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SYNCHRONIZATION AND FLUCTUATIONS FOR INTERACTING STOCHASTIC SYSTEMS WITH INDIVIDUAL AND COLLECTIVE REINFORCEMENT

P.-Y. Louis¹, M. Mirebrahimi^{2,3}

ABSTRACT. The Pólya urn is the paradigmatic example of a reinforced stochastic process. It leads to a random (non degenerated) time-limit. The Friedman urn is a natural generalization whose a.s. time-limit is not random anymore. In this work, in the stream of previous recent works, we introduce a new family of (finite) systems of reinforced stochastic processes, interacting through an additional collective reinforcement of mean field type. The two reinforcement rules strengths (one componentwise, one collective) are tuned through (possibly) different rates $n^{-\gamma}$. In the case the reinforcement rates are like n^{-1} , these reinforcements are of Pólya or Friedman type as in urn contexts and may thus lead to limits which may be random or not. Different parameter regimes needs to be considered.

We state two kind of results. First, we study the time-asymptotics and show that L^2 and almost sure convergence always holds. Moreover all the components share the same time-limit (so called *synchronization*). We study the nature of the limit (random/deterministic) according to the parameters' regime considered. Second, we study fluctuations by proving central limit theorems. Scaling coefficients vary according to the regime considered. This gives insights into the different rates of convergence.

Keywords. Reinforced stochastic processes; Interacting random systems; Almost sure convergence; Central limit theorems; Synchronisation; Fluctuations

MSC2010 Classification. Primary 60K35, 60F15, 60F05; Secondary 62L20, 62P35

1. INTRODUCTION

In urn models, it is well known that the bicolor Pólya reinforcement rule [19] (reinforcement of the chosen color) leads to a random limiting a.s. proportion where as the Friedman rule (reinforcement of the chosen color as well as the non chosen color) leads to a deterministic limiting proportion. This somewhat surprising fact is explained hereafter through a theorem stated in [12]. This work is motivated by the study of asymptotic time behavior of models of (discrete time) stochastic processes interacting through a reinforcement rule. When two rules, one individual, one collective (meaning all the components are involved), compete (in a sense precised hereafter) through different rates, is there one leading ? In particular, is there *loss of synchronization* ? Moreover, in the case were Pólya and Friedman reinforcement rule compete, through the system, in case of synchronization, may the shared time-limit be random ?

The first model of this class was introduced in [7] as a finite system of interacting Pólya urns, where each urn is updated through the Pólya urn using its own proportion or using the proportion averaged over the system. Synchronization was proved in the sense of the a.s. convergence (in

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time) of each system's component to a common random limit. Some variations are considered in [16]. Fluctuations were then proved through central limit theorems (CLT) in [6]. The case of interacting Friedman urn was considered in [20]. Generalizing the reinforcement rate r_n from $r_n \sim n^{-1}$ to $r_n \sim_{n \rightarrow \infty} n^{-\gamma}$ leads to systems of interacting stochastic algorithms. In [5] several cases of reinforcement (like Pólya/Friedman) were considered. A.s. synchronization was stated and speed of convergence studied through functional central limit theorems (FCLT). In parameters' regime where the time limit is random, synchronization happens quicker than convergence to the time limit. The kind of interaction was generalized from mean field to network based interaction in [1]. The empirical means are studied in [2, 3].

As emphasized in the previous works, there are many applicative contexts these models may be useful for. Urn models are well known [17] to have applications in economy, in clinical trials adaptive design [15], random networks [13]. In the general case $r_n \sim_{n \rightarrow \infty} n^{-\gamma}$, each component dynamics is nothing but a stochastic algorithm [4] with many applications in the framework of stochastic optimisation (see for instance [10]). In [5] and in [1] an interpretation of these processes as opinion dynamics was presented.

In the family of models we are introducing (1) and studying in this paper, we are considering a (finite) system of reinforced stochastic processes. There are two kind of reinforcement, one depending only on the component $\xi_{n+1}^l(i)$, one creating the interaction ξ_{n+1}^g and depending on the average over all components. This is modeling a collective reinforcement effect to be confronted with an individual reinforcement. For the sake of simplicity, we choose to consider a mean field interaction. Each reinforcement has its own rate r_n^l (resp. r_n^g). Each rate may have its own asymptotic behaviour: $r_n^l \sim n^{-\gamma_1}$ (resp. $r_n^g \sim n^{-\gamma_2}$) up to constants. If $\gamma_1 < \gamma_2$ (for instance), one may expect the collective reinforcement to be negligible. The system could behave like a system with independent components, leading to a possible *loss of synchronization*. We prove later this is not happening: L^2 /a.s. synchronization holds (meaning, each component dynamics share the same random time limit) like in [7] Additional issues we are addressing are: *nature* of the almost sure time limit distribution according to the type of reinforcement (deterministic/random, diffuse or atomic), *fluctuations* with respect to this limit, which are studied through CLT. We prove, according to the parameters' cases, that the rate of synchronization is quicker or the same as the speed of convergence to the limit. In the models considered in [5], synchronization quicker than convergence towards the asymptotics value Z_∞ holds only in cases where $\text{Var}(Z_\infty) > 0$. In the following models it may happen even when Z_∞ is deterministic.

The paper is organized as follows. In section 2 we define the new family of models. In section 3 we prove that L^2 and a.s. convergence holds towards a limit Z_∞ shared by all the components (synchronization). Two main cases are to be distinguished : Theorems 3.1 and 3.2 deals with cases where $Z_\infty = \frac{1}{2}$ (the word synchronisation is abusive in this situation) where as Theorem 3.4 deals with Z_∞ random. In section 4, in the different cases, we prove central limit theorems about the fluctuations of $Z_n - Z_\infty$ and $Z_n - Z_n(i)$. Scaling factors are worth of interest. Th. 4.1 consider the case where each individual and collective reinforcement leads to a deterministic limiting value. Theorem 4.2 consider the special case when $\gamma_1 = \gamma_2 = 1$ reminiscent of the Friedman urn context, in the regime where fluctuations are not gaussian ($\lambda_1 + \lambda_2 < \frac{1}{4}$). Theorem 4.3 deals with the mixed cases where individual and reinforcement type are of different nature. Th. 4.4 consider the case where the individual and the collective reinforcement leads to a random limit. Section 5 is dedicated to the proof of the synchronization. Sections 6, 7, 8 and 9 deal respectively with the proofs of the CLTs. An appendix A holds for the sake of completeness.

2. MODEL DEFINITION AND FRAMEWORK

Let us define the following new model. For $i \in \{1, \dots, N\}$ and $n \in \mathbb{N}$, we consider the stochastic dynamics defined through the recursive relation

$$Z_{n+1}(i) = (1 - r_n^l - r_n^g)Z_n(i) + r_n^l \xi_{n+1}^l(i) + r_n^g \xi_{n+1}^g, \quad (1)$$

where $Z_0(i) = \frac{1}{2}$ and where $\xi_{n+1}^l(i)$ and ξ_{n+1}^g denote local and collective reinforcements random variables. Given \mathcal{F}_n , they have independent Bernoulli distributions with

$$\begin{aligned} \mathbb{P}(\xi_{n+1}^l(i) = 1 | \mathcal{F}_n) &= \psi_1(Z_n(i)) := (1 - 2\lambda_1)Z_n(i) + \lambda_1, \\ \mathbb{P}(\xi_{n+1}^g = 1 | \mathcal{F}_n) &= \psi_2(Z_n) := (1 - 2\lambda_2)Z_n + \lambda_2, \end{aligned} \quad (2)$$

where $\psi_k : [0, 1] \rightarrow [0, 1]$ ($k \in \{1, 2\}$) with $Z_n := \frac{1}{N} \sum_{i=1}^N Z_n(i)$ (so called *mean field*) and where $\lambda_1, \lambda_2 \in [0, 1]$ are parameters. The local (resp. collective) reinforcement rate are such that $r_n^l \sim \frac{D_1}{n^{\gamma_1}}$ (resp. $r_n^g \sim \frac{D_2}{n^{\gamma_2}}$). The symbol \sim has the usual meaning $\lim_{n \rightarrow \infty} n^\gamma r_n = 1$ whereas we will use the symbol \simeq to mean $\lim_{n \rightarrow \infty} n^\gamma r_n$ is a constant.

In all the paper, we assume that $\frac{1}{2} < \gamma_i \leq 1$ for $i \in \{1, 2\}$ in order to satisfy the usual assumptions about r_n^l (resp. r_n^g):

$$\sum r_n^l = +\infty, \quad \sum (r_n^l)^2 < +\infty.$$

We are choosing the transformations ψ_1 and ψ_2 as linear. For the sake of simplicity, we are choosing to symetrize the model with respect to $\frac{1}{2}$. For the same reason, according to the previously cited works, it is enough to consider the starting conditions all equal to $\frac{1}{2}$.

Remark 2.1. *In the particular case when $\gamma_1 = \gamma_2 = \gamma$, we can rewrite the model such as*

$$Z_{n+1}(i) = (1 - 2r_n)Z_n(i) + r_n \tilde{\xi}_{n+1}(i),$$

where $\tilde{\xi}_{n+1}(i) = \xi_{n+1}^l(i) + \xi_{n+1}^g$, therefore $\tilde{\xi}_{n+1}(i) \in \{0, 1, 2\}$. The other probabilities may be computed in an analogous way. The reinforcement rate remains $r_n \sim D n^{-\gamma}$.

Remark 2.2. *As mentioned earlier, please notice that when $\gamma_1 = 1$, we find a reinforcement of Pólya type ($\lambda_1 = 0$) or of Friedman type ($\lambda_1 \neq 0$). The same remark hold for the collective reinforcement effect tuned through γ_2, λ_2 .*

Remark 2.3. *In this paper the parameters λ_1, λ_2 are kept fixed. Cases where λ_j may converge to 0 depending on n, N will be considered in a forthcoming work. Remark, in the particular case when $\lambda_1 = \lambda_2 = \frac{1}{2}$, there is no reinforcement in the model anymore. Moreover, when $\lambda_1 \neq \frac{1}{2}$ and $\lambda_2 = \frac{1}{2}$, although we still have reinforcement at each component individually, we are loosing the interaction.*

It is then straightforward, to compute the following relationships that will be used frequently in this paper:

$$\mathbb{E}(Z_{n+1}(i) - Z_n(i) | \mathcal{F}_n) = \lambda_1 r_n^l (1 - 2Z_n(i)) + \lambda_2 r_n^g (1 - 2Z_n) + r_n^g (Z_n - Z_n(i)), \quad (3)$$

and by averaging over i in $\{1, \dots, N\}$, we have

$$\mathbb{E}[Z_{n+1} | \mathcal{F}_n] = (1 - 2\lambda_1 r_n^l - 2\lambda_2 r_n^g)Z_n + \lambda_1 r_n^l + \lambda_2 r_n^g. \quad (4)$$

3. MAIN RESULTS: CONVERGENCE AND SYNCHRONIZATION

In this section we study convergence of $(Z_n)_n$ and the synchronization phenomenon. Indeed we obtain different kind of time-limit (deterministic or random) for $(Z_n)_n$ according to the nullness of λ_1, λ_2 . Moreover L^2 and a.s. synchronization are proven to always hold.

3.1. Case of a deterministic time-asymptotics. We call *deterministic*, the case when the time limit Z_∞ ($n \rightarrow \infty$) is not random. This behavior corresponds to cases where at least one of the following assumptions is true $\lambda_1 > 0$ or $\lambda_2 > 0$. The mean field process $(Z_n)_n$ is not a martingale. In order to investigate the behavior of the interacting system, we first consider the time limits of the variances $\text{Var}(Z_n(i))$ and $\text{Var}(Z_n)$. Second we show that L^2 -synchronization holds *i.e.* $\lim_{n \rightarrow \infty} \text{Var}(Z_n(i) - Z_n) = 0$. We get the rates of convergence too. Finally, we prove that the synchronization holds almost surely and the deterministic limit is $Z_\infty := \frac{1}{2}$

Theorem 3.1. *For any $\lambda_1 > 0$ and $\lambda_2 > 0$ following results hold, where C denotes a generic constant:*

- i) *asymptotics of variances ($n \rightarrow \infty$):*
 $\text{Var}(Z_n) \sim \frac{C}{n^\gamma}$ and $\text{Var}(Z_n(i)) \sim \frac{C}{n^\gamma}$ where $\gamma := \min(\gamma_1, \gamma_2)$;
- ii) *behavior of the L^2 -distance between Z_n and $Z_n(i)$ when $n \rightarrow \infty$:*
 - a) *if $\gamma_1 \leq \gamma_2$, then $\mathbb{E}[Z_n - Z_n(i)]^2 \sim \frac{C}{n^{\gamma_1}}$,*
 - b) *if $\gamma_2 < \gamma_1$, then $\mathbb{E}[Z_n - Z_n(i)]^2 \sim \frac{C}{n^{2\gamma_1 - \gamma_2}}$;*
- iii) *almost sure convergence holds i.e.*
 $\forall i \in \{1, \dots, N\}, \quad \lim_{n \rightarrow +\infty} Z_n(i) = \lim_{n \rightarrow +\infty} Z_n = \frac{1}{2} := Z_\infty$ *a.s.*

Two others choices of parameters λ_1, λ_2 lead to the following results.

Theorem 3.2. *In the following cases: either ($\lambda_1 > 0$ and $\lambda_2 = 0$) or ($\lambda_1 = 0$ and $\lambda_2 > 0$) it holds $\lim_{n \rightarrow +\infty} Z_n(i) = \lim_{n \rightarrow +\infty} Z_n = \frac{1}{2}$ a.s. Moreover, the following table summarizes the L^2 speed of convergence with $\gamma := \min(\gamma_1, \gamma_2)$ and where C denotes a generic constant.*

	$\lambda_1 \neq 0, \lambda_2 = 0$	$\lambda_1 = 0, \lambda_2 \neq 0$
$\gamma_1 \leq \gamma_2$	$\text{Var}(Z_n) \sim \frac{C}{n^{\gamma_1}}$ $\mathbb{E}[Z_n - Z_n(i)]^2 \sim \frac{C}{n^{\gamma_1}}$	$\text{Var}(Z_n) \sim \frac{C}{n^{2\gamma_1 - \gamma_2}}$ $\mathbb{E}[Z_n - Z_n(i)]^2 \sim \frac{C}{n^{2\gamma_1 - \gamma_2}}$
$\gamma_2 < \gamma_1$	$\text{Var}(Z_n) \sim \frac{C}{n^{2\gamma_2 - \gamma_1}}$ $\mathbb{E}[Z_n - Z_n(i)]^2 \sim \frac{C}{n^{2\gamma_1 - \gamma_2}}$	$\text{Var}(Z_n) \sim \frac{C}{n^{\gamma_2}}$ $\mathbb{E}[Z_n - Z_n(i)]^2 \sim \frac{C}{n^{2\gamma_1 - \gamma_2}}$

Remark 3.3. *(Comparison of convergence and synchronization rates)*

In the case $\lambda_1 > 0, \lambda_2 > 0$, When $\gamma_1 < \gamma_2$, the L^2 convergence rate of $(Z_n)_n$ to $\frac{1}{2}$ and the L^2 rate of convergence of $(Z_n(i) - Z_n)_n$ to 0 are the same. However, when $\gamma_2 < \gamma_1$, we obtain that synchronization happen faster than convergence.

Moreover in the case $\lambda_1 > 0, \lambda_2 = 0$ and when $\gamma_1 < \gamma_2$, the speed of convergence and synchronization are the same ($n^{-\gamma_1}$). While when $\gamma_2 < \gamma_1$, the synchronization is faster than convergence.

Similarly, in the case $\lambda_1 = 0, \lambda_2 > 0$ and when $\gamma_2 < \gamma_1$, the speed of convergence and synchronization are the same ($n^{-(2\gamma_1 - \gamma_2)}$), while when $\gamma_2 < \gamma_1$, the speed of synchronization is faster than convergence $n^{-(2\gamma_1 - \gamma_2)}$ and $n^{-\gamma_2}$ respectively).

3.2. Case of a common shared random time-asymptotics. Differently to the previous cases, the case $\lambda_1 = \lambda_2 = 0$ yields $(Z_n)_n$ is a martingale. We will prove it leads to a random time-asymptotics Z_∞ ($\text{Var}(Z_\infty) > 0$). We will study the system's time-asymptotics behaviour in a similar way as in the previous cases. First we show that $\lim_{n \rightarrow \infty} \text{Var}(Z_n) \neq 0$. Second we prove that L^2 -synchronization holds. Third we state the almost sure synchronization holds.

Theorem 3.4. *For $\lambda_1 = \lambda_2 = 0$,*

- (i) *it holds ($n \rightarrow \infty$) $\text{Var}(Z_n) \sim e^{-\zeta(2\gamma_2)} \sim \frac{C}{n^{2\gamma_2 - 1}}$, where $\gamma := \min(\gamma_1, \gamma_2)$ and C denotes a generic constant, and $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ is Riemann's zeta function. In particular $(Z_n)_n$ converges a.s. to a non-degenerated random limit denoted by Z_∞ ($\text{Var}(Z_\infty) > 0$).*

(ii) The L^2 -distance between the mean field Z_n and each component $Z_n(i)$ behaves as follows,

$$\mathbb{E}[Z_n(i) - Z_n]^2 \sim \frac{C}{n^{2\gamma_1 - \gamma_2}}$$

and synchronization holds almost surely. It means, for all $i \in \{1, \dots, N\}$,
 $\lim_{n \rightarrow \infty} Z_n(i) = Z_\infty$ a.s.

4. MAIN RESULTS: FLUCTUATIONS THROUGH CLT

In this section we study the fluctuations of $(Z_n(i) - Z_n)_n$ (synchronization) w.r.t 0 and also fluctuations of $(Z_n)_n$ w.r.t its limit Z_∞ . These are studied by stating Central Limit Theorems. Pay attention different scaling hold according to (γ_1, γ_2) relationship. We follow the proof's techniques initiated for these models in [6] based on Theorem A.3 in Appendix leading to stable convergence results.

We first study cases where $Z_\infty = \frac{1}{2}$. The Theorems 4.1, 4.2 deals with the case $\lambda_1 > 0$ and $\lambda_2 > 0$. Moreover, we show that there is a non gaussian limit distribution for some special regime when $0 < (\lambda_1 + \lambda_2) < \frac{1}{4}$. The Theorem 4.3 describe the results of the cases where exactly one of the λ_j is 0.

Finally we state the behaviour when $\text{Var}(Z_\infty) > 0$. with Theorem 4.4.

Theorem 4.1. *Let $\lambda_1 > 0, \lambda_2 > 0$; let $\gamma := \min(\gamma_1, \gamma_2)$. The following statements hold, where C denotes a generic constant (depending on N and λ_j).*

i) It holds

- a) when $\gamma_1 \leq \gamma_2$, $n^{\frac{\gamma_1}{2}}(Z_n - Z_n(i)) \xrightarrow[n \rightarrow \infty]{\text{stably}} \mathcal{N}(0, C)$,
- b) when $\gamma_2 < \gamma_1$, $n^{\frac{2\gamma_1 - \gamma_2}{2}}(Z_n - Z_n(i)) \xrightarrow[n \rightarrow \infty]{\text{stably}} \mathcal{N}(0, C)$.

ii) For $\gamma < 1$, it holds

$$n^{\frac{\gamma}{2}}(Z_n - \frac{1}{2}) \xrightarrow[n \rightarrow \infty]{\text{stably}} \mathcal{N}(0, C).$$

iii) For $\gamma = 1$ (meaning $\gamma_1 = \gamma_2 = 1$),

- a) for $(\lambda_1 + \lambda_2) > \frac{1}{4}$, $\sqrt{n}(Z_n - \frac{1}{2}) \xrightarrow[n \rightarrow \infty]{\text{stably}} \mathcal{N}(0, C)$.
- b) for $(\lambda_1 + \lambda_2) = \frac{1}{4}$, $\frac{\sqrt{n}}{\sqrt{\ln n}}(Z_n - \frac{1}{2}) \xrightarrow[n \rightarrow \infty]{\text{stably}} \mathcal{N}(0, C)$.

Theorem 4.2. *Let $\lambda_1 > 0, \lambda_2 > 0$. When $\gamma_1 = \gamma_2 = 1$ and when $(\lambda_1 + \lambda_2) < \frac{1}{4}$, the following statement holds*

$$n^{4(\lambda_1 + \lambda_2)}(Z_n - \frac{1}{2}) \xrightarrow[n \rightarrow \infty]{\text{a.s./}\mathcal{L}^1} \tilde{X},$$

for some real random variable \tilde{X} such that $\mathbb{P}(\tilde{X} \neq 0) > 0$.

Two other main cases leads to following results.

Theorem 4.3. *In the following cases: either $(\lambda_1 > 0, \lambda_2 = 0)$ or $(\lambda_1 = 0, \lambda_2 > 0)$, the stable convergence towards some Gaussian distribution holds for the quantities $(Z_n - Z_n(i))_n$ and $(Z_n - \frac{1}{2})_n$. The following tables summarizes the different scaling of convergence according to the relationship between γ_1, γ_2 and where C denotes a generic constant. The first table deals with $\gamma := \min(\gamma_1, \gamma_2) < 1$.*

	$\lambda_1 \neq 0, \lambda_2 = 0$	$\lambda_1 = 0, \lambda_2 \neq 0$
$\gamma_1 \leq \gamma_2$	$n^{\frac{\gamma_1}{2}} (Z_n - Z_n(i)) \xrightarrow{\text{stably}} \mathcal{N}(0, C)$ $n^{\frac{\gamma_1}{2}} (Z_n - \frac{1}{2}) \xrightarrow{\text{stably}} \mathcal{N}(0, C)$	$n^{\frac{2\gamma_1 - \gamma_2}{2}} (Z_n - Z_n(i)) \xrightarrow{\text{stably}} \mathcal{N}(0, C)$ $n^{\frac{2\gamma_1 - \gamma_2}{2}} (Z_n - \frac{1}{2}) \xrightarrow{\text{stably}} \mathcal{N}(0, C)$
$\gamma_2 < \gamma_1$	$n^{\frac{2\gamma_1 - \gamma_2}{2}} (Z_n - Z_n(i)) \xrightarrow{\text{stably}} \mathcal{N}(0, C)$ $n^{\frac{2\gamma_2 - \gamma_1}{2}} (Z_n - \frac{1}{2}) \xrightarrow{\text{stably}} \mathcal{N}(0, C)$	$n^{\frac{2\gamma_1 - \gamma_2}{2}} (Z_n - Z_n(i)) \xrightarrow{\text{stably}} \mathcal{N}(0, C)$ $n^{\frac{\gamma_2}{2}} (Z_n - \frac{1}{2}) \xrightarrow{\text{stably}} \mathcal{N}(0, C)$

The following second table deals with $\gamma = 1$.

$\lambda_i = 0, \lambda_j > \frac{1}{4}$	$\lambda_i = 0, \lambda_j = \frac{1}{4}$	$\lambda_i = 0, \lambda_j < \frac{1}{4}$
$\sqrt{n}(Z_n - \frac{1}{2}) \xrightarrow{\text{stably}} \mathcal{N}(0, C)$	$\frac{\sqrt{n}}{\sqrt{\ln n}}(Z_n - \frac{1}{2}) \xrightarrow{\text{stably}} \mathcal{N}(0, C)$	$n^{4(\lambda_1 + \lambda_2)}(Z_n - \frac{1}{2}) \xrightarrow{\text{a.s./}\mathcal{L}^1} \tilde{X}$

Theorem 4.4. Assume $\lambda_1 = \lambda_2 = 0$.

(i) It holds

$$n^{\frac{2\gamma_1 - \gamma_2}{2}} (Z_n - Z_n(i)) \xrightarrow[n \rightarrow \infty]{\text{stably}} \mathcal{N}(0, C Z_\infty(1 - Z_\infty)).$$

(ii) With $\gamma := \min(\gamma_1, \gamma_2)$, it holds

$$n^{\frac{2\gamma - 1}{2}} (Z_n - Z_\infty) \xrightarrow[n \rightarrow \infty]{\text{stably}} \mathcal{N}(0, C Z_\infty(1 - Z_\infty)).$$

Remark 4.5. (analogous to Theorem 3.2 in [6]). We have $\mathbb{P}(Z_\infty = 0) + \mathbb{P}(Z_\infty = 1) < 1$ and $\mathbb{P}(Z_\infty = z) = 0$ for each $z \in (0, 1)$. Indeed, it guarantees that these limit Gaussian kernels are not degenerate.

Proof. (of Remark 4.5) The first part immediately follows from the relation $\mathbb{E}[Z_\infty^2] < \mathbb{E}[Z_\infty]$ by Lemma 5.3. The second part is a consequence of the almost sure conditional convergence stated in Th. 4.4 (ii) (for details see proof of Theorem 2.5 in [5]). \square

5. PROOF OF A.S. SYNCHRONIZATION AND RATES OF CONVERGENCES

This section is devoted to the proofs of Th. 3.1, Th. 3.2, Th. 3.4. Cases need indeed to be distinguished according to the nullness of λ_1, λ_2 .

5.1. First results about the variances. Using (3) and (4), we compute the recursive equations satisfied by variance of $Z_n(i)$ and Z_n :

$$\begin{aligned} \text{Var}(Z_{n+1}(i)|\mathcal{F}_n) &= \text{Var} \left[(1 - r_n^l - r_n^g)Z_n + r_n^l \xi_{n+1}^l(i) + r_n^g \xi_{n+1}^g \mid \mathcal{F}_n \right] \\ &= (r_n^l)^2 [(1 - 2\lambda_1)^2 (Z_n(i) - Z_n(i)^2) + \lambda_1 - \lambda_1^2] + (r_n^g)^2 [(1 - 2\lambda_2)^2 (Z_n - Z_n^2) + \lambda_2 - \lambda_2^2], \end{aligned}$$

then using the usual decomposition (*), we have

$$\begin{aligned} \text{Var}(Z_{n+1}(i)) &\stackrel{*}{=} \mathbb{E}[\text{Var}(Z_{n+1}(i)|\mathcal{F}_n)] + \text{Var}[\mathbb{E}(Z_{n+1}(i)|\mathcal{F}_n)] \\ &= (r_n^l)^2 [(1 - 2\lambda_1)^2 (Z_n(i) - Z_n(i)^2) + \lambda_1 - \lambda_1^2] + (r_n^g)^2 [(1 - 2\lambda_2)^2 (Z_n - Z_n^2) + \lambda_2 - \lambda_2^2] \\ &\quad + (1 - 2\lambda_1 r_n^l - r_n^g)^2 \text{Var}(Z_n(i)) + (1 - 2\lambda_2)^2 (r_n^g)^2 \text{Var}(Z_n) + 2(1 - 2\lambda_2)(1 - 2\lambda_1 r_n^l - r_n^g) r_n^g \text{Var}(Z_n), \end{aligned} \tag{5}$$

where in the last equation we used the fact that $\text{Cov}(Z_n(i), Z_n) = \text{Var}(Z_n)$ by symmetry. Moreover,

$$\begin{aligned} \text{Var}(Z_{n+1}|\mathcal{F}_n) &= \frac{(r_n^l)^2}{N^2} \sum_{i=1}^N \text{Var}(\xi_{n+1}^l(i)|\mathcal{F}_n) + (r_n^g)^2 \text{Var}(\xi_{n+1}^g|\mathcal{F}_n) \\ &= \frac{(r_n^l)^2}{N} \left[(1 - 2\lambda_1)^2 \left(Z_n - \frac{1}{N} \sum_{i=1}^N Z_n(i)^2 \right) + \lambda_1 - \lambda_1^2 \right] + (r_n^g)^2 \left[(1 - 2\lambda_2)^2 (Z_n - Z_n^2) + \lambda_2 - \lambda_2^2 \right] \end{aligned}$$

thus,

$$\begin{aligned} \text{Var}(Z_{n+1}) &= (1 - 2\lambda_1 r_n^l - 2\lambda_1 r_n^g)^2 \text{Var}(Z_n) \\ &\quad + \frac{(r_n^l)^2}{N} \left[(1 - 2\lambda_1)^2 \left(\mathbb{E}(Z_n) - \frac{1}{N} \sum_{i=1}^N \mathbb{E} Z_n^2(i) \right) + \lambda_1 - \lambda_1^2 \right] \\ &\quad + (r_n^g)^2 \left[(1 - 2\lambda_2)^2 (\mathbb{E}(Z_n) - \mathbb{E}(Z_n^2)) + \lambda_2 - \lambda_2^2 \right]. \quad (6) \end{aligned}$$

Furthermore we can rewrite the equation (1) as

$$Z_{n+1} - Z_n = r_n^l \left(\frac{1}{N} \sum_{i=1}^N \xi_{n+1}^l(i) - Z_n \right) + r_n^g (\xi_{n+1}^g - Z_n), \quad (7)$$

also we can obtain easily

$$\mathbb{E} \left(\frac{1}{N} \sum_{i=1}^N \xi_{n+1}^l(i) \middle| \mathcal{F}_n \right) = (1 - 2\lambda_1) Z_n + \lambda_1,$$

$$\mathbb{E}(\xi_{n+1}^g | \mathcal{F}_n) = (1 - 2\lambda_2) Z_n + \lambda_2.$$

Finally, remark that defining

$$\Delta M_{n+1}^l := \frac{1}{N} \sum_{i=1}^N \xi_{n+1}^l(i) - \mathbb{E} \left(\frac{1}{N} \sum_{i=1}^N \xi_{n+1}^l(i) \middle| \mathcal{F}_n \right)$$

$$\Delta M_{n+1}^g := \xi_{n+1}^g - \mathbb{E}(\xi_{n+1}^g | \mathcal{F}_n),$$

leads to

$$\Delta M_{n+1}^l = \frac{1}{N} \sum_{i=1}^N \xi_{n+1}^l(i) - [(1 - 2\lambda_1) Z_n(i) + \lambda_1], \quad (8)$$

$$\Delta M_{n+1}^g = \xi_{n+1}^g - [(1 - 2\lambda_2) Z_n(i) + \lambda_2]. \quad (9)$$

Using (7), (8) and (9), the following relationship holds

$$Z_{n+1} - \frac{1}{2} = \left(Z_n - \frac{1}{2} \right) \left[1 - 2r_n^l \lambda_1 - 2r_n^g \lambda_2 \right] + r_n^l \Delta M_{n+1}^l + r_n^g \Delta M_{n+1}^g. \quad (10)$$

Remark 5.1. *The assumption $\forall i \in \{1, \dots, N\}$, $Z_0(i) = \frac{1}{2}$ leads to $\forall n \in \mathbb{N}$, $\mathbb{E}(Z_n) = \mathbb{E}(Z_n(i)) = \frac{1}{2}$ since*

$$\mathbb{E}(Z_{n+1}(i)) = (1 - 2\lambda_1 r_n^l - r_n^g) \mathbb{E}(Z_n(i)) + (1 - 2\lambda_2) r_n^g \mathbb{E}(Z_n) + (\lambda_1 r_n^l + \lambda_2 r_n^g).$$

Thus from (6) we obtain

$$\begin{aligned} \text{Var}(Z_{n+1}) &= \left[1 - 4 \left(\lambda_1 r_n^l + \lambda_2 r_n^g - 2\lambda_1 \lambda_2 r_n^l r_n^g - \lambda_1^2 (r_n^l)^2 - \lambda_2^2 (r_n^g)^2 + \frac{(r_n^g)^2}{4} (1 - 2\lambda_2) \right) \right] \text{Var}(Z_n) \\ &+ \frac{(r_n^l)^2}{N} \left[(1 - 2\lambda_1)^2 \left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n(i)^2) \right) + \lambda_1 - \lambda_1^2 \right] + \frac{(r_n^g)^2}{4}. \end{aligned} \quad (11)$$

and from (5) it follows

$$\begin{aligned} \text{Var}(Z_{n+1}(i)) &= \left[(1 - 2\lambda_1 r_n^l - r_n^g)^2 - (r_n^l)^2 (1 - 2\lambda_1)^2 \right] \text{Var}(Z_n(i)) \\ &+ (r_n^l)^2 \left[\frac{1}{2} - \lambda_1 + \lambda_1^2 - \frac{(1 - 2\lambda_1)^2}{4} \right] + (r_n^g)^2 \left[\frac{1}{2} - \lambda_2 + \lambda_2^2 - \frac{(1 - 2\lambda_2)^2}{4} \right] \\ &+ 2(1 - 2\lambda_1 r_n^l - r_n^g) r_n^g (1 - 2\lambda_2) \text{Var}(Z_n). \end{aligned} \quad (12)$$

Remark 5.2. When $\gamma_1 = \gamma_2 = \gamma$,

$$\begin{aligned} \mathbb{E}(Z_{n+1}(i)|\mathcal{F}_n) &= (1 - 2r_n)Z_n(i) + r_n \mathbb{E}[\tilde{\xi}_{n+1}(i)|\mathcal{F}_n] \\ &= (1 - (1 + 2\lambda_1)r_n)Z_n(i) + (1 - 2\lambda_2)r_n Z_n + (\lambda_1 + \lambda_2)r_n \end{aligned}$$

and

$$\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = (1 - 2r_n)Z_n + \frac{r_n}{N} \sum_{i=1}^n \mathbb{E}(\tilde{\xi}_{n+1}(i)|\mathcal{F}_n) = \left(1 - 2(\lambda_1 + \lambda_2)r_n \right) Z_n + (\lambda_1 + \lambda_2)r_n.$$

Moreover,

$$\begin{aligned} \text{Var}(Z_{n+1}(i)) &= r_n^2 \left(\frac{1}{2} - \lambda_1 + \lambda_2 - \frac{(1 - 2\lambda_1)^2}{4} + \frac{1}{2} - \lambda_2 + \lambda_2^2 - \frac{(1 - 2\lambda_2)^2}{4} \right) \\ &+ [(1 - r_n(2\lambda_1 + 1))^2 - r_n(1 - 2\lambda_1)^2] \text{Var}(Z_n(i)) \\ &+ [(1 - 2\lambda_2)^2 + r_n(1 - r_n(2\lambda_1 + 1))(1 - 2\lambda_1)] \text{Var}(Z_n). \end{aligned} \quad (13)$$

Furthermore,

$$\begin{aligned} \text{Var}(Z_{n+1}) &= (1 - 4[(\lambda_1 + \lambda_2)r_n - (\lambda_1 + \lambda_2)^2 r_n^2] - (1 - 2\lambda_2)^2 r_n^2) \text{Var}(Z_n) \\ &+ \frac{r_n^2}{N} \left[(1 - 2\lambda_1)^2 \left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n(i)^2) \right) + \lambda_1 - \lambda_1^2 + \frac{N}{4} \right]. \end{aligned} \quad (14)$$

Finally we get

$$Z_{n+1} - \frac{1}{2} = \left(Z_n - \frac{1}{2} \right) [1 - 2r_n(\lambda_1 + \lambda_2)] + r_n \Delta \tilde{M}_{n+1}(i), \quad (15)$$

where $\Delta \tilde{M}_{n+1}(i) = \Delta M_{n+1}^l(i) + \Delta M_{n+1}^g$.

Lemma 5.3. When $\lambda_1 = \lambda_2 = 0$, for every $n \geq 0$, $\text{Var}(Z_n) < \frac{1}{4}$. Moreover, it holds

$$\sup_n \mathbb{E}(Z_n^2) = \mathbb{E}(Z_\infty^2) < \frac{1}{2}.$$

Proof. Obviously $\text{Var}(Z_n) \leq \frac{1}{4}$. We need to show that $\text{Var}(Z_n) < \frac{1}{4}$. For this purpose, using (11) with $\lambda_1 = \lambda_2 = 0$ gives:

$$\text{Var}(Z_{n+1}) = \left(1 - (r_n^g)^2 \right) \text{Var}(Z_n) + \frac{(r_n^g)^2}{4} + \frac{(r_n^l)^2}{2N} - (r_n^l)^2 \left(\frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n(i)^2) \right).$$

Since $\mathbb{E}(Z_n^2(i)) \leq \frac{1}{2}$ we get,

$$\text{Var}(Z_{n+1}) \geq \left(1 - (r_n^g)^2\right) \text{Var}(Z_n) + \frac{(r_n^g)^2}{4}.$$

Defining $x_n := \frac{1}{4} - \text{Var}(Z_n) \geq 0$, the goal is equivalent to

$$x_{n+1} \leq \left(1 - (r_n^g)^2\right) x_n$$

from which it follows

$$x_n \leq x_0 \prod_{k=0}^{n-1} \left(1 - (r_k^g)^2\right) x_n.$$

Since by assumption, $\sum_n (r_n^g)^2 < +\infty$, we obtain $\lim_{n \rightarrow \infty} x_n \neq 0$. Moreover, from $\mathbb{E}(Z_{n+1}^2 | \mathcal{F}_n) = Z_n^2 + \text{Var}(Z_{n+1} | \mathcal{F}_n)$, we get $\mathbb{E}(Z_{n+1}^2 | \mathcal{F}_n) \geq Z_n^2$. Thus, Z_n^2 is submartingale. Consequently $\sup_n \mathbb{E}(Z_n^2) = \lim_n \mathbb{E}(Z_n^2) = \mathbb{E}(Z_\infty^2) < \frac{1}{2}$. \square

5.2. Proofs of L^2 and a.s. convergence results. We now prove the theorems of section 3 about convergence and synchronization. Recall, we use the notation $a_n \sim b_n$ when $\lim_n \frac{a_n}{b_n} = 1$.

Proof. (i) of Theorem 3.1

First consider the equation (11) which behaves like

$$\text{Var}(Z_{n+1}) = \left(1 - \frac{4\lambda}{n^\gamma}\right) \text{Var}(Z_n) + \frac{K_n}{n^{2\gamma}} + \mathbf{o}\left(\frac{1}{n^{2\gamma}}\right),$$

where $\lambda = \begin{cases} \lambda_1 & \text{if } \gamma_1 < \gamma_2 \\ \lambda_2 & \text{if } \gamma_1 > \gamma_2 \end{cases}$.

- If $\gamma_1 < \gamma_2$ then $C_{N,\lambda} = K_n/4\lambda_1$ where

$$K_n = \frac{1}{N} \left((1 - 2\lambda_1)^2 \left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n^2(i)) \right) + \lambda_1 - \lambda_1^2 \right)$$

is bounded and not equal zero. Indeed, using $\mathbb{E}(Z_n^2) < \frac{1}{2}$, we get

$$\sum_{i=1}^N \mathbb{E}(Z_n(i)^2) = \mathbb{E}(Z_n^2(j)) + \sum_{i=1, i \neq j}^{N-1} \mathbb{E}(Z_n(i)^2) < \frac{1}{2} + \frac{N-1}{2} = \frac{N}{2}$$

thus, by Lemma A.1, $\text{Var}(Z_n) \sim \frac{C_{N,\lambda}}{n^{\gamma_1}}$.

- If $\gamma_1 > \gamma_2$ then $C_{N,\lambda} = K_n/4\lambda_2$ where $K_n = \frac{1}{4}$ that is bounded clearly, so we obtain that $\text{Var}(Z_n) \sim \frac{C_{N,\lambda}}{n^{\gamma_2}}$.

- Hence in both cases $\lim_{n \rightarrow \infty} \text{Var}(Z_n) = 0$.

In order to investigate the behaviour of $\text{Var}(Z_n(i))$, consider (12) in the three following cases and use

$$\begin{aligned} \text{Var}(Z_{n+1}(i)) &= \left[1 - 4\lambda_1 r_n^l - 2r_n^g + 4\lambda^2 (r_n^l)^2 + 4\lambda_1 r_n^l r_n^g - (r_n^l)^2 (1 - 2\lambda_1)^2 \right] \text{Var}(Z_n(i)) \\ &\quad + (r_n^l)^2 \left[\frac{1}{2} - \lambda_1 + \lambda_1^2 - \frac{(1 - 2\lambda_1)^2}{4} \right] + (r_n^g)^2 \left[\frac{1}{2} - \lambda_2 + \lambda_2^2 - \frac{(1 - 2\lambda_2)^2}{4} \right] \\ &\quad + \left[\left(1 - 2\lambda_2 r_n^g - 2\lambda_1 (1 - 2\lambda_2) r_n^l r_n^g \right) (1 - 2\lambda_2) r_n^g \right] \text{Var}(Z_n). \end{aligned}$$

- When $\gamma_1 < \gamma_2$ we have $\text{Var}(Z_n) \sim \frac{C_{N,\lambda}}{n^{\gamma_1}}$ so,

$$\begin{aligned} \text{Var}(Z_{n+1}(i)) &= \left[1 - 4\lambda_1 r_n^l + \mathbf{o}(r_n^l)\right] \text{Var}(Z_n(i)) \\ &+ (r_n^l)^2 \left[\frac{1}{2} - \lambda_1 + \lambda_1^2 - \frac{(1-2\lambda_1)^2}{4}\right] + \mathbf{o}((r_n^l)^2) \end{aligned}$$

then $C_{N,\lambda} = K_n/4\lambda_1$ where $K_n = \left(\frac{1}{2} - \lambda_1 + \lambda_1^2 - \frac{(1-2\lambda_1)^2}{4}\right)$ which implies $\text{Var}(Z_n(i)) \sim \frac{C_{N,\lambda}}{n^{\gamma_1}}$.

- When $\gamma_2 < \gamma_1$ we have $\text{Var}(Z_n) \sim \frac{1}{n^{\gamma_2}}$ so,

$$\begin{aligned} \text{Var}(Z_{n+1}(i)) &= [1 - 2r_n^g + \mathbf{o}(r_n^g)] \text{Var}(Z_n(i)) \\ &+ (r_n^g)^2 \left[\frac{1}{2} - \lambda_2 + \lambda_2^2 - \frac{(1-2\lambda_2)^2}{4}\right] + (1-2\lambda_2) \frac{1}{n^{2\gamma_2}} \end{aligned}$$

then $C_{N,\lambda} = K_n/2$ in which $K_n = \left[\frac{1}{2} - \lambda_2 + \lambda_2^2 - \frac{(1-2\lambda_2)^2}{4} + (1-2\lambda_2)\right]$. It implies by Lemma A.1 $\text{Var}(Z_n(i)) \sim \frac{C_{N,\lambda}}{n^{\gamma_2}}$.

- When $\gamma_1 = \gamma_2$, using (14) we have

$$\text{Var}(Z_{n+1}) = (1 - 2(\lambda_1 + \lambda_2)r_n - N(1 - 2\lambda_2)^2 r_n^2) \text{Var}(Z_n) + r_n^2 K_n$$

where $C_{N,\lambda_1,\lambda_2} = K_n/2(\lambda_1 + \lambda_2)$ where

$$K_n = \frac{1}{N} \left((1 - 2\lambda_1)^2 \left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n(i)^2) \right) + \lambda_1 - \lambda_1^2 + \frac{N}{4} \right),$$

which implies $\text{Var}(Z_n) \sim \frac{C_{N,\lambda_1,\lambda_2}}{n^{\gamma}}$. Moreover, using (13) and the fact that $\text{Var}(Z_n) \sim \frac{1}{n^{\gamma}}$,

$$\text{Var}(Z_{n+1}(i)) = \left[(1 - r_n(2\lambda_1 + 1))^2 - r_n(1 - 2\lambda_1)^2 \right] \text{Var}(Z_n(i)) + r_n^2 K_n$$

then $C_{N,\lambda_1,\lambda_2}$ where

$$K_n = \left[\frac{1}{2} - \lambda_1 + \lambda_2 - \frac{(1-2\lambda_1)^2}{4} + \frac{1}{2} - \lambda_2 + \lambda_2^2 - \frac{(1-2\lambda_2)^2}{4} \right] + [(1-2\lambda_2)^2 + r_n(1-r_n(2\lambda_1+1))(1-2\lambda_1)]$$

which implies $\text{Var}(Z_n(i)) \sim \frac{1}{n^{\gamma}}$. □

Proof. (proof of (ii) Theorem 3.1)

Consider the following recursive equation satisfied, for any $i \in \{1, \dots, N\}$ by the L^2 -distance between one component and the mean field. With

$$x_n := \mathbb{E}[(Z_n(i) - Z_n)^2] = \text{Var}(Z_n(i) - Z_n),$$

it holds

$$\begin{aligned}
x_{n+1} &\stackrel{*}{=} \mathbb{E} \left(\text{Var} \left[(1 - r_n^l - r_n^g)(Z_n(i) - Z_n) + r_n^l(\xi_n^l(i) - \frac{1}{N} \sum_j \xi_n^l(j)) \middle| \mathcal{F}_n \right] \right) \\
&+ \text{Var} \left[Z_n(i) - 2\lambda_1 r_n^l Z_n(i) + r_n^g(Z_n(i) - (1 - 2\lambda_2)Z_n) - Z_n(1 - 2\lambda_1 r_n^l - r_n^g) \right] \\
&= (r_n^l)^2 \mathbb{E} \left[\text{Var}(\xi_n^l(i) - \frac{1}{N} \sum_i \xi_n^l(i) | \mathcal{F}_n) \right] + \text{Var} \left[(1 - 2\lambda_1 r_n^l - r_n^g)(Z_n(i) - Z_n) \right] \\
&= (1 - 2\lambda_1 r_n^l - r_n^g)^2 \text{Var}(Z_n(i) - Z_n) \\
&+ (r_n^l)^2 \left((1 - \frac{1}{N})^2 + (\frac{N-1}{N^2}) \right) \mathbb{E} \left[\text{Var}(\xi_n^l(i) | \mathcal{F}_n) \right] \\
&= (1 - 2\lambda_1 r_n^l - r_n^g)^2 x_n \\
&+ \frac{N-1}{N} (r_n^l)^2 \left[[(1 - 2\lambda_1) \mathbb{E}(Z_n(i)) + \lambda_1] - \left[\frac{(1 - 2\lambda_1)^2}{N} \mathbb{E}(Z_n(i)^2) + \lambda_1^2 + 2\lambda_1(1 - 2\lambda_1) \mathbb{E}(Z_n) \right] \right].
\end{aligned}$$

Therefore we obtain

$$x_{n+1} = \left(1 - 4\lambda_1 r_n^l - 2r_n^g + 4\lambda_1^2 (r_n^l)^2 + (r_n^g)^2 + 2\lambda_1 r_n^l r_n^g \right) x_n + (r_n^l)^2 \varepsilon_n, \quad (16)$$

where $\varepsilon_n = \frac{N-1}{N} \left(\frac{1}{2} - \left[\frac{(1-2\lambda_1)^2}{N} \mathbb{E}(Z_n(i)^2) + \lambda_1 - \lambda_1^2 \right] \right)$ is bounded and not equal zero for $N > 1$.

(a) For $\gamma_1 < \gamma_2$ the relation (16) gives $x_{n+1} = [1 - 4\lambda_1 r_n^l - \mathbf{o}(r_n^l)]x_n + (r_n^l)^2 \varepsilon_n$ that can be written as $x_{n+1} = (1 - \frac{4\lambda_1}{n^{\gamma_1}})x_n + \frac{\varepsilon_n}{n^{2\gamma_1}}$ which implies $x_n \sim \frac{C_{N,\lambda_1}}{n^{\gamma_1}}$ where $C_{N,\lambda_1} = \varepsilon_n/4\lambda_1$.

Also for $\gamma_1 = \gamma_2$ we have from (16)

$$x_{n+1} = [(1 - r_n - 2\lambda_1 r_n)^2]x_n + r_n^2 \varepsilon_n = [1 - (2 + 4\lambda_1)r_n + \mathbf{o}(r_n)]x_n + r_n^2 \varepsilon_n$$

which implies $x_n \sim \frac{C_{N,\lambda_1}}{n^\gamma}$ where $C_{N,\lambda_1} = \varepsilon_n/2 + 4\lambda_1$.

(b) When $\gamma_2 < \gamma_1$, $x_{n+1} = (1 - \frac{2}{n^{\gamma_2}} + \mathbf{o}(\frac{1}{n^{\gamma_2}}))x_n + \frac{\varepsilon_n}{n^{2\gamma_1}}$ where $C_{N,\lambda_1} = \varepsilon_n/2$ implies $x_n \sim \frac{C_{N,\lambda_1}}{n^{2\gamma_1 - \gamma_2}}$.

Thus in all cases, $\lim_{n \rightarrow \infty} \mathbb{E}[(Z_n(i) - Z_n)^2] = 0$. \square

Proof. (iii) Theorem 3.1

Now to prove that the convergence is towards $\frac{1}{2}$ a.s, we use (10) and consider

$$\begin{aligned}
&\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2 | \mathcal{F}_n] \\
&= \left(Z_n - \frac{1}{2} \right)^2 [1 + 4(r_n^l)^2 \lambda_1^2 + 4(r_n^g)^2 \lambda_2^2 - 4r_n^l \lambda_1 - 4r_n^g \lambda_2 + 4r_n^l r_n^g \lambda_1 \lambda_2] \\
&\quad + (r_n^l)^2 \mathbb{E}[(\Delta M_{n+1}^l)^2 | \mathcal{F}_n] + (r_n^g)^2 \mathbb{E}[(\Delta M_{n+1}^g)^2 | \mathcal{F}_n] \\
&= \left(Z_n - \frac{1}{2} \right)^2 [1 - 4r_n^l \lambda_1 - 4r_n^g \lambda_2 + \mathbf{o}(r_n^l) + \mathbf{o}(r_n^g)] \\
&\quad + (r_n^l)^2 \left[4\lambda_1^2 (Z_n - \frac{1}{2})^2 + \mathbb{E}[(\Delta M_{n+1}^l)^2 | \mathcal{F}_n] \right] + (r_n^g)^2 \left[4\lambda_2^2 (Z_n - \frac{1}{2})^2 + \mathbb{E}[(\Delta M_{n+1}^g)^2 | \mathcal{F}_n] \right].
\end{aligned}$$

Thus, $\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2 | \mathcal{F}_n] \leq (Z_n - \frac{1}{2})^2 + (r_n^l)^2 W_n^l + (r_n^g)^2 W_n^g$ where

$$W_n^l := (r_n^l)^2 \left[4\lambda_1^2 \left(Z_n - \frac{1}{2} \right)^2 + \mathbb{E}[(\Delta M_{n+1}^l)^2 | \mathcal{F}_n] \right],$$

$$W_n^g := (r_n^g)^2 \left[4\lambda_2^2 \left(Z_n - \frac{1}{2} \right)^2 + \mathbb{E}[(\Delta M_{n+1}^g)^2 | \mathcal{F}_n] \right].$$

By assumption $\frac{1}{2} < \gamma_1 \leq 1$ and $\frac{1}{2} < \gamma_2 \leq 1$, so $(Z_{n+1} - \frac{1}{2})^2$ is a positive super-martingale and almost sure convergence holds. It is enough to consider L^2 convergence in order to identify the (deterministic) limit. We consider

$$\begin{aligned} \mathbb{E} \left(\mathbb{E} \left((Z_{n+1} - \frac{1}{2})^2 | \mathcal{F}_n \right) \right) &= \mathbb{E} \left(\left(Z_n - \frac{1}{2} \right)^2 \left[1 - 4r_n^l \lambda_1 - 4r_n^g \lambda_2 + 4r_n^l r_n^g \lambda_1 \lambda_2 \right] \right) \\ &\quad + (r_n^l)^2 K_n^l + (r_n^g)^2 K_n^g \end{aligned}$$

and let $y_n := \mathbb{E}(Z_n - \frac{1}{2})^2$, thus

$$y_{n+1} = \left(1 - 4r_n^l \lambda_1 - 4r_n^g \lambda_2 + \lambda_1^2 (r_n^l)^2 + \lambda_2^2 (r_n^g)^2 + 4r_n^l r_n^g \lambda_1 \lambda_2 \right) y_n + (r_n^l)^2 K_{n+1}^l + (r_n^g)^2 K_{n+1}^g$$

where $0 \leq K_{n+1}^l := \mathbb{E}[(\Delta M_{n+1}^l)^2] \leq K$, $0 \leq K_{n+1}^g := \mathbb{E}[(\Delta M_{n+1}^g)^2] \leq K$, by $\lambda_i \leq K$ and by lemma A.1, $\lim_{n \rightarrow \infty} x_n = 0$. So, $\lim \mathbb{E}(Z_n - \frac{1}{2})^2 = 0$. Using the fact that $(Z_n)_n$ converges a.s., then $Z_n \xrightarrow{\text{a.s.}} \frac{1}{2}$. The proof of the case when $\gamma_1 = \gamma_2$ is essentially the same as above using (15). Indeed,

$$\begin{aligned} \mathbb{E} \left[\left(Z_{n+1} - \frac{1}{2} \right)^2 | \mathcal{F}_n \right] &= \left(Z_n - \frac{1}{2} \right)^2 [1 - 2r_n(\lambda_1 + \lambda_2)]^2 + r_n^2 \mathbb{E}[\Delta \tilde{M}_{n+1}(i)^2 | \mathcal{F}_n] \\ &\quad + 2 \left(Z_n - \frac{1}{2} \right) [1 - 2r_n(\lambda_1 + \lambda_2)] r_n \mathbb{E}[\Delta \tilde{M}_{n+1}(i) | \mathcal{F}_n]. \end{aligned}$$

Thus $\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2 | \mathcal{F}_n] \leq (Z_n - \frac{1}{2})^2 + r_n^2 \tilde{W}_n$, where

$$\tilde{W}_n = r_n^2 \left(4(\lambda_1 + \lambda_2)^2 (Z_n - \frac{1}{2})^2 + \mathbb{E}[(\Delta \tilde{M}_{n+1}(i))^2 | \mathcal{F}_n] \right).$$

To prove the a.s. synchronization, since L^2 -synchronization holds, it is enough to show a.s convergence exists for $(Z_n(i))_n$. We can obtain that $(Z_n(i))_n$ is quasi-martingale, *i.e.* satisfies

$$\sum_{n=0}^{+\infty} \mathbb{E} | \mathbb{E}[Z_{n+1}(i) | \mathcal{F}_n] - Z_n(i) | < +\infty. \quad (17)$$

Indeed using (3) and the fact that $(Z_n(i))_n$ goes to $\frac{1}{2}$ in L^2 , we obtain the convergence using the following inequality and bounding the expectation with the second moment,

$$\sum_n \mathbb{E} | \mathbb{E}(Z_{n+1}(i) - Z_n(i) | \mathcal{F}_n) | \leq 2\lambda_1 r_n^l \mathbb{E} | Z_n(i) - \frac{1}{2} | + 2\lambda_2 r_n^g \mathbb{E} | Z_n - \frac{1}{2} | + r_n^g \mathbb{E} | Z_n - Z_n(i) |.$$

□

Proof. (Theorem 3.2)

We shall consider two different situations of λ_j and different relationships between γ_j .

- **Case** $\lambda_1 \neq 0, \lambda_2 = 0$.

First consider to the recursive equation (11) of $\text{Var}(Z_n)$.

- When $\gamma_1 < \gamma_2$,

$$\text{Var}(Z_{n+1}) = \left(1 - \frac{4\lambda_1}{n^{\gamma_1}} + \mathbf{o}\left(\frac{1}{n^{\gamma_1}}\right)\right) \text{Var}(Z_n) + \frac{K_n}{n^{2\gamma_1}},$$

where

$$K_n = \frac{1}{N} \left(\frac{1}{2} - \frac{(1-2\lambda_1)^2}{N} \sum_{i=1}^N \mathbb{E}(Z_n(i)^2) - (\lambda_1 - \lambda_1^2) \right),$$

where $C_{N,\lambda_1} = K_n/4\lambda_1$, it implies $\text{Var}(Z_n) \sim \frac{C_{N,\lambda_1}}{n^{\gamma_1}}$. It thus means $(Z_n)_n$ converges to a constant. Using similar argument in proof of Theorem 3.1(ii), we can proof that $\lim_{n \rightarrow \infty} Z_n = \frac{1}{2}$ a.s. To show it, using (10) and let $\lambda_2 = 0$, then the result is obtained.

To study the synchronization, consider to the L^2 -distance (16) which behaves as follows

$$x_{n+1} = \left(1 - \frac{4\lambda_1}{n^{\gamma_1}} + \mathbf{o}\left(\frac{1}{n^{\gamma_1}}\right)\right) x_n + \frac{\varepsilon_n}{n^{2\gamma_1}},$$

where $\varepsilon_n = \frac{N-1}{N} \left(\frac{1}{2} - \left[\frac{(1-2\lambda_1)^2}{N} \mathbb{E}(Z_n(i)^2) + \lambda_1 - \lambda_1^2 \right] \right)$ and where $C_{N,\lambda_1} = \varepsilon_n/4\lambda_1$. One can then derive $x_n \sim \frac{C_{N,\lambda_1}}{n^{\gamma_1}}$ which implies $\lim_{n \rightarrow \infty} x_n = 0$. Using a similar argument as in Theorem 3.1(ii), we get that synchronization holds a.s.

- When $\gamma_1 = \gamma_2 (=:\gamma)$, so (11)

$$\text{Var}(Z_{n+1}) = \left(1 - \frac{4\lambda_1}{n^\gamma} + \mathbf{o}\left(\frac{1}{n^\gamma}\right)\right) \text{Var}(Z_n) + \frac{K_n}{n^{2\gamma}},$$

where $C_{N,\lambda_1} = K_n/4\lambda_1$ which implies $\text{Var}(Z_n) \sim \frac{C_{N,\lambda_1}}{n^\gamma}$. Using (15) and let $\lambda_2 = 0$ then the conclusion of a.s. follows.

To study the L^2 -distance's behavior, consider (16)

$$x_{n+1} = \left(1 - \frac{(1-2\lambda_1)}{n^\gamma} + \mathbf{o}\left(\frac{1}{n^\gamma}\right)\right) x_n + \frac{\varepsilon_n}{n^{2\gamma}},$$

where $C_{N,\lambda_1} = \varepsilon_n/(1-2\lambda_1)$ which implies $x_n \sim \frac{C_{N,\lambda_1}}{n^\gamma}$, synchronization holds a.s.

- When $\gamma_2 < \gamma_1$. Let us consider the recursive equation (11),

$$\text{Var}(Z_{n+1}) = \left(1 - \frac{4\lambda_1}{n^{\gamma_1}}\right) \text{Var}(Z_n) + \frac{K_n}{n^{2\gamma_2}},$$

where $C_{N,\lambda_1} = K_n/4\lambda_1$ which implies $\text{Var}(Z_n) \sim \frac{C_{N,\lambda_1}}{n^{2\gamma_2-\gamma_1}}$ which implies $(Z_n)_n$ converges to a constant. Again by similar argument as before, we get $\lim_{n \rightarrow \infty} Z_n = \frac{1}{2}$ a.s. Moreover considering the L^2 -distance's (16) behaves,

$$x_{n+1} = \left(1 - \frac{2}{n^{\gamma_2}} + \mathbf{o}\left(\frac{1}{n^{\gamma_2}}\right)\right) x_n + \frac{\varepsilon_n}{n^{2\gamma_1}},$$

where $C_{N,\lambda_1} = \varepsilon_n/2$ which implies $x_n \sim \frac{C_{N,\lambda_1}}{n^{2\gamma_1-\gamma_2}}$, synchronization holds a.s.

- **Case** $\lambda_1 = 0, \lambda_2 \neq 0$.
- When $\gamma_1 < \gamma_2$,

$$\text{Var}(Z_{n+1}) = \left(1 - \frac{4\lambda_2}{n^{\gamma_2}} + \mathbf{o}\left(\frac{1}{n^{\gamma_2}}\right)\right) \text{Var}(Z_n) + \frac{K_n}{n^{2\gamma_1}},$$

where $C_{N,\lambda_2} = K_n/4\lambda_2$ and $K_n = \frac{1}{N} \left\{ \frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n(i)^2) \right\}$ which implies $\text{Var}(Z_n) \sim \frac{C_{N,\lambda_2}}{n^{2\gamma_1-\gamma_2}}$, thus means Z_n converges to constant. Using similar argument in proof of Th. 3.1(ii), we can prove

that $\lim_{n \rightarrow \infty} Z_n = \frac{1}{2}$ a.s. To show it, using (10) and let $\lambda_1 = 0$, then the result is obtained. To study the synchronization, consider the L^2 -distance which behaves as follows

$$x_{n+1} = \left(1 - \frac{2}{n^{\gamma_2}} + \mathbf{o}\left(\frac{1}{n^{\gamma_2}}\right)\right)x_n + \frac{\varepsilon_n}{n^{2\gamma_1}},$$

where $C_{N,\lambda_2} = \varepsilon_n/2$ which implies $x_n \sim \frac{C_{N,\lambda_2}}{n^{2\gamma_1-\gamma_2}}$, synchronization holds a.s.

- When $\gamma_2 < \gamma_1$. Let us consider the recursive equation (11),

$$\text{Var}(Z_{n+1}) = \left(1 - \frac{4\lambda_2}{n^{\gamma_2}} + \mathbf{o}\left(\frac{1}{n^{\gamma_2}}\right)\right) \text{Var}(Z_n) + \frac{K_n}{n^{2\gamma_2}},$$

where $C_{N,\lambda_2} = K_n/4\lambda_2$ which implies $\text{Var}(Z_n) \sim \frac{C_{N,\lambda_2}}{n^{\gamma_2}}$ which implies Z_n converges to constant then again by similar argument Th. 3.1(ii) hence $\lim_{n \rightarrow \infty} Z_n = \frac{1}{2}$ a.s. To study the synchronization, consider the L^2 -distance which behaves as follows

$$x_{n+1} = \left(1 - \frac{2}{n^{\gamma_2}}\right)x_n + \frac{\varepsilon_n}{n^{2\gamma_1}},$$

where $C_{N,\lambda_2} = \varepsilon_n/2$ which implies $x_n \sim \frac{C_{N,\lambda_2}}{n^{2\gamma_1-\gamma_2}}$, synchronization holds almost surely.

- When $\gamma_1 = \gamma_2$. It holds

$$\text{Var}(Z_{n+1}) = \left(1 - \frac{4\lambda_2}{n^\gamma} + \mathbf{o}\left(\frac{1}{n^\gamma}\right)\right) \text{Var}(Z_n) + \frac{K_n}{n^{2\gamma}},$$

where $C_{N,\lambda_2} = K_n/4\lambda_2$ which implies $\text{Var}(Z_n) \sim \frac{C_{N,\lambda_2}}{n^\gamma}$. Using (14) and let $\lambda_1 = 0$ then the conclusion of a.s. is obtained.

To study the L^2 -distance's behavior,

$$x_{n+1} = \left(1 - \frac{1}{n^\gamma} + \mathbf{o}\left(\frac{1}{n^\gamma}\right)\right)x_n + \frac{\varepsilon_n}{n^{2\gamma}},$$

which implies $x_n \sim \frac{C_{N,\lambda_2}}{n^\gamma}$, synchronization holds a.s. □

Proof. ((i) of Theorem 3.4)

If $\lambda_1 = \lambda_2 = 0$ then $(Z_n)_n$ is a bounded martingale. Therefore, it converges a.s. to a random variable Z_∞ . Let us consider

$$\text{Var}(Z_{n+1}) = \left(1 - \frac{1}{4N n^{2\gamma_2}}\right) \text{Var}(Z_n) + \frac{K_n}{n^{2\min(\gamma_1, \gamma_2)}},$$

where

- a) If $\gamma_1 < \gamma_2$ then $C_N = 4NK_n$ and $K_n = \frac{1}{N} \left[\left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n(i)^2) \right) \right]$. Using lemma A.1, we get

$$\text{Var}(Z_n) - e^{-\sum_{n=0}^{\infty} \frac{1}{n^{2\gamma_2}}} \sim \frac{C_N}{n^{2\gamma_1-1}}.$$

- b) $C_N = 4NK_n$ and $K_n = \frac{1}{4}$ when $\gamma_2 < \gamma_1$, so $\text{Var}(Z_n) - e^{-\sum_{n=0}^{\infty} \frac{1}{n^{2\gamma_2}}} \sim \frac{C_N}{n^{2\gamma_2-1}}$.

Also when $\gamma_1 = \gamma_2$,

$$\text{Var}(Z_{n+1}) = \left(1 - \frac{1}{n^{2\gamma}}\right) \text{Var}(Z_n) + \frac{K_n}{n^{2\gamma}},$$

where $K_n = \left[\left(\frac{1}{2} - \frac{1}{N} \sum_{i=1}^N \mathbb{E}(Z_n(i)^2) \right) + \frac{N}{4} \right]$ which implies $\text{Var}(Z_n) - e^{-\sum_{n=0}^{\infty} \frac{1}{n^{2\gamma}}} \sim \frac{C_N}{n^{2\gamma-1}}$ where $C_N = K_n$. □

Proof. ((ii) Theorem 3.4)

To study the synchronization phenomenon, consider the L^2 -distance

$$x_{n+1} = (1 - 2r_n^g + (r_n^g)^2)x_n + (r_n^l)^2\varepsilon_n = \left(1 - \frac{2}{n^{\gamma_2}} + \mathbf{o}\left(\frac{1}{n^{\gamma_2}}\right)\right)x_n + \frac{\varepsilon_n}{n^{2\gamma_1}},$$

where $\varepsilon_n = \frac{N-1}{N}\{\mathbb{E}(Z_n) - [\frac{1}{N}\mathbb{E}(Z_n(i)^2)]\}$ and then $C_N = \varepsilon_n/2$. Thus, $x_n \sim \frac{C_N}{n^{2\gamma_1 - \gamma_2}}$, hence the L^2 -synchronization holds as $n \rightarrow \infty$. Moreover when $\gamma_1 = \gamma_2$, $x_{n+1} = (1 - \frac{2}{n^\gamma})x_n + \frac{\varepsilon_n}{n^{2\gamma}}$ which implies $x_n \sim \frac{C_N}{n^\gamma}$. To show that synchronization holds a.s., we shall show that the a.s. limit of $(Z_n(i))_n$ exists. We observe that $-Z_n(i)_n$ is a quasi-martingale, since $\mathbb{E}(Z_{n+1}(i) - Z_n(i)|\mathcal{F}_n) = r_n^g(Z_n - Z_n(i))$. As bounded quasi-martingale, it converges a.s. \square

6. PROOFS OF THE CLTs (THEOREM 4.1)

We now prove the central limit theorems to study fluctuations. Recall we are using the notation $a_n \simeq b_n$ when $\lim_n \frac{a_n}{b_n} = cst$.

Define $X_k = Z_k - Z_k(i)$. Set $L_0 = X_0$ and define

$$L_n := X_n - \sum_{k=0}^{n-1} (\mathbb{E}[X_{k+1}|\mathcal{F}_k] - X_k). \quad (18)$$

Then it follows:

$$X_{n+1} = [1 - 2\lambda_1 r_n^l - r_n^g]X_n + \Delta L_{n+1} \quad (19)$$

where $\Delta L_{n+1} = L_{n+1} - L_n$. Note that L_n is an \mathcal{F}_n -martingale by construction. Iterating the above relation, we can write

$$X_n = c_{1,n}X_1 + \sum_{k=1}^{n-1} c_{k+1,n}\Delta L_{k+1} \quad (20)$$

where $c_{1,n} = 1$ and $c_{k,n} = \prod_{h=k}^{n-1} [1 - 2\lambda_1 r_h^l - r_h^g]$ for $k < n$.

Proof. ((i)(a) Theorem 4.1)

If $\gamma_1 < \gamma_2$, it is easy to check that $\lim_{n \rightarrow \infty} n^{\frac{\gamma_1}{2}} c_{1,n} = 0$ since,

$$c_{1,n} = \prod_{h=1}^{n-1} [1 - 2\lambda_1 r_h^l - \mathbf{o}(r_h^l)] \sim \exp\left(\frac{-2\lambda_1}{1-\gamma_1} n^{1-\gamma_1}\right).$$

$$\lim_{k \rightarrow \infty} \sup_{n \geq k} \left| \frac{c_{k,n}}{\exp\left[\frac{-2\lambda_1}{1-\gamma_1} (n^{1-\gamma_1} - k^{1-\gamma_1})\right]} - 1 \right| = 0. \quad (21)$$

So it is enough to prove the convergence $n^{\frac{\gamma_1}{2}} \sum_k c_{k+1,n} \Delta L_{k+1} \rightarrow \mathcal{N}(0, (1 - 1/N)/16\lambda_1)$. First, let us define $U_{n,k} = n^{\frac{\gamma_1}{2}} c_{k+1,n} \Delta L_{k+1}$ and $\mathcal{G}_{n,k} = \mathcal{F}_{k+1}$. Thus $\{U_{n,k}, \mathcal{G}_{n,k} : 1 \leq k \leq n\}$ is a square-integrable martingale difference array.

Indeed we have $\mathbb{E}(U_{n,k}^2) < +\infty$ and $\mathbb{E}(U_{n,k+1}|\mathcal{G}_{n,k}) = n^{\frac{\gamma_1}{2}} c_{k+1,n} \mathbb{E}(\Delta L_{k+1}|\mathcal{F}_{k+1}) = 0$. Then, using the Theorem recalled as A.3, through the following statements for $U_{n,k} = n^{\frac{\gamma_1}{2}} c_{k+1,n} \Delta L_{k+1}$.

- 1) $\max_{1 \leq k \leq n} |U_{n,k}| \rightarrow 0$.
- 2) $\mathbb{E}[\max_{1 \leq k \leq n} U_{n,k}^2]$ is bounded in n .
- 3) $\sum_{k=1}^n U_{n,k}^2 \rightarrow (1 - 1/N)/16\lambda_1$ a.s.

Considering 1), since $\Delta L_{n+1} - (X_{n+1} - X_n) \sim 2\lambda_1 X_n/n^{\gamma_1}$, $|\Delta L_{n+1}| = \mathcal{O}(n^{-\gamma_1})$.

For 2), use (21) and 1) to obtain

$$\begin{aligned}
\mathbb{E}[\max_{1 \leq k \leq n} U_{n,k}^2] &\leq \mathbb{E}[\sum_{k=1}^n U_{n,k}^2] \\
&= n^{\gamma_1} \sum_{k=1}^n c_{k+1,n}^2 \mathbb{E}[(\Delta L_{k+1})^2] \\
&\simeq n^{\gamma_1} \sum_{k=1}^n e^{\frac{-4\lambda_1}{1-\gamma_1}(n^{1-\gamma_1} - k^{1-\gamma_1})} \mathcal{O}(k^{-2\gamma_1}) \\
&= n^{\gamma_1} e^{\frac{-4\lambda_1}{1-\gamma_1} n^{1-\gamma_1}} \sum_{k=1}^{n-1} e^{\frac{4\lambda_1}{1-\gamma_1} k^{1-\gamma_1}} \mathcal{O}(k^{-2\gamma_1}) + \frac{n^2 \mathcal{O}(n^{-2\gamma_1})}{n}.
\end{aligned}$$

Thus, $\mathbb{E}[\max_{1 \leq k \leq n} U_{n,k}^2]$ is bounded in n .

Let us now consider 3). We have

$$\sum_{k=1}^n U_{n,k}^2 = n^{\gamma_1} \sum_{k=1}^n c_{k+1,n}^2 (\Delta L_{n+1})^2 \simeq n^{\gamma_1} \sum_{k=1}^n \frac{k^{-2\gamma_1} e^{\frac{4\lambda_1}{1-\gamma_1} k^{1-\gamma_1}}}{e^{\frac{4\lambda_1}{1-\gamma_1} n^{1-\gamma_1}}} (\Delta L_{k+1})^2 k^{2\gamma_1}.$$

From 1) we obtain

$$\begin{aligned}
\Delta L_{k+1}^2 &= (X_{k+1} - X_k + 2\lambda_1 r_k^l X_k)^2 \\
&= ((Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i)) + 2\lambda_1 r_k^l (Z_k - Z_k(i)))^2 \\
&= [(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))]^2 + 4\lambda_1^2 (r_k^l)^2 (Z_k - Z_k(i))^2 \\
&\quad + (r_k^l)(Z_k - Z_k(i))[(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))].
\end{aligned}$$

Since $Z_n \sim Z_n(i)$ a.s. and $X_k^2 = \mathcal{O}(k^{-2\gamma_1})$ so,

$$\sum_{k=1}^n U_{n,k}^2 = n^{\gamma_1} \sum_{k=1}^n c_{k+1,n}^2 [(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))].$$

Let $V_k = k^{2\gamma_1} [(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$ and setting the

$b_n = \frac{1}{n^{\gamma_1}} e^{+4\lambda_1 \frac{n^{1-\gamma_1}}{1-\gamma_1}}$ and $a_k = \frac{k^{2\gamma_1}}{c_{1,n}^2} e^{-4\lambda_1 \frac{k^{1-\gamma_1}}{1-\gamma_1}}$. Hence, by Lemma A.2, it holds

$\frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} \xrightarrow{n \rightarrow \infty} \frac{1}{4\lambda_1}$. Indeed,

$$\begin{aligned}
\frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} &= \frac{n^{\gamma_1}}{e^{\frac{4\lambda_1}{1-\gamma_1} n^{1-\gamma_1}}} \sum_{k=1}^n k^{-2\gamma_1} e^{\frac{4\lambda_1}{1-\gamma_1} k^{1-\gamma_1}} \\
&\simeq n^{\gamma_1} e^{\frac{-4\lambda_1}{1-\gamma_1} n^{1-\gamma_1}} \int_1^n u^{-2\gamma_1} e^{4\lambda_1 \frac{u^{1-\gamma_1}}{1-\gamma_1}} du \\
&= n^{\gamma_1} e^{\frac{-4\lambda_1}{1-\gamma_1} n^{1-\gamma_1}} \int_1^n \frac{u^{-\gamma_1}}{4\lambda_1} 4\lambda_1 u^{-\gamma_1} e^{4\lambda_1 \frac{u^{1-\gamma_1}}{1-\gamma_1}} du \\
&= n^{\gamma_1} e^{\frac{-4\lambda_1}{1-\gamma_1} n^{1-\gamma_1}} \left[\left(\frac{u^{-\gamma_1}}{4\lambda_1} e^{4\lambda_1 \frac{u^{1-\gamma_1}}{1-\gamma_1}} \right)_1^n + \frac{\gamma_1}{4\lambda_1} \int_1^n u^{-\gamma_1-1} e^{4\lambda_1 \frac{u^{1-\gamma_1}}{1-\gamma_1}} du \right] \\
&= \frac{1}{4\lambda_1} + \frac{\gamma_1 n^{\gamma_1}}{4\lambda_1} e^{-4\lambda_1 \frac{n^{1-\gamma_1}}{1-\gamma_1}} \int_1^n \frac{1}{u^{1+\gamma_1}} e^{4\lambda_1 \frac{u^{1-\gamma_1}}{1-\gamma_1}} du = \frac{1}{4\lambda_1}
\end{aligned}$$

implies that $\sum_{k=1}^n U_{n,k}^2$ converges to $\frac{V}{4\lambda_1}$ a.s., where V is deterministic such that $\mathbb{E}(V_{k+1}|\mathcal{F}_k) \rightarrow V$. Indeed, we compute

$$\begin{aligned} & \mathbb{E}(k^{2\gamma_1}(Z_{k+1}(i) - Z_k(i))^2|\mathcal{F}_k) = \\ & k^{2\gamma_1} \left((r_k^l)^2 \mathbb{E} \left((\xi_{k+1}^l(i) - Z_k(i))^2|\mathcal{F}_k \right) + (r_k^g)^2 \mathbb{E} \left((\xi_{k+1}^g - Z_k(i))^2|\mathcal{F}_k \right) \right. \\ & \quad \left. + 2r_k^l r_k^g \mathbb{E} \left((\xi_{k+1}^l(i) - Z_k(i))(\xi_{k+1}^g - Z_k(i))|\mathcal{F}_k \right) \right) \\ & = k^{2\gamma_1} \left((r_k^l)^2 \left(\text{Var}(\xi_{k+1}^l(i)|\mathcal{F}_k) + \mathbb{E}((\xi_{k+1}^l(i) - Z_k(i))^2|\mathcal{F}_k) \right) + (r_k^g)^2 \left(\text{Var}(\xi_{k+1}^g|\mathcal{F}_k) + \mathbb{E}((\xi_{k+1}^g - Z_k(i))^2|\mathcal{F}_k) \right) \right. \\ & \quad \left. + 2r_k^l r_k^g \mathbb{E} \left((\xi_{k+1}^l(i) - Z_k(i))(\xi_{k+1}^g - Z_k(i))|\mathcal{F}_k \right) \right) \end{aligned}$$

behaves like $k^{2\gamma_1} \left(\frac{(r_k^l)^2}{4} + \frac{(r_k^g)^2}{4} \right)$ when $k \rightarrow \infty$. Similarly,

$$\begin{aligned} \mathbb{E}[k^{2\gamma_1}(Z_{k+1} - Z_k)^2|\mathcal{F}_k] & = k^{2\gamma_1} \left((r_k^l)^2 \mathbb{E} \left[\left(\frac{1}{N} \sum_i \xi_{k+1}^l(i) - Z_k \right)^2|\mathcal{F}_k \right] + (r_k^g)^2 \mathbb{E} \left[(\xi_{k+1}^g - Z_k)^2|\mathcal{F}_k \right] \right. \\ & \quad \left. + 2r_k^l r_k^g \mathbb{E} \left[\left(\frac{1}{N} \sum_i \xi_{k+1}^l(i) - Z_k \right) (\xi_{k+1}^g - Z_k) |\mathcal{F}_k \right] \right) \\ & = k^{2\gamma_1} \left((r_k^l)^2 \left(\text{Var} \left[\frac{1}{N} \sum_i \xi_{k+1}^l(i) |\mathcal{F}_k \right] + \mathbb{E}^2 \left(\frac{1}{N} \sum_i \xi_{k+1}^l(i) - Z_k |\mathcal{F}_k \right) \right) \right. \\ & \quad \left. + (r_k^g)^2 \left(\text{Var}[\xi_{k+1}^g|\mathcal{F}_k] + \mathbb{E}^2(\xi_{k+1}^g - Z_k|\mathcal{F}_k) \right) \right. \\ & \quad \left. + 2r_k^l r_k^g \mathbb{E} \left[\left(\frac{1}{N} \sum_i \xi_{k+1}^l(i) - Z_k \right) (\xi_{k+1}^g - Z_k) |\mathcal{F}_k \right] \right) \end{aligned}$$

behaves like $k^{2\gamma_1} \left(\frac{(r_k^l)^2}{4N} + \frac{(r_k^g)^2}{4} \right)$ when $k \rightarrow \infty$; and

$$\begin{aligned} & \mathbb{E}[k^{2\gamma_1}(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))|\mathcal{F}_k] = \\ & k^{2\gamma_1} \left((r_k^l)^2 \mathbb{E} \left[(\xi_{k+1}^l(i) - Z_k(i)) \left(\frac{1}{N} \sum_i \xi_{k+1}^l(i) - Z_k \right) |\mathcal{F}_k \right] + (r_k^g)^2 \mathbb{E} \left[(\xi_{k+1}^g - Z_k(i)) (\xi_{k+1}^g - Z_k) |\mathcal{F}_k \right] \right) \end{aligned}$$

behaves like $k^{2\gamma_1} \left(\frac{(r_k^l)^2}{4N} + \frac{(r_k^g)^2}{4} \right)$. It follows

$$\begin{aligned} \mathbb{E}(V_{k+1}|\mathcal{F}_k) & = k^{2\gamma_1} \left[(r_k^l)^2 \left(\text{Var}[\xi_{k+1}^l(i)|\mathcal{F}_k] + \text{Var} \left[\frac{1}{N} \sum_i \xi_{k+1}^l(i) |\mathcal{F}_k \right] \right. \right. \\ & \quad \left. \left. - 2 \mathbb{E} \left[(\xi_{k+1}^l(i) - Z_k(i)) \left(\frac{1}{N} \sum_i \xi_{k+1}^l(i) - Z_k \right) |\mathcal{F}_k \right] \right] \right] \xrightarrow{a.s.} \frac{1}{4} \left(1 - \frac{1}{N} \right). \end{aligned}$$

Thus, $V_k \xrightarrow{a.s.} \frac{1}{4} \left(1 - \frac{1}{N} \right)$.

The proof of next parts and the other theorems follows along the same lines as above. We sketch the essential arguments below.

- **Case** $\gamma_1 = \gamma_2 (= \gamma)$. We obtain that $c_{1,n} \sim \exp \left[\frac{-(1+2\lambda_1)}{1-\gamma} n^{1-\gamma} \right]$.

Therefore $\lim_{n \rightarrow \infty} n^{\frac{\gamma}{2}} c_{1,n} = 0$. So,

$$\limsup_{k \rightarrow \infty} \sup_{n \geq k} \left| \frac{c_{k,n}}{\exp\left[\frac{-(1+2\lambda_1)}{1-\gamma}(n^{1-\gamma} - k^{1-\gamma})\right]} - 1 \right| = 0$$

and that 1), 2) (as in proof of (i)(a)) hold.

So it is enough to prove that $\sum_{k=1}^n U_{n,k}^2 \rightarrow (1 - 1/N)/4(1 + 2\lambda_1)$. By Lemma A.2 and letting $b_n = \frac{1}{n^\gamma} e^{\frac{2(1+2\lambda_1)}{1-\gamma} n^{1-\gamma}}$ and $a_k = \frac{k^{2\gamma}}{c_{1,n}^2} e^{-\frac{2(1+2\lambda_1)}{1-\gamma} k^{1-\gamma}}$, thus $\frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} \rightarrow \frac{1}{2(1+2\lambda_1)}$.

Then consider

$$\begin{aligned} \mathbb{E}(k^{2\gamma}(Z_{k+1}(i) - Z_k(i))^2 | \mathcal{F}_k) &= k^{2\gamma} r_k^{2\gamma} \mathbb{E}[(\tilde{\xi}_{k+1}(i) - Z_k(i))^2 | \mathcal{F}_k] \xrightarrow{a.s.} \\ &\frac{(1 - 2\lambda_1)}{2} + \lambda_1(1 - \lambda_1) - \frac{(1 - 2\lambda_1)^2}{4} - \lambda_1(1 - 2\lambda_1) \\ &+ \frac{(1 - 2\lambda_2)}{2} + \lambda_2(1 - \lambda_2) - \frac{(1 - 2\lambda_2)^2}{4} - \lambda_2(1 - 2\lambda_2) = \frac{1}{2}. \end{aligned}$$

Similarly, $\mathbb{E}(k^{2\gamma}(Z_{k+1} - Z_k)^2 | \mathcal{F}_k) = k^{2\gamma} r_k^{2\gamma} \mathbb{E}[(\frac{1}{N} \sum_i \tilde{\xi}_{k+1}(i) - Z_k)^2 | \mathcal{F}_k] \xrightarrow{a.s.} \frac{1}{2N}$, and

$\mathbb{E}(k^{2\gamma}(Z_{k+1}(i) - Z_k(i))(Z_{k+1} - Z_k) | \mathcal{F}_k) = k^{2\gamma} r_k^{2\gamma} \mathbb{E}[(\tilde{\xi}_{k+1}(i) - Z_k(i))(\frac{1}{N} \sum_i \tilde{\xi}_{k+1}(i) - Z_k) | \mathcal{F}_k] \xrightarrow{a.s.} \frac{1}{2N}$, thus $V_k \xrightarrow{a.s.} \frac{1}{2}(1 - \frac{1}{N})$.

• **Case** $\gamma_1 = \gamma_2 = 1$.

We obtain $c_{1,n} := \prod_{h=1}^n [1 - (1 + 2\lambda_1)r_h] \sim n^{-(1+2\lambda_1)}$. Then $\sqrt{n}c_{1,n} \sim n^{-1-2\lambda_1+\frac{1}{2}} \rightarrow 0$. So it is enough to prove that $\sqrt{n} \sum_k c_{k+1,n} \Delta L_{n+1} \rightarrow \mathcal{N}(0, (1 - 1/N)/2(1 + 4\lambda_1))$. This can be proved by usual three conditions for $U_{n,k+1} = \sqrt{n} \sum_k c_{k+1,n} \Delta L_{n+1}$: 1), 2) (as in previous proofs) and 3) $\sum_{k=1}^n U_{n,k}^2 \rightarrow (1 - 1/N)/2(1 + 4\lambda_1)$.

To investigate these conditions, first consider to 1). Since $\Delta L_{n+1} = X_{n-1} - X_n + (1 + 2\lambda_1)X_n/n$, $|\Delta L_{n+1}| = \mathcal{O}(n^{-1})$.

For 2), use (21) and 1) to obtain

$$\begin{aligned} \mathbb{E}[\max_{1 \leq k \leq n} U_{n,k}^2] &\leq \mathbb{E}[\sum_{k=1}^n U_{n,k}^2] \\ &\simeq \frac{1}{n^{1+4\lambda_1}} \sum_{k=1}^{n-1} \frac{k^2 \mathcal{O}(k^{-2})}{k^{-4\lambda_1}} + \frac{n^2 \mathcal{O}(n^{-2})}{n}. \end{aligned}$$

Thus, $\mathbb{E}[\max_{1 \leq k \leq n} U_{n,k}^2]$ is bounded in n . Let us now consider 3). We have

$$\sum_{k=1}^n U_{n,k}^2 = n \sum_k c_{k+1,n}^2 (\Delta L_{n+1})^2 \simeq \frac{1}{n^{1+4\lambda_1}} \sum_{k=1}^n \frac{k^2 (\Delta L_{n+1})^2}{k^{-4\lambda_1}}.$$

From 1) we obtain

$$\begin{aligned} \Delta L_{n+1}^2 &\simeq [(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))]^2 + r_k^2 (Z_k - Z_k(i))^2 \\ &\quad + r_k^2 (Z_k - Z_k(i)) [(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))] \end{aligned}$$

Since $Z_n \xrightarrow{a.s.} Z_n(i)$ and $X_k^2 = \mathcal{O}(k^{-2})$ so,

$$\sum_{k=1}^n U_{n,k}^2 \xrightarrow{a.s.} n \sum_{k=1}^n c_{k+1,n}^2 [(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))].$$

We use Lemma A.2 with $b_n = n^{1+4\lambda_1}$ and $a_k = k^{-4\lambda_1}$.

Let $U_k = k^2[(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$.

So $\frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} \rightarrow \frac{1}{1+4\lambda_1}$. This implies that $\sum_{k=1}^n U_{n,k}^2$ converges to $\frac{U}{1+4\lambda_1}$ a.s., where V is deterministic such that $\mathbb{E}(V_{k+1}|\mathcal{F}_k) \rightarrow V$. Indeed, $\mathbb{E}(k^2(Z_{k+1}(i) - Z_k(i))^2|\mathcal{F}_k) \xrightarrow{a.s.} \frac{1}{2}$. Similarly, $\mathbb{E}(k^2(Z_{k+1} - Z_k)^2|\mathcal{F}_k) \xrightarrow{a.s.} \frac{1}{2N}$, and $\mathbb{E}(k^2(Z_{k+1}(i) - Z_k(i))(Z_{k+1} - Z_k)|\mathcal{F}_k) \xrightarrow{a.s.} \frac{1}{2N}$.

Thus, $V_k \xrightarrow{a.s.} \frac{1}{2}(1 - \frac{1}{N})$.

[(i)(b) Theorem 4.1]

• **Case** $\gamma_2 < \gamma_1$. Since $c_{1,n} = \prod_{h=1}^{n-1} [1 - r_h^g - \mathbf{o}(r_h^g)] \sim \exp[\frac{-1}{1-\gamma_2} n^{1-\gamma_2}]$ therefore, $n^{\gamma_1 - \frac{\gamma_2}{2}} c_{1,n} \rightarrow 0$. Thus

$$\limsup_{k \rightarrow \infty} \sup_{n \geq k} \left| \frac{c_{k,n}}{\exp[\frac{-1}{1-\gamma_2}(n^{1-\gamma_2} - k^{1-\gamma_2})]} - 1 \right| = 0$$

and that 1) and 2) (as in proof of theorem) hold. So it is enough to prove that

$\sum_{k=1}^n U_{n,k}^2 \rightarrow (1 - 1/N)/4$. We have

$$\begin{aligned} (\Delta L_{n+1})^2 &\simeq [(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))]^2 + (r_k^g)^2 (Z_k - Z_k(i))^2 \\ &\quad + (r_k^g)^2 (Z_k - Z_k(i))[(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))]. \end{aligned}$$

Since $Z_n \xrightarrow{a.s.} Z_n(i)$ and $X_k^2 = \mathcal{O}(k^{-2\gamma_1 + \gamma_2})$ so,

$$\sum_{k=1}^n U_{n,k}^2 = n^{2\gamma_1 - \gamma_2} \sum_{k=1}^n c_{k+1,n}^2 [(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))].$$

We use Lemma A.2 in Appendix with $b_n = \frac{1}{n^{2\gamma_1 - \gamma_2}} e^{\frac{2}{1-\gamma_2} n^{1-\gamma_2}}$ and $a_k = \frac{k^{2\gamma_1}}{c_{1,n}^2} e^{-\frac{2}{1-\gamma_2} k^{1-\gamma_2}}$ thus,

$\frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} \rightarrow \frac{1}{2}$.

Let $V_k = k^{2\gamma_1} [(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$. This implies that $\sum_{k=1}^n U_{n,k}^2$ converges to V a.s., where V is deterministic such that $\mathbb{E}(V_{k+1}|\mathcal{F}_k) \rightarrow V$. Since we completed this computation in the proof of the previous part of theorem, we know that in this case $V_k \xrightarrow{a.s.} \frac{1}{4}(1 - \frac{1}{N})$.

[(ii) Theorem 4.1]

When $\gamma_1 < \gamma_2$, let $X_k = Z_k - \frac{1}{2}$ so,

$$L_n = X_n - \sum_{k=0}^{n-1} \left(\mathbb{E}(Z_{k+1} - \frac{1}{2}|\mathcal{F}_k) - (Z_k - \frac{1}{2}) \right) = X_n + 2(\lambda_1 r_n^l + \lambda_2 r_n^g) \sum_{k=0}^{n-1} X_k$$

and $X_{n+1} \simeq [1 - 2\lambda_1 r_n^l] X_n + \Delta L_{n+1}$. So $c_{1,n} \sim \exp[\frac{-2\lambda_1}{1-\gamma_1} n^{1-\gamma_1}]$ and therefore $n^{\frac{\gamma_1}{2}} c_{1,n} \rightarrow 0$. Then

$$\limsup_{k \rightarrow \infty} \sup_{n \geq k} \left| \frac{c_{k,n}}{\exp[\frac{-2\lambda_1}{1-\gamma_1}(n^{1-\gamma_1} - k^{1-\gamma_1})]} - 1 \right| = 0.$$

It is enough to show that $\sum_{k=1}^n U_{n,k}^2 = n^{\gamma_1} \sum_{k=1}^n c_{k+1,n}^2 k^{-2\gamma_1} (\Delta L_{k+1})^2 k^{2\gamma_1}$ is a constant. Using Lemma A.2, $b_n = \frac{1}{n^{\gamma_1}} e^{\frac{4\lambda_1}{1-\gamma_1} n^{1-\gamma_1}}$ and $a_k = \frac{k^{2\gamma_1}}{c_{1,n}^2} e^{-\frac{4\lambda_1}{1-\gamma_1} k^{1-\gamma_1}}$. Therefore $\frac{1}{b_n} \sum_k \frac{1}{a_k} \rightarrow \frac{1}{4\lambda_1}$. Also

$(\Delta L_{n+1})^2 = (Z_{k+1} - Z_k + 2\lambda_1 r_n^l (Z_k - \frac{1}{2}))^2 = (Z_{k+1} - Z_k)^2$. Then $k^{2\gamma_1} \mathbb{E}((Z_{k+1} - Z_k)^2|\mathcal{F}_k) = \frac{1}{4}$.

When $\gamma_2 < \gamma_1$, set $X_k = Z_k - \frac{1}{2}$ then $L_n = X_n + 2(\lambda_1 r_n^l + \lambda_2 r_n^g) \sum_{k=0}^{n-1} X_k$. So $X_{n+1} \simeq [1 - 2\lambda_2 r_n^g] X_n + \Delta L_{n+1}$. Thus, $c_{1,n} \sim \exp[\frac{-2\lambda_2}{1-\gamma_2} n^{1-\gamma_2}]$ and therefore $n^{\frac{\gamma_2}{2}} c_{1,n} \rightarrow 0$. Then

$$\limsup_{k \rightarrow \infty} \sup_{n \geq k} \left| \frac{c_{k,n}}{\exp[\frac{-2\lambda_2}{1-\gamma_2} (n^{1-\gamma_2} - k^{1-\gamma_2})]} - 1 \right| = 0.$$

It is enough to show that $\sum_{k=1}^n U_{n,k}^2 = n^{\gamma_2} \sum_{k=1}^n c_{k+1,n}^2 k^{-2\gamma_2} (\Delta L_{k+1})^2 k^{2\gamma_2}$ is a constant. Using Lemma A.2 with $b_n = \frac{1}{n^{\gamma_2}} e^{\frac{4\lambda_2}{1-\gamma_2} n^{1-\gamma_2}}$ and $a_k = \frac{k^{2\gamma_2}}{c_{1,n}^2} e^{\frac{-4\lambda_2}{1-\gamma_2} k^{1-\gamma_2}}$. Therefore $\frac{1}{b_n} \sum_k \frac{1}{a_k} \rightarrow \frac{1}{4\lambda_2}$. Also

$$\begin{aligned} (\Delta L_{n+1})^2 &= (X_{n+1} X_n - 2\lambda_2 r_n^g X_n)^2 \\ &= (Z_{k+1} - Z_k + 2\lambda_2 r_n^g (Z_k - \frac{1}{2}))^2 \\ &= (Z_{k+1} - Z_k)^2. \end{aligned}$$

Thus, $k^{2\gamma_2} \mathbb{E}((Z_{k+1} - Z_k)^2 | \mathcal{F}_k) = \frac{1}{4}$.

When $\gamma_1 = \gamma_2 (= \gamma)$, set $X_k = Z_k - \frac{1}{2}$ then $X_{n+1} = [1 - 2r_n(\lambda_1 + \lambda_2)] X_n + \Delta L_{n+1}$ and $c_{1,n} \sim \exp[\frac{-2(\lambda_1 + \lambda_2)}{1-\gamma} n^{1-\gamma}]$ and therefore $n^{\frac{\gamma}{2}} c_{1,n} \rightarrow 0$. Then

$$\limsup_{k \rightarrow \infty} \sup_{n \geq k} \left| \frac{c_{k,n}}{\exp[\frac{-2(\lambda_1 + \lambda_2)}{1-\gamma} (n^{1-\gamma} - k^{1-\gamma})]} - 1 \right| = 0.$$

It is enough to show that $\sum_{k=1}^n U_{n,k}^2 = n^\gamma \sum_{k=1}^n c_{k+1,n}^2 k^{-2\gamma} (\Delta L_{k+1})^2 k^{2\gamma}$ is a constant. Using Lemma A.2 with $b_n = \frac{1}{n^\gamma} e^{\frac{4(\lambda_1 + \lambda_2)}{1-\gamma} n^{1-\gamma}}$ and $a_k = \frac{k^{2\gamma}}{c_{1,n}^2} e^{\frac{-4(\lambda_1 + \lambda_2)}{1-\gamma} k^{1-\gamma}}$.

Therefore $\frac{1}{b_n} \sum_k \frac{1}{a_k} \rightarrow \frac{1}{4(\lambda_1 + \lambda_2)}$. Also $(\Delta L_{n+1})^2 = (Z_{k+1} - Z_k)^2$ and so $k^{2\gamma} \mathbb{E}((Z_{k+1} - Z_k)^2 | \mathcal{F}_k) = \frac{1}{4}$.

[(iii) Theorem 4.1]

When $\gamma_1 = \gamma_2 = 1$, it holds $c_{1,n} = \prod_h^n [1 - 2(\lambda_1 + \lambda_2)r_h] \sim \exp[\ln n^{-2(\lambda_1 + \lambda_2)}] \sim n^{-2(\lambda_1 + \lambda_2)}$ so, $\sqrt{n} c_{1,n} = n^{-2(\lambda_1 + \lambda_2) + \frac{1}{2}} \rightarrow 0$ for $(\lambda_1 + \lambda_2) > \frac{1}{4}$. Then we can obtain

$$\limsup_{k \rightarrow \infty} \sup_{n \geq k} \left| \frac{c_{k,n}}{\left(\frac{k}{n}\right)^{2(\lambda_1 + \lambda_2)}} - 1 \right| = 0.$$

Moreover, $\sum_k U_{k,n}^2 = n \sum_k \left(\frac{k}{n}\right)^{4(\lambda_1 + \lambda_2)} (\Delta L_{k+1})^2 k^2 k^{-2}$ and therefore by using A.2 with taking a_n and b_n , $\frac{1}{b_n} \sum_k \frac{1}{a_k} \rightarrow \frac{1}{1-4(\lambda_1 + \lambda_2)}$ and thus, $(\Delta L_{n+1})^2 = (Z_{k+1} - Z_k)^2$ then $k^2 \mathbb{E}((Z_{k+1} - Z_k)^2 | \mathcal{F}_k) = \frac{1}{4}$.

When $(\lambda_1 + \lambda_2) = \frac{1}{4}$, $\sqrt{n}(\ln n)^{-\frac{1}{2}} c_{1,n} \rightarrow 0$. So $c_{k,n} = \left(\frac{k}{n}\right)^{\frac{1}{2}}$ and $U_{k,n} = \frac{\sqrt{n}}{\sqrt{\ln n}} c_{k+1,n} \Delta L_{k+1}$ so $\sum_k U_{k,n}^2 = \frac{n}{\ln n} \sum_k \left(\frac{k}{n}\right) (\Delta L_{k+1})^2 = \frac{1}{\ln n} \sum_k \frac{1}{k} k^2 (\Delta L_{k+1})^2 = k^2 (\Delta L_{k+1})^2$ therefore $b_n = \ln n$ and $a_k = k$, so $\frac{1}{\ln n} \sum_k \frac{1}{k} \rightarrow 1$. \square

7. PROOFS OF THE CLTs (THEOREM 4.2)

Proof. (Theorem 4.2)

Define $\tilde{X}_n = n^{4(\lambda_1 + \lambda_2)}(Z_n - \frac{1}{2})$. It is therefore enough to show that \tilde{X}_n is a quasimartingale. Indeed, we have

$$\begin{aligned} \sum_k \mathbb{E} \left(\left| \mathbb{E}[\tilde{X}_{k+1} | \mathcal{F}_k] - \tilde{X}_k \right| \right) &= \sum_k \mathbb{E} \left[\left| (k+1)^{4(\lambda_1 + \lambda_2)} (1 - 2(\lambda_1 + \lambda_2)r_k) X_k - \tilde{X}_k \right| \right] \\ &= \sum_k \mathcal{O}\left(\frac{1}{k^2}\right) 8(\lambda_1 + \lambda_2)^2 \mathbb{E}(|\tilde{X}_k|) < +\infty. \end{aligned}$$

So $(\tilde{X}_n)_n$ is an \mathcal{F} -quasi-martingale. Moreover, from the computations carried out in the proof of Theorem 4.1, $\mathbb{E}(\tilde{X}_n^2) < +\infty$ and so it converges *a.s* and in mean to some real random variable \tilde{X} . In order to prove that $\mathbb{P}(\tilde{X} \neq 0) > 0$, we will prove that $(\tilde{X}_n^2)_n$ is bounded in L^p for a suitable $p > 1$. Indeed this fact implies that \tilde{X}_n^2 converges in mean to \tilde{X}^2 and so we obtain $\mathbb{E}(\tilde{X}^2) = \lim_n \mathbb{E}(\tilde{X}_n^2) = \lim_n n^{4(\lambda_1 + \lambda_2)} \mathbb{E}(X_n^2) > 0$. To this purpose, we set $p = 1 + \epsilon/2$, with $\epsilon > 0$ and $x_n = \mathbb{E}(|X_n|^{2+\epsilon})$. Using the following recursive equation:

$$X_{n+1} = (1 - 2r_n)Z_n + \frac{r_n}{N} \sum_{i=1}^k \tilde{\xi}_{k+1}(i) - \frac{1}{2}$$

$$\begin{aligned} x_{n+1} &= \mathbb{E}(|X_n|^{2+\epsilon}) - (2 + \epsilon)r_n 2Z_n \mathbb{E}(|X_n|^{1+\epsilon}) \\ &\quad + (2 + \epsilon)r_n \mathbb{E} \left[|X_n|^{1+\epsilon} \operatorname{sign}(X_n) (X_n) \left(\frac{1}{N} \sum_i \tilde{\xi}_{k+1}(i) \right) \right] + R_n \end{aligned}$$

where $R_n = \mathcal{O}(n^{-2})$. Now since $\mathbb{E}[\frac{1}{N} \sum_i \tilde{\xi}_{k+1}(i) | \mathcal{F}_n] = 2Z_n - 2(\lambda_1 + \lambda_2)(Z_n - \frac{1}{2})$, we have

$$\begin{aligned} x_{n+1} &= \mathbb{E}(|X_n|^{2+\epsilon}) - 2(2 + \epsilon)r_n Z_n \mathbb{E}(|X_n|^{1+\epsilon}) \\ &\quad + (2 + \epsilon)r_n \mathbb{E}[|X_n|^{1+\epsilon} \operatorname{sign}(X_n) (2Z_n - 2(\lambda_1 + \lambda_2))X_n] + R_n \\ &= \mathbb{E}(|X_n|^{2+\epsilon}) - (2 + \epsilon)r_n 2(\lambda_1 + \lambda_2) \mathbb{E}[|X_n|^{1+\epsilon} \operatorname{sign}(X_n) (X_n)X_n] + R_n \\ &= \mathbb{E}(|X_n|^{2+\epsilon}) - (2 + \epsilon)r_n 2(\lambda_1 + \lambda_2) \mathbb{E}(|X_n|^{2+\epsilon}) + R_n \\ &= \left(1 - 2(\lambda_1 + \lambda_2)(2 + \epsilon)r_n\right)x_n + g(n) \end{aligned}$$

with $g(n) = \mathcal{O}(n^{-2})$. Therefore, we have

$$x_{n+1} = \left(1 - 2(\lambda_1 + \lambda_2)(2 + \epsilon)r_n\right)x_n + g(n).$$

$$\begin{aligned} \prod_{k=0}^{n-1} \left(1 - 2(\lambda_1 + \lambda_2)(2 + \epsilon)r_n\right) &= \exp \left(\sum_{k=0}^{n-1} \ln \left(1 - 2(\lambda_1 + \lambda_2)(2 + \epsilon)r_n\right) \right) \\ &= \exp \left(-2(\lambda_1 + \lambda_2)(2 + \epsilon) \sum_{k=0}^{n-1} r_n \right) \\ &= n^{-2(\lambda_1 + \lambda_2)(2 + \epsilon)}. \end{aligned}$$

Thus,

$$\mathbb{E}[|X_n|^{(2+\epsilon)}] = \mathcal{O}\left(\frac{1}{n^{2(\lambda_1 + \lambda_2)(2+\epsilon)}}\right)$$

and it implies that \tilde{X}^2 is bounded in $\mathcal{L}^{1+\frac{\epsilon}{2}}$. □

8. PROOFS OF THE CLTs (THEOREM 4.3)

Proof. We separate the proof as two main situations according to two λ_j 's values.

- **Case** $\lambda_1 \neq 0, \lambda_2 = 0$.

In order to study the evolution of $(Z_n - Z_n(i))$, we consider two cases.

- When $\gamma_1 \leq \gamma_2$, $X_{n+1} = (1 - 2\lambda_1 r_n^l)X_n + \Delta L_{n+1}$ and the proof follows essentially the same as the part (i)(a) of Theorem 4.1.
- When $\gamma_2 < \gamma_1$, $X_{n+1} = (1 - r_n^g)X_n + \Delta L_{n+1}$, then the proof follows essentially the same as part (i)(b) of Theorem 4.1.

In order to study the $(Z_n - \frac{1}{2})_n$:

- When $\gamma_1 \leq \gamma_2$, $X_{n+1} = (1 - r_n^g)X_n + \Delta L_{n+1}$, then the proof follows essentially the same as the part (ii) of Theorem 4.1.
- When $\gamma_2 < \gamma_1$, the proof follows along the same lines as previous. We sketch essential argument below. We have

$$X_{n+1} = (1 - 2\lambda_1 r_n^l)X_n + \Delta L_{n+1}.$$

therefore, $c_{1,n} \sim e^{\frac{-2\lambda_1}{1-\gamma_1}n^{1-\gamma_1}}$ and thus, $n^{\gamma_2 - \frac{\gamma_1}{2}} c_{1,n} \rightarrow 0$. Following the same steps as in the previous proof, it can be verified that 1) and 2) hold. Only showing that

$$\sum_{k=1}^n U_{n,k}^2 = n^{2\gamma_2 - \gamma_1} e^{\frac{-4\lambda_1}{1-\gamma_1}n^{1-\gamma_1}} \sum_k^n k^{-2\gamma_2} e^{\frac{4\lambda_1 k^{1-\gamma_1}}{1-\gamma_1}} k^{2\gamma_2} (\Delta L_{k+1})^2$$

tends to a constant.

It is easy to derive by Lemma A.2 that $\frac{1}{b_n} \sum_k \frac{1}{a_k} \rightarrow \frac{1}{4\lambda_1}$ and $k^{2\gamma_2} (\Delta L_{k+1})^2 \rightarrow k^{2\gamma_2} (Z_{k+1} - Z_k)^2 = \frac{1}{4}$.

- **Case** $\lambda_1 = 0, \lambda_2 \neq 0$.

Concerning the evolution of $(Z_n - Z_n(i))$, for both cases $\gamma_1 \leq \gamma_2$ and $\gamma_2 < \gamma_1$, it is proved analogously as the part (i)(b) of Theorem 4.1.

For $(Z_n - \frac{1}{2})_n$:

- When $\gamma_1 \leq \gamma_2$, the proof follows along the same lines as previous. We sketch essential argument below. We have

$$X_{n+1} = (1 - 2\lambda_2 r_n^g)X_n + \Delta L_{n+1},$$

therefore it holds $c_{1,n} \sim e^{\frac{-2\lambda_2}{1-\gamma_2}n^{1-\gamma_2}}$ and thus, $n^{\gamma_1 - \frac{\gamma_2}{2}} c_{1,n} \rightarrow 0$.

Following the same steps as in the previous proof, it can be verified that 1) and 2) hold. It is enough to show that

$$\sum_{k=1}^n U_{n,k}^2 = n^{2\gamma_1 - \gamma_2} e^{\frac{-4\lambda_2}{1-\gamma_2}n^{1-\gamma_2}} \sum_k^n k^{-2\gamma_1} e^{\frac{4\lambda_2 k^{1-\gamma_2}}{1-\gamma_2}} k^{2\gamma_1} (\Delta L_{k+1})^2.$$

tends to a constant.

It is easy to derive by Lemma A.2 that $\frac{1}{b_n} \sum_k \frac{1}{a_k} \rightarrow \frac{1}{4\lambda_2}$ and $k^{2\gamma_1} (\Delta L_{k+1})^2 \rightarrow k^{2\gamma_1} (Z_{k+1} - Z_k)^2 = \frac{1}{4}$ is a constant.

The proof when $\gamma_2 < \gamma_1$ goes the same as the part (ii) of Theorem 4.1. ($\gamma_2 < \gamma_1$)

- **The case** $\gamma_1 = \gamma_2 = 1$ is proven similarly as in part (iii) Theorem 4.1 and Theorem 4.2. □

9. PROOFS OF THE CLTs (THEOREM 4.4)

Proof. (Theorem 4.4)

[(i) Theorem 4.4]

• **Case** $\gamma_1 \neq \gamma_2$. Define $X_k = Z_k - Z_k(i)$. Set $L_0 = X_0$ and let us rewrite

$$\begin{aligned} L_n &= X_n - \sum_{k=0}^{n-1} (\mathbb{E}[X_{k+1} | \mathcal{F}_n] - X_k) \\ &= X_n - \sum_{k=0}^{n-1} ([1 - r_k^g](Z_k - Z_k(i)) - (Z_k - Z_k(i))) = X_n + \sum_{k=0}^{n-1} r_k^g X_k. \end{aligned}$$

Then $X_{n+1} = [1 - r_n^g]X_n + \Delta L_{n+1}$. Note that $(L_n)_n$ is an \mathcal{F}_n -martingale by construction. Iterating the above relation, we can write $X_n = c_{1,n}X_1 + \sum_{k=1}^n c_{k+1,n}\Delta L_{n+1}$ where $c_{n+1,n} = 1$ and $c_{k,n} = \prod_{h=k}^n [1 - r_h^g]$ for $k \leq n$. $c_{1,n} = \prod_{h=1}^n [1 - r_h^g] \sim \exp[\frac{-1}{1-\gamma_2}n^{1-\gamma_2}]$.

Then $n^{\gamma_1 - \frac{\gamma_2}{2}} c_{1,n} \sim n^{\gamma_1 - \frac{\gamma_2}{2}} \exp[\frac{-1}{1-\gamma_2}n^{1-\gamma_2}] \rightarrow 0$.

$$\limsup_{k \rightarrow \infty} \sup_{n \geq k} \left| \frac{c_{k,n}}{\exp[\frac{-1}{1-\gamma_2}(n^{1-\gamma_2} - k^{1-\gamma_2})]} - 1 \right| = 0.$$

So it is enough to prove that $n^{\gamma_1 - \frac{\gamma_2}{2}} \sum_k c_{k+1,n} \Delta L_{n+1} \rightarrow \mathcal{N}(0, (1 - 1/N)Z_\infty - Z_\infty^2/2)$. Again, this can be proved using following conditions for $U_{n,k+1} = n^{\gamma_1 - \frac{\gamma_2}{2}} \sum_k c_{k+1,n} \Delta L_{n+1}$.

- 1) $\max_{1 \leq k \leq n} |U_{n,k}| \rightarrow 0$.
- 2) $\mathbb{E}[\max_{1 \leq k \leq n} U_{n,k}^2]$ is bounded in n .
- 3) $\sum_{k=1}^n U_{n,k}^2 \rightarrow (1 - 1/N)Z_\infty - Z_\infty^2/2$.

It is easy to check that conditions 1) and 2) holds. Let us now consider 3). We have

$$\sum_{k=1}^n U_{n,k}^2 = n^{2\gamma_1 - \gamma_2} \sum_k c_{k+1,n}^2 (\Delta L_{n+1})^2 \simeq n^{2\gamma_1 - \gamma_2} \sum_{k=1}^n \frac{k^{-2\gamma_1} e^{\frac{1}{1-\gamma_2}k^{1-\gamma_2}}}{e^{\frac{1}{1-\gamma_2}n^{1-\gamma_2}}} (\Delta L_{n+1})^2 k^{2\gamma_1}.$$

From 1) we obtain

$$\begin{aligned} (\Delta L_{n+1})^2 &\simeq [(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))]^2 + (r_k^g)^2 (Z_k - Z_k(i))^2 \\ &\quad + (r_k^g)^2 (Z_k - Z_k(i)) [(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))]. \end{aligned}$$

Since $Z_n - Z_n(i) \xrightarrow{a.s.} 0$ and $X_k^2 = \mathcal{O}(k^{-2\gamma_1 + \gamma_2})$ so,

$$\sum_{k=1}^n U_{n,k}^2 = n^{2\gamma_1 - \gamma_2} \sum_{k=1}^n c_{k+1,n}^2 [(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$$

where we use Lemma A.2 with $b_n = \frac{1}{n^{2\gamma_1 - \gamma_2}} e^{\frac{2}{1-\gamma_2}n^{1-\gamma_2}}$ and $a_k = \frac{k^{2\gamma_1}}{c_{1,n}^2} e^{-\frac{2}{1-\gamma_2}k^{1-\gamma_2}}$.

Let $V_k = k^{2\gamma_1} [(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$.

Thus $\lim_{n \rightarrow \infty} \frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} = \frac{1}{2}$. This implies that $\sum_{k=1}^n U_{n,k}^2$ converges to V a.s., where V is deterministic such that $\mathbb{E}(V_{k+1} | \mathcal{F}_k) \rightarrow V$. Indeed,

$$\begin{aligned} \mathbb{E}(k^{2\gamma_1}(Z_{k+1}(i) - Z_k(i))^2 | \mathcal{F}_k) &= k^{2\gamma_1}(r_k^l)^2 \mathbb{E}[(\xi_{k+1}^l(i) - Z_k(i))^2 | \mathcal{F}_k] \\ &= k^{2\gamma_1}(r_k^l)^2 \text{Var}[\xi_{k+1}^l(i) | \mathcal{F}_k] \\ &= k^{2\gamma_1}(r_k^l)^2 (Z_k - Z_k^2) \xrightarrow{a.s.} Z_\infty - Z_\infty^2. \end{aligned}$$

Similarly, $\mathbb{E}(k^{2\gamma_1}(Z_{k+1} - Z_k)^2 | \mathcal{F}_k) \xrightarrow{a.s.} Z_\infty - Z_\infty^2$, and

$$\mathbb{E}(k^{2\gamma_1}(Z_{k+1}(i) - Z_k(i))(Z_{k+1} - Z_k) | \mathcal{F}_k) \xrightarrow{a.s.} \frac{Z_\infty - Z_\infty^2}{N}.$$

Thus, $\lim_{k \rightarrow \infty} U_k = 2(Z_\infty - Z_\infty^2)(1 - \frac{1}{N})$ a.s.

Consider the case $\gamma_1 = \gamma_2 (= \gamma)$. Since $L_n = X_n + \sum_{k=0}^{n-1} r_n X_k$, it holds $L_{n+1} - L_n = X_{n+1} - (1 - r_n)X_n$. So $X_{n+1} = (1 - r_n)X_n + \Delta L_{n+1}$. Note that $(L_n)_n$ is an \mathcal{F}_n -martingale by construction. Iterating the above relation, we can write

$X_n = c_{1,n}X_1 + \sum_{k=1}^n c_{k+1,n}\Delta L_{n+1}$ where $c_{n+1,n} = 1$ and $c_{k,n} = \prod_{h=k}^n (1 - r_h)$ for $k \leq n$. We get $c_{1,n} = \prod_{h=1}^n [1 - r_h] \sim \exp[\frac{-1}{1-\gamma}n^{1-\gamma}]$. Then $n^{\frac{\gamma}{2}}c_{1,n} \sim n^{\frac{\gamma}{2}} \exp[\frac{-1}{1-\gamma}n^{1-\gamma}] \rightarrow 0$.

Moreover $\lim_{n \rightarrow \infty} \frac{1}{b_n} \sum_k \frac{1}{a_k} = \frac{1}{2}$,

$$\begin{aligned} \mathbb{E}(k^{2\gamma}(Z_{k+1}(i) - Z_k(i))^2 | \mathcal{F}_k) &= k^{2\gamma}r_k^{2\gamma} \mathbb{E}[(\tilde{\xi}_{k+1}(i) - Z_k(i))^2 | \mathcal{F}_k] \\ &= k^{2\gamma}r_k^{2\gamma} \text{Var}[\tilde{\xi}_{k+1}(i) | \mathcal{F}_k] \xrightarrow{a.s.} 2(Z_\infty - Z_\infty^2). \end{aligned}$$

Similarly, $\mathbb{E}(k^{2\gamma}(Z_{k+1} - Z_k)^2 | \mathcal{F}_k) \xrightarrow{a.s.} 2(Z_\infty - Z_\infty^2)$, and

$$\mathbb{E}(k^{2\gamma}(Z_{k+1}(i) - Z_k(i))(Z_{k+1} - Z_k) | \mathcal{F}_k) \xrightarrow{a.s.} \frac{2(Z_\infty - Z_\infty^2)}{N}$$

Thus, $U_k \xrightarrow{a.s.} 4(Z_\infty - Z_\infty^2)(1 - \frac{1}{N})$.

• **Case** $\gamma_1 = \gamma_2 = 1$. Consider $L_n = X_n - \sum_{k=0}^{n-1} (Z_k - Z_k(i))(-r_n) = X_n + \sum_{k=0}^{n-1} r_n X_k$. Then $L_{n+1} - L_n = X_{n+1} - [1 - r_n]X_n$ so, $X_{n+1} = [1 - r_n]X_n + \Delta L_{n+1}$. Note that L_n is an \mathcal{F}_n -martingale by construction. Iterating the above relation, we can write $X_n = c_{1,n}X_1 + \sum_{k=1}^n c_{k+1,n}\Delta L_{n+1}$ where $c_{n+1,n} = 1$ and $c_{k,n} = \prod_{h=k}^n [1 - r_h]$ for $k \leq n$. $c_{1,n} = \prod_{h=1}^n [1 - r_h] \sim n^{-1}$. Then $\sqrt{n} c_{1,n} \sim \sqrt{n} n^{-1}$. Choosing $b_n = n$ and $a_k = 1$, $\lim_{n \rightarrow \infty} \frac{1}{b_n} \sum_k \frac{1}{a_k} = 1$. It holds

$$\mathbb{E}(k^{2\gamma}(Z_{k+1}(i) - Z_k(i))^2 | \mathcal{F}_k) \xrightarrow{a.s.} 2(Z_\infty - Z_\infty^2).$$

Similarly, $\mathbb{E}(k^{2\gamma}(Z_{k+1} - Z_k)^2 | \mathcal{F}_k) \xrightarrow{a.s.} 2(Z_\infty - Z_\infty^2)$, and

$$\mathbb{E}(k^{2\gamma}(Z_{k+1}(i) - Z_k(i))(Z_{k+1} - Z_k) | \mathcal{F}_k) \xrightarrow{a.s.} \frac{2(Z_\infty - Z_\infty^2)}{N}.$$

Thus, $\lim_{k \rightarrow \infty} U_k = 4(Z_\infty - Z_\infty^2)(1 - \frac{1}{N})$ a.s.

(i) Theorem 4.4

• **Case** $\gamma_1 < \gamma_2$. The process $(Z_n)_n$ is a martingale. Indeed, by

$$\mathbb{E}(Z_{n+1} | \mathcal{F}_n) = (1 - 2\lambda_1 r_n^l - 2\lambda_2 r_n^g)Z_n + \lambda_1 r_n^l + \lambda_2 r_n^g,$$

we have $\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = Z_n$ when $\lambda_1 = 0, \lambda_2 = 0$, therefore Z_n converges a.s. To this purpose, satisfies the following two conditions:

- (1) $\mathbb{E} \left[\sup_k k^{\gamma_1 - \frac{1}{2}} |Z_{k+1} - Z_k| \right] < +\infty$;
- (2) $n^{2\gamma_1 - 1} \sum_{k \geq n} (Z_{k+1} - Z_k)^2 \xrightarrow{a.s.} \frac{1}{N(2\gamma_1 - 1)} (Z_\infty - Z_\infty^2)$.

Indeed, the first condition immediately follows from

$$|Z_{k+1} - Z_k| = |r_n^l \left(\frac{1}{N} \sum_i \xi_{k+1}^l(i) - Z_k \right) + r_n^g (\xi_{k+1}^g - Z_k)| = \mathcal{O}(k^{-\gamma_1}).$$

Regarding the second condition, we observe that

$$n^{2\gamma_1 - 1} \sum_{k \geq n} (Z_{k+1} - Z_k)^2 = n^{2\gamma_1 - 1} \sum_{k \geq n} k^{-2\gamma_1} (r_k^l)^2 \left(\frac{\sum_i \xi_{k+1}^l(i)}{N} - Z_k \right)^2 k^{2\gamma_1}$$

and so the desired convergence follows by lemma A.2 with $a_k = k^{-2\gamma_1 + 2}$, $b_n = n^{2\gamma_1 - 1}$ and $U_k = k^{2\gamma_1} (r_k^l)^2 \left(\frac{\sum_k \xi_{k+1}^l(i)}{N} - Z_k \right)^2$, $\lim_{n \rightarrow \infty} b_n \sum_{k \geq n} \frac{1}{a_k b_k^2} = -\frac{1}{1 - 2\gamma_1}$ so,

$$\mathbb{E} \left(\left(\frac{\sum_k \xi_{k+1}^l(i)}{N} - Z_k \right)^2 | \mathcal{F} \right) = \text{Var} \left(\frac{\sum_k \xi_{k+1}^l(i)}{N} | \mathcal{F}_n \right) = \frac{1}{N} (Z_\infty - Z_\infty^2).$$

• **Case** $\gamma_2 < \gamma_1$. To this purpose, satisfies the following two conditions:

- 1) $\mathbb{E} \left(\sup_k k^{\gamma_2 - \frac{1}{2}} |Z_{k+1} - Z_k| \right) < +\infty$
- 2) $n^{2\gamma_2 - 1} \sum_{k \geq n} (Z_{k+1} - Z_k)^2 \xrightarrow{a.s.} \frac{1}{(2\gamma_2 - 1)} (Z_\infty - Z_\infty^2)$

Indeed, the first condition immediately follows from

$$|Z_{k+1} - Z_k| = |r_n^l \left(\frac{1}{N} \sum_i \xi_{k+1}^l(i) - Z_k \right) + r_n^g (\xi_{k+1}^g - Z_k)| = \mathcal{O}(k^{-\gamma_2}).$$

Regarding the second condition, we observe that

$$n^{2\gamma_2 - 1} \sum_{k \geq n} (Z_{k+1} - Z_k)^2 = n^{2\gamma_2 - 1} \sum_{k \geq n} (r_k^l)^2 k^{-2\gamma_2} \left(\frac{\sum_i \xi_{k+1}^l(i)}{N} - Z_k \right)^2 k^{2\gamma_2}$$

and the desired convergence follows by lemma A.2 with $a_k = k^{2\gamma_2 + 2}$, $b_n = n^{2\gamma_2 - 1}$ and $U_k = k^{2\gamma_2} (r_k^l)^2 (\xi_{k+1}^l - Z_k)^2$, $\lim_{n \rightarrow \infty} b_n \sum_{k \geq n} \frac{1}{a_k b_k^2} = -\frac{1}{1 - 2\gamma_2}$ and

$$\mathbb{E}(\xi_{k+1}^l(i) - Z_k)^2 | \mathcal{F} = \text{Var}(\xi_{k+1}^l | \mathcal{F}) = (Z_\infty - Z_\infty^2).$$

• **Case** $\gamma_1 = \gamma_2 (= \gamma)$. The process $(Z_n)_n$ is a martingale and converges a.s. Indeed,

$$\mathbb{E}(Z_{n+1} | \mathcal{F}_n) = (1 - 2r_n)Z_n + r_n \mathbb{E} \left(\frac{\sum_i \tilde{\xi}_{n+1}(i)}{N} | \mathcal{F}_n \right) = Z_n.$$

To this purpose, we need to check the following two conditions:

- 1) $\mathbb{E} \left[\sup_k k^{\gamma - \frac{1}{2}} |Z_{k+1} - Z_k| \right] < +\infty$;
- 2) $n^{2\gamma - 1} \sum_{k \geq n} (Z_{k+1} - Z_k)^2 \xrightarrow{a.s.} \frac{2}{N(2\gamma - 1)} (Z_\infty - Z_\infty^2)$.

Indeed, the first condition immediately follows from

$$|Z_{k+1} - Z_k| = |r_n \left(\frac{1}{N} \sum_i \tilde{\xi}_{k+1}(i) - 2Z_k \right)| = \mathcal{O}(k^{-\gamma}).$$

Regarding the second condition, we observe that

$$n^{2\gamma-1} \sum_{k \geq n} (Z_{k+1} - Z_k)^2 = n^{2\gamma-1} \sum_{k \geq n} r_k^2 k^{-2\gamma} \left(\frac{\sum_i \tilde{\xi}_{k+1}(i)}{N} - Z_k \right)^2 k^{2\gamma}$$

and so the desired convergence follows by lemma with $a_k = k^{-2\gamma+2}$, $b_n = n^{2\gamma-1}$ and

$$U_k = k^{2\gamma} r_k^2 \left(\frac{\sum_i \tilde{\xi}_{k+1}(i)}{N} - 2Z_k \right)^2,$$

$$\lim_{n \rightarrow \infty} b_n \sum_{k \geq n} \frac{1}{a_k b_k^2} = -\frac{1}{1-2\gamma}, \quad \mathbb{E} \left(\left(\frac{\sum_i \tilde{\xi}_{k+1}(i)}{N} - 2Z_k \right)^2 | \mathcal{F}_k \right) = \frac{2}{N} (Z_\infty - Z_\infty^2).$$

• **Case** $\gamma_1 = \gamma_2 = 1$. To this purpose, satisfies the following two conditions:

- 1) $\mathbb{E} \left[\sup_k k^{\frac{1}{2}} |Z_{k+1} - Z_k| \right] < +\infty$;
- 2) $n \sum_{k \geq n} (Z_{k+1} - Z_k)^2 \xrightarrow{a.s.} \frac{2}{N} (Z_\infty - Z_\infty^2)$.

Indeed, the first condition immediately follows from

$$|Z_{k+1} - Z_k| = |r_n \left(\frac{1}{N} \sum_i \tilde{\xi}_{k+1}(i) - 2Z_k \right)| = \mathcal{O}(k^{-1}).$$

To deal with the second condition, we observe that

$$n \sum_{k \geq n} (Z_{k+1} - Z_k)^2 = n \sum_{k \geq n} r_k^2 k^{-2} k^2 \left(\frac{\sum_i \tilde{\xi}_{k+1}(i)}{N} - Z_k \right)^2$$

and so the desired convergence follows by lemma A.2 with $a_k = 1$, $b_n = n$ and

$$U_k = k^2 r_k^2 \left(\frac{\sum_i \tilde{\xi}_{k+1}(i)}{N} - 2Z_k \right)^2, \quad \lim_{n \rightarrow \infty} b_n \sum_{k \geq n} \frac{1}{a_k b_k^2} = 1.$$

$$\mathbb{E} \left(\left(\frac{\sum_i \tilde{\xi}_{k+1}(i)}{N} - 2Z_k \right)^2 | \mathcal{F} \right) = \frac{2}{N} Z_\infty (1 - Z_\infty).$$

□

APPENDIX A. APPENDIX

Lemma A.1. *Let $(x_n)_n$ be a sequence of positive reals satisfying the following equation:*

$$x_{n+1} = \left(1 - \frac{A}{n^{a\gamma_i}} \right) x_n + \frac{K_n}{n^{b\gamma_j}},$$

where $n, a, b \in \mathbb{N}$, $b \geq a > 0$, $b > 1$, $\frac{1}{2} < (\gamma_i, \gamma_j) \leq 1$, $(i, j) \in \{1, 2\}$, $A > 0$ and $\forall n \in \mathbb{N}$, $0 < K_n \leq K$. it holds,

- if $\sum_n \frac{1}{n^{a\gamma_i}} = +\infty$ then, $x_n \xrightarrow{n \rightarrow \infty} \frac{C}{n^{b\gamma_j - \gamma_i}}$ (where C denotes a generic constant),
- if $\sum_n \frac{1}{n^{a\gamma_i}} < +\infty$ then,

$$\begin{cases} \lim_{n \rightarrow \infty} \left(x_n - \frac{K_n/A}{n^{b\gamma_j - \gamma_i}} \right) = e^{-\sum_{n=0}^{\infty} \frac{1}{n^{a\gamma_i}}} & \text{for } a = 1; \\ \lim_{n \rightarrow \infty} \left(x_n - \frac{K_n/A}{n^{b\gamma_j - 1}} \right) = e^{-\sum_{n=0}^{\infty} \frac{1}{n^{a\gamma_i}}} & \text{for } a \geq 2. \end{cases}$$

Proof. Let $x_{n+1} = (1 - \varepsilon_n)x_n + \delta_n$ where $\varepsilon_n = \frac{A}{n^{a\gamma_i}}$ and $\delta_n = \frac{K_n}{n^{b\gamma_j}}$. It holds, for any $0 \leq l \leq n-1$,

$$x_n = x_l \prod_{i=l}^{n-1} (1 - \varepsilon_i) + \sum_{i=l}^{n-1} \delta_i \prod_{k=i+1}^{n-1} (1 - \varepsilon_k). \quad (22)$$

- If $\sum_n \varepsilon_n = +\infty$, it implies $\prod_{u=l}^{n-1} (1 - \varepsilon_u) \rightarrow 0$.
- If $\sum_n \varepsilon_n < +\infty$, it follows that $\prod_{u=l}^{n-1} (1 - \varepsilon_u) = e^{-\sum_{u=l}^{n-1} \frac{1}{u^{a\gamma_i}}}$.

Let $y_n := \sum_{i=l}^{n-1} \delta_i \prod_{k=i+1}^{n-1} (1 - \varepsilon_k)$, so y_n behaves for $n \rightarrow \infty$ like

$$\int_l^n \delta_s \exp\left(-\int_s^n \varepsilon(u) du\right) ds = \frac{K_n}{n^{b\gamma_j}} \frac{\int_l^n s^{-b\gamma_j} e^{\frac{A}{(1-a\gamma_i)s^{a\gamma_i-1}} ds}}{n^{-b\gamma_j} e^{\frac{A}{(1-a\gamma_i)n^{a\gamma_i-1}}}}.$$

Letting $n \rightarrow \infty$, using de L'Hôpital rule, it holds

$$y_n \sim_{n \rightarrow \infty} \frac{K}{A} \frac{1}{n^{b\gamma_j - a\gamma_i}} \left(\frac{1}{1 - \frac{b\gamma_j n^{a\gamma_i}}{nA}} \right).$$

Conclusion follows. \square

Lemma A.2. *Let \mathcal{G} be an (increasing) filtration and $(Y_k)_k$ be an \mathcal{G} -adapted sequence of real random variables such that $\lim_{k \rightarrow \infty} \mathbb{E}[Y_k | \mathcal{G}_{k-1}] = Y$ a.s. for some real random variable Y . Moreover, let $(a_k)_k$ and $(b_k)_k$ be two sequences of strictly positive real numbers such that*

$$b_k \nearrow +\infty, \quad \sum_{k=1}^{\infty} \frac{\mathbb{E}[Y_k^2]}{a_k^2 b_k^2} < +\infty.$$

Then we have:

- if $\lim_{n \rightarrow \infty} \frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} = \sigma$ for some constant σ , then $\lim_{n \rightarrow \infty} \frac{1}{b_n} \sum_{k=1}^n \frac{Y_k}{a_k} = \sigma Y$;
- if $\lim_{n \rightarrow \infty} b_n \sum_{k \geq n} \frac{1}{a_k b_k^2} = \sigma$ for some constant σ , then $\lim_{n \rightarrow \infty} b_n \sum_{k \geq n} \frac{Y_k}{a_k b_k^2} = \sigma Y$.

Theorem A.3. (Theorem 3.2 in [11])

Let $\{S_{n,k}, \mathcal{F}_{n,k} : 1 \leq k \leq k_n, n \geq 1\}$ be a zero-mean, square-integrable martingale array with differences $Y_{n,k}$, and let η^2 be an a.s. finite random variable. Assume

- $\max_{1 \leq k \leq k_n} |Y_{n,k}| \xrightarrow[n \rightarrow \infty]{P} 0$;
- $\mathbb{E}(\max_{1 \leq k \leq k_n} Y_{n,k}^2)$ is bounded in n ;
- $\sum_{k=1}^{k_n} Y_{n,k}^2 \xrightarrow[n \rightarrow \infty]{P} \eta^2$;

and the σ -fields are nested, i.e. for $1 \leq k \leq k_n$, $n \geq 1$, $\mathcal{F}_{n,k} \subseteq \mathcal{F}_{n+1,k}$. Then $S_{n,k_n} = \sum_{k=1}^{k_n} Y_{n,k}$ converges stably to a random variable with characteristic function $\varphi(u) = \mathbb{E}[\exp(-\eta^2 u^2 / 2)]$, i.e. to the Gaussian kernel $\mathcal{N}(0, \eta^2)$.

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