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Synchronization and Fluctuations for Interacting Stochastic Systems with Individual and Collective Reinforcement

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Abstract. The Pólya urn is the most representative example of a reinforced stochastic process. It leads to a random (non degenerated) time-limit. The Friedman urn is a natural generalization whose almost sure (a.s.) time-limit is not random any more. In this work, in the stream of previous recent works, we introduce a new family of (finite size) systems of reinforced stochastic processes, interacting through an additional collective reinforcement of mean field type. The two reinforcement rules strengths (one component-wise, one collective) are tuned through (possibly) two different rates. In special cases, these reinforcements are of Pólya or Friedman type as in urn contexts and may thus lead to limits which may be random or not. Different parameter regimes need to be considered.

We state two kind of results. First, we study the time-asymptotics and show that L^2 and a.s. convergence always holds. Moreover all the components share the same time-limit (so called *synchronization* phenomenon). We study the nature of the limit (random/deterministic) according to the parameters' regime considered. Second, we study fluctuations by proving central limit theorems. Scaling coefficients vary according to the regime considered. This gives insights into many different rates of convergence. In particular, we identify the regimes where synchronization is faster than convergence towards the shared time-limit.

1. Introduction and model's definition

In urn models, it is well known that the bicolor Pólya reinforcement rule (reinforcement of the chosen color) leads to a random limiting a.s. proportion whereas the Friedman rule (reinforcement of the chosen color as well as the non chosen color) leads to a deterministic time-asymptotics proportion. This somewhat surprising fact is explained for instance through a fix point [Hill et al. \(1980\)](#). See [Pemantle \(2007\)](#) too. Following many recent works (see [Section 3](#) for details), this paper is motivated by the study of asymptotics time behavior of models of (discrete time) stochastic processes interacting through a reinforcement rule.

Let us define the following new model. Let $(Z_n(i), 1 \leq i \leq N)_{n \geq 0}$ be an N -dimensional stochastic process with values in $[0, 1]^N$ where $N \geq 2$. For each component indexed by i ($1 \leq i \leq N$) we

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consider the stochastic dynamics defined through the recursive equation, for any $n \in \mathbb{N}$,

$$Z_{n+1}(i) = (1 - r_n^\ell - r_n^g)Z_n(i) + r_n^\ell \xi_{n+1}^\ell(i) + r_n^g \xi_{n+1}^g, \quad (1.1)$$

where $Z_0(i) = \frac{1}{2}$ and where $\xi_{n+1}^\ell(i)$ and ξ_{n+1}^g denote local (component-wise) and collective reinforcements. Let $(\mathcal{F}_n)_n$ be the natural filtration associated with $(Z_n(i), 1 \leq i \leq N)_{n \geq 0}$. Given the σ -field \mathcal{F}_n , they have independent Bernoulli distributions with

$$\mathbb{P}(\xi_{n+1}^\ell(i) = 1 | \mathcal{F}_n) = \psi_1(Z_n(i)) := (1 - 2\lambda_1)Z_n(i) + \lambda_1, \quad (1.2)$$

$$\mathbb{P}(\xi_{n+1}^g = 1 | \mathcal{F}_n) = \psi_2(Z_n) := (1 - 2\lambda_2)Z_n + \lambda_2, \quad (1.3)$$

where $\psi_k : [0, 1] \rightarrow [0, 1]$ ($k \in \{1, 2\}$) are linear maps with $Z_n := \frac{1}{N} \sum_{i=1}^N Z_n(i)$ (so called *mean field*) and where $\lambda_1, \lambda_2 \in [0, 1]$ are parameters. The local (resp. the collective) reinforcement rate $(r_n^\ell)_n$ (resp. $(r_n^g)_n$) are real sequences such that $0 \leq r_n^\ell < 1$ and $0 \leq r_n^g < 1$, and

$$\lim_n n^{\gamma_1} r_n^\ell = c_1 > 0 \quad \text{and} \quad \lim_n n^{\gamma_2} r_n^g = c_2 > 0 \quad (1.4)$$

where $\gamma_j \in (\frac{1}{2}, 1]$ for $j \in \{1, 2\}$. This assumption gives that $(r_n^\ell)_n$ (resp. $(r_n^g)_n$) satisfy the following usual assumptions for processes defined through recursive equations like (1.1)

$$\sum r_n^\ell = +\infty, \quad \sum (r_n^\ell)^2 < +\infty. \quad (1.5)$$

Note $(r_n^\ell)^2$ stand for the square of r_n^ℓ . As emphasized in [Aletti et al. \(2019\)](#), in order to state the CLTs, we actually assume the slightly stricter following assumptions.

Assumption 1.1. There exist real constants γ_1, γ_2 and $c_1 > 0, c_2 > 0$ with $(\gamma_1, \gamma_2) \in (\frac{1}{2}, 1]^2$, such that when $n \rightarrow \infty$

$$r_n^\ell = \frac{c_1}{n^{\gamma_1}} + \mathcal{O}\left(\frac{1}{n^{2\gamma_1}}\right) \quad \text{and} \quad r_n^g = \frac{c_2}{n^{\gamma_2}} + \mathcal{O}\left(\frac{1}{n^{2\gamma_2}}\right). \quad (1.6)$$

When the two reinforcement rules compete through the different rates r_n^ℓ, r_n^g , one individual rule, one collective (in the sense all the components are involved), is there one leading? In particular, is there *loss of synchronization* which means, is there an almost sure limit depending on the component? Moreover, in the case were Pólya and Friedman type of reinforcement rule do compete through the system, and if there is synchronization, may the shared time-limit be random?

As emphasized in the previous works, there are many applications these stochastic models may be useful for. Urn models are well known [Mahmoud \(2008\)](#) to have applications in economy, in contagion models [Harrington et al. \(2019\)](#), in clinical trials adaptive design [Laruelle and Pagès \(2014\)](#), random networks [Hofstad \(van der\)](#). In general, the reinforcement rate $(r_n)_n$, may be such that $\lim_n n^\gamma r_n = c > 0$. The dynamics is nothing but a vector-valued stochastic algorithm [Benaïm \(1999\)](#); [Duflo \(1997\)](#). Such processes have many applications like in the framework of stochastic optimization (see for instance [Delyon et al., 1999](#); [Gadat and Younès, 2007](#)). In [Crimaldi et al. \(2019\)](#) and in [Aletti et al. \(2017\)](#) an application of these processes as opinion dynamics was introduced. We will briefly explain it in our context in Section 3 : each random variable $Z_n(i)$ may be considered as an agent's inclination towards one of two possible choices.

In the new family of models we introduce and study in this paper, following previous recent works [Dai Pra et al. \(2014\)](#); [Sahasrabudhe \(2016\)](#); [Crimaldi et al. \(2016\)](#); [Aletti et al. \(2017, 2019, 2020\)](#); [Crimaldi et al. \(2019\)](#); [Mirebrahimi \(2019\)](#), we consider a (finite) system of reinforced stochastic processes defined though recursive equations (1.1). Two kind of reinforcement are involved. One depending only on the component $(\xi_{n+1}^\ell(i))_n$ (see next section for the notations), one creating the interaction $(\xi_{n+1}^g)_n$ and depending on the average over all components. *The interaction holds through the reinforcement.* This is modeling a collective reinforcement effect that may compete with the individual/local/component-wise reinforcement. For the sake of simplicity, we choose to consider a mean field interaction, in the sense that collective reinforcement depends on the arithmetic

mean over the system at previous time step. Each reinforcement has its own rate r_n^ℓ (resp. r_n^g). Each rate may have its own asymptotic behavior: $r_n^\ell \sim c_1 n^{-\gamma_1}$ (resp. $r_n^g \sim c_2 n^{-\gamma_2}$). If $1/2 < \gamma_1 < \gamma_2 \leq 1$ (for instance), one may expect the collective reinforcement to become negligible in the long run. A naive guess could be, the system behaves for large time like a system with independent components, thus leading to a possible *loss of synchronization*. We prove later this does not happen: L^2 /a.s. synchronization holds (meaning, each component dynamics shares the same random time limit) like in Dai Pra et al. (2014). Additional issues we address, are: *nature* (deterministic/random, diffuse or atomic) of the almost sure (a.s.) time limit distribution according to the type of reinforcement, scales of *fluctuations* with respect to this limit, which are stated by proving central limit theorems (CLT) w.r.t stable convergence. We prove, according to the parameters' cases, that the rate of synchronization is quicker, or the same, as the speed of convergence to the limit. In the models considered in Crimaldi et al. (2019), synchronization quicker than convergence towards the asymptotic value Z_∞ holds only in cases where $\text{Var}(Z_\infty) > 0$. In the following models it may happen even when Z_∞ is deterministic.

The paper is organized as follows. In Section 2 we give some preliminary remarks about the model's definition. In Section 3 we compare the model with related families studied in previous works and give some interpretation as models for opinion dynamics. In Section 4 we prove that L^2 and a.s. convergence holds towards a limiting value Z_∞ in \mathbb{R} shared by all the components (synchronization). Two main cases are to be distinguished: Theorems 4.1 and 4.2 deals with cases where $Z_\infty = 1/2$ (the word synchronization is abusive in this situation) whereas Theorem 4.4 deals with Z_∞ random. In Section 5, in the different cases, we state central limit theorems about the fluctuations of $(Z_n - Z_\infty)$ and $(Z_n - Z_n(i))$. Scaling factors are worth of interest. Th. 5.1 consider the case where each individual and collective reinforcement leads to a deterministic limiting value. Theorem 5.2 consider the special case when $\gamma_1 = \gamma_2 = 1$ reminiscent of the Friedman urn context, in the regime where fluctuations are known not to be gaussian ($c_1 \lambda_1 + c_2 \lambda_2 < \frac{1}{4}$). Theorem 5.3 deals with the mixed cases where individual and reinforcement type are of different nature. Th. 5.4 consider the case where both the individual and the collective reinforcement lead to a random limit. Section 6 is dedicated to comments on the model from a stochastic approximation perspective. Section 7 is dedicated to the proof of the synchronization. Section 8 deals with the proofs of the CLTs. An appendix A states and recalls for the sake of completeness some technical results.

2. Preliminary remarks

The model is define through the recursive equations (1.1), with (1.2), and general assumption (1.2) about the rates' asymptotic behavior. We refer to Crimaldi et al. (2019) for a discussion on the case $0 < \gamma \leq 1/2$ in another model. We believe that the case $0 < \gamma \leq 1/2$ leads to a drastically different behavior, and therefore leave this case for further studies.

For the sake of simplicity, we choose to have some symmetry in the model with respect to $1/2$. For the same reason, according to the previous works cited in the Section 3, we consider the starting conditions all equal to $1/2$ without loss of generality.

Remark 2.1. In the particular case when $\gamma_1 = \gamma_2 =: \gamma$, we can rewrite the model as

$$Z_{n+1}(i) = (1 - 2r_n)Z_n(i) + r_n \tilde{\xi}_{n+1}(i),$$

where $\tilde{\xi}_{n+1}(i) = \xi_{n+1}^\ell(i) + \xi_{n+1}^g$, therefore $\tilde{\xi}_{n+1}(i) \in \{0, 1, 2\}$. The other probabilities in (1.2) and their subsequent relations may be computed in an analogous way. The reinforcement rate remains such that $r_n \sim cn^{-\gamma}$ when $n \rightarrow \infty$.

Remark 2.2. In this paper the parameters λ_1, λ_2 are kept fixed. Cases where λ_1, λ_2 may converge to 0 depending on n, N will be considered in a forthcoming work. Note that when $\lambda_1 = \lambda_2 = 1/2$

(see (1.2)), there is no reinforcement. Moreover, when $\lambda_1 \neq 1/2$ and $\lambda_2 = 1/2$, although we still have reinforcement on each component individually, we lose the interaction between components.

Remark 2.3. If $0 \leq \lambda_1 < 1/2$ and $0 \leq \lambda_2 < 1/2$, then ψ_1 and ψ_2 are increasing maps and occurrence of events $\{\xi_n(i) = 1\}$ increases the probability of having $\{\xi_{n+1}(i) = 1\}$ at next time step. This is the basic original meaning of reinforcement.

The following relationships hold.

$$\mathbb{E}(Z_{n+1}(i) - Z_n(i) | \mathcal{F}_n) = \lambda_1 r_n^\ell (1 - 2Z_n(i)) + \lambda_2 r_n^g (1 - 2Z_n) + r_n^g (Z_n - Z_n(i)), \quad (2.1)$$

and by averaging over i in $\{1, \dots, N\}$, we have

$$\mathbb{E}(Z_{n+1} - Z_n | \mathcal{F}_n) = (\lambda_1 r_n^\ell + \lambda_2 r_n^g) (1 - 2Z_n). \quad (2.2)$$

3. Related models and application motivations

As emphasized in [Crimaldi et al. \(2019\)](#), the evolution of proportions in urn models satisfies recursive equation like (1.1) with $r_n^g = 0$ (*i.e.* without interaction) and with $\gamma_1 = 1$. This family of models we introduce is related to some other models that were studied recently. We briefly present them in this section.

3.1. Interacting urns models. The models considered in this paper were introduced in [Mirebrahimi \(2019\)](#). When $r_n^g \equiv 0$ and $\gamma_1 = 1$, the model is a non interacting system where each component's value $Z_n(i)$ can be interpreted as the proportion of balls of a given color in a bicolor balanced (same deterministic number of balls are added whatever the chosen color is) urn classic model. The case $\lambda_1 = 0$ leads to $\psi_1(x) = x$ which is the basic Pólya reinforcement rule, where a fix number of balls is added, whose color is the same as the chosen one. It is well known $\lim_{n \rightarrow \infty} Z_n(i)$ exists a.s. and defines a Beta distributed random variable $Z_\infty(i)$, whose parameters depend on the initial number of balls of each colors. The case $\lambda_1 \neq 0$ leads to a Friedman urn model. The unique fix point of ψ_1 is $1/2$. It is known, $\lim_{n \rightarrow \infty} Z_n(i)$ exists a.s. and is equal to this fix point $Z_\infty(i) = 1/2 := Z_\infty$. See for instance [Hill et al. \(1980\)](#) where very interesting cases of less regular ψ_1 maps are considered. This can be proven using stochastic approximation results [Benaïm \(1999\)](#); [Duflo \(1997\)](#). In our context we use the terminology Pólya type when $\lambda_1 = 0$ and Friedman type when $\lambda_1 \neq 0$. A similar remark holds for the collective reinforcement effect ruled by ψ_2 and tuned through γ_2 for the asymptotic behavior of the reinforcement rate r_n^g and λ_2 for the reinforcement's type.

In [Dai Pra et al. \(2014\)](#), the system introduced and studied is related to the case $\gamma_1 = \gamma_2 = 1$ and $\lambda_1 = \lambda_2 = 0$. It solves Equation (1.1) with $r_n^g \equiv 0$ and with

$$\mathbb{P}(\xi_{n+1}^\ell(i) = 1 | \mathcal{F}_n) = (1 - \alpha)Z_n(i) + \alpha Z_n$$

where $\alpha \in [0, 1]$ is a parameter. A.s. synchronization towards a random shared limit Z_∞ ($\text{Var}(Z_\infty) > 0$) was proved as soon as $\alpha > 0$. Fluctuations were studied in [Crimaldi et al. \(2016\)](#) by proving central limit theorems. In [Sahasrabudhe \(2016\)](#), a similar interacting model was studied, whose components dynamics can be interpreted as urn models (reinforcement's rate behaving like n^{-1}) with a Friedman reinforcement rule. A.s. convergence holds towards a unique deterministic value. Moreover, it is known that Friedman urns can exhibit non gaussian fluctuations [Janson \(2004\)](#); [Pemantle \(2007\)](#). See [Flajolet et al. \(2006\)](#); [Chauvin et al. \(2015\)](#); [Lasmar et al. \(2018\)](#); [Mailler \(2018\)](#); [Chauvin et al. \(2011\)](#) for more specific recent results about urn models and generalizations. In [Sahasrabudhe \(2016\)](#), this was proven to have consequences for the mean-field interacting system, where different speed of convergence may happen. In relationship with systems of interacting urns, some variations with different kind of urns/reinforcements bias towards one or the other color, were considered in [Louis and Minelli \(2018\)](#).

For the model considered in this paper, similar interpretation as bicolour balanced urn model can be made, when $\gamma_1 = \gamma_2 = 1$. Each $Z_n(i)$ can be interpreted as the proportion of one chosen color in an urn i . Two reinforcement mechanism hold which can be related to the following one, applied to each urn i of the systems, independently, between two iterations. One (for instance) ball is chosen uniformly at random in the urn i and one (for instance) ball of the chosen color is added into urn i (Pólya reinforcement type, $\lambda_1 = 0$); resp. one ball of the non chosen color, Friedman type, $\lambda_1 \neq 0$). Additionally, one ball is chosen uniformly at random in the *whole* system (proportion Z_n at previous time step) and one ball of the chosen color is added (collective reinforcement of Pólya type, $\lambda_2 = 0$) into urn i ; resp. one ball of the non chosen color, collective reinforcement of Friedman type, $\lambda_2 \neq 0$). In fact adding a ball of the non chosen color is similar in general to add a ball of the chosen color as well as the non chosen color. The reinforcement matrices (for local reinforcement, resp. for collective reinforcement) defining these numbers are then giving the exact values of λ_1 (resp. λ_2).

3.2. General reinforcements' rates. Generalizing the reinforcement rate r_n asymptotic behavior from $r_n \sim cn^{-1}$ to $r_n \sim_{n \rightarrow \infty} cn^{-\gamma}$ leads to systems of stochastic processes with reinforcement which can be considered as interacting stochastic algorithms of Robbins-Monro type. In [Crimaldi et al. \(2019\)](#) several cases of reinforcement (like Pólya/Friedman) were considered. A.s. synchronization was stated towards different kind of limit Z_∞ (deterministic or not) and speed of convergence studied through *functional* central limit theorems (FCLT) for $Z_n(i) - Z_n$ and $Z_n - Z_\infty$. It was proved that in parameters' regime where the time limit Z_∞ is random (in the sense $\mathbb{V}ar(Z_\infty) > 0$), *synchronization happens quicker than convergence* to the time limit.

Building a reinforcement with the average proportion Z_n (mean field) helps in these interacting systems since it is enough to deal with closed uni-dimensional recursive equations for $(Z_n)_n$ and $(Z_n - Z_n(i))_n$. The interaction was generalized from mean field to *network based* interaction in [Aletti et al. \(2017\)](#), with a reinforcement of Pólya type. The system dynamics is defined, for $i \in V := \{1, \dots, N\}$, through

$$Z_{n+1}(i) = (1 - r_n)Z_n(i) + r_n \xi_{n+1}(i) \quad (3.1)$$

where for any $n \geq 0$, the random variables $\{\xi_{n+1}(i) : i \in V\}$ are conditionally independent given \mathcal{F}_n with

$$\mathbb{P}(\xi_{n+1}(i) = 1 \mid \mathcal{F}_n) = \sum_{j=1}^N w_{j,i} Z_n(j) \quad (3.2)$$

with $\mathcal{F}_n := \sigma(Z_n(i))$. The non negative matrix $W = [w_{j,i}]_{j,i \in V \times V}$ is considered as a weighted adjacency matrix of the graph $G = (V, E)$ with $V = \{1, \dots, N\}$ as the set of vertices and $E \subseteq V \times V$ as the set of directed edges. Each edge $(j, i) \in E$ represents the fact that the vertex j has a direct influence on the vertex i . The weight $w_{i,j} \geq 0$ quantifies how much j can influence i . The weights are assumed to be normalized $W^\top \mathbf{1} = \mathbf{1}$ where $\mathbf{1}$ denotes $(1, \dots, 1) \in \mathbb{R}^N$. The matrix W is assumed to be irreducible and diagonalizable. The reinforcement rate r_n is assumed to satisfy (1.4) with $\gamma \in (1/2, 1]$ or a more restrictive condition as (1.6). Synchronization is proven to hold and CLT's were stated. The empirical means $N_n(i) := N^{-1} \sum_{i=1}^N \xi_n(i)$ are studied in [Aletti et al. \(2019\)](#): a.s. synchronization toward Z_∞ and CLT are proven. Weighted empirical means are studied analogously in [Aletti et al. \(2020\)](#).

As considered in [Crimaldi et al. \(2019\)](#); [Aletti et al. \(2019\)](#), we may think about following context for the random evolutions we consider. Let us state in the case where $S = \{0, 1\}$ represents two possible choices or actions made by "individuals" or agents $i \in V$. To each vertex $i \in V$ is associated a value $Z_n(i) \in [0, 1]$. The quantity $Z_n(i)$ (resp. $1 - Z_n(i)$) is interpreted as the *inclination* to adopt the choice 1 (resp. 0) at time n . The recursive equation (1.1) means the inclination of agent i at the next time step, is a convex combination of

- the inclination $Z_n(i)$ with self-reinforcement weight $1 - r_n^\ell - r_n^g$,
- a choice $\xi_{n+1}^\ell(i)$ made with a probability $\psi_1(Z_n(i))$ for opinion 1 related to the personal inclination, with a weight r_n^ℓ ,
- and a collective choice ξ_{n+1}^g made with a probability $\psi_2(Z_n)$ for opinion 1 related to the collective inclination Z_n , average of the personal inclinations, with a weight r_n^g .

As time goes, the rates r_n^ℓ and r_n^g vanish. For larger n , the self-reinforcement leads. The different speed of convergence towards 0 for r_n^ℓ and r_n^g , tuned by γ_1, γ_2 could mean the influence of collective actions may disappear quicker than the influence of individual choices. On the contrary, the a.s. synchronization phenomenon towards a shared inclination Z_∞ could be interpreted as the emergence of a *consensus*, in the sense every individual shares the same inclination. The special case $Z_\infty = 1/2$ may be interpreted as a complete undetermined "fifty-fifty" inclination towards the two actions. Issues we addressed at the beginning of this paper may be reconsidered through this interpretation.

4. Main results: convergence and a.s. synchronization

In this section we state result on the convergence of $(Z_n)_n$ and the synchronization phenomenon. Indeed we obtain different kind of time-limit (deterministic or (truly-)random) for $(Z_n)_n$ according to the nullity of λ_1, λ_2 . Moreover L^2 and a.s. synchronization are stated in all the cases. As previously mentioned, the choice of the mean-field instead of a network-based interaction allows us to address the proofs by studying Z_n and $Z_n - Z_n(i)$ instead of dealing with \mathbb{R}^N valued recursive equations.

4.1. *Case of a deterministic time-asymptotics.* We call the case *deterministic*, if the time limit Z_∞ ($n \rightarrow \infty$) is not random ($\text{Var}(Z_\infty) = 0$). This behavior corresponds to cases where at least one of the following assumptions is true: $\lambda_1 > 0$ or $\lambda_2 > 0$. The mean field process $(Z_n)_n$ is not a martingale. In order to investigate the behavior of the interacting system, we first consider the time limits of $\text{Var}(Z_n)$. Second we show that L^2 -synchronization holds *i.e.* $\lim_{n \rightarrow \infty} \text{Var}(Z_n(i) - Z_n) = 0$. We get the rates of convergence too. Finally, we prove that the synchronization holds almost surely and the deterministic limit is $Z_\infty := \frac{1}{2}$.

Theorem 4.1. *For any $\lambda_1 > 0$ and $\lambda_2 > 0$ the following results hold:*

- i) *Asymptotics of variance as $n \rightarrow \infty$:*
 $\text{Var}(Z_n) = \mathcal{O}(\frac{1}{n^{\gamma_*}})$ where $\gamma_* := \min(\gamma_1, \gamma_2)$;
- ii) *behavior of the L^2 -distance between Z_n and $Z_n(i)$ when $n \rightarrow \infty$:*
 - a) *if $\gamma_1 \leq \gamma_2$, then $\mathbb{E}([Z_n - Z_n(i)]^2) = \mathcal{O}(\frac{1}{n^{\gamma_1}})$,*
 - b) *if $\gamma_2 < \gamma_1$, then $\mathbb{E}([Z_n - Z_n(i)]^2) = \mathcal{O}(\frac{1}{n^{2\gamma_1 - \gamma_2}})$;*
- iii) *almost sure convergence holds i.e.*
 $\forall i \in \{1, \dots, N\}, \quad \lim_{n \rightarrow +\infty} Z_n(i) = \lim_{n \rightarrow +\infty} Z_n = \frac{1}{2} =: Z_\infty$ a.s.

Two others choices of parameters λ_1, λ_2 lead to the following results.

Theorem 4.2. *In the following cases: either ($\lambda_1 > 0$ and $\lambda_2 = 0$) or ($\lambda_1 = 0$ and $\lambda_2 > 0$) it holds $\lim_{n \rightarrow +\infty} Z_n(i) = \lim_{n \rightarrow +\infty} Z_n = \frac{1}{2}$ a.s. Moreover, the following table summarizes the L^2 speed of convergence.*

	$\lambda_1 \neq 0, \lambda_2 = 0$	$\lambda_1 = 0, \lambda_2 \neq 0$
$\gamma_1 \leq \gamma_2$	$\text{Var}(Z_n) = \mathcal{O}(\frac{1}{n^{\gamma_1}})$ $\mathbb{E}([Z_n - Z_n(i)]^2) = \mathcal{O}(\frac{1}{n^{\gamma_1}})$	$\text{Var}(Z_n) = \mathcal{O}(\frac{1}{n^{2\gamma_1 - \gamma_2}})$ $\mathbb{E}([Z_n - Z_n(i)]^2) = \mathcal{O}(\frac{1}{n^{2\gamma_1 - \gamma_2}})$
$\gamma_2 < \gamma_1$	$\text{Var}(Z_n) = \mathcal{O}(\frac{1}{n^{2\gamma_2 - \gamma_1}})$ $\mathbb{E}([Z_n - Z_n(i)]^2) = \mathcal{O}(\frac{1}{n^{2\gamma_1 - \gamma_2}})$	$\text{Var}(Z_n) = \mathcal{O}(\frac{1}{n^{\gamma_2}})$ $\mathbb{E}([Z_n - Z_n(i)]^2) = \mathcal{O}(\frac{1}{n^{2\gamma_1 - \gamma_2}})$

□

Remark 4.3. (Comparison of convergence and synchronization rates)

In the case $\lambda_1 > 0$, $\lambda_2 > 0$, when $\gamma_1 < \gamma_2$, the L^2 convergence rate of $(Z_n)_n$ to $\frac{1}{2}$ and the L^2 rate of convergence of $(Z_n(i) - Z_n)_n$ to 0 are the same. However, when $\gamma_2 < \gamma_1$, we obtain that synchronization happen faster than convergence.

Moreover in the case $\lambda_1 > 0$, $\lambda_2 = 0$ and when $\gamma_1 < \gamma_2$, the speed of convergence and synchronization are the same ($n^{-\gamma_1}$). While when $\gamma_2 < \gamma_1$, the synchronization is faster than convergence.

Similarly, in the case $\lambda_1 = 0$, $\lambda_2 > 0$ and when $\gamma_1 \leq \gamma_2$, the speed of convergence and synchronization are the same ($n^{-(2\gamma_1-\gamma_2)}$), while when $\gamma_2 < \gamma_1$, the speed of synchronization is faster than convergence ($n^{-(2\gamma_1-\gamma_2)}$ and $n^{-\gamma_2}$ respectively).

4.2. Case of a shared random time-asymptotics. Unlike to the previous cases, the case $\lambda_1 = \lambda_2 = 0$ yields $(Z_n)_n$ is a martingale. We will prove it leads to a random time-asymptotics Z_∞ ($\text{Var}(Z_\infty) > 0$). We will study the system's time-asymptotics behavior in a similar way as in the previous cases. First we show that $\lim_{n \rightarrow \infty} \text{Var}(Z_n) \neq 0$. Second we prove that L^2 -synchronization holds. Third we state the almost sure synchronization holds.

Theorem 4.4. *When $\lambda_1 = \lambda_2 = 0$,*

- i) *it holds ($n \rightarrow \infty$) $\text{Var}(Z_n) > 0$. In particular $(Z_n)_n$ converges a.s. to a non-degenerated random limit denoted by Z_∞ ($\text{Var}(Z_\infty) > 0$).*
- ii) *The L^2 -distance between the mean field Z_n and each component $Z_n(i)$ behaves as follows,*

$$\mathbb{E}[(Z_n(i) - Z_n)^2] = \mathcal{O}\left(\frac{1}{n^{2\gamma_1-\gamma_2}}\right)$$

and synchronization holds almost surely. It means, for all $i \in \{1, \dots, N\}$,

$$\lim_{n \rightarrow \infty} Z_n(i) = Z_\infty \text{ a.s.}$$

□

5. Main results: fluctuations through CLT

In this section we study the fluctuations of $(Z_n(i) - Z_n)_n$ (synchronization) w.r.t 0 and also fluctuations of $(Z_n)_n$ w.r.t its limit Z_∞ . These are studied by stating Central Limit Theorems. Pay attention to the fact that different scalings hold according to (γ_1, γ_2) relationship. We follow the proof's techniques initiated for these models in [Crimaldi et al. \(2016\)](#) based on [Theorem A.5](#) in [Appendix](#), which leads to stable convergence results.

We first study cases where $Z_\infty = \frac{1}{2}$. [Theorems 5.1, 5.2](#) deal with the case $\lambda_1 > 0$ and $\lambda_2 > 0$. Moreover, we show that there is some special regime when $0 < (c_1\lambda_1 + c_2\lambda_2) < \frac{1}{4}$. [Theorem 5.3](#) describes the results of the cases where exactly one of the λ_j is 0.

Finally we state the behavior when $\text{Var}(Z_\infty) > 0$ in [Theorem 5.4](#).

The following statements hold, where the generic symbol σ^2 denotes the variances (depending on N and λ_1, λ_2) make more precise in proofs. In the proofs of [Sections 7 and 8](#) we used $c_1 = c_2 = 1$ to simplify. Following statements are nevertheless formulated in full generality.

Theorem 5.1. *Let $\lambda_1 > 0, \lambda_2 > 0$; let $\gamma_* := \min(\gamma_1, \gamma_2)$.*

i) *It holds*

a) *when $\gamma_1 \leq \gamma_2$, $n^{\frac{\gamma_1}{2}}(Z_n - Z_n(i)) \xrightarrow[n \rightarrow \infty]{\text{stably}} \mathcal{N}\left(0, \tilde{\sigma}_1^2\right)$,*

b) *when $\gamma_2 < \gamma_1$, $n^{\frac{2\gamma_1-\gamma_2}{2}}(Z_n - Z_n(i)) \xrightarrow[n \rightarrow \infty]{\text{stably}} \mathcal{N}\left(0, \tilde{\sigma}_2^2\right)$.*

ii) *When $\gamma_* < 1$, it holds*

$$n^{\frac{\gamma_*}{2}}\left(Z_n - \frac{1}{2}\right) \xrightarrow[n \rightarrow \infty]{\text{stably}} \mathcal{N}\left(0, \hat{\sigma}^2\right).$$

iii) *When $\gamma_* = 1$,*

- a) if $(c_1\lambda_1 + c_2\lambda_2) > \frac{1}{4}$, $\sqrt{n}(Z_n - \frac{1}{2}) \xrightarrow[n \rightarrow \infty]{\text{stably}} \mathcal{N}(0, \sigma_1^{*2})$.
b) if $(c_1\lambda_1 + c_2\lambda_2) = \frac{1}{4}$, $\frac{\sqrt{n}}{\sqrt{\ln n}}(Z_n - \frac{1}{2}) \xrightarrow[n \rightarrow \infty]{\text{stably}} \mathcal{N}(0, \sigma_2^{*2})$.

□

Theorem 5.2. Let $\lambda_1 > 0, \lambda_2 > 0$. When $\gamma_* = 1$ and when $(c_1\lambda_1 + c_2\lambda_2) < \frac{1}{4}$, the following statement holds

$$n^{4(c_1\lambda_1 + c_2\lambda_2)}(Z_n - \frac{1}{2}) \xrightarrow[n \rightarrow \infty]{\text{a.s./}L^1} \tilde{X},$$

for some real random variable \tilde{X} such that $\mathbb{P}(\tilde{X} \neq 0) > 0$.

This regime is related to the known non gaussian fluctuation regime of the Friedman urn (see for instance Th. 2.9 (ii) in [Crimaldi et al. \(2019\)](#) or Th. 4 and 5 in [Sahasrabudhe \(2016\)](#) where additive assumptions need to be used).

Two other main cases leads to following results. For the sake of readability, the asymptotic variances are detailed in the proofs.

Theorem 5.3. In the following cases: either $(\lambda_1 > 0, \lambda_2 = 0)$ or $(\lambda_1 = 0, \lambda_2 > 0)$, the stable convergence towards some Gaussian distribution holds for the quantities $(Z_n - Z_n(i))_n$ and $(Z_n - \frac{1}{2})_n$. The following tables summarizes the different scales according to the relationship between γ_1, γ_2 . The first table deals with $\gamma_* := \min(\gamma_1, \gamma_2) < 1$.

	$\lambda_1 \neq 0, \lambda_2 = 0$	$\lambda_1 = 0, \lambda_2 \neq 0$
$\gamma_1 \leq \gamma_2$	$n^{\frac{\gamma_1}{2}}(Z_n - Z_n(i)) \xrightarrow{\text{stably}} \mathcal{N}(0, \tilde{\sigma}_3^2)$ $n^{\frac{\gamma_1}{2}}(Z_n - \frac{1}{2}) \xrightarrow{\text{stably}} \mathcal{N}(0, \hat{\sigma}_1^2)$	$n^{\frac{2\gamma_1 - \gamma_2}{2}}(Z_n - Z_n(i)) \xrightarrow{\text{stably}} \mathcal{N}(0, \tilde{\sigma}_4^2)$ $n^{\frac{2\gamma_1 - \gamma_2}{2}}(Z_n - \frac{1}{2}) \xrightarrow{\text{stably}} \mathcal{N}(0, \hat{\sigma}_2^2)$
$\gamma_2 < \gamma_1$	$n^{\frac{2\gamma_1 - \gamma_2}{2}}(Z_n - Z_n(i)) \xrightarrow{\text{stably}} \mathcal{N}(0, \tilde{\sigma}_5^2)$ $n^{\frac{2\gamma_2 - \gamma_1}{2}}(Z_n - \frac{1}{2}) \xrightarrow{\text{stably}} \mathcal{N}(0, \hat{\sigma}_3^2)$	$n^{\frac{2\gamma_1 - \gamma_2}{2}}(Z_n - Z_n(i)) \xrightarrow{\text{stably}} \mathcal{N}(0, \tilde{\sigma}_6^2)$ $n^{\frac{\gamma_2}{2}}(Z_n - \frac{1}{2}) \xrightarrow{\text{stably}} \mathcal{N}(0, \hat{\sigma}_4^2)$

The following second table holds when $\gamma_* = 1$. The indices i and j are different and belongs to $\{1, 2\}$.

$\lambda_i = 0, \lambda_j > \frac{1}{4}$	$\lambda_i = 0, \lambda_j = \frac{1}{4}$	$\lambda_i = 0, \lambda_j < \frac{1}{4}$
$\sqrt{n}(Z_n - \frac{1}{2}) \xrightarrow{\text{stably}} \mathcal{N}(0, \sigma_3^{*2})$	$\frac{\sqrt{n}}{\sqrt{\ln n}}(Z_n - \frac{1}{2}) \xrightarrow{\text{stably}} \mathcal{N}(0, \sigma_4^{*2})$	$n^{4(\lambda_1 + \lambda_2)}(Z_n - \frac{1}{2}) \xrightarrow{\text{a.s./}L^1} \tilde{X}$

Theorem 5.4. Assume $\lambda_1 = \lambda_2 = 0$. The stable convergence towards some Gaussian kernel holds for the quantities $(Z_n - Z_n(i))_n$ and $(Z_n - \frac{1}{2})_n$ with the following scalings.

(i) It holds

$$n^{\frac{2\gamma_1 - \gamma_2}{2}}(Z_n - Z_n(i)) \xrightarrow[n \rightarrow \infty]{\text{stably}} \mathcal{N}(0, \vartheta Z_\infty(1 - Z_\infty)).$$

(ii) With $\gamma_* := \min(\gamma_1, \gamma_2)$, it holds

$$n^{\frac{2\gamma_* - 1}{2}}(Z_n - Z_\infty) \xrightarrow[n \rightarrow \infty]{\text{stably}} \mathcal{N}(0, \vartheta Z_\infty(1 - Z_\infty)),$$

where ϑ denotes a constant, whose dependency according to N, γ_1, γ_2 is given in the proofs. □

Remark 5.5. (analogous to Theorem 3.2 in [Crimaldi et al. \(2016\)](#)).

We have $\mathbb{P}(Z_\infty = 0) + \mathbb{P}(Z_\infty = 1) < 1$ and $\mathbb{P}(Z_\infty = z) = 0$ for each $z \in (0, 1)$. Indeed, it guarantees that these asymptotic Gaussian kernels are not degenerate.

Proof: The first part immediately follows from the relation $\mathbb{E}[Z_\infty^2] < \mathbb{E}[Z_\infty]$ by Lemma 7.2. The second part is a consequence of the almost sure conditional convergence stated in Th. 5.4 (ii) (for details see proof of Theorem 2.5 in [Crimaldi et al., 2019](#)). \square

6. Stochastic approximation point of view

The recursive equations (1.1) may be written in the following stochastic approximation forms:

$$\begin{aligned} Z_{n+1}(i) &= Z_n(i) + r_n^\ell \lambda_1 (1 - 2Z_n(i)) + r_n^\ell \Delta \hat{M}_{n+1}^\ell(i) \\ &\quad + r_n^g \lambda_2 (1 - 2Z_n(i)) + r_n^g (Z_n - Z_n(i)) + r_n^g \Delta M_{n+1}^g \end{aligned} \quad (6.1)$$

and

$$Z_{n+1} = Z_n + (r_n^\ell \lambda_1 + r_n^g \lambda_2)(1 - 2Z_n) + r_n^\ell \Delta M_{n+1}^\ell + r_n^g \Delta M_{n+1}^g \quad (6.2)$$

where

$$\Delta \hat{M}_{n+1}^\ell(i) := \xi_{n+1}^\ell(i) - \mathbb{E}(\xi_{n+1}^\ell(i) | \mathcal{F}_n), \quad (6.3)$$

$$\Delta M_{n+1}^\ell := \frac{1}{N} \sum_{i=1}^N \Delta \hat{M}_{n+1}^\ell(i), \quad (6.4)$$

$$\Delta M_{n+1}^g := \xi_{n+1}^g - \mathbb{E}(\xi_{n+1}^g | \mathcal{F}_n) \quad (6.5)$$

are martingale differences.

Similarly, it holds for $X_n(i) := Z_n - Z_n(i)$,

$$X_{n+1}(i) = X_n(i) - (2\lambda_1 r_n^\ell + r_n^g) X_n(i) + r_n^\ell (\Delta M_{n+1}^\ell - \Delta \hat{M}_{n+1}^\ell(i)). \quad (6.6)$$

We refer to the general theorems about asymptotic behavior as stated in [Laruelle and Pagès \(2013\)](#); [Lasmar et al. \(2018\)](#); [Laruelle and Pagès \(2019\)](#) and classical references therein like [Dufflo \(1997\)](#); [Benaïm \(1999\)](#). According to the cases either $\gamma_1 \leq \gamma_2$ or $\gamma_2 < \gamma_1$ and $\lambda_i = 0$ or not ($i \in \{1, 2\}$), then system $\mathbf{Z}_n = (Z_n(1), \dots, Z_n(N))^\top$ satisfies the following framework.

Let $\mathbf{Z} = (\mathbf{Z}_n)_{n \geq 0}$ be an N -dimensional stochastic process with values in $[0, 1]^N$, adapted to a filtration $\mathcal{F} = (\mathcal{F}_n)_{n \geq 0}$. Suppose that \mathbf{Z} satisfies

$$\mathbf{Z}_{n+1} = \mathbf{Z}_n + r_n \mathbf{F}(\mathbf{Z}_n) + r_n \Delta \mathbf{M}_{n+1} + r_n \zeta_{n+1}, \quad (6.7)$$

where $(r_n)_n$ is such that (1.5) hold; \mathbf{F} is a bounded \mathcal{C}^1 vector-valued function on an open subset \mathcal{O} of \mathbb{R}^N , with $[0, 1]^N \subset \mathcal{O}$; $(\Delta \mathbf{M}_n)_n$ is a bounded martingale difference with respect to \mathcal{F} ; and $(\zeta_n)_n$ is a $[0, 1]^N$ -valued \mathcal{F}_{n+1} -adapted term such that $\lim_{n \rightarrow \infty} \zeta_n = 0$ a.s. Thus a.s. convergence towards zeros of F gives the a.s. convergence towards $1/2$ when $\lambda_1 + \lambda_2 > 0$ or towards a value belonging to the diagonal $\{\mathbf{z} = (z_1, \dots, z_N) \in [0, 1]^N : \forall i \in \{1, \dots, N\}, z_i = z_1\}$ when $\lambda_1 = \lambda_2 = 0$. The case $\lambda_1 = \lambda_2 = 0$ leads to non isolated zeros of \mathbf{F} which is not a case covered by the general stochastic approximation theorems. The methods developed here, following [Dai Pra et al. \(2014\)](#); [Crimaldi et al. \(2016\)](#) covers all parameters' cases, including the one when $\lambda_1 = \lambda_2 = 0$; and they give L^2 rates. These are useful to prove the scales of fluctuations stated in Section 5 thanks to CLT's w.r.t stable convergence [Crimaldi \(2016\)](#); [Häusler and Luschgy \(2015\)](#).

7. Proof of a.s. synchronization and rates of convergences

This section is devoted to the proofs of Th. 4.1, Th. 4.2, Th. 4.4. As discussed in Section 6 cases need indeed to be distinguished according to the nullity of λ_1, λ_2 ($\lambda_1 + \lambda_2 > 0$ or $\lambda_1 = \lambda_2 = 0$).

7.1. *First results about the variances.* First remark the assumption $\forall i \in \{1, \dots, N\}$, $Z_0(i) = \frac{1}{2}$ leads to $\forall n \in \mathbb{N}$, $\mathbb{E}(Z_n) = \mathbb{E}(Z_n(i)) = \frac{1}{2}$ thanks to (2.1) and (2.2). We then state the following relationships.

Proposition 7.1. *The following recursive equation hold:*

$$\begin{aligned} \mathbb{V}ar(Z_{n+1}) &= [1 - 4(\lambda_1 r_n^\ell + \lambda_2 r_n^g - 2\lambda_1 \lambda_2 r_n^\ell r_n^g - \lambda_1^2 (r_n^\ell)^2 - \lambda_2^2 (r_n^g)^2 + \frac{(r_n^g)^2}{4} (1 - 2\lambda_2)^2)] \mathbb{V}ar(Z_n) \\ &\quad + \frac{(r_n^\ell)^2}{N} [(1 - 2\lambda_1)^2 \left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n^2(i))\right) + \lambda_1 - \lambda_1^2] + \frac{(r_n^g)^2}{4}. \end{aligned} \quad (7.1)$$

Proof: From (1.1), we get

$$Z_{n+1} = (1 - r_n^\ell - r_n^g) Z_n + \frac{r_n^\ell}{N} \sum_i \xi_{n+1}^\ell(i) + r_n^g \xi_{n+1}^g.$$

Using (1.1) and (1.2), we compute:

$$\begin{aligned} \mathbb{V}ar(Z_{n+1} | \mathcal{F}_n) &= \mathbb{V}ar \left[(1 - r_n^\ell - r_n^g) Z_n + \frac{r_n^\ell}{N} \sum_i \xi_{n+1}^\ell(i) + r_n^g \xi_{n+1}^g \middle| \mathcal{F}_n \right] \\ &= \frac{(r_n^\ell)^2}{N^2} \sum_{i=1}^N \mathbb{V}ar(\xi_{n+1}^\ell(i) | \mathcal{F}_n) + (r_n^g)^2 \mathbb{V}ar(\xi_{n+1}^g | \mathcal{F}_n), \end{aligned}$$

then using (2.1), (2.2) and the *law of total variance* (\star), we have

$$\begin{aligned} \mathbb{V}ar(Z_{n+1}) &\stackrel{\star}{=} \mathbb{E}[\mathbb{V}ar(Z_{n+1} | \mathcal{F}_n)] + \mathbb{V}ar[\mathbb{E}(Z_{n+1} | \mathcal{F}_n)] \\ &= \frac{(r_n^\ell)^2}{N} \left[(1 - 2\lambda_1)^2 \left(\mathbb{E}(Z_n) - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n^2(i)) \right) + \lambda_1 - \lambda_1^2 \right] \\ &\quad + (r_n^g)^2 \left[(1 - 2\lambda_2)^2 \left(\mathbb{E}(Z_n) - \mathbb{E}(Z_n^2) \right) + \lambda_2 - \lambda_2^2 \right] + (1 - 2\lambda_1 r_n^\ell - 2\lambda_2 r_n^g)^2 \mathbb{V}ar(Z_n) \end{aligned}$$

and using the fact $Z_0(i) = 1/2$ leading to the result. \square

Lemma 7.2. *When $\lambda_1 = \lambda_2 = 0$, it holds $\lim_{n \rightarrow \infty} \mathbb{V}ar(Z_n) < \frac{1}{4}$. Moreover, $\sup_n \mathbb{E}(Z_n^2) < \frac{1}{2}$.* \square

Remark this implies

$$\lim_{n \rightarrow \infty} \left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n^2(i)) \right) > 0. \quad (7.2)$$

Proof: Since for all i , $\mathbb{E}(Z_n^2(i)) \leq \mathbb{E}(Z_n(i)) = \frac{1}{2}$, it holds obviously $\mathbb{V}ar(Z_n) \leq \frac{1}{4}$. Using (7.1) with $\lambda_1 = \lambda_2 = 0$ gives:

$$\mathbb{V}ar(Z_{n+1}) = \left(1 - (r_n^g)^2 \right) \mathbb{V}ar(Z_n) + \frac{(r_n^g)^2}{4} + \frac{(r_n^\ell)^2}{N} \left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n^2(i)) \right).$$

Then using the fact that

$$\left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n^2(i)) \right) \leq \frac{1}{2} \quad (7.3)$$

we have

$$\mathbb{V}ar(Z_{n+1}) \leq \left(1 - (r_n^g)^2 \right) \mathbb{V}ar(Z_n) + \frac{(r_n^g)^2}{4} + \frac{(r_n^\ell)^2}{2N}.$$

Let $x_n := \mathbb{V}ar(Z_n) - \frac{1}{4} \leq 0$, one gets $x_{n+1} \leq (1 - (r_n^g)^2)x_n + \frac{(r_n^\ell)^2}{2N}$. With $K_n = \frac{1}{2N}$, take $x_{n+1} = (1 - (r_n^g)^2)x_n + K_n(r_n^\ell)^2$. Since $\sum_n (r_n^g)^2 < +\infty$, we obtain $\lim_{n \rightarrow \infty} x_n \neq 0$ by lemma A.2. As claimed, $\lim_{n \rightarrow \infty} \mathbb{V}ar(Z_n) < \frac{1}{4}$, that equivalents to $\lim_n \mathbb{E}(Z_n^2) < \frac{1}{2}$.

Moreover, since $\mathbb{E}(Z_{n+1}^2 | \mathcal{F}_n) = Z_n^2 + \mathbb{V}ar(Z_{n+1} | \mathcal{F}_n)$, it holds $\mathbb{E}(Z_{n+1}^2 | \mathcal{F}_n) \geq Z_n^2$ so, $(Z_n^2)_n$ is a sub-martingale. Consequently, $\sup_n \mathbb{E}(Z_n^2) = \lim_n \mathbb{E}(Z_n^2) < \frac{1}{2}$. \square

To treat the cases when $(\lambda_1, \lambda_2) \neq (0, 0)$, we need the following lemma:

Lemma 7.3. *If $\lambda_1 > 0$ or $\lambda_2 > 0$, then it holds $\lim_{n \rightarrow \infty} \mathbb{V}ar(Z_n) = 0$.*

In particular, $\lim_{n \rightarrow \infty} \mathbb{V}ar(Z_n) < 1/4$ and

$$\lim_{n \rightarrow \infty} \left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n^2(i)) \right) > 0.$$

\square

Proof: We re-write (7.1) as:

$$\mathbb{V}ar(Z_{n+1}) = (1 - 4\varepsilon_n)\mathbb{V}ar(Z_n) + K_n^1(r_n^\ell)^2 + \frac{(r_n^g)^2}{4},$$

where

$$\varepsilon_n := \lambda_1 r_n^\ell + \lambda_2 r_n^g - 2\lambda_1 \lambda_2 r_n^\ell r_n^g - \lambda_1^2 (r_n^\ell)^2 - \lambda_2^2 (r_n^g)^2 + \frac{(r_n^g)^2}{4} (1 - 2\lambda_2)^2,$$

and

$$K_n^1 := \frac{1}{N} \left[(1 - 2\lambda_1)^2 \left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n^2(i)) \right) + \lambda_1 - \lambda_1^2 \right].$$

In all the considered cases, $\sum_n \varepsilon_n = +\infty$. Using (7.3), we get

$$0 \leq K_n^1 \leq \frac{1}{2} (1 - 2\lambda_1)^2 + \lambda_1 - \lambda_1^2 \leq 1.$$

It follows $\mathbb{V}ar(Z_n) \leq y_n$ where $(y_n)_n$ is the sequence defined in the proof of lemma A.1 (see (A.3)) with the same ε_n and $K\delta_n$ is $\frac{1}{N}(r_n^\ell)^2 + \frac{(r_n^g)^2}{4}$. Thus, using Lemma A.1, we get $\lim_{n \rightarrow \infty} \mathbb{V}ar(Z_n) = 0$.

Remark that using the same argument as previously, $(Z_n^2)_{n \geq 0}$ is a sub-martingale. Thus we get $\sup_n \mathbb{E}(Z_n^2) < \frac{1}{2}$ as claimed. \square

7.2. Proofs of L^2 and a.s. convergence. We now prove the theorems of Section 4 about convergence and synchronization.

Proof: Theorem 4.1 (i)

♣ First consider the equation (7.1) summarized as

$$\mathbb{V}ar(Z_{n+1}) = (1 - 4\lambda r_n^* + o(r_n^*))\mathbb{V}ar(Z_n) + K_n(r_n^*)^2 + o((r_n^*)^2),$$

where $\lambda = \begin{cases} \lambda_1 & \text{if } \gamma_1 < \gamma_2 \\ \lambda_2 & \text{if } \gamma_1 > \gamma_2 \end{cases}$. and $r_n^* = \frac{1}{n^{\gamma^*}}$.

We now give the values of A and K_n , depending on $\gamma_1, \gamma_2, \lambda_1$ and λ_2 for which this equation holds, and show how one can apply Lemma A.1 in all different cases.

• **When** $\gamma_1 < \gamma_2$, then $A = 4\lambda_1$ and

$$K_n = \frac{1}{N} \left((1 - 2\lambda_1)^2 \left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n^2(i)) \right) + \lambda_1 - \lambda_1^2 \right)$$

is bounded and $\lim_{n \rightarrow \infty} K_n > 0$. Indeed, since $\mathbb{E}(Z_n^2) < \frac{1}{2}$, we get

$$\sum_{i=1}^N \mathbb{E}(Z_n^2(i)) = \mathbb{E}(Z_n^2(j)) + \sum_{i=1, i \neq j}^{N-1} \mathbb{E}(Z_n^2(i)) < \frac{1}{2} + \frac{N-1}{2} = \frac{N}{2}.$$

By Lemma A.1 we get $\lim_{n \rightarrow \infty} \mathbb{V}ar(Z_n) = 0$. Moreover, by Lemma A.2, it holds $\mathbb{V}ar(Z_n) = \mathcal{O}(\frac{1}{n^{\gamma_1}})$.

• **When** $\gamma_1 > \gamma_2$, it holds $A = 4\lambda_2$ and $K_n = \frac{1}{4}$ thus, by Lemma A.1 it holds $\lim_{n \rightarrow \infty} \mathbb{V}ar(Z_n) = 0$ and $\mathbb{V}ar(Z_n) = \mathcal{O}(\frac{1}{n^{\gamma_2}})$ by Lemma A.2.

• **When** $\gamma_1 = \gamma_2 (= \gamma)$, we have

$$\mathbb{V}ar(Z_{n+1}) = \left(1 - 4(\lambda_1 + \lambda_2)r_n - ((2\lambda_1 + 2\lambda_2)^2 - (1 - 2\lambda_2)^2)r_n^2\right) \mathbb{V}ar(Z_n) + r_n^2 K_n$$

where $A = 4(\lambda_1 + \lambda_2)$ and

$$K_n = \frac{1}{N} \left((1 - 2\lambda_1)^2 \left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n^2(i)) \right) + \lambda_1 - \lambda_1^2 + \frac{N}{4} \right),$$

which is bounded and $\lim_{n \rightarrow \infty} K_n > 0$, which implies by Lemma A.1 $\lim_{n \rightarrow \infty} \mathbb{V}ar(Z_n) = 0$ where by Lemma A.2, it holds $\mathbb{V}ar(Z_n) = \mathcal{O}(\frac{1}{n^\gamma})$. In the case $\gamma = 1$ and $\lambda_1 + \lambda_2 = \frac{1}{4}$, $\mathbb{V}ar(Z_n) = \mathcal{O}(\frac{\log n}{n})$. \square

Proof: Theorem 4.1 (ii)

Consider the following recursive equation satisfied, for any $i \in \{1, \dots, N\}$, by the L^2 -distance between one component and the mean field. For symmetry reasons, the following quantity x_n does not depend on the specific choice of the component i . With

$$x_n := \mathbb{E}[(Z_n(i) - Z_n)^2] = \mathbb{V}ar(Z_n(i) - Z_n),$$

recall \star means law of total variation and it holds

$$\begin{aligned} x_{n+1} &\stackrel{\star}{=} \mathbb{E}(\mathbb{V}ar[(1 - r_n^\ell - r_n^g)(Z_n(i) - Z_n) + r_n^\ell(\xi_n^\ell(i) - \frac{1}{N} \sum_j \xi_n^\ell(j)) | \mathcal{F}_n]) \\ &+ \mathbb{V}ar \left[Z_n(i) - 2\lambda_1 r_n^\ell Z_n(i) + r_n^g(Z_n(i) - (1 - 2\lambda_2)Z_n) - Z_n(1 - 2\lambda_1 r_n^\ell - r_n^g) \right] \\ &= (r_n^\ell)^2 \mathbb{E} \left[\mathbb{V}ar(\xi_n^\ell(i) - \frac{1}{N} \sum_i \xi_n^\ell(i) | \mathcal{F}_n) \right] + \mathbb{V}ar \left((1 - 2\lambda_1 r_n^\ell - r_n^g)(Z_n(i) - Z_n) \right) \\ &= (1 - 2\lambda_1 r_n^\ell - r_n^g)^2 \mathbb{V}ar(Z_n(i) - Z_n) + (r_n^\ell)^2 \left(\left(1 - \frac{1}{N}\right)^2 + \left(\frac{N-1}{N^2}\right) \right) \mathbb{E} \left[\mathbb{V}ar(\xi_n^\ell(i) | \mathcal{F}_n) \right] \\ &= (1 - 2\lambda_1 r_n^\ell - r_n^g)^2 x_n \\ &+ \frac{N-1}{N} (r_n^\ell)^2 \left[(1 - 2\lambda)^2 [\mathbb{E}(Z_n(i)) - \mathbb{E}(Z_n^2(i))] + \lambda_1 - \lambda_1^2 \right] \end{aligned}$$

Therefore we obtain

$$x_{n+1} = \left(1 - 4\lambda_1 r_n^\ell - 2r_n^g + 4\lambda_1^2 (r_n^\ell)^2 + (r_n^g)^2 + 2\lambda_1 r_n^\ell r_n^g\right) x_n + (r_n^\ell)^2 J_n, \quad (7.4)$$

where $J_n = \frac{N-1}{N} \left((1 - 2\lambda_1)^2 \left[\frac{1}{2} - \mathbb{E}(Z_n^2(i)) \right] + \lambda_1 - \lambda_1^2 \right)$ is bounded and not equal zero for $N > 1$.

(a) **When** $\gamma_1 < \gamma_2$ the relation (7.4) gives $x_{n+1} = [1 - 4\lambda_1 r_n^\ell - o(r_n^\ell)] x_n + (r_n^\ell)^2 J_n$. It implies by Lemma A.1 $\lim_{n \rightarrow \infty} x_n = 0$ and it holds, by Lemma A.2, $x_n = \mathcal{O}(\frac{1}{n^{\gamma_1}})$ where $A = 4\lambda_1 c_1$.

When $\gamma_1 = \gamma_2$, we have from (7.4)

$$x_{n+1} = [(1 - r_n - 2\lambda_1 r_n)^2] x_n + r_n^2 J_n = [1 - (2 + 4\lambda_1)r_n + o(r_n)] x_n + r_n^2 J_n$$

which implies by Lemma A.1 $\lim_{n \rightarrow \infty} x_n = 0$ and it holds, by Lemma A.2, $x_n = \mathcal{O}(\frac{1}{n^\gamma})$ where $A = 2 + 4\lambda_1$.

(b) **When** $\gamma_2 < \gamma_1$, $x_{n+1} = (1 - 2r_n^g + o(r_n^g))x_n + J_n(r_n^\ell)^2$ where $A = 2$ implies by Lemma A.1 $\lim_{n \rightarrow \infty} x_n = 0$ and it holds, by Lemma A.2, $x_n = \mathcal{O}(\frac{1}{n^{2\gamma_1 - \gamma_2}})$.

□

Proof: (iii) **Theorem 4.1**

• To prove that, in this case, a.s. convergence holds towards $1/2$, we use (6.2) and consider

$$\begin{aligned} \mathbb{E}[(Z_{n+1} - \frac{1}{2})^2 | \mathcal{F}_n] &= (Z_n - \frac{1}{2})^2 [1 + 4(r_n^\ell)^2 \lambda_1^2 + 4(r_n^g)^2 \lambda_2^2 - 4r_n^\ell \lambda_1 - 4r_n^g \lambda_2 + 4r_n^\ell r_n^g \lambda_1 \lambda_2] \\ &\quad + (r_n^\ell)^2 \mathbb{E}[(\Delta M_{n+1}^\ell)^2 | \mathcal{F}_n] + (r_n^g)^2 \mathbb{E}[(\Delta M_{n+1}^g)^2 | \mathcal{F}_n] \\ &= (Z_n - \frac{1}{2})^2 \left[1 - 4r_n^\ell \lambda_1 - 4r_n^g \lambda_2 + o(r_n^\ell) + o(r_n^g) \right] + (r_n^\ell)^2 W_n^\ell + (r_n^g)^2 W_n^g. \end{aligned}$$

with $W_n^\ell := 4\lambda_1^2 (Z_n - \frac{1}{2})^2 + \mathbb{E}[(\Delta M_{n+1}^\ell)^2 | \mathcal{F}_n]$ and $W_n^g := 4\lambda_2^2 (Z_n - \frac{1}{2})^2 + \mathbb{E}[(\Delta M_{n+1}^g)^2 | \mathcal{F}_n]$.

Thus, $\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2 | \mathcal{F}_n] \leq (Z_n - \frac{1}{2})^2 + (r_n^\ell)^2 W_n^\ell + (r_n^g)^2 W_n^g$. Since $(\gamma_1, \gamma_2) \in (1/2, 1]^2$, we get that $((Z_{n+1} - \frac{1}{2})^2)_n$ is a positive almost super-martingale and a.s. convergence holds. It is enough to consider L^2 convergence in order to identify the (deterministic) limit.

$$\begin{aligned} \mathbb{E}(\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2 | \mathcal{F}_n]) &= \mathbb{E}(Z_n - \frac{1}{2})^2 \left[1 - 4r_n^\ell \lambda_1 - 4r_n^g \lambda_2 + 4r_n^\ell r_n^g \lambda_1 \lambda_2 \right] \\ &\quad + (r_n^\ell)^2 K_n^\ell + (r_n^g)^2 K_n^g. \end{aligned}$$

With $y_n := \mathbb{E}(Z_n - \frac{1}{2})^2$, one gets

$$y_{n+1} = \left(1 - 4r_n^\ell \lambda_1 - 4r_n^g \lambda_2 + \lambda_1^2 (r_n^\ell)^2 + \lambda_2^2 (r_n^g)^2 + 4r_n^\ell r_n^g \lambda_1 \lambda_2 \right) y_n + (r_n^\ell)^2 K_{n+1}^\ell + (r_n^g)^2 K_{n+1}^g \quad (7.5)$$

where $0 < K_{n+1}^\ell := \mathbb{E}[(\Delta M_{n+1}^\ell)^2] \leq 1$, and $0 < K_{n+1}^g := \mathbb{E}[(\Delta M_{n+1}^g)^2] \leq 1$. By lemma A.1 we get $\lim_{n \rightarrow \infty} y_n = 0$.

• When $\gamma_1 = \gamma_2$, the proof holds similarly. Indeed,

$$\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2 | \mathcal{F}_n] = (Z_n - \frac{1}{2})^2 [1 - 2r_n(\lambda_1 + \lambda_2)]^2 + r_n^2 \mathbb{E}[\Delta \tilde{M}_{n+1}(i)^2 | \mathcal{F}_n].$$

Thus, $\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2 | \mathcal{F}_n] \leq (Z_n - \frac{1}{2})^2 + r_n^2 \tilde{W}_n$, where $\tilde{W}_n = 4(\lambda_1 + \lambda_2)^2 (Z_n - \frac{1}{2})^2 + \mathbb{E}[(\Delta \tilde{M}_{n+1}(i))^2 | \mathcal{F}_n]$.

To prove the a.s. synchronization, use (6.6) with $X_n(i) := Z_n(i) - Z_n$ which means in the considered case $\mathbb{E}(X_{n+1}(i) | \mathcal{F}_n) = (1 - 2\lambda_1 r_n^\ell - r_n^g) X_n(i)$. Thus, we obtain $\mathbb{E}(X_{n+1}(i) | \mathcal{F}_n) \leq X_n(i)$ and therefore $(Z_n(i) - Z_n)_n$ is a bounded super-martingale and its a.s. limit exists. □

Proof: **Theorem 4.2**

As expected, we shall consider two different situations of nullity or not for λ_1, λ_2 and different relationships between γ_1 and γ_2 .

The proof of following cases follows along the same arguments as above. We only sketch the proof and list the essential elements below. Then it is easy to get the results.

♣ **Case** $\lambda_1 \neq 0, \lambda_2 = 0$.

Consider the recursive equations (7.1) and (7.4) and letting $\lambda_2 = 0$ satisfied by $\mathbb{V}ar(Z_n)$ and L^2 -distance respectively.

- **Case** $\gamma_1 < \gamma_2$. To study the variance,

$$\mathbb{V}ar(Z_{n+1}) = [1 - 4\lambda_1 r_n^\ell + o(r_n^\ell)]\mathbb{V}ar(Z_n) + K_n (r_n^\ell)^2,$$

where $K_n = \frac{1}{N} \left((1 - 2\lambda_1)^2 \left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n^2(i)) \right) + \lambda_1 - \lambda_1^2 \right)$ and with $A = 4\lambda_1$.

To study the synchronization,

$$x_{n+1} = (1 - 4\lambda_1 r_n^\ell + o(r_n^\ell))x_n + J_n (r_n^\ell)^2,$$

where $J_n = \frac{N-1}{N} \left((1 - 2\lambda_1)^2 \left[\frac{1}{2} - \mathbb{E}(Z_n^2(i)) \right] + \lambda_1 - \lambda_1^2 \right)$ and with $A = 4\lambda_1$.

- **Case** $\gamma_2 < \gamma_1$. To study the variance

$$\mathbb{V}ar(Z_{n+1}) = (1 - 4\lambda_1 r_n^\ell)\mathbb{V}ar(Z_n) + K_n (r_n^g)^2,$$

where $K_n = \frac{1}{4}$ with $A = 4\lambda_1$.

To study the L^2 -distance's behavior, $x_{n+1} = (1 - 2r_n^g + o(r_n^g))x_n + J_n (r_n^\ell)^2$ with $A = 2$.

To prove that a.s. convergence holds towards $1/2$ consider to equation (6.2) and let $\lambda_2 = 0$,

$$\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2 | \mathcal{F}_n] = \left(Z_n - \frac{1}{2} \right)^2 [1 - 4r_n^\ell \lambda_1] + (r_n^\ell)^2 W_n^\ell + (r_n^g)^2 W_n^g$$

where $W_n^\ell := 4\lambda_1^2 \left(Z_n - \frac{1}{2} \right)^2 + \mathbb{E}[(\Delta M_{n+1}^\ell)^2 | \mathcal{F}_n]$ and $W_n^g := \mathbb{E}[(\Delta M_{n+1}^g)^2 | \mathcal{F}_n]$.

- **Case** $\gamma_1 = \gamma_2 (= \gamma)$. To study the variance,

$$\mathbb{V}ar(Z_{n+1}) = (1 - 4\lambda_1 r_n + o(r_n))\mathbb{V}ar(Z_n) + K_n r_n^2,$$

where $K_n = \frac{1}{N} \left((1 - 2\lambda_1)^2 \left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n^2(i)) \right) + \lambda_1 - \lambda_1^2 + \frac{N}{4} \right)$ and with $A = 4\lambda_1$.

To study the L^2 -distance's, $x_{n+1} = (1 - (2 + 4\lambda_1)r_n + o(r_n))x_n + J_n r_n^2$ with $A = (2 + 4\lambda_1)$.

To prove $\lim_{n \rightarrow \infty} Z_n = \frac{1}{2}$ a.s., $\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2 | \mathcal{F}_n] = (Z_n - \frac{1}{2})^2 [1 - 2r_n \lambda_1]^2 + r_n^2 \mathbb{E}[\Delta \tilde{M}_{n+1}(i)^2 | \mathcal{F}_n]$.

Finally, to prove the a.s. synchronization, using (6.6), $\mathbb{E}(X_{n+1}(i) | \mathcal{F}_n) = (1 - 2\lambda_1 r_n^\ell - r_n^g)X_n(i)$.

♣ **Case** $\lambda_1 = 0, \lambda_2 \neq 0$.

- **Case** $\gamma_1 < \gamma_2$. To study the variance,

$$\mathbb{V}ar(Z_{n+1}) = (1 - 4\lambda_2 r_n^g + o(r_n^g))\mathbb{V}ar(Z_n) + K_n (r_n^\ell)^2,$$

where $K_n = \frac{1}{N} \left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n^2(i)) \right)$ and with $A = 4\lambda_2$.

To study the L^2 -distance, $x_{n+1} = (1 - 2r_n^g + o(r_n^g))x_n + J_n (r_n^\ell)^2$, with $A = 2$.

- **Case** $\gamma_2 < \gamma_1$. To study the variance,

$$\mathbb{V}ar(Z_{n+1}) = (1 - 4\lambda_2 r_n^g + o(r_n^g))\mathbb{V}ar(Z_n) + K_n (r_n^g)^2,$$

where $K_n = \frac{1}{4}$ and with $A = 4\lambda_2$.

To study the L^2 -distance, $x_{n+1} = (1 - 2r_n^g)x_n + J_n (r_n^\ell)^2$, where $A = 2$.

To prove $\lim_{n \rightarrow \infty} Z_n = \frac{1}{2}$ a.s.,

$$\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2 | \mathcal{F}_n] = \left(Z_n - \frac{1}{2} \right)^2 [1 - 4r_n^g \lambda_2] + (r_n^\ell)^2 W_n^\ell + (r_n^g)^2 W_n^g.$$

where $W_n^\ell := \mathbb{E}[(\Delta M_{n+1}^\ell)^2 | \mathcal{F}_n]$ and $W_n^g := 4\lambda_2^2 (Z_n - \frac{1}{2})^2 + \mathbb{E}[(\Delta M_{n+1}^g)^2 | \mathcal{F}_n]$.

- **Case** $\gamma_1 = \gamma_2 =: \gamma$. To study the variance

$$\mathbb{V}ar(Z_{n+1}) = (1 - 4\lambda_2 r_n + o(r_n))\mathbb{V}ar(Z_n) + K_n r_n^2,$$

where $K_n = \frac{1}{N} \left(\left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n^2(i)) \right) + \frac{N}{4} \right)$ and with $A = 4\lambda_2$.

To study the L^2 -distance, $x_{n+1} = (1 - 2r_n + o(r_n))x_n + J_n r_n^2$.

To prove $\lim_{n \rightarrow \infty} Z_n = \frac{1}{2}$ a.s. $\mathbb{E} \left[\left(Z_{n+1} - \frac{1}{2} \right)^2 | \mathcal{F}_n \right] = \left(Z_n - \frac{1}{2} \right)^2 [1 - 2r_n \lambda_2]^2 + r_n^2 \mathbb{E}[\Delta \tilde{M}_{n+1}(i)^2 | \mathcal{F}_n]$.

Finally, to prove the a.s. synchronization, $\mathbb{E}(X_{n+1}(i) | \mathcal{F}_n) = (1 - r_n^g) X_n(i)$. \square

Proof: Theorem 4.4 (i)

When $\lambda_1 = \lambda_2 = 0$, $(Z_n)_n$ is a bounded martingale which therefore converges a.s. to a random variable Z_∞ . On the other hand, by Lemma 7.2, $\text{Var}(Z_\infty)_n < \frac{1}{4}$. Remark, it means we do not have $Z_\infty \in \{0, 1\}$ a.s. which is a behavior that may happen with some reinforcements like in reinforced random walks.

Let us use $\gamma_* = \min(\gamma_1, \gamma_2)$ and $r_n^* := \max(r_n^\ell, r_n^g)$,

$$\text{Var}(Z_{n+1}) = \left(1 - \frac{(r_n^g)^2}{4}\right) \text{Var}(Z_n) + K_n (r_n^*)^2,$$

with the following developments.

• **When** $\gamma_1 < \gamma_2$, then $A = 1$ and $K_n = \frac{1}{N} \left[\left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n^2(i)) \right) \right]$ is bounded and not equal zero. Indeed, since $\mathbb{E}(Z_n^2) < \frac{1}{2}$ by Lemma 7.2, we get

$$\sum_{i=1}^N \mathbb{E}(Z_n^2(i)) = \mathbb{E}(Z_n^2(j)) + \sum_{i=1, i \neq j}^{N-1} \mathbb{E}(Z_n^2(i)) < \frac{1}{2} + \frac{N-1}{2} = \frac{N}{2}.$$

Using the first part of Lemma A.2, since $\sum_n (r_n^g)^2 < +\infty$ we get $\text{Var}(Z_n) > 0$.

• **When** $\gamma_2 < \gamma_1$ then, $A = 1$ and $K_n = \frac{1}{4}$, thus, by the first part of Lemma A.2, we get $\text{Var}(Z_n) > 0$.

• **When** $\gamma_1 = \gamma_2 (= \gamma)$,

$$\text{Var}(Z_{n+1}) = (1 - r_n^2) \text{Var}(Z_n) + K_n r_n^2,$$

where $K_n = \frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n^2(i)) + \frac{N}{4}$ which using Lemma A.1 implies $\text{Var}(Z_n) > 0$, where $A = 1$. \square

Proof: Theorem 4.4 (ii)

To study the synchronization phenomenon, we consider the L^2 -distance x_n between $Z_n(i)$ and Z_n .

$$x_{n+1} = (1 - 2r_n^g + (r_n^g)^2)x_n + (r_n^\ell)^2 J_n$$

where $J_n = \frac{N-1}{N} \left(\frac{1}{2} - \frac{1}{N} \mathbb{E}(Z_n^2(i)) \right)$ is bounded and not equal zero for $N > 1$ and then $A = 2$. Thus, by Lemma A.1 it holds $\lim_{n \rightarrow \infty} x_n = 0$ and Lemma A.2 yields $x_n = \mathcal{O}\left(\frac{1}{n^{2\gamma_1 - \gamma_2}}\right)$, meaning in particular that the L^2 -synchronization holds as $n \rightarrow \infty$.

Moreover when $\gamma_1 = \gamma_2$, $x_{n+1} = (1 - 2r_n)x_n + J_n r_n^2$ we get $x_n = \mathcal{O}\left(\frac{1}{n^\gamma}\right)$.

Finally, using (6.6) where $X_n(i) := Z_n(i) - Z_n$, it follows $\mathbb{E}(X_{n+1}(i) | \mathcal{F}_n) = (1 - r_n^g) X_n(i)$. Thus, we get $\mathbb{E}(X_{n+1}(i) | \mathcal{F}_n) \leq X_n(i)$. As bounded super-martingale, $(Z_n(i) - Z_n)_n$ converges a.s. \square

8. Proofs of the CLTs

We now prove the central limit theorems in order to study the scales of the fluctuations. Recall we are using the notation $a_n \simeq b_n$ when $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ exists and is a constant. We will use Th. A.5 in order to prove the CLT's w.r.t. stable convergence.

8.1. *Proofs of the CLTs (Theorem 5.1).* Consider the following definitions. Define $X_k(i) := Z_k - Z_k(i)$. Set $L_0(i) = X_0(i)$ and define

$$L_n(i) := X_n(i) - \sum_{k=0}^{n-1} (\mathbb{E}[X_{k+1}(i)|\mathcal{F}_k] - X_k(i)). \quad (8.1)$$

As (6.6), we get

$$X_{n+1}(i) = [1 - 2\lambda_1 r_n^\ell - r_n^g] X_n(i) + \Delta L_{n+1}(i) \quad (8.2)$$

where $\Delta L_{n+1}(i) := L_{n+1}(i) - L_n(i)$. Note that $(L_n)_n$ is an \mathcal{F} -martingale by construction. Iterating the above relation, we can write

$$X_n(i) = c_{1,n} X_1(i) + \sum_{k=1}^{n-1} c_{k+1,n} \Delta L_{k+1}(i) \quad (8.3)$$

where $c_{n,n} = 1$ and $c_{k,n} = \prod_{h=k}^{n-1} (1 - 2\lambda_1 r_h^\ell - r_h^g)$ for $k < n$.

Proof: Theorem 5.1 (i-a)

Case $\gamma_1 < \gamma_2$. Note that in this case $\gamma_1 \neq 1$ because $(\gamma_1, \gamma_2) \in (1/2, 1]$. We get $\lim_{n \rightarrow \infty} n^{\frac{\gamma_1}{2}} c_{1,n} = 0$ since,

$$\begin{aligned} c_{1,n} &= \prod_{h=1}^{n-1} [1 - 2\lambda_1 r_h^\ell - r_h^g] = \prod_{h=1}^{n-1} \left[1 - \frac{2\lambda_1 c_1}{h^{\gamma_1}} - \frac{c_2}{h^{\gamma_2}} - \mathcal{O}\left(\frac{1}{h^{2\gamma_1}}\right) \right] \\ &= \exp\left[-\sum_{h=1}^{n-1} \left(\frac{2\lambda_1 c_1}{h^{\gamma_1}} - \frac{c_2}{h^{\gamma_2}} + \mathcal{O}(1) \right)\right] \\ &= \mathcal{O}\left(\exp\left[-\frac{2\lambda_1 c_1}{1-\gamma_1} n^{1-\gamma_1} \left(1 - \frac{c_2}{1-\gamma_2} \frac{1-\gamma_1}{2\lambda_1 c_1} \frac{1}{n^{\gamma_2-\gamma_1}}\right)\right]\right) \\ &= \mathcal{O}\left(\exp\left(\frac{-2\lambda_1}{1-\gamma_1} n^{1-\gamma_1}\right)\right). \end{aligned}$$

Therefore, using the same argument as previously for $c_{k,n}$ we obtain

$$\begin{aligned} c_{k,n} &= \prod_{h=k}^{n-1} [1 - 2\lambda_1 r_h^\ell - r_h^g] = \mathcal{O}\left(\exp\left[-\frac{2\lambda_1 c_1}{1-\gamma_1} (n^{1-\gamma_1} - k^{1-\gamma_1}) \left(1 - \frac{c_2}{1-\gamma_2} \frac{1-\gamma_1}{2\lambda_1 c_1} \frac{1}{n^{\gamma_2-\gamma_1}}\right)\right]\right) \\ &= \mathcal{O}\left(\exp\left[\frac{-2\lambda_1}{1-\gamma_1} (n^{1-\gamma_1} - k^{1-\gamma_1})\right]\right) \end{aligned}$$

(agree with Lemma A.3).

It is then enough to prove the convergence $n^{\frac{\gamma_1}{2}} \sum_k c_{k+1,n} \Delta L_{k+1}(i) \rightarrow \mathcal{N}(0, (1 - 1/N)/16\lambda_1)$. First, let us define $U_{n,k} = n^{\frac{\gamma_1}{2}} c_{k+1,n} \Delta L_{k+1}(i)$ and $\mathcal{G}_{n,k} = \mathcal{F}_{k+1}$. Thus $\{U_{n,k}, \mathcal{G}_{n,k} : 1 \leq k \leq n\}$ is a square-integrable martingale difference array.

Indeed we have $\mathbb{E}(U_{n,k}^2) < +\infty$ and $\mathbb{E}(U_{n,k+1} | \mathcal{G}_{n,k}) = n^{\frac{\gamma_1}{2}} c_{k+1,n} \mathbb{E}(\Delta L_{k+1}(i) | \mathcal{F}_{k+1}) = 0$. In order to conclude, we use the Theorem recalled as Th. A.5. We will prove the following three statements for $U_{n,k} := n^{\frac{\gamma_1}{2}} c_{k+1,n} \Delta L_{k+1}(i)$.

- a) $\max_{1 \leq k \leq n} |U_{n,k}| \rightarrow 0$.
- b) $\mathbb{E}[\max_{1 \leq k \leq n} U_{n,k}^2]$ is bounded in n .
- c) $\sum_{k=1}^n U_{n,k}^2 \rightarrow (1 - 1/N)/16\lambda_1$ a.s.

- It holds a) since $\Delta L_{n+1}(i) - (X_{n+1}(i) - X_n(i)) = 2\lambda_1 X_n(i) n^{-\gamma_1}$, $|\Delta L_{n+1}(i)| = \mathcal{O}(n^{-\gamma_1})$.
- To state b), we use a) and

$$\begin{aligned}
\mathbb{E}[\max_{1 \leq k \leq n} U_{n,k}^2] &\leq \mathbb{E}[\sum_{k=1}^n U_{n,k}^2] \\
&= n^{\gamma_1} \sum_{k=1}^n c_{k+1,n}^2 \mathbb{E}[(\Delta L_{k+1}(i))^2] \\
&\simeq n^{\gamma_1} \sum_{k=1}^n e^{\frac{-4\lambda_1}{1-\gamma_1}(n^{1-\gamma_1} - k^{1-\gamma_1})} \mathcal{O}(k^{-2\gamma_1}) \\
&= n^{\gamma_1} e^{\frac{-4\lambda_1}{1-\gamma_1} n^{1-\gamma_1}} \sum_{k=1}^{n-1} e^{\frac{4\lambda_1}{1-\gamma_1} k^{1-\gamma_1}} \mathcal{O}(k^{-2\gamma_1}) + \frac{n^2 \mathcal{O}(n^{-2\gamma_1})}{n}.
\end{aligned}$$

Thus, $\mathbb{E}[\max_{1 \leq k \leq n} U_{n,k}^2]$ is bounded in n .

- Finally, in order to prove c), we have

$$\sum_{k=1}^n U_{n,k}^2 = n^{\gamma_1} \sum_k c_{k+1,n}^2 (\Delta L_{n+1}(i))^2 \simeq n^{\gamma_1} \sum_{k=1}^n \frac{k^{-2\gamma_1} e^{\frac{4\lambda_1}{1-\gamma_1} k^{1-\gamma_1}}}{e^{\frac{4\lambda_1}{1-\gamma_1} n^{1-\gamma_1}}} (\Delta L_{k+1}(i))^2 k^{2\gamma_1}.$$

From a) we obtain

$$\begin{aligned}
\Delta L_{k+1}(i)^2 &= (X_{k+1} - X_k + 2\lambda_1 r_k^\ell X_k)^2 \\
&= [(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))]^2 + 4\lambda_1^2 (r_k^\ell)^2 (Z_k - Z_k(i))^2 \\
&\quad + (r_k^\ell)(Z_k - Z_k(i))[(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))].
\end{aligned}$$

Since $Z_n - Z_n(i) \rightarrow 0$ a.s. and $(r_k^\ell)^2 X_k^2 = \mathcal{O}(k^{-2\gamma_1})$ thus,

$$\sum_{k=1}^n U_{n,k}^2 = n^{\gamma_1} \sum_{k=1}^n c_{k+1,n}^2 [(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))].$$

Let $V_k = k^{2\gamma_1} [(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$ and setting the sequences $b_n := \frac{1}{n^{\gamma_1}} e^{+4\lambda_1 \frac{n^{1-\gamma_1}}{1-\gamma_1}}$ and $a_k := \frac{k^{2\gamma_1}}{c_{k+1,n}^2} e^{-4\lambda_1 \frac{k^{1-\gamma_1}}{1-\gamma_1}}$. Hence, it holds

$$\frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} \xrightarrow{n \rightarrow \infty} \frac{1}{4\lambda_1}. \text{ Indeed,}$$

$$\begin{aligned}
\limsup_n \frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} &= \limsup_n n^{\gamma_1} e^{\frac{-4\lambda_1}{1-\gamma_1} n^{1-\gamma_1}} \int_1^n u^{-2\gamma_1} e^{4\lambda_1 \frac{u^{1-\gamma_1}}{1-\gamma_1}} du \\
&= \limsup_n \left(\frac{1}{4\lambda_1} + \frac{\gamma_1 n^{\gamma_1}}{4\lambda_1} e^{-4\lambda_1} \frac{n^{1-\gamma_1}}{1-\gamma_1} \int_1^n \frac{1}{u^{1+\gamma_1}} e^{4\lambda_1 \frac{u^{1-\gamma_1}}{1-\gamma_1}} du \right) = \frac{1}{4\lambda_1}.
\end{aligned}$$

The same holds for the limit inferior. Then $\lim_n \frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} = \frac{1}{4\lambda_1}$. It implies by Lemma A.4, that $\sum_{k=1}^n U_{n,k}^2$ converges to $\frac{V}{4\lambda_1}$ a.s., where V is (deterministic random variable) defined as $\lim_k \mathbb{E}(V_{k+1} | \mathcal{F}_k) =$

V. Indeed, we compute

$$\begin{aligned}
& \mathbb{E}(k^{2\gamma_1}(Z_{k+1}(i) - Z_k(i))^2 | \mathcal{F}_k) = \\
& k^{2\gamma_1} \left((r_k^\ell)^2 \mathbb{E}((\xi_{k+1}^\ell(i) - Z_k(i))^2 | \mathcal{F}_k) + (r_k^g)^2 \mathbb{E}((\xi_{k+1}^g - Z_k(i))^2 | \mathcal{F}_k) \right. \\
& \quad \left. + 2r_k^\ell r_k^g \mathbb{E} \left((\xi_{k+1}^\ell(i) - Z_k(i))(\xi_{k+1}^g - Z_k(i)) | \mathcal{F}_k \right) \right) \\
& = k^{2\gamma_1} \left((r_k^\ell)^2 (\text{Var}(\xi_{k+1}^\ell(i) | \mathcal{F}_k) + \mathbb{E}((\xi_{k+1}^\ell(i) - Z_k(i))^2 | \mathcal{F}_k)) + (r_k^g)^2 (\text{Var}(\xi_{k+1}^g | \mathcal{F}_k) + \mathbb{E}((\xi_{k+1}^g - Z_k(i))^2 | \mathcal{F}_k)) \right. \\
& \quad \left. + 2r_k^\ell r_k^g \mathbb{E} \left((\xi_{k+1}^\ell(i) - Z_k(i))(\xi_{k+1}^g - Z_k(i)) | \mathcal{F}_k \right) \right)
\end{aligned}$$

which behaves like $k^{2\gamma_1} \left(\frac{(r_k^\ell)^2}{4} + \frac{(r_k^g)^2}{4} \right)$ when $k \rightarrow \infty$. Similarly,

$$\begin{aligned}
\mathbb{E}[k^{2\gamma_1}(Z_{k+1} - Z_k)^2 | \mathcal{F}_k] &= k^{2\gamma_1} \left((r_k^\ell)^2 \mathbb{E} \left[\left(\frac{1}{N} \sum_i \xi_{k+1}^\ell(i) - Z_k \right)^2 | \mathcal{F}_k \right] + (r_k^g)^2 \mathbb{E} \left[(\xi_{k+1}^g - Z_k)^2 | \mathcal{F}_k \right] \right. \\
& \quad \left. + 2r_k^\ell r_k^g \mathbb{E} \left[\left(\frac{1}{N} \sum_i \xi_{k+1}^\ell(i) - Z_k \right) (\xi_{k+1}^g - Z_k) | \mathcal{F}_k \right] \right) \\
& = k^{2\gamma_1} \left((r_k^\ell)^2 \left(\text{Var} \left[\frac{1}{N} \sum_i \xi_{k+1}^\ell(i) | \mathcal{F}_k \right] + \mathbb{E}^2 \left(\frac{1}{N} \sum_i \xi_{k+1}^\ell(i) - Z_k | \mathcal{F}_k \right) \right) \right. \\
& \quad \left. + (r_k^g)^2 \left(\text{Var}[\xi_{k+1}^g | \mathcal{F}_k] + \mathbb{E}^2(\xi_{k+1}^g - Z_k | \mathcal{F}_k) \right) \right. \\
& \quad \left. + 2r_k^\ell r_k^g \mathbb{E} \left[\left(\frac{1}{N} \sum_i \xi_{k+1}^\ell(i) - Z_k \right) (\xi_{k+1}^g - Z_k) | \mathcal{F}_k \right] \right)
\end{aligned}$$

behaves like $k^{2\gamma_1} \left(\frac{(r_k^\ell)^2}{4N} + \frac{(r_k^g)^2}{4} \right)$ when $k \rightarrow \infty$. And it holds

$$\begin{aligned}
& \mathbb{E}[k^{2\gamma_1}(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i)) | \mathcal{F}_k] = \\
& k^{2\gamma_1} \left((r_k^\ell)^2 \mathbb{E} \left[(\xi_{k+1}^\ell(i) - Z_k(i)) \left(\frac{1}{N} \sum_i \xi_{k+1}^\ell(i) - Z_k \right) | \mathcal{F}_k \right] + (r_k^g)^2 \mathbb{E} \left[(\xi_{k+1}^g - Z_k(i)) (\xi_{k+1}^g - Z_k) | \mathcal{F}_k \right] \right)
\end{aligned}$$

which behaves like $k^{2\gamma_1} \left(\frac{(r_k^\ell)^2}{4N} + \frac{(r_k^g)^2}{4} \right)$. It follows

$$\begin{aligned}
\mathbb{E}(V_{k+1} | \mathcal{F}_k) &= k^{2\gamma_1} \left[(r_k^\ell)^2 \left(\text{Var}[\xi_{k+1}^\ell(i) | \mathcal{F}_k] + \text{Var} \left[\frac{1}{N} \sum_i \xi_{k+1}^\ell(i) | \mathcal{F}_k \right] \right) \right. \\
& \quad \left. - 2\mathbb{E} \left[(\xi_{k+1}^\ell(i) - Z_k(i)) \left(\frac{1}{N} \sum_i \xi_{k+1}^\ell(i) - Z_k \right) | \mathcal{F}_k \right] \right] \xrightarrow{a.s.} \frac{1}{4} \left(1 - \frac{1}{N} \right).
\end{aligned}$$

Thus, $V_k \xrightarrow{a.s.} \frac{1}{4} \left(1 - \frac{1}{N} \right)$ and therefore, $\tilde{\sigma}_1^2 = \frac{(1 - \frac{1}{N})}{16\lambda_1}$.

The proof of next parts follows similarly as previously. We sketch the proofs and mention the essential steps below.

- **Case** $\gamma_1 = \gamma_2 (=:\gamma)$.

We obtain $c_{1,n} = \mathcal{O} \left(\exp \left[\frac{-(1+2\lambda_1)}{1-\gamma} n^{1-\gamma} \right] \right)$. Therefore $\lim_{n \rightarrow \infty} n^{\frac{\gamma}{2}} c_{1,n} = 0$.

Let $b_n = \frac{1}{n^\gamma} e^{\frac{2(1+2\lambda_1)}{1-\gamma} n^{1-\gamma}}$, $a_k = \frac{k^{2\gamma}}{c_{1,n}^2} e^{-\frac{2(1+2\lambda_1)}{1-\gamma} k^{1-\gamma}}$ and

$V_k = k^{2\gamma}[(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$. Therefore, $\tilde{\sigma}_1^2 = \frac{(1-\frac{1}{N})}{4(1+2\lambda_1)}$.

• **Case** $\gamma_1 = \gamma_2 = 1$.

We obtain $c_{1,n} := \prod_{h=1}^n [1 - (1 + 2\lambda_1)r_h] = \mathcal{O}(n^{-(1+2\lambda_1)})$. Then $\sqrt{n}c_{1,n} \rightarrow 0$.

Let $b_n := n^{1+4\lambda_1}$, $a_k := k^{-4\lambda_1}$ and

$V_k = k^2[(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$. Therefore, $\tilde{\sigma}_1^2 = \frac{(1-\frac{1}{N})}{2(1+4\lambda_1)}$.

Proof of Theorem 5.1 (i-b)

Since $c_{1,n} = \prod_{h=1}^{n-1} [1 - 2\lambda_1 r_h^\ell - r_h^g] = \mathcal{O}(\exp[\frac{-1}{1-\gamma_2} n^{1-\gamma_2}])$ therefore, $n^{\gamma_1 - \frac{\gamma_2}{2}} c_{1,n} \rightarrow 0$.

Let $b_n := n^{-(2\gamma_1 - \gamma_2)} \exp(\frac{2}{1-\gamma_2} n^{1-\gamma_2})$ and $a_k := k^{2\gamma_1} c_{1,n}^{-2} \exp(-\frac{2}{1-\gamma_2} k^{1-\gamma_2})$ and

$V_k = k^{2\gamma_1}[(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$.

Therefore, $\tilde{\sigma}_2^2 = \frac{1}{8}(1 - \frac{1}{N})$.

Proof of Theorem 5.1 (ii)

• **When** $\gamma_1 < \gamma_2$, let $X_k := Z_k - \frac{1}{2}$ so,

$$L_n = X_n - \sum_{k=0}^{n-1} \left(\mathbb{E}(Z_{k+1} - \frac{1}{2} | \mathcal{F}_k) - (Z_k - \frac{1}{2}) \right) = X_n + 2(\lambda_1 r_n^\ell + \lambda_2 r_n^g) \sum_{k=0}^{n-1} X_k$$

and $X_{n+1} = [1 - 2\lambda_1 r_n^\ell - 2\lambda_2 r_n^g] X_n + \Delta L_{n+1}$. So $c_{1,n} = \mathcal{O}(\exp[\frac{-2\lambda_1}{1-\gamma_1} n^{1-\gamma_1}])$ and therefore $n^{\frac{\gamma_2}{2}} c_{1,n} \rightarrow 0$.

Let $b_n := \frac{1}{n^{\gamma_1}} e^{\frac{4\lambda_1}{1-\gamma_1} n^{1-\gamma_1}}$ and $a_k := \frac{k^{2\gamma_1}}{c_{1,n}^2} e^{\frac{-4\lambda_1}{1-\gamma_1} k^{1-\gamma_1}}$ and $V_k = k^{2\gamma_1} (Z_{k+1} - Z_k)^2$ therefore, $\hat{\sigma}^2 = \frac{1}{16\lambda_1}$.

• **When** $\gamma_2 < \gamma_1$, it holds $c_{1,n} = \mathcal{O}(\exp[\frac{-2\lambda_2}{1-\gamma_2} n^{1-\gamma_2}])$ and therefore $n^{\frac{\gamma_2}{2}} c_{1,n} \rightarrow 0$.

Let $b_n := \frac{1}{n^{\gamma_2}} e^{\frac{4\lambda_2}{1-\gamma_2} n^{1-\gamma_2}}$, $a_k := \frac{k^{2\gamma_2}}{c_{1,n}^2} e^{\frac{-4\lambda_2}{1-\gamma_2} k^{1-\gamma_2}}$ and $V_k = k^{2\gamma_2} (Z_{k+1} - Z_k)^2$ therefore, $\hat{\sigma}^2 = \frac{1}{16\lambda_2}$.

• **When** $\gamma_1 = \gamma_2 (= \gamma)$, it holds $c_{1,n} = \mathcal{O}(\exp[\frac{-2(\lambda_1 + \lambda_2)}{1-\gamma} n^{1-\gamma}])$ and therefore $n^{\frac{\gamma}{2}} c_{1,n} \rightarrow 0$.

Let $b_n = \frac{1}{n^\gamma} e^{\frac{4(\lambda_1 + \lambda_2)}{1-\gamma} n^{1-\gamma}}$, $a_k = \frac{k^{2\gamma}}{c_{1,n}^2} e^{\frac{-4(\lambda_1 + \lambda_2)}{1-\gamma} k^{1-\gamma}}$ and $V_k = k^{2\gamma} (Z_{k+1} - Z_k)^2$.

Therefore, $\hat{\sigma}^2 = \frac{1}{16(\lambda_1 + \lambda_2)}$.

Proof of Theorem 5.1 (iii)

• **When** $\gamma_1 = \gamma_2 = 1$, it holds $c_{1,n} = \prod_{h=1}^n [1 - 2(\lambda_1 + \lambda_2)r_h] = \mathcal{O}(n^{-2(\lambda_1 + \lambda_2)})$. We then consider the following sub-cases.

- When $(\lambda_1 + \lambda_2) > \frac{1}{4}$, $\sqrt{n} c_{1,n} = n^{-2(\lambda_1 + \lambda_2) + \frac{1}{2}} \rightarrow 0$ then we get

$$c_{k,n} = \mathcal{O}\left(\left(\frac{k}{n}\right)^{2(\lambda_1 + \lambda_2)}\right).$$

Moreover, $\sum_k U_{k,n}^2 = n \sum_k \left(\frac{k}{n}\right)^{4(\lambda_1 + \lambda_2)} (\Delta L_{k+1})^2 k^2 k^{-2}$ and therefore using A.4 taking suitable $(a_n)_n$ and $(b_n)_n$, $\frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} \rightarrow \frac{1}{1-4(\lambda_1 + \lambda_2)}$ and thus, $(\Delta L_{n+1})^2 = (Z_{k+1} - Z_k)^2$ then

$\lim_{k \rightarrow \infty} k^2 \mathbb{E}((Z_{k+1} - Z_k)^2 | \mathcal{F}_k) = \frac{1}{4}$ a.s. and therefore, $\sigma_1^{*2} = \frac{1}{4(1-4(\lambda_1 + \lambda_2))}$.

- When $\lambda_1 + \lambda_2 = \frac{1}{4}$, it holds $\sqrt{n}(\log n)^{-\frac{1}{2}} c_{1,n} \rightarrow 0$. So $c_{k,n} = \left(\frac{k}{n}\right)^{\frac{1}{2}}$ and $U_{k,n} = \frac{\sqrt{n}}{\sqrt{\log n}} c_{k+1,n} \Delta L_{k+1}$ and $\sum_{k=1}^n U_{k,n}^2 = \frac{n}{\log n} \sum_{k=1}^n \left(\frac{k}{n}\right) (\Delta L_{k+1})^2 = \frac{1}{\log n} \sum_k \frac{1}{k} k^2 (\Delta L_{k+1})^2$. Using Lemma A.4 leads to conclusion with $\sigma_2^{*2} = \frac{1}{4}$. \square

8.2. *Proofs of the CLTs (Theorem 5.2).* We now prove Theorem 5.2.

Proof: Let us define $\tilde{X}_n := n^{4(\lambda_1 + \lambda_2)}(Z_n - \frac{1}{2})$. Recall we state $c_1 = c_2 = 1$ for simplicity. Since $\mathbb{E}[\tilde{X}_n^2] < \infty$, it is therefore enough to show that $(\tilde{X}_n)_n$ is a quasi-martingale. Indeed, we have

$$\begin{aligned} \sum_k \mathbb{E}(|\mathbb{E}[\tilde{X}_{k+1}|\mathcal{F}_k] - \tilde{X}_k|) &= \sum_k \mathbb{E}\left(\left| \left[\left(1 + \frac{1}{k}\right)^{4(\lambda_1 + \lambda_2)} (1 - 2(\lambda_1 + \lambda_2)r_k) - 1 \right] \tilde{X}_k \right|\right) \\ &= \sum_k \mathcal{O}\left(\frac{1}{k^2}\right) 8(\lambda_1 + \lambda_2)^2 \mathbb{E}(|\tilde{X}_k|) < +\infty. \end{aligned}$$

Moreover, from the computations carried out in the proof of Theorem 5.1, $\mathbb{E}(\tilde{X}_n^2) < +\infty$ and so it converges *a.s* and in L^1 to some real random variable \tilde{X} .

In order to prove that $\mathbb{P}(\tilde{X} \neq 0) > 0$, we will prove that $(\tilde{X}_n^2)_n$ is bounded in L^p for a suitable $p > 1$. Indeed this fact implies that \tilde{X}_n^2 converges to \tilde{X}^2 and so we obtain $\mathbb{E}(\tilde{X}^2) = \lim_n \mathbb{E}(\tilde{X}_n^2) = \lim_n n^{4(\lambda_1 + \lambda_2)} \mathbb{E}(X_n^2) > 0$. To this purpose, we set $p = 1 + \epsilon/2$, with $\epsilon > 0$ and $z_n := \mathbb{E}(|X_n|^{2+\epsilon})$. Using the following recursive equation:

$$X_{n+1} = (1 - 2r_n)Z_n + \frac{r_n}{N} \sum_{i=1}^N \tilde{\xi}_{k+1} - \frac{1}{2}$$

one gets

$$\begin{aligned} z_{n+1} &= \mathbb{E}(|X_n|^{2+\epsilon}) - (2 + \epsilon)r_n 2Z_n \mathbb{E}(|X_n|^{1+\epsilon}) \\ &\quad + (2 + \epsilon)r_n \mathbb{E}\left[|X_n|^{1+\epsilon} \text{sign}(X_n) (X_n) \left(\frac{1}{N} \sum_i \tilde{\xi}_{k+1}(i)\right)\right] + R_n \end{aligned}$$

where $R_n = \mathcal{O}(n^{-2})$. Now since $\mathbb{E}[\frac{1}{N} \sum_i \tilde{\xi}_{n+1}(i)|\mathcal{F}_n] = 2Z_n - 2(\lambda_1 + \lambda_2)(Z_n - \frac{1}{2})$, we have

$$\begin{aligned} z_{n+1} &= \mathbb{E}(|X_n|^{2+\epsilon}) - 2(2 + \epsilon)r_n Z_n \mathbb{E}(|X_n|^{1+\epsilon}) \\ &\quad + (2 + \epsilon)r_n \mathbb{E}[|X_n|^{1+\epsilon} \text{sign}(X_n) (2Z_n - 2(\lambda_1 + \lambda_2))X_n] + R_n \\ &= \mathbb{E}(|X_n|^{2+\epsilon}) - (2 + \epsilon)r_n 2(\lambda_1 + \lambda_2) \mathbb{E}[|X_n|^{1+\epsilon} \text{sign}(X_n) (X_n)X_n] + R_n \\ &= \mathbb{E}(|X_n|^{2+\epsilon}) - (2 + \epsilon)r_n 2(\lambda_1 + \lambda_2) \mathbb{E}\left(|X_n|^{2+\epsilon}\right) + R_n \\ &= \left(1 - 2(\lambda_1 + \lambda_2)(2 + \epsilon)r_n\right)z_n + g(n) \end{aligned}$$

with $g(n) = \mathcal{O}(n^{-2})$. Therefore, we have

$$z_{n+1} = \left(1 - 2(\lambda_1 + \lambda_2)(2 + \epsilon)r_n\right)z_n + g(n).$$

Since, for $\epsilon > 0$ sufficiently small, we have $\alpha(2 + \epsilon) < 1$ and for n large,

$$\begin{aligned} \prod_{k=0}^{n-1} \left(1 - 2(\lambda_1 + \lambda_2)(2 + \epsilon)r_k\right) &= \exp\left[\sum_{k=0}^{n-1} \left(\ln\left(1 - 2(\lambda_1 + \lambda_2)(2 + \epsilon)\frac{c}{k^\gamma} + \mathcal{O}\left(\frac{1}{k^{2\gamma}}\right)\right)\right)\right] \\ &= \mathcal{O}(\exp[-2(\lambda_1 + \lambda_2)(2 + \epsilon) \ln n]) \\ &= \mathcal{O}(n^{-2(\lambda_1 + \lambda_2)(2 + \epsilon)}). \end{aligned}$$

Thus,

$$\mathbb{E}[|X_n|^{(2+\epsilon)}] = \mathcal{O}\left(\frac{1}{n^{2(\lambda_1 + \lambda_2)(2 + \epsilon)}}\right)$$

which it implies that \tilde{X}^2 is bounded in $L^{1+\frac{\epsilon}{2}}$. \square

8.3. Proofs of the CLTs from Theorem 5.3.

Proof: We organize the proof in two main cases according to nullity of λ_1 and λ_2 .

♣ Case $\lambda_1 \neq 0, \lambda_2 = 0$.

In order to study the evolution of $X_n(i) := (Z_n - Z_n(i))$, we consider two sub-cases.

- **When** $\gamma_1 \leq \gamma_2$, $X_{n+1}(i) = (1 - 2\lambda_1 r_n^\ell)X_n(i) + \Delta L_{n+1}(i)$ and the proof follows like the part (i)(a) of Theorem 5.1 with $\tilde{\sigma}_3^2 = \frac{(1-\frac{1}{N})}{16\lambda_1}$ when $\gamma_1 < \gamma_2$, $\tilde{\sigma}_3^2 = \frac{(1-\frac{1}{N})}{4(1+2\lambda_1)}$ when $\gamma_1 = \gamma_2$ (denoted by γ) and $\tilde{\sigma}_3^2 = \frac{(1-\frac{1}{N})}{2(1+4\lambda_1)}$ when $\gamma_1 = \gamma_2 = 1$.

- **When** $\gamma_2 < \gamma_1$, $X_{n+1}(i) = (1 - r_n^g)X_n(i) + \Delta L_{n+1}(i)$, then the proof follows like part (i-b) of Theorem 5.1 with $\tilde{\sigma}_5^2 = \frac{1}{8}(1 - \frac{1}{N})$.

We further consider $X_n := (Z_n - \frac{1}{2})_n$.

- When $\gamma_1 \leq \gamma_2$, $X_{n+1} = (1 - r_n^g)X_n + \Delta L_{n+1}$, then the proof follows in a similar way as the part (ii) of Theorem 5.1 with $\hat{\sigma}_1^2 = \frac{1}{16\lambda_1}$ when $\gamma_1 < \gamma_2$ and $\hat{\sigma}_1^2 = \frac{1}{16(\lambda_1 + \lambda_2)}$ when $\gamma_1 = \gamma_2 = \gamma$.

- When $\gamma_2 < \gamma_1$, the proof follows along the same lines as previously. We sketch the essential arguments in the following. We have

$$X_{n+1} = (1 - 2\lambda_1 r_n^\ell)X_n + \Delta L_{n+1}.$$

therefore, $c_{1,n} = \mathcal{O}(e^{\frac{-2\lambda_1}{1-\gamma_1}n^{1-\gamma_1}})$ and thus, $n^{\gamma_2 - \frac{\gamma_1}{2}}c_{1,n} \rightarrow 0$. Following the same steps as in the previous proof. Only showing that

$$\sum_{k=1}^n U_{n,k}^2 = n^{2\gamma_2 - \gamma_1} e^{\frac{-4\lambda_1}{1-\gamma_1}n^{1-\gamma_1}} \sum_{k=1}^n k^{-2\gamma_2} e^{\frac{4\lambda_1 k^{1-\gamma_1}}{1-\gamma_1}} k^{2\gamma_2} (\Delta L_{k+1})^2$$

goes to a constant with $\lim_{n \rightarrow \infty} \frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} = \frac{1}{4\lambda_1}$ and $k^{2\gamma_2} (\Delta L_{k+1})^2 \simeq k^{2\gamma_2} (Z_{k+1} - Z_k)^2 \simeq \frac{1}{4}$. Therefore, $\hat{\sigma}_3^2 = \frac{1}{16\lambda_1}$.

♣ Case $\lambda_1 = 0, \lambda_2 \neq 0$.

Concerning the evolution of $(Z_n - Z_n(i))$, for both cases $\gamma_1 \leq \gamma_2$ and $\gamma_2 < \gamma_1$, it is proved analogously as part (i)(b) of Theorem 5.1 with $\tilde{\sigma}_4^2 = \tilde{\sigma}_6^2 = \frac{1}{8}(1 - \frac{1}{N})$.

We now consider $X_n := (Z_n - \frac{1}{2})_n$.

- **When** $\gamma_1 \leq \gamma_2$, the proof follows in a similar way. We sketch essential arguments below. We have

$$X_{n+1} = (1 - 2\lambda_2 r_n^g)X_n + \Delta L_{n+1},$$

therefore it holds $c_{1,n} = \mathcal{O}(e^{\frac{-2\lambda_2}{1-\gamma_2}n^{1-\gamma_2}})$ and thus, $n^{\gamma_1 - \frac{\gamma_2}{2}}c_{1,n} \rightarrow 0$. It is enough to show that

$$\sum_{k=1}^n U_{n,k}^2 = n^{2\gamma_1 - \gamma_2} e^{\frac{-4\lambda_2}{1-\gamma_2}n^{1-\gamma_2}} \sum_{k=1}^n k^{-2\gamma_1} e^{\frac{4\lambda_2 k^{1-\gamma_2}}{1-\gamma_2}} k^{2\gamma_1} (\Delta L_{k+1})^2.$$

tends to a constant with $\lim_n \frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} = \frac{1}{4\lambda_2}$ and $k^{2\gamma_1} (\Delta L_{k+1})^2 \simeq k^{2\gamma_1} (Z_{k+1} - Z_k)^2 \simeq \frac{1}{4}$. Therefore, $\hat{\sigma}_2^2 = \frac{1}{16\lambda_2}$.

The proof when $\gamma_2 < \gamma_1$ follows as in part (ii) of Theorem 5.1 with $\hat{\sigma}_4^2 = \frac{1}{16\lambda_2}$.

• **The case** $\gamma_1 = \gamma_2 = 1$ is proven similarly as in part (iii) Theorem 5.1 with $\sigma_3^{*2} = \frac{1}{4(1-4(\lambda_1+\lambda_2))}$ when $\lambda_1 + \lambda_2 > \frac{1}{4}$, $\sigma_4^{*2} = \frac{1}{4}$ when $\lambda_1 + \lambda_2 = \frac{1}{4}$ and Theorem 5.2 when $\lambda_1 + \lambda_2 < \frac{1}{4}$. \square

8.4. Proofs of the CLTs from Theorem 5.4.

Proof:

Proof of Theorem 5.4 (i)

• **Case** $\gamma_1 \neq \gamma_2$. Define $X_k(i) := Z_k - Z_k(i)$. Set $L_0(i) = X_0(i)$ and let us rewrite

$$\begin{aligned} L_n(i) &= X_n(i) - \sum_{k=0}^{n-1} ([1 - r_k^g](Z_k - Z_k(i)) - (Z_k - Z_k(i))) \\ &= X_n(i) + \sum_{k=0}^{n-1} r_k^g X_k(i). \end{aligned}$$

Then $X_{n+1}(i) = [1 - r_n^g]X_n(i) + \Delta L_{n+1}(i)$. Note that $(L_n(i))_n$ is an \mathcal{F} -martingale by construction. Iterating the above relation, we can write $X_n(i) = c_{1,n}X_1(i) + \sum_{k=1}^n c_{k+1,n}\Delta L_{n+1}(i)$ where $c_{n+1,n} = 1$ and $c_{k,n} = \prod_{h=k}^n [1 - r_h^g]$ for $k \leq n$. It holds $c_{1,n} = \prod_{h=1}^n [1 - r_h^g] = \mathcal{O}(\exp[\frac{-1}{1-\gamma_2}n^{1-\gamma_2}])$.

Then $n^{\gamma_1 - \frac{\gamma_2}{2}}c_{1,n} \rightarrow 0$ and $c_{k,n} = \mathcal{O}(\exp[\frac{-1}{1-\gamma_2}(n^{1-\gamma_2} - k^{1-\gamma_2})])$.

So it is enough to prove that $n^{\gamma_1 - \frac{\gamma_2}{2}} \sum_k c_{k+1,n} \Delta L_{n+1}(i) \rightarrow \mathcal{N}\left(0, (1 - 1/N)(Z_\infty - Z_\infty^2)\right)$.

Again, this can be proved using Theorem A.5 for $U_{n,k+1} = n^{\gamma_1 - \frac{\gamma_2}{2}} \sum_k c_{k+1,n} \Delta L_{n+1}(i)$ and proving a), b) and c). It is easy to check that conditions a) and b) hold. Let us now consider 3). We have

$$\sum_{k=1}^n U_{n,k}^2 = n^{2\gamma_1 - \gamma_2} \sum_k c_{k+1,n}^2 (\Delta L_{n+1}(i))^2 \simeq n^{2\gamma_1 - \gamma_2} \sum_{k=1}^n \frac{k^{-2\gamma_1} e^{\frac{1}{1-\gamma_2}k^{1-\gamma_2}}}{e^{\frac{1}{1-\gamma_2}n^{1-\gamma_2}}} (\Delta L_{n+1}(i))^2 k^{2\gamma_1}.$$

From 1) we obtain

$$\begin{aligned} (\Delta L_{n+1}(i))^2 &\simeq [(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))]^2 + (r_k^g)^2 (Z_k - Z_k(i))^2 \\ &\quad + (r_k^g)^2 (Z_k - Z_k(i)) [(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))]. \end{aligned}$$

Since $Z_n - Z_n(i) \xrightarrow{a.s.} 0$ and $(r_k^g)^2 X_k(i)^2 = \mathcal{O}(k^{-2\gamma_2})$ so,

$$\sum_{k=1}^n U_{n,k}^2 = n^{2\gamma_1 - \gamma_2} \sum_{k=1}^n c_{k+1,n}^2 [(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$$

where we use Lemma A.4 with $b_n := \frac{1}{n^{2\gamma_1 - \gamma_2}} e^{\frac{2}{1-\gamma_2}n^{1-\gamma_2}}$ and $a_k := \frac{k^{2\gamma_1}}{c_{1,n}^2} e^{-\frac{2}{1-\gamma_2}k^{1-\gamma_2}}$.

Let $V_k = k^{2\gamma_1} [(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$. Thus $\lim_{n \rightarrow \infty} \frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} = \frac{1}{2}$. This implies that $\sum_{k=1}^n U_{n,k}^2$ converges to $\frac{V}{2}$ a.s., where V is such that $\lim_{k \rightarrow \infty} \mathbb{E}(V_{k+1} | \mathcal{F}_k) = V$. Indeed,

$$\begin{aligned} \mathbb{E}(k^{2\gamma_1} (Z_{k+1}(i) - Z_k(i))^2 | \mathcal{F}_k) &= k^{2\gamma_1} (r_k^\ell)^2 \mathbb{E}[(\xi_{k+1}^\ell(i) - Z_k(i))^2 | \mathcal{F}_k] \\ &= k^{2\gamma_1} (r_k^\ell)^2 \text{Var}[\xi_{k+1}^\ell(i) | \mathcal{F}_k] \\ &= k^{2\gamma_1} (r_k^\ell)^2 (Z_k - Z_k^2) \xrightarrow{a.s.} Z_\infty - Z_\infty^2. \end{aligned}$$

Similarly, $\mathbb{E}(k^{2\gamma_1}(Z_{k+1} - Z_k)^2 | \mathcal{F}_k) \xrightarrow{a.s.} Z_\infty - Z_\infty^2$, and

$$\mathbb{E}(k^{2\gamma_1}(Z_{k+1}(i) - Z_k(i))(Z_{k+1} - Z_k) | \mathcal{F}_k) \xrightarrow{a.s.} \frac{Z_\infty - Z_\infty^2}{N}.$$

Thus, $\lim_{k \rightarrow \infty} U_k^2 = \vartheta 2(1 - \frac{1}{N})(Z_\infty - Z_\infty^2)$ a.s. where $\vartheta = \frac{1}{2}$.

• **Case** $\gamma_1 = \gamma_2 (= \gamma)$. We get $c_{1,n} = \prod_{h=1}^n [1 - r_h] = \mathcal{O}(\exp[\frac{-1}{1-\gamma} n^{1-\gamma}])$. Then $n^{\frac{\gamma}{2}} c_{1,n} \rightarrow 0$. Moreover $\lim_{n \rightarrow \infty} \frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} = \frac{1}{2}$ and $V_k = k^{2\gamma}[(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$.

Thus, $\lim_{k \rightarrow \infty} U_k^2 = \vartheta 4(1 - \frac{1}{N})(Z_\infty - Z_\infty^2)$ a.s. where $\vartheta = \frac{1}{2}$.

• **Case** $\gamma_1 = \gamma_2 = 1$. $c_{1,n} = \prod_{h=1}^n [1 - r_h] = \mathcal{O}(n^{-1})$. Then $\sqrt{n} c_{1,n} \rightarrow 0$. Choosing $b_n := n$ and $a_k := 1$ and let $V_k = k^2[(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$. Thus, $\lim_{k \rightarrow \infty} U_k^2 = \vartheta 4(1 - \frac{1}{N})(Z_\infty - Z_\infty^2)$ a.s. where $\vartheta = 1$.

Proof of Theorem 5.4 (ii)

• **Case** $\gamma_1 < \gamma_2$. The process $(Z_n)_n$ is a (bounded) martingale. Therefore $(Z_n)_n$ converges a.s. We want to prove the following two statements

- (1) $\mathbb{E} \left[\sup_k k^{\gamma_1 - \frac{1}{2}} |Z_{k+1} - Z_k| \right] < +\infty$;
- (2) $n^{2\gamma_1 - 1} \sum_{k \geq n} (Z_{k+1} - Z_k)^2 \xrightarrow{a.s.} \frac{1}{N(2\gamma_1 - 1)} (Z_\infty - Z_\infty^2)$.

Indeed, the first condition immediately follows from

$$|Z_{k+1} - Z_k| = |r_n^\ell \left(\frac{1}{N} \sum_i \xi_{k+1}^\ell(i) - Z_k \right) + r_n^g (\xi_{k+1}^g - Z_k)| = \mathcal{O}(k^{-\gamma_1}).$$

Concerning the condition 2), we observe that

$$n^{2\gamma_1 - 1} \sum_{k \geq n} (Z_{k+1} - Z_k)^2 = n^{2\gamma_1 - 1} \sum_{k \geq n} k^{-2\gamma_1} (r_k^\ell)^2 \left(\frac{\sum_i \xi_{k+1}^\ell(i)}{N} - Z_k \right)^2 k^{2\gamma_1}$$

and so the desired convergence follows by lemma A.4 with $a_k := k^{-2\gamma_1 + 2}$, $b_n := n^{2\gamma_1 - 1}$ and $U_k = k^{2\gamma_1} (r_k^\ell)^2 \left(\frac{\sum_k \xi_{k+1}^\ell(i)}{N} - Z_k \right)^2$, $\lim_{n \rightarrow \infty} b_n \sum_{k \geq n} \frac{1}{a_k b_k^2} = -\frac{1}{1 - 2\gamma_1}$ so,

$$\mathbb{E} \left(\left(\frac{\sum_k \xi_{k+1}^\ell(i)}{N} - Z_k \right)^2 | \mathcal{F} \right) = \text{Var} \left(\frac{\sum_k \xi_{k+1}^\ell(i)}{N} | \mathcal{F}_n \right) = \frac{1}{N} (Z_\infty - Z_\infty^2).$$

Finally, we take $\vartheta = \frac{1}{(2\gamma_1 - 1)}$.

• **Case** $\gamma_2 < \gamma_1$. We want to prove the following two statements

- 1) $\mathbb{E} \left(\sup_k k^{\gamma_2 - \frac{1}{2}} |Z_{k+1} - Z_k| \right) < +\infty$;
- 2) $n^{2\gamma_2 - 1} \sum_{k \geq n} (Z_{k+1} - Z_k)^2 \xrightarrow{a.s.} \frac{1}{(2\gamma_2 - 1)} (Z_\infty - Z_\infty^2)$.

The first result immediately follows from

$$|Z_{k+1} - Z_k| = |r_n^\ell \left(\frac{1}{N} \sum_i \xi_{k+1}^\ell(i) - Z_k \right) + r_n^g (\xi_{k+1}^g - Z_k)| = \mathcal{O}(k^{-\gamma_2}).$$

To prove the second point, we observe that

$$n^{2\gamma_2 - 1} \sum_{k \geq n} (Z_{k+1} - Z_k)^2 = n^{2\gamma_2 - 1} \sum_{k \geq n} (r_k^\ell)^2 k^{-2\gamma_2} \left(\frac{\sum_{i=1}^N \xi_{k+1}^\ell(i)}{N} - Z_k \right)^2 k^{2\gamma_2}$$

and the desired convergence follows by lemma A.4 with $a_k := k^{2\gamma_2+2}$, $b_n := n^{2\gamma_2-1}$ and $U_k = k^{2\gamma_2} (r_k^g)^2 (\xi_{k+1}^g - Z_k)^2$, $\lim_{n \rightarrow \infty} b_n \sum_{k \geq n} \frac{1}{a_k b_k^2} = -\frac{1}{1-2\gamma_2}$ and

$$\mathbb{E}(\xi_{k+1}^g(i) - Z_k)^2 | \mathcal{F} = \text{Var}(\xi_{k+1}^g | \mathcal{F}) = (Z_\infty - Z_\infty^2).$$

Finally, we take $\vartheta = \frac{1}{(2\gamma_2-1)}$.

- **Case** $\gamma_1 = \gamma_2 (=:\gamma)$. The process $(Z_n)_n$ is a martingale and converges a.s. Indeed,

$$\mathbb{E}(Z_{n+1} | \mathcal{F}_n) = (1 - 2r_n)Z_n + r_n \mathbb{E}\left(\frac{\sum_{i=1}^N \tilde{\xi}_{n+1}(i)}{N} | \mathcal{F}_n\right) = Z_n.$$

We want to check the following two conditions:

- 1) $\mathbb{E}\left[\sup_k k^{\gamma-\frac{1}{2}} |Z_{k+1} - Z_k|\right] < +\infty$;
- 2) $n^{2\gamma-1} \sum_{k \geq n} (Z_{k+1} - Z_k)^2 \xrightarrow{a.s.} \frac{2}{N(2\gamma-1)} (Z_\infty - Z_\infty^2)$.

The first result follows from

$$|Z_{k+1} - Z_k| = \left| r_n \left(\frac{1}{N} \sum_{i=1}^N \tilde{\xi}_{k+1}(i) - 2Z_k \right) \right| = \mathcal{O}(k^{-\gamma}).$$

And for the second point, we observe that

$$n^{2\gamma-1} \sum_{k \geq n} (Z_{k+1} - Z_k)^2 = n^{2\gamma-1} \sum_{k \geq n} r_k^2 k^{-2\gamma} \left(\frac{\sum_i \tilde{\xi}_{k+1}(i)}{N} - Z_k \right)^2 k^{2\gamma}$$

and so the desired convergence follows by lemma A.5 with $a_k := k^{-2\gamma+2}$, $b_n := n^{2\gamma-1}$ and

$$U_k = k^{2\gamma} r_k^2 \left(\frac{\sum_i \tilde{\xi}_{k+1}(i)}{N} - 2Z_k \right)^2,$$

$\lim_{n \rightarrow \infty} b_n \sum_{k \geq n} \frac{1}{a_k b_k^2} = -\frac{1}{1-2\gamma}$, and $\mathbb{E}\left[\left(\frac{\sum_{i=1}^N \tilde{\xi}_{k+1}(i)}{N} - 2Z_k\right)^2 | \mathcal{F}_k\right] = \frac{2}{N} (Z_\infty - Z_\infty^2)$. Finally, we have $\vartheta = \frac{1}{(2\gamma-1)}$.

- **Case** $\gamma_1 = \gamma_2 = 1$. As usual, we prove

- 1) $\mathbb{E}\left[\sup_k k^{\frac{1}{2}} |Z_{k+1} - Z_k|\right] < +\infty$;
- 2) $n \sum_{k \geq n} (Z_{k+1} - Z_k)^2 \xrightarrow{a.s.} \frac{2}{N} (Z_\infty - Z_\infty^2)$.

First result follows from

$$|Z_{k+1} - Z_k| = \left| r_n \left(\frac{1}{N} \sum_{i=1}^N \tilde{\xi}_{k+1}(i) - 2Z_k \right) \right| = \mathcal{O}(k^{-1}).$$

Second result comes from

$$n \sum_{k \geq n} (Z_{k+1} - Z_k)^2 = n \sum_{k \geq n} r_k^2 k^{-2} k^2 \left(\frac{\sum_i \tilde{\xi}_{k+1}(i)}{N} - Z_k \right)^2$$

and the desired convergence follows then by lemma A.4 with $a_k := 1$, $b_n := n$ and

$$U_k = k^2 r_k^2 \left(\frac{\sum_i \tilde{\xi}_{k+1}(i)}{N} - 2Z_k \right)^2, \lim_{n \rightarrow \infty} b_n \sum_{k \geq n} \frac{1}{a_k b_k^2} = 1.$$

Moreover, $\mathbb{E}\left(\frac{\sum_i \tilde{\xi}_{k+1}(i)}{N} - 2Z_k\right)^2 | \mathcal{F} = \frac{2}{N} Z_\infty (1 - Z_\infty)$. Finally, we have $\vartheta = 1$. \square

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Appendix A. Appendix

In this section, we prove and recall some technical results. The following Lemma is adapted from [Crimaldi et al. \(2019\)](#) to the more general cases considered in this work. It is used with $\varepsilon_n = ar_n^\ell$ or $\varepsilon_n = ar_n^g$ and $\delta_n = (r_n^\ell)^2$ or $\delta_n = (r_n^g)^2$.

Lemma A.1. *Let (x_n) be a sequence of positive such that following equation holds:*

$$x_{n+1} = (1 - \varepsilon_n)x_n + K_n\delta_n \quad (\text{A.1})$$

where $a > 0$, $r_n \geq 0$ and $0 \leq K_n \leq K$. Assume that $(\varepsilon_n)_n$ and $(\delta_n)_n$ are positive sequences of reals

$$\sum_n \varepsilon_n = +\infty, \quad \sum_n \varepsilon_n^2 < +\infty, \quad \text{and} \quad \sum_n \delta_n < +\infty. \quad (\text{A.2})$$

Then $\lim_{n \rightarrow +\infty} x_n = 0$. □

Proof. The case $K = 0$ is well-known. We will prove the statement when $K > 0$. Let m_0 be such that $\varepsilon_n < 1$ for all $n \geq m_0$. Then for $n \geq m_0$ we have $x_n \leq y_n$, where

$$\begin{cases} y_{n+1} = (1 - \varepsilon_n)y_n + K\delta_n \\ y_\ell = x_\ell \end{cases} \quad (\text{A.3})$$

It holds

$$y_n = y_\ell \prod_{i=\ell}^{n-1} (1 - \varepsilon_i) + K \sum_{i=\ell}^{n-1} \delta_i \prod_{j=i+1}^{n-1} (1 - \varepsilon_j). \quad (\text{A.4})$$

Using the assumptions (A.2) about $(\varepsilon_n)_n$, it follows that

$$\prod_{i=\ell}^{n-1} (1 - \varepsilon_i) \longrightarrow 0.$$

Moreover, for every $m \geq m_0$,

$$\begin{aligned} \sum_{i=\ell}^{n-1} \delta_i \prod_{j=i+1}^{n-1} (1 - \varepsilon_j) &= \sum_{i=\ell}^{m-1} \delta_i \prod_{j=i+1}^{n-1} (1 - \varepsilon_j) + \sum_{i=m}^{n-1} \delta_i \prod_{j=i+1}^{n-1} (1 - \varepsilon_j) \\ &\leq \prod_{j=m}^{n-1} (1 - \varepsilon_j) \sum_{i=\ell}^{m-1} \delta_i + \sum_{i=m}^{+\infty} \delta_i. \end{aligned} \quad (\text{A.5})$$

Using that $\prod_{j=m}^{n-1} (1 - \varepsilon_j) \longrightarrow 0$ and that $\sum_n \delta_n < +\infty$, letting first $n \rightarrow +\infty$ and then $m \rightarrow +\infty$ in (A.5), the conclusion follows. □

We now present an extended version of the previous result, stating the rate of convergence. Following Lemma is adapted from [Crimaldi et al. \(2019\)](#). This is in agreement with [Aletti et al. \(2020, Lemma A.1\)](#), [Aletti et al. \(2019, Lemma A.1\)](#) given here as Lemma A.3 for completeness.

Lemma A.2. *Let $(z_n)_n$ be a sequence of positive reals satisfying the following equation:*

$$z_{n+1} = (1 - A\varepsilon_n)z_n + K_n\delta_n, \quad (\text{A.6})$$

where $A > 0$ and $\forall n \in \mathbb{N}$, $0 < K_n \leq K$. Assume that $(\varepsilon_n)_n$ and $(\delta_n)_n$ are positive sequences of reals

$$\sum_n \varepsilon_n^2 < +\infty \quad \text{and} \quad \sum_n \delta_n < +\infty$$

Then it holds,

$$\lim_{n \rightarrow +\infty} z_n = 0 \Leftrightarrow \sum_n \varepsilon_n = +\infty.$$

In particular, assuming $\liminf_n K_n > 0$ and

$$\varepsilon_n = \frac{c_1}{n^{\kappa_1}} + \mathcal{O}\left(\frac{1}{n^{2\kappa_1}}\right),$$

$$\lim_n n^{\kappa_2} \delta_n = c_2 > 0$$

where $\frac{1}{2} < \kappa_1 \leq 1 < \kappa_2$ then, we get that

$$z_n = \begin{cases} \mathcal{O}\left(\frac{1}{n^{\kappa_2 - \kappa_1}}\right) & \text{if } \kappa_1 < 1, \\ \mathcal{O}\left(\frac{\log n}{n^A}\right) & \text{if } \kappa_1 = 1 \quad \text{and} \quad \kappa_2 - A = 1, \\ \mathcal{O}\left(\frac{1}{n^{\kappa_2 - 1}}\right) & \text{if } \kappa_1 = 1 \quad \text{and} \quad \kappa_2 - A < 1, \\ \mathcal{O}\left(\frac{1}{n^A}\right) & \text{if } \kappa_1 = 1 \quad \text{and} \quad \kappa_2 - A > 1. \end{cases}$$

□

Proof: The case $K = 0$ is well-known and we will prove the statement $K > 0$. Let ℓ be such that $A\varepsilon_n < 1$ for all $n \geq \ell$. Then for $n \geq \ell$ we have $z_n \leq y_n$, where

$$\begin{cases} y_{n+1} = (1 - A\varepsilon_n)y_n + K\delta_n, \\ y_\ell = z_\ell. \end{cases}$$

Set $\varepsilon'_n = A\varepsilon_n$ and $\delta'_n = K\delta_n$. It holds

$$y_n = y_\ell \prod_{h=\ell}^{n-1} (1 - \varepsilon'_h) + \sum_{h=\ell}^{n-1} \delta'_h \prod_{k=h+1}^{n-1} (1 - \varepsilon'_k).$$

Since $\sum_n \varepsilon_n = +\infty$, then $\lim_{n \rightarrow \infty} \prod_{h=\ell}^{n-1} (1 - \varepsilon'_h) = 0$. Moreover, for every $m \geq \ell$,

$$\begin{aligned} \sum_{h=\ell}^{n-1} \delta'_h \prod_{k=h+1}^{n-1} (1 - \varepsilon'_k) &= \sum_{h=\ell}^{m-1} \delta'_h \prod_{k=h+1}^{n-1} (1 - \varepsilon'_k) + \sum_{h=m}^{n-1} \delta'_h \prod_{k=h+1}^{n-1} (1 - \varepsilon'_k) \\ &\leq \prod_{k=m}^{n-1} (1 - \varepsilon'_k) \sum_{h=\ell}^{m-1} \delta'_h + \sum_{h=m}^{+\infty} \delta'_h. \end{aligned}$$

Using the fact that $\prod_{k=m}^{n-1} (1 - \varepsilon'_k) \rightarrow 0$ and that $\sum_n \delta_n < +\infty$, letting first $n \rightarrow +\infty$ and then $m \rightarrow +\infty$, the conclusion follows. We are left to prove if $\sum_n \varepsilon_n < +\infty$ then $\lim_n z_n \neq 0$.

From (A.6) we have

$$z_{n+1} \geq (1 - \varepsilon'_n)z_n$$

from which it follows

$$z_n \geq z_0 \prod_{k=0}^{n-1} (1 - \varepsilon'_k).$$

Since by assumption, $\sum_n \varepsilon_n < +\infty$, we obtain $\lim_{n \rightarrow \infty} z_n > 0$.

Thus, $\lim_{n \rightarrow +\infty} z_n = 0 \Leftrightarrow \sum_n \varepsilon_n = +\infty$ ($\kappa_1 \leq 1$). Otherwise, if $\sum_n \varepsilon_n < +\infty$ ($\kappa_1 > 1$), then $\lim_{n \rightarrow +\infty} z_n \neq 0$.

More specifically, one gets.

• When $\kappa_1 < 1$. Let $z_{\ell,n} := \sum_{h=\ell}^{n-1} \delta'_h \prod_{k=h+1}^{n-1} (1 - \varepsilon'_k)$, thus, assuming ℓ is large enough to replace ε_n and δ_n with their asymptotics, and using the monotony of their asymptotics,

$$\begin{aligned} z_{\ell,n} &= \mathcal{O}\left(\int_{\ell}^n \frac{1}{s^{\kappa_2}} \exp\left(-\int_s^n \frac{1}{u^{\kappa_1}} du\right) ds\right) \\ &= \mathcal{O}\left(\int_{\ell}^n \frac{1}{s^{\kappa_2}} \exp\left[-\frac{1}{(1-\kappa_1)u^{\kappa_1-1}}\right]_s^n ds\right) \\ &= \mathcal{O}\left(\int_{\ell}^n \frac{1}{s^{\kappa_2}} \exp\left[\frac{1}{1-\kappa_1}\left(\frac{1}{s^{\kappa_1-1}} - \frac{1}{n^{\kappa_1-1}}\right)\right] ds\right) \\ &= \mathcal{O}\left(e^{-\frac{1}{(1-\kappa_1)n^{\kappa_1-1}}} \int_{\ell}^n \frac{1}{s^{\kappa_2}} e^{\frac{1}{(1-\kappa_1)s^{\kappa_1-1}}} ds\right) \\ &= \mathcal{O}\left(\frac{1}{n^{\kappa_2}} \frac{\int_{\ell}^n s^{-\kappa_2} e^{\frac{1}{(1-\kappa_1)s^{\kappa_1-1}}} ds}{n^{-\kappa_2} e^{\frac{1}{(1-\kappa_1)n^{\kappa_1-1}}}}\right). \end{aligned}$$

Letting $n \rightarrow \infty$, using L'Hôpital rule, we obtain

$$\begin{aligned} z_{\ell,n} &= \mathcal{O}\left(\frac{1}{n^{\kappa_2}} \frac{n^{-\kappa_2} e^{\frac{1}{(1-\kappa_1)n^{\kappa_1-1}}}}{[(-\kappa_2)n^{-\kappa_2-1} + n^{-\kappa_2} n^{-\kappa_1}] e^{\frac{1}{(1-\kappa_1)s^{\kappa_1-1}}}}\right) \\ &= \mathcal{O}\left(\frac{1}{n^{\kappa_2}} \frac{1}{\frac{-\kappa_2}{n} + \frac{1}{n^{\kappa_1}}}\right) \\ &= \mathcal{O}\left(\frac{1}{n^{\kappa_2-\kappa_1}} \left(\frac{1}{1 - \frac{\kappa_2 n^{\kappa_1}}{n}}\right)\right). \end{aligned}$$

• When $\kappa_1 = 1$, set

$$f_n := \frac{z_n}{\prod_{k=0}^{n-1} (1 - \varepsilon'_k)}.$$

By (A.6) we obtain,

$$f_{n+1} = f_n + F(n)$$

where $F(n) = \frac{\delta'_n}{\prod_{k=0}^n (1 - \varepsilon'_k)}$. So, observing that $f_0 = z_0 = 0$, we obtain

$$f_n = \sum_{h=0}^{n-1} F(h),$$

or equivalently,

$$z_n = \left[\prod_{k=0}^{n-1} (1 - \varepsilon'_k) \right] \sum_{h=0}^{n-1} F(h).$$

Since $\prod_{k=0}^{n-1}(1 - \varepsilon'_k) = \mathcal{O}(\frac{1}{n^A})$ and therefore $F(n) = \mathcal{O}(n^{A-\kappa_2})$, then

$$\begin{aligned} z_n &= \left[\prod_{k=0}^{n-1} (1 - \varepsilon'_k) \right] \sum_{h=0}^{n-1} F(h) &= \mathcal{O}\left(\frac{\sum_{h=0}^{n-1} \frac{1}{h^{\kappa_2-A}}}{n^A}\right) \\ & &= \mathcal{O}\left(\frac{\int_1^n h^{A-\kappa_2}}{n^A}\right) \\ & \stackrel{n \rightarrow \infty}{\cong} \begin{cases} \mathcal{O}\left(\frac{\log n}{n^A}\right) & \text{if } \kappa_2 - A = 1, \\ \mathcal{O}\left(\frac{1}{n^{b-1}}\right) & \text{if } \kappa_2 - A < 1, \\ \mathcal{O}\left(\frac{1}{n^A}\right) & \text{if } \kappa_2 - A > 1. \end{cases} \end{aligned}$$

The conclusion follows. \square

As mentioned, previous result agrees with the next lemma which is proved as Lemma A.4 in [Aletti et al. \(2017\)](#).

Lemma A.3. *Let γ be a real in $(\frac{1}{2}, 1]$, and $c > 0$. Let $(r_n)_n$ be a sequence of real numbers such that $0 < r_n < 1$. Assume that*

$$r_n = \frac{c}{n^\gamma} + \mathcal{O}\left(\frac{1}{n^{2\gamma}}\right)$$

as $n \rightarrow \infty$.

Let $a > 0$. Denote with $m_0 \geq 2$ an integer such that $\forall m \geq m_0, a < \frac{1}{r_m}$. Let

$$p_{m_0, n} := \prod_{m=m_0}^n (1 - ar_m) \text{ and } \ell_{m_0, n} = p_{m_0, n}^{-1}.$$

It holds

$$p_{m_0, n} = \begin{cases} \mathcal{O}(\exp[-\frac{ca}{1-\gamma}n^{1-\gamma}]) & \text{if } \frac{1}{2} < \gamma < 1 \\ \mathcal{O}(n^{-ca}) & \text{if } \gamma = 1 \end{cases}.$$

and

$$\ell_{m_0, n} = \begin{cases} \mathcal{O}(\exp[\frac{ca}{1-\gamma}n^{1-\gamma}]) & \text{if } \frac{1}{2} < \gamma < 1 \\ \mathcal{O}(n^{ca}) & \text{if } \gamma = 1 \end{cases}.$$

Thus, setting

$$F_{k+1, n} := \frac{p_{m_0, n}}{p_{m_0, k}} \quad \text{for } m_0 \leq k \leq n,$$

one gets

$$F_{k+1, n} = \begin{cases} \mathcal{O}(\exp(\frac{a}{1-\kappa_1}(k^{1-\kappa_1} - n^{1-\kappa_1}))) & \text{for } 1/2 < \kappa_1 < 1 \\ \mathcal{O}(\left(\frac{k}{n}\right)^a) & \text{for } \kappa_1 = 1. \end{cases}$$

\square

Lemma A.4. *Let \mathcal{G} be an (increasing) filtration and (V_k) be an \mathcal{G} -adapted sequence of real random variables such that $\mathbb{E}[V_k | \mathcal{G}_{k-1}] \rightarrow V$ a.s. for some real random variable V . Moreover, let (a_k) and (b_k) be two sequences of strictly positive real numbers such that*

$$b_k \uparrow +\infty, \quad \sum_{k=1}^{\infty} \frac{\mathbb{E}[V_k^2]}{a_k^2 b_k^2} < +\infty.$$

Then we have:

- a) If $\frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} \rightarrow \vartheta$ for some constant ϑ , then $\frac{1}{b_n} \sum_{k=1}^n \frac{V_k}{a_k} \rightarrow \vartheta V$.
- b) If $b_n \sum_{k \geq n} \frac{1}{a_k b_k^2} \rightarrow \vartheta$ for some constant ϑ , then $b_n \sum_{k \geq n} \frac{V_k}{a_k b_k^2} \rightarrow \vartheta V$. \square

Theorem A.5. (Theorem 3.2 in *Hall and Heyde, 1980*)

Let $\{S_{n,k}, \mathcal{F}_{n,k} : 1 \leq k \leq k_n, n \geq 1\}$ be a zero-mean, square-integrable martingale array with differences $U_{n,k}$, and let σ^2 be an a.s. finite random variable. Suppose that

- 1) $\max_{1 \leq k \leq k_n} |U_{n,k}| \xrightarrow{P} 0$;
- 2) $\mathbb{E}[\max_{1 \leq k \leq k_n} U_{n,k}^2]$ is bounded in n ;
- 3) $\sum_{k=1}^{k_n} U_{n,k}^2 \xrightarrow{P} \sigma^2$

and the σ -fields are nested, i.e. $\mathcal{F}_{n,k} \subseteq \mathcal{F}_{n+1,k}$ for $1 \leq k \leq k_n$, $n \geq 1$. Then $S_{n,k_n} = \sum_{k=1}^{k_n} U_{n,k}$ converges stably to a random variable with characteristic function $\phi(u) = \mathbb{E}[\exp(-\sigma^2 u^2/2)]$, i.e. to the Gaussian kernel $\mathcal{N}(0, \sigma^2)$. \square

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