcal{A} \rightarrow 1 + \mathcal{A}$ .*

**Notation:** for each  $u \in \mathcal{A}$ , we write  $[u]_n$  for the  $A_n$ -component of  $u$ .

**Theorem 16.** [25] *Let*

$$1 \longrightarrow K \xrightarrow{i} G \xrightarrow{p} H \longrightarrow 1$$

*be an exact sequence of Frölicher Lie groups, such that there is a smooth section  $s : H \rightarrow G$ , and such that the trace diffeology from  $G$  on  $i(K)$  coincides with the push-forward diffeology from  $K$  to  $i(K)$ . We consider also the corresponding sequence of Lie algebras*

$$0 \longrightarrow \mathfrak{k} \xrightarrow{i'} \mathfrak{g} \xrightarrow{p} \mathfrak{h} \longrightarrow 0.$$

*Then,*

- *The Lie algebras  $\mathfrak{k}$  and  $\mathfrak{h}$  are integral if and only if the Lie algebra  $\mathfrak{g}$  is integral*
- *The Frölicher Lie groups  $K$  and  $H$  are regular if and only if the Frölicher Lie group  $G$  is regular.*

Similar results, as in theorem 16, are valid for Fréchet Lie groups, see [21].

### 3. PRELIMINARIES ON FOURIER INTEGRAL AND PSEUDO-DIFFERENTIAL OPERATORS

We introduce the groups and algebras of (non-formal!) pseudo-differential operators needed to set up a KP hierarchy and to prove a non-formal Birkhoff-Mulase decomposition. In this section  $E$  is a real or complex finite-dimensional vector bundle over  $S^1$ ; below we will specialize our considerations to the case  $E = S^1 \times V$  in which  $V$  is a finite-dimensional vector space. The following definition appears in [2, Section 2.1].



**Definition 17.** *The graded algebra of differential operators acting on the space of smooth sections  $C^\infty(S^1, E)$  is the algebra  $DO(E)$  generated by:*

- *Elements of  $End(E)$ , the group of smooth maps  $E \rightarrow E$  leaving each fibre globally invariant and which restrict to linear maps on each fibre. This group acts on sections of  $E$  via (matrix) multiplication;*
- *The differentiation operators*

$$\nabla_X : g \in C^\infty(S^1, E) \mapsto \nabla_X g$$

where  $\nabla$  is a connection on  $E$  and  $X$  is a vector field on  $S^1$ .

Multiplication operators are operators of order 0; differentiation operators and vector fields are operators of order 1. In local coordinates, a differential operator of order  $k$  has the form  $P(u)(x) = \sum p_{i_1 \dots i_r} \nabla_{x_{i_1}} \dots \nabla_{x_{i_r}} u(x)$ ,  $r \leq k$ , in which the coefficients  $p_{i_1 \dots i_r}$  can be matrix-valued. We note by  $DO^k(S^1)$ ,  $k \geq 0$ , the differential operators of order less or equal than  $k$ .

The algebra  $DO(E)$  is graded by order. It is a subalgebra of the algebra of classical pseudo-differential operators  $Cl(S^1, E)$ , an algebra that contains, for example, the square root of the Laplacian, its inverse, and all trace-class operators on  $L^2(S^1, E)$ . Basic facts on pseudo-differential operators defined on a vector bundle  $E \rightarrow S^1$  can be found in [14]. A global symbolic calculus for pseudo-differential operators has been defined independently by J. Bokobza-Haggiag, see [3] and H. Widom, see [43]. In these papers is shown how the geometry of the base manifold  $M$  furnishes an obstruction to generalizing local formulas of composition and inversion of symbols; we do not recall these formulas here since they are not involved in our computations.

Following [26], we assume henceforth that  $S^1$  is equipped with charts such that the changes of coordinates are translations.

**Notations.** We note by  $PDO(S^1, E)$  (resp.  $PDO^o(S^1, E)$ , resp.  $Cl(S^1, E)$ ) the space of pseudo-differential operators (resp. pseudo-differential operators of order  $o$ , resp. classical pseudo-differential operators) acting on smooth sections of  $E$ , and by  $Cl^o(S^1, E) = PDO^o(S^1, E) \cap Cl(S^1, E)$  the space of classical pseudo-differential operators of order  $o$ . We also denote by  $Cl^{o,*}(S^1, E)$  the group of units of  $Cl^o(S^1, E)$  and by  $\mathcal{F}Cl^{o,*}(S^1, E)$  the group of units of the algebra  $\mathcal{F}Cl^o(S^1, E)$ . If the vector bundle  $E$  is trivial, i.e.  $E = S^1 \times V$  or  $E = S^1 \times \mathbb{K}^p$  with  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , we use the notation  $Cl(S^1, V)$  or  $Cl(S^1, \mathbb{K}^p)$  instead of  $Cl(S^1, E)$ .

A topology on spaces of classical pseudo differential operators has been described by Kontsevich and Vishik in [19]; see also [4, 37, 39] for other descriptions. We use all along this work the Kontsevich-Vishik topology. This is a Fréchet topology such that each space  $Cl^o(S^1, E)$  is closed in  $Cl(S^1, E)$ . We set

$$PDO^{-\infty}(S^1, E) = \bigcap_{o \in \mathbb{Z}} PDO^o(S^1, E).$$

It is well-known that  $PDO^{-\infty}(S^1, E)$  is a two-sided ideal of  $PDO(S^1, E)$ , see e.g. [14, 39]. Therefore, we can define the quotients

$$\begin{aligned} \mathcal{F}PDO(S^1, E) &= PDO(S^1, E)/PDO^{-\infty}(S^1, E), \\ \mathcal{F}Cl(S^1, E) &= Cl(S^1, E)/PDO^{-\infty}(S^1, E), \\ \mathcal{F}Cl^o(S^1, E) &= Cl^o(S^1, E)/PDO^{-\infty}(S^1, E). \end{aligned}$$

The script font  $\mathcal{F}$  stands for *formal pseudo-differential operators*. The quotient  $FPDO(S^1, E)$  is an algebra isomorphic to the set of formal symbols, see [3], and the identification is a morphism of  $\mathbb{C}$ -algebras for the usual multiplication on formal symbols (see e.g. [14]).

**Theorem 18.** *The groups  $Cl^{0,*}(S^1, E)$ ,  $Diff_+(S^1)$  and  $\mathcal{F}Cl^{0,*}(S^1, E)$  are regular Frölicher Lie groups.*

Even more, it has been noticed in [23], by applying the results of [15], that the group  $Cl^{0,*}(S^1, V)$  (resp.  $\mathcal{F}Cl^{0,*}(S^1, V)$ ) is open in  $Cl^0(S^1, V)$  (resp.  $\mathcal{F}Cl^0(S^1, V)$ ) and also that it is a regular Fréchet Lie group. It follows from [10, 36] that  $Diff_+S^1$  is open in the Fréchet manifold  $C^\infty(S^1, S^1)$ . This fact makes it a Fréchet manifold and, following [36], a regular Fréchet Lie group.

**Definition 19.** *A classical pseudo-differential operator  $A$  on  $S^1$  is called odd class if and only if for all  $n \in \mathbb{Z}$  and all  $(x, \xi) \in T^*S^1$  we have:*

$$\sigma_n(A)(x, -\xi) = (-1)^n \sigma_n(A)(x, \xi) ,$$

in which  $\sigma_n$  is the symbol of  $A$ .

This particular class of pseudo-differential operators has been introduced in [19, 20]. Odd class operators are also called “even-even class” operators, see [39]. We choose to follow the terminology of the first two references. Hereafter, the notation  $Cl_{odd}$  will refer to odd class classical pseudo-differential operators.

**Proposition 20.** *The algebra  $Cl_{odd}^0(S^1, V)$  is a closed subalgebra of  $Cl^0(S^1, V)$ . Moreover,  $Cl_{odd}^{0,*}(S^1, V)$  is*

- an open subset of  $Cl^0(S^1, V)$  and,
- a regular Fréchet Lie group.

*Proof.* We note by  $\sigma(A)(x, \xi)$  the total formal symbol of  $A \in Cl^0(S^1, V)$ . Let  $\phi : Cl^0(S^1, V) \rightarrow \mathcal{F}Cl^0(S^1, V)$  defined by

$$\phi(A) = \sum_{n \in \mathbb{N}} \sigma_{-n}(x, \xi) - (-1)^n \sigma_{-n}(x, -\xi).$$

This map is smooth, and

$$Cl_{odd}^0(S^1, V) = Ker(\phi),$$

which shows that  $Cl_{odd}^0(S^1, V)$  is a closed subalgebra of  $Cl^0(S^1, V)$ . Moreover, if  $H = L^2(S^1, V)$ ,

$$Cl_{odd}^{0,*}(S^1, V) = Cl_{odd}^0(S^1, V) \cap GL(H),$$

which proves that  $Cl_{odd}^{0,*}(S^1, V)$  is open in the Fréchet algebra  $Cl^0(S^1, V)$ , and it follows that it is a regular Fréchet Lie group by arguing along the lines of [15, 35].  $\square$

In addition to groups of pseudo-differential operators, we also need a restricted class of groups of Fourier integral operators which we will call  $Diff_+(S^1)$ -pseudo-differential operators following [26]. These groups appear as central extensions of  $Diff_+(S^1)$  by groups of (often bounded) pseudo-differential operators. We do not state the basic facts on Fourier integral operators here (they can be found in the classical paper [16]), but we recall the following theorem, which was stated in [26] for a general base manifold  $M$ .

**Theorem 21.** [26, Theorem 4] *Let  $H$  be a regular Lie group of pseudo-differential operators acting on smooth sections of a trivial bundle  $E \sim V \times S^1 \rightarrow S^1$ . The group  $Diff(S^1)$  acts smoothly on  $C^\infty(S^1, V)$ , and it is assumed to act smoothly on  $H$  by adjoint action. If  $H$  is stable under the  $Diff(S^1)$ -adjoint action, then there exists a regular Lie group  $G$  of Fourier integral operators defined through the exact sequence:*

$$0 \rightarrow H \rightarrow G \rightarrow Diff(S^1) \rightarrow 0.$$

*If  $H$  is a Frölicher Lie group, then  $G$  is a Frölicher Lie group.*

This result follows from Theorem 16. The pseudo-differential operators considered in this theorem can be classical, odd class, or anything else. Applying the formulas of “changes of coordinates” (which can be understood as adjoint actions of diffeomorphisms) of e.g. [14], we obtain that odd class pseudo-differential operators are stable under the adjoint action of  $Diff(S^1)$ . Thus, we can define the following group:

**Definition 22.** *The group  $FCl_{Diff(S^1), odd}^{0,*}(S^1, V)$  is the regular Frölicher Lie group  $G$  obtained in Theorem 21 with  $H = Cl_{odd}^{0,*}(S^1, V)$ .*

Following [26], we remark that operators  $A$  in this group can be understood as operators in  $Cl_{odd}^{0,*}(S^1, V)$  twisted by diffeomorphisms, this is,

$$(3) \quad A = B \circ g,$$

where  $g \in Diff(S^1)$  and  $B \in Cl_{odd}^{0,*}(S^1, V)$ . We note that the diffeomorphism  $g$  is the phase of the operators, but here the phase (and hence the decomposition (3)) is unique, which is not the case for general Fourier integral operators, see e.g. [16].

**Remark 23.** *This construction of phase functions of  $Diff(M)$ -pseudo-differential operators differs from the one described by Omori [36] and Adams, Ratiu and Schmid [1] for the groups of Fourier integral operators; the exact relation among these constructions still needs to be investigated.*

We finish our constructions by using again Theorem 21: we note that the group  $Diff(S^1)$  decomposes into two connected components  $Diff(S^1) = Diff_+(S^1) \cup Diff_-(S^1)$ , where the connected component of the identity,  $Diff_+(S^1)$ , is the group of orientation preserving diffeomorphisms of  $S^1$ . We make the following definition:

**Definition 24.** *The group  $FCl_{Diff_+(S^1), odd}^{0,*}(S^1, V)$  is the regular Frölicher Lie group of all operators in  $FCl_{Diff(S^1), odd}^{0,*}(S^1, V)$  whose phase diffeomorphisms lie in  $Diff_+(S^1)$ .*

We observe that the central extension  $FCl_{Diff(S^1), odd}^{0,*}(S^1, V)$  contains as a subgroup the group

$$(4) \quad DO^{0,*}(S^1, V) \rtimes Diff_+(S^1),$$

where  $DO^{0,*}(S^1, V) = C^\infty(S^1, GL(V))$  is the loop group on  $GL(V)$ .

Let us now assume that  $V$  is a complex vector space. By the symmetry property stated in Definition 19, an odd class pseudo-differential operator  $A$  has a partial symbol of non-negative order  $n$  that reads

$$(5) \quad \sigma_n(A)(x, \xi) = \gamma_n(x)(i\xi)^n,$$

where  $\gamma_n \in C^\infty(S^1, L(V))$ . This consequence of Definition 19 allows us to check the following direct sum decomposition:

**Proposition 25.**

$$Cl_{odd}(S^1, V) = Cl_{odd}^{-1}(S^1, V) \oplus DO(S^1, V).$$

The second summand of this expression will be specialized in the sequel to differential operators having symbols of order 1. Because of (5), we can understand these symbols as elements of  $Vect(S^1) \otimes Id_V$ .

#### 4. $FCl_{Diff_+, odd}^{0,*}(S^1, V)$ AND THE BIRKHOFF-MULASE DECOMPOSITION

Mulase's factorization theorem (see [33]) alluded to in Section 1, tells us that an appropriate group of formal pseudo-differential operators of infinite order decomposes uniquely as the product of a group of differential operators of infinite order and a group of the form  $1 + \mathcal{I}$ , in which  $\mathcal{I}$  is the algebra of formal pseudo-differential operators of order at most  $-1$ . In our context, we consider the Fréchet Lie group  $Cl_{odd}^{-1,*}(S^1, V)$  of invertible operators of the form  $A = Id + A_{-1}$ , where  $A_{-1} \in Cl_{odd}^{-1}(S^1, V)$ . We have the following non-formal version of the Birkhoff-Mulase decomposition:

**Theorem 26.** *Let  $U \in FCl_{Diff_+, odd}^{0,*}(S^1, V)$ . There exists a unique pair*

$$(S, Y) \in Cl_{odd}^{-1,*}(S^1, V) \times (DO^{0,*}(S^1, V) \rtimes Diff_+(S^1))$$

such that

$$U = SY.$$

Moreover, the map  $U \mapsto (S, Y)$  is smooth and, there is a short exact sequence of Lie groups:

$$0 \rightarrow Cl_{odd}^{-1,*}(S^1, V) \rightarrow FCl_{Diff_+, odd}^{0,*}(S^1, V) \rightarrow DO^0(S^1, V) \rtimes Diff_+(S^1) \rightarrow 0$$

for which the  $Y$ -part defines a smooth global section, and which is a morphism of groups (canonical inclusion).

*Proof.* We already know that  $U$  splits in a unique way as  $U = A_0 \cdot g$ , in which  $g \in Diff_+(S^1)$  and  $A_0 \in Cl_{odd}^{0,*}(S^1, V)$ . By Proposition 25, the pseudo-differential operator  $A_0$  can be written uniquely as a sum,  $A = A_I + A_D$ , in which  $A_D \in DO^0(S^1, V) \subset Cl_{odd}(S^1, V)$ . Since  $A_0$  is invertible,  $\sigma_0(A_0) \in C^\infty(S^1, GL(V))$  and hence  $A_D \in DO^{0,*}(S^1, V)$ . In this way, we have

$$U = A_0 \cdot A_D^{-1} \cdot A_D \cdot g.$$

We get  $Y = A_D \cdot g \in DO^{0,*}(S^1, V) \rtimes Diff_+(S^1)$  and  $S = A_0 \cdot A_D^{-1} \in Cl_{odd}^{0,*}(S^1, V)$  (the inverse of an odd class operator is an odd class operator). Let us compute the principal symbol  $\sigma_0(S)$ :

$$\sigma_0(S) = \sigma_0(A_0)\sigma_0(A_D^{-1}) = \sigma_0(A_0)\sigma_0(A_0)^{-1} = Id_V.$$

Thus,  $S \in Cl_{odd}^{-1,*}(S^1, V)$ . Moreover, the maps  $U \mapsto g$  and  $A_0 \mapsto A_D$  are smooth, which ends the proof.  $\square$

We have already stated that the Lie groups  $Cl_{odd}^{-1,*}(S^1, V)$ ,  $FCl_{Diff_+, odd}^{0,*}(S^1, V)$  and  $DO^0(S^1, V) \rtimes Diff_+(S^1)$  are regular. Let us take advantage of the existence of exponential mappings. Let us consider a curve

$$L(t) \in C^\infty([0; 1], Cl_{odd}^1(S^1, V)) ;$$

we compare the exponential  $\exp(L)(t) \in C^\infty([0; 1], FCl_{Diff_+}^{0,*}(S^1, V))$  with

$$\exp(L_D)(t) \in C^\infty([0; 1], DO^{0,*}(S^1, V) \rtimes Diff_+(S^1))$$

and

$$\exp(L_S)(t) \in C^\infty([0; 1], Cl_{odd}^{-1,*}(S^1, V)) .$$

On one hand, we can write

$$\exp(L)(t) = S(t)Y(t)$$

according to Theorem 26, and we know that the paths  $t \mapsto S(t)$  and  $t \mapsto Y(t)$  are smooth. On the other hand, using the definition of the left exponential map, we get

$$\frac{d}{dt} \exp(L)(t) = \exp(L)(t) \cdot L(t) .$$

Thus, gathering the last two expressions we obtain

$$\begin{aligned} \frac{d}{dt} \exp(L)(t) &= \frac{d}{dt} (S(t)Y(t)) \\ &= \left( \frac{d}{dt} S(t) \right) S^{-1}(t) S(t) Y(t) + S(t) Y(t) Y^{-1}(t) \left( \frac{d}{dt} Y(t) \right) \\ &= \left( \frac{d}{dt} S(t) S^{-1}(t) \right) \exp(L)(t) + \exp(L)(t) Y^{-1}(t) \left( \frac{d}{dt} Y(t) \right) \\ &= \exp(L)(t) \left( Ad_{\exp(L)(t)^{-1}} \left( \left( \frac{d}{dt} S(t) S^{-1}(t) \right) \right) + Y^{-1}(t) \left( \frac{d}{dt} Y(t) \right) \right) . \end{aligned}$$

Now,  $Y^{-1}(t) \frac{d}{dt} Y(t)$  is a smooth path on the space of differential operators of order 1, and we have

$$Ad_{\exp(L)(t)^{-1}} \left( \left( \frac{d}{dt} S(t) S^{-1}(t) \right) \right) \in Cl_{odd}^{-1}(S^1, V) .$$

In this way we obtain:

**Proposition 27.** *Let us assume that  $L(t) \in C^\infty([0; 1], Cl_{odd}^1(S^1, V))$ ,  $L = L_S + L_D$  with  $L_S \in Cl_{odd}^{-1}(S^1, V)$  and  $L_D \in DO^1(S^1, V)$ , and that  $\exp(L)(t) = S(t)Y(t)$ . Then,*

$$Y(t) = \exp(L_D)(t)$$

and

$$S(t) = \exp \left( Ad_{\exp(L)(t)} (L_S) \right) (t) .$$

*Proof.* We have obtained that

$$L_D = Y(t)^{-1} \frac{d}{dt} Y(t)$$

and that

$$L_S = Ad_{\exp(L)(t)^{-1}} \left( \left( \frac{d}{dt} S(t) S^{-1}(t) \right) \right)$$

by the uniqueness of the decomposition

$$L = L_S + L_D .$$

We obtain the result by passing to the exponential maps on the groups  $Cl_{odd}^{-1,*}(S^1, V)$  and  $DO^{0,*}(S^1, V) \times Diff_+(S^1)$ .  $\square$

### 5. THE H-KP HIERARCHY WITH NON FORMAL ODD-CLASS OPERATORS

We make the following definition, along the lines of the theory developed in [25] for formal pseudo-differential operators:

**Definition 28.** *Let  $h$  be a formal parameter. The set of odd class  $h$ -pseudo-differential operators is the set of formal series*

$$(6) \quad Cl_{h,odd}(S^1, V) = \left\{ \sum_{n \in \mathbb{N}} a_n h^n \mid a_n \in Cl_{odd}^n(S^1, V) \right\} .$$

We state the following result on the structure of  $Cl_{h,odd}(S^1, V)$ :

**Theorem 29.** *The set  $Cl_{h,odd}(S^1, V)$  is a Fréchet algebra, and its group of units given by*

$$(7) \quad Cl_{h,odd}^*(S^1, V) = \left\{ \sum_{n \in \mathbb{N}} a_n h^n \mid a_n \in Cl_{odd}^n(S^1, V), a_0 \in Cl_{odd}^{0,*}(S^1, V) \right\} ,$$

*is a regular Fréchet Lie group.*

*Proof.* This result is mostly an application of Theorem 15, hence the growth conditions on the coefficients  $a_n$  appearing in (6) and (7).

From the work [15] by Glöckner, we know that  $Cl_{odd}^{0,*}(S^1, V)$  is a regular Fréchet Lie group since it is open in  $Cl_{odd}^0(S^1, V)$ . According to classical properties of composition of pseudo-differential operators [39], see also [19], the natural multiplication on  $Cl_{odd}^{0,*}(S^1, V)$  is smooth for the product topology inherited from the classical topology on classical pseudo-differential operators, and inversion is smooth using the classical formulas of inversion of series. In this way we conclude that  $Cl_{h,odd}(S^1, V)$  is a Fréchet algebra.

Moreover, the series  $\sum_{n \in \mathbb{N}} a_n h^n \in Cl_{h,odd}(S^1, V)$  is invertible if and only if  $a_0 \in Cl_{odd}^{0,*}(S^1, V)$ , which shows that  $Cl_{h,odd}^*(S^1, V)$  is open in  $Cl_{h,odd}(S^1, V)$ . The same result as before, from [15], ends the proof.  $\square$

**Remark 30.** *The assumption  $a_n \in Cl_{odd}^n$  in Definition 28 and Theorem 29 can be relaxed to the condition*

$$a_0 \in Cl_{odd}^{0,*} \text{ and } \forall n \in \mathbb{N}^*, a_n \in Cl_{odd} ;$$

*this is sufficient for having a regular Lie group. However, the Birkhoff-Mulase decomposition seems to fail in this context. Following [28], we find that the growth conditions imposed in (6) and (7) will ensure both regularity and existence of a Birkhoff-Mulase decomposition.*

The decomposition  $L = L_S + L_D$ ,  $L_S \in Cl_{odd}^{-1}(S^1, V)$ ,  $L_D \in DO^1(S^1, V)$ , which is valid on  $Cl_{odd}(S^1, V)$ , see Proposition 25, extends straightforwardly to the algebra  $Cl_{h,odd}(S^1, V)$ . We now introduce the  $h$ -KP hierarchy with non-formal pseudo-differential operators. Let us assume that  $t_1, t_2, \dots, t_n, \dots$ , are an infinite number

of different formal variables. Then, again adapting work carried out in [25], we make the following definition:

**Definition 31.** Let  $S_0 \in Cl_{odd}^{-1,*}(S^1, V)$  and let  $L_0 = S_0(h\frac{d}{dx})S_0^{-1}$ . We say that an operator

$$L(t_1, t_2, \dots) \in Cl_{h,odd}(S^1, V)[[ht_1, \dots, h^n t_n \dots]]$$

satisfies the  $h$ -deformed KP hierarchy if and only if

$$(8) \quad \begin{cases} L(0) = L_0 \\ \frac{d}{dt_n} L = [(L^n)_D, L] \end{cases} .$$

We recall from [25] that the  $h$ -KP hierarchy is obtained from the classical KP hierarchy by means of the  $h$ -scaling

$$\begin{cases} t_n & \mapsto h^n t_n \\ \frac{d}{dx} & \mapsto h \frac{d}{dx} \end{cases} ,$$

and we also recall that formal series in  $t_1, \dots, t_n, \dots$  can be also understood as smooth functions on the algebraic sum

$$T = \bigoplus_{n \in \mathbb{N}^*} (\mathbb{R}t_n)$$

for the product topology and product Frölicher structure, see Proposition 5 and [25]. Now we solve the initial value problem for (8).

**Theorem 32.** Let  $U_h(t_1, \dots, t_n, \dots) = \exp(\sum_{n \in \mathbb{N}^*} h^n t_n (L_0)^n) \in Cl_{h,odd}(S^1, V)$ . Then:

- There exists a unique pair  $(S, Y)$  such that
  - (1)  $U_h = S^{-1}Y$ ,
  - (2)  $Y \in Cl_{h,odd}^*(S^1, V)_D$
  - (3)  $S \in Cl_{h,odd}^*(S^1, V)$  and  $S - 1 \in Cl_{h,odd}(S^1, V)_S$ .

Moreover, the map

$$(S_0, t_1, \dots, t_n, \dots) \in Cl_{odd}^{0,*}(S^1, V) \times T \mapsto (U_h, Y) \in (Cl_{h,odd}^*(S^1, V))^2$$

is smooth.

- The operator  $L \in Cl_{h,odd}(S^1, V)[[ht_1, \dots, h^n t_n \dots]]$  given by  $L = SL_0 S^{-1} = YL_0 Y^{-1}$ , is the unique solution to the hierarchy of equations

$$(9) \quad \begin{cases} \frac{d}{dt_n} L & = [(L^n)_D(t), L(t)] = -[(L^n)_S(t), L(t)] \\ L(0) & = L_0 \end{cases} ,$$

in which the operators in this infinite system are understood as formal operators.

- The operator  $L \in Cl_{h,odd}(S^1, V)[[ht_1, \dots, h^n t_n \dots]]$  given by  $L = SL_0 S^{-1} = YL_0 Y^{-1}$  is the unique solution of the hierarchy of equations

$$(10) \quad \begin{cases} \frac{d}{dt_n} L & = [(L^n)_D(t), L(t)] = -[(L^n)_S(t), L(t)] \\ L(0) & = L_0 \end{cases}$$

in which the operators in this infinite system are understood as odd class, non-formal operators.

*Proof.* The part of the theorem on formal operators is proved in [25] or, it can be derived from the results of [28] by the  $h$ -scaling defined before. We now have to pass from decompositions and equations for formal pseudo-differential operators to the same properties for non-formal, odd class operators. For this, we denote by  $\sigma(A)$  the formal operator (or equivalently the asymptotic expansion of the symbol) corresponding to the operator  $A$ . Following [25], existence and uniqueness of the decomposition holds on formal operators, that is, there exist non-formal odd class operators  $Y$  and  $W$  defined up to smoothing operators such that

$$\sigma(U_h) = \sigma(W)^{-1} \sigma(Y).$$

Now,  $\sigma(Y)$  is a formal series in  $h, t_1, \dots, t_n, \dots$  of symbols of differential operators, which are in one-to-one correspondence with a series of (non-formal) differential operators. Thus, the operator  $Y$  is uniquely defined, not up to a smoothing operator; it depends smoothly on  $U_h$ , and so does  $W = YU_h^{-1}$ . This ends the proof of the first point.

The second point on the  $h$ -deformed KP hierarchy is proven along the lines of [25].

The proof of third point is similar to the proof of the first point. We have that  $L = YL_0Y^{-1}$  is well-defined and, following classical computations which can be found in e.g. [11], see also [28], we have:

- (1)  $L^k = YL_0^kY^{-1}$
- (2)  $U_h L_0^k U_h^{-1} = L_0^k$  since  $L_0$  commutes with  $U_h = \exp(\sum_k h^k t_k L_0^k)$ .

It follows that  $L^k = YL_0^kY^{-1} = WW^{-1}YL_0^kY^{-1}WW^{-1} = WL_0^kW^{-1}$ .

We take  $t_k$ -derivative of  $U$  for each  $k \geq 1$ . We get the equation

$$\frac{dU_h}{dt_k} = -W^{-1} \frac{dW}{dt_k} W^{-1} Y + S^{-1} \frac{dY}{dt_k}$$

and so, using  $U_h = S^{-1}Y$ , we obtain the decomposition

$$WL_0^k W^{-1} = -\frac{dW}{dt_k} W^{-1} + \frac{dY}{dt_k} Y^{-1}.$$

Since  $\frac{dW}{dt_k} W^{-1} \in Cl_{h,odd}(S^1, V)_S$  and  $\frac{dY}{dt_k} Y^{-1} \in Cl_{h,odd}(S^1, V)_D$ , we conclude that

$$(L^k)_D = \frac{dY}{dt_k} Y^{-1} \quad \text{and} \quad (L^k)_S = -\frac{dW}{dt_k} W^{-1}.$$

Now we take  $t_k$ -derivative of  $L$ :

$$\begin{aligned} \frac{dL}{dt_k} &= \frac{dY}{dt_k} L_0 Y^{-1} - Y L_0 Y^{-1} \frac{dY}{dt_k} Y^{-1} \\ &= \frac{dY}{dt_k} Y^{-1} Y L_0 Y^{-1} - Y L_0 Y^{-1} \frac{dY}{dt_k} Y^{-1} \\ &= (L^k)_D L - L (L^k)_D \\ &= [(L^k)_D, L]. \end{aligned}$$

It remains to check the initial condition: We have  $L(0) = Y(0)L_0Y(0)^{-1}$ , but  $Y(0) = 1$  by the definition of  $U_h$ .

Smoothness with respect to the variables  $(S_0, t_1, \dots, t_n, \dots)$  is already proved by construction, and we have established smoothness of the map  $L_0 \mapsto Y$  at the beginning of the proof. Thus, the map

$$L_0 \mapsto L(t) = Y(t)L_0Y^{-1}(t)$$



is smooth. The corresponding equation

$$\frac{d}{dt_k} L = - [(L^k)_S, L]$$

is obtained the same way.

Let us finish by checking that the announced solution is the unique solution to the non-formal hierarchy (10). This is still true at the formal level, but two solutions which differ by a smoothing operator may appear at this step of the proof. Let  $(L + K)(t_1, \dots)$  be another solution, in which  $K$  is a smoothing operator depending on the variables  $t_1, \dots$ , and  $L$  is the solution derived from  $U_h$ . Then, for each  $n \in \mathbb{N}^*$  we have

$$(L + K)_D^n = L_D^n,$$

which implies that  $K$  satisfies the *linear* equation

$$\frac{dK}{dt_n} = [L_D^n, K]$$

with initial conditions  $K|_{t=0} = 0$ . We can construct the unique solution  $K$  by induction on  $n$ , beginning with  $n = 1$ . Let  $g_n$  be such that

$$(g_n^{-1} dg_n)(t_n) = L_D^n(t_1, \dots, t_{n-1}, t_n, 0, \dots).$$

Then we get that

$$K(t_1, \dots, t_n, 0, \dots) = Ad_{g_n(t_n)}(K(t_1, t_{n-1}, 0, \dots)),$$

and hence, by induction,

$$K(0) = 0 \Rightarrow K(t_1, 0, \dots) = 0 \Rightarrow \dots \Rightarrow K(t_1, \dots, t_n, 0, \dots) = 0 \Rightarrow \dots,$$

which implies that  $K = 0$ . □

## 6. KP EQUATIONS AND $Diff_+(S^1)$

Let  $A_0 \in Cl_{odd}^{-1}(S^1, V)$ , and set  $S_0 = \exp(A_0)$ . The operator  $S_0 \in Cl_{odd}^{-1,*}(S^1, V)$  is our version of the dressing operator of standard KP theory, see for instance [7, Chapter 6]. We define the operator  $L_0$  by

$$f \mapsto L_0(f) = h \left( S_0 \circ \frac{d}{dx} \circ S_0^{-1} \right) (f)$$

for  $f \in C^\infty(S^1, V)$ . We note that  $L_0^k(f) = h^k S_0 \frac{d^k}{dx^k} (S_0^{-1}(f))$ , a formula which we will use presently. Our aim is to connect the operator

$$U_h = \exp \left( \sum_{n \in \mathbb{N}^*} h^n t_n L_0^n \right),$$

which generates the solutions of the  $h$ -deformed KP hierarchy described in Theorem 32, with the Taylor expansion of functions in the image of the twisted operator

$$A : f \in C^\infty(S^1, V) \mapsto S_0^{-1}(f) \circ g,$$

in which  $g \in Diff_+(S^1)$ . We remark that  $A \in FCl_{Diff_+, odd}^{0,*}(S^1, V)$  for each  $g \in Diff_+(S^1)$ , and that it is smooth with respect to  $g$  due to our Birkhoff-Mulase decomposition theorem.

For convenience, we identify  $S^1$  with  $[0; 2\pi[ \sim \mathbb{R}/2\pi\mathbb{Z}$ , assuming implicitly that all the values under consideration are up to terms of the form  $2k\pi$ , for  $k \in \mathbb{Z}$ . Set  $c = S_0^{-1}(f) \circ g \in C^\infty(S^1, V)$ . We compute:

$$\begin{aligned} c(x_0 + h) &= (S_0^{-1}(f) \circ g)(x_0 + h) \\ &\sim_{x_0} (S_0^{-1}(f) \circ g)(x_0) + \sum_{n \in \mathbb{N}^*} \left[ \frac{h^n}{n!} \frac{d^n}{dx^n} (S_0^{-1}(f) \circ g) \right] (x_0) \\ &= (S_0^{-1}(f) \circ g)(x_0) + \\ &\quad \sum_{n \in \mathbb{N}^*} \left[ \frac{h^n}{n!} \sum_{k=1}^n B_{n,k}(u_1(x_0), \dots, u_{n-k+1}(x_0)) \frac{d^k}{dx^k} (S_0^{-1}(f) \circ g)(x_0) \right], \end{aligned}$$

in which we have used the classical Faà de Bruno formula for the higher chain rule in terms of Bell's polynomials  $B_{n,k}$ , and  $u_i(x_0) = g^{(i)}(x_0)$  for  $i = 1, \dots, n - k + 1$ . We can rearrange the last sum and write

$$\begin{aligned} c(x_0 + h) &\sim_{x_0} (S_0^{-1}(f) \circ g)(x_0) + \\ &\quad \sum_{k \in \mathbb{N}^*} \sum_{n \geq k} \left[ \frac{h^n}{n!} B_{n,k}(u_1(x_0), \dots, u_{n-k+1}(x_0)) \frac{d^k}{dx^k} (S_0^{-1}(f)) \right] (g(x_0)) \end{aligned}$$

or,

$$(11) \quad c(x_0 + h) \sim_{x_0} \sum_{k \in \mathbb{N}} \left[ a_k h^k \frac{d^k}{dx^k} (S_0^{-1}(f)) \right] (g(x_0))$$

in which  $a_0 = 1$  and

$$a_k = \sum_{n \geq k} \frac{h^{n-k}}{n!} B_{n,k}(u_1(x_0), \dots, u_{n-k+1}(x_0))$$

for  $k \geq 1$ . In terms of the operator  $L_0$ , Equation (11) means that

$$(12) \quad c(x_0 + h) \sim_{x_0} S_0^{-1} \sum_{k \in \mathbb{N}} [a_k L_0^k(f)] (g(x_0)).$$

We now define the sequence  $(t_n)_{n \in \mathbb{N}^*}$  by the formula

$$(13) \quad \log \left( \sum_{k \in \mathbb{N}} a_k X^k \right) = \sum_{n \in \mathbb{N}^*} t_n X^n,$$

so that both,  $a_k$  and  $t_n$ , are series in the variable  $h$ . We obtain

$$c(x_0 + h) \sim_{x_0} S_0^{-1} \exp \left( \sum_{n \in \mathbb{N}^*} \frac{t_n}{h^n} L_0^n(f) \right) (g(x_0)).$$

We state the following theorem:

**Theorem 33.** *Let  $f \in C^\infty(S^1, V)$  and set  $c = S_0^{-1}(f) \circ g \in C^\infty(S^1, V)$ . The Taylor series at  $x_0$  of the function  $c$  is given by*

$$c(x_0 + h) \sim_{x_0} S_0^{-1} (U_h(t_1/h, t_2/h^2, \dots)(f)) (g(x_0)),$$

in which the times  $t_i$  are related to the derivatives of  $g$  via Equation (13).

The coefficients of the series  $a_k$  and  $t_n$  appearing in (13) depend smoothly on  $g \in \text{Diff}_+(S^1)$  and  $x_0 \in S^1$ . Indeed, the map

$$(x, g) \in S^1 \times \text{Diff}_+(S^1) \mapsto (g(x), (u_n(x))_{n \in \mathbb{N}^*}) \in S^1 \times \mathbb{R}^{\mathbb{N}^*}$$

is smooth due to Proposition 5 (more precisely, due to the generalization of Proposition 5 to infinite products); smoothness  $a_k$  then follows, while smoothness of  $t_n$  is consequence of Equation (13).

**Remark 34.** *As a by-product of the foregoing computations, we notice the following relation. If  $f \in C^\infty(S^1, V)$ , we can write*

$$f(x_0+h) \sim_{x_0} f(x_0) + \sum_{n \in \mathbb{N}^*} \left( \frac{h^n}{n!} \left( \frac{d}{dx} \right)^n f \right) (x_0) = \left( \exp \left( h \frac{d}{dx} \right) f \right) (x_0) \in J^\infty(S^1, V)$$

for  $x_0 \in S^1$ . Thus, the operator  $\exp \left( h \frac{d}{dx} \right)$  belongs to the space  $Cl_h(S^1, V)$ .

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