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SYNCHRONIZATION AND FLUCTUATIONS FOR INTERACTING STOCHASTIC SYSTEMS WITH INDIVIDUAL AND COLLECTIVE REINFORCEMENT

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Abstract

The Pólya urn is the paradigmatic example of a reinforced stochastic process. It leads to a random time-limit. The Friedman urn is a generalization whose a.s. time-limit is not random anymore. In this work, following previous recent works, we introduce a new family of (finite) systems of reinforced stochastic processes, interacting through an additive collective reinforcement of mean field type. The two reinforcement rules strengths are tuned through different rates $n^{-\gamma}$. In the case the reinforcement rates are like n^{-1}, these reinforcements are of Pólya or Friedman type and may thus lead to random limits or not. Different parameter regimes needs to be considered.
We state two kind of results. First, we study the time-asymptotics and show that L^2 and almost sure convergence always holds. In particular, all the components share the same time-limit. We show, the nature of the limit (random/deterministic) depends on the parameters' regime considered. Second, we study fluctuations and prove central limit theorems whose scaling coefficient vary according to the regime considered.

Keywords: Reinforced stochastic processes; Interacting random systems; Almost sure convergence/Synchronization; Central limit theorems

2010 Mathematics Subject Classification: Primary 60K35; 60F15, 60F05
Secondary 62L20, 62P35

1. Introduction

This work is motivated by systems of interacting stochastic processes (discrete time) related to the one considered in the publications [4] and [6]. These are systems of interacting reinforced stochastic processes where the interaction holds through a reinforcement scheme. In some special cases, these are models of interacting urns. We generalize a new model type (1) which is able to behaves both random or deterministic in time limit distribution according to the type of reinforcement. Issues are including:

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nature of the almost sure time limit distribution (deterministic or random) according to the type of reinforcement (Pólya or Friedman like in case of urn model, see [5, 13]); existence of a.s. synchronization in the system (in the sens, each component dynamics share the same random time-limit); fluctuations with respect to this limit (see [4, 6, 16]) through central limit theorems with adapted scaling. As for *stochastic algorithms*, these dynamics are defined through recursive equations and a step size is defined. This work is considering generalization with competing local/collective aspects. To be more specific with the motivation, *synchronization* phenomenon for stochastic systems have received a considerable interest recently (Kuramoto model for instance), both for their impact in applications and their intrinsic mathematical value. Synchronization occurs in many natural contexts and is a common topic of different scientific fields. This is a general concept for a phenomenon observed in multicomponent dynamical evolutions. The following are constituting aspect: notion of unit (cell, component, individual) with a proper dynamics; finite (possibly large) number of units (here N); interaction among units which influences their dynamics (here, mean field interaction); the units after some time adopt the same kind of behavior, each individual behavior being coordinated to a collective common characteristic. A basic model of *self-reinforcement* is the well known Pólya urn. In Physics, Biology or social science and economy, reinforcement is defined as an action which increases the frequency of a certain behavior. We may define a reinforced process as a stochastic process where an event which has occurred many times in the past has a higher probability to occur in the future. To survey on this kind of processes see [15]. There is a big variety of reinforced processes, urn models belong to them. The Pólya urn is the simplest. We briefly describe it below. See Chap. 7 in [3] too for an introduction. At time 0 an urn contains a red balls and b blue balls. At each discrete time $n > 0$, a ball is drawn out and it is replaced in the urn together with c balls of the same color. We denote by Z_n the proportion of red balls in the urn at time n, namely, the conditional probability of choosing a red ball at time $n + 1$, given the proportion of the red ball at time n. We are interested in the distribution of Z_n when n is large. An easy calculation shows that $(Z_n)_{n\geq 0}$ is bounded *martingale*, thus it converges almost surely to a (non degenerate) random variable Z_∞. Moreover it can be proved that Z_∞ has Beta distribution with parameters a/c and b/c [11]. *Generalized Pólya urns* may exhibit very different behaviors, even when some seemingly slight changes in the reinforcement scheme is made. An example is given by the Friedman urn [7]: at each step the ball selected is replaced by α balls of the same color and β balls of the color not drawn, where $\alpha > \beta > 0$. A theorem was stated in [8] proving the proportion Z_n of red balls converges a.s. to $1/2$. Thus, modifying the reinforcement scheme may leads to a deterministic limit distribution.
If we be interested in studying, as main component of an interacting system, reinforced stochastic processes (urn models) can defined through the following recursive equation:

$$\forall n \in \mathbb{N}, \qquad Z_{n+1} = (1 - r_n)\ Z_n + r_n\ \xi_{n+1}, \quad Z_0 = \frac{1}{2}$$

where the Bernoulli law of the random variable ξ_{n+1} is a transformation of Z_n such that $\mathbb{P}(\xi_{n+1} = 1|\mathcal{F}_n) = \varphi(Z_n)$ where φ is a map from $[0, 1]$ to $[0, 1]$, $r_n = n + 3^{-1}$ and $\mathcal{F}_n := \sigma\{Z_1, \ldots, Z_n\}$. Regarding this point of view, a well known result [10] states that: Assume φ is continuous then, $(Z_n)_n$ converges a.s. and its limit Z_∞ is such that $\varphi(Z_\infty) = Z_\infty$ a.s;

- Case $\varphi(x) = x$ is the well known evolution of the proportion of one color in a two-color Pólya urn context. The random time-asymptotics proportion Z_∞ is beta-distributed.
- Case $\varphi(x) = 1 - x$ corresponds to the proportion when a Friedman replacement scheme is used, meaning at each time step, a balls of the chosen color are added to the urn and $b > 0$ balls from the not chosen color. $Z_\infty = \frac{1}{2}$ a.s.

We are interested in studying finite size systems $\{(Z_n(i))_n,\ 1 \leq i \leq N\}$ of such processes for $N > 1$ where an interaction takes place through the reinforcement mechanism. Such systems proved to have very interesting properties, as it was stated in previous recent works [6, 5, 16]. Motivated by [4, 1], we want to study different strength of reinforcement considering $r_n \sim \frac{D}{n^\gamma}$. In the case $\gamma = 1$, we come back to the urn model as known as Pólya and Friedman. The objects are organized as two main concepts. The first one is dedicated to the asymptotic behavior of reinforced interacting systems such as convergence and synchronization similar to those were studied in [4] and [6]. The second is study to fluctuations of these type of model which in the tools and methods are based on [5] and [16].

2. Framework

Let us define the following new model. For $i \in \{1, ..., N\}$ and $n \in \mathbb{N}$, we consider the stochastic dynamics defined through the recursive relation

$$Z_{n+1}(i) = (1 - r_n^l - r_n^g)Z_n(i) + r_n^l \xi_{n+1}^l(i) + r_n^g \xi_{n+1}^g, \tag{1}$$

where $Z_0(i) = \frac{1}{2}$ and where $\xi_{n+1}^l(i)$ and ξ_{n+1}^g denote local and collective reinforcements schema, sequences of random variables which are conditionally independent given $\mathcal{F}_n := \sigma(Z_0, Z_1, ..., Z_n)$ as a Borel σ-algebra. Given $\mathcal{F}_n$, they are Bernoulli distribution such that

$$\begin{aligned}\mathbb{P}(\xi_{n+1}^l(i) = 1|\mathcal{F}_n) = \psi_1(Z_n(i)) = (1 - 2\lambda_1)Z_n(i) + \lambda_1,\\ \mathbb{P}(\xi_{n+1}^g = 1|\mathcal{F}_n) = \psi_2(Z_n) = (1 - 2\lambda_2)Z_n + \lambda_2,\end{aligned}$$

for $\psi_k : [0,1] \longrightarrow [0,1]$ $(k \in \{1,2\})$ where $Z_n = \frac{1}{N}\sum_{i=1}^N Z_n(i)$ and $\lambda_1, \lambda_2 \in [0,1]$ are parameters. The local and collective reinforcement rates are such that $r_n^l \sim \frac{D_1}{n^{\gamma_1}}$ and $r_n^g \sim \frac{D_2}{n^{\gamma_2}}$, with $\sim$ meaning $\lim_{n\to\infty} n^{\gamma \cdot} r_n^{\cdot}$ is a constant (like $D_1,\ D_2$)

Remark 1. In the specific case when $\gamma_1 = \gamma_2 = \gamma$, we can rewrite the model such as

$$Z_{n+1}(i) = (1 - 2r_n)Z_n(i) + r_n\tilde{\xi}_{n+1}(i),$$

where $\tilde{\xi}_{n+1}(i) = \xi_{n+1}^l(i) + \xi_{n+1}^g$, therefore $\tilde{\xi}_{n+1}(i) \in \{0, 1, 2\}$ such that

$$\begin{aligned}\mathbb{P}(\tilde{\xi}_{n+1}(i) = 2|\mathcal{F}_n) &= \mathbb{P}(\xi_{n+1}^l(i) = 1|\mathcal{F}_n) \times \mathbb{P}(\xi_{n+1}^g = 1|\mathcal{F}_n)\\ &= [(1 - 2\lambda_1)Z_n(i) + \lambda_1] \times [(1 - 2\lambda_2)Z_n + \lambda_2].\end{aligned}$$

The reinforcement rate is $r_n \sim \frac{1}{2n^\gamma}$,.

Remark 2. As mentioned earlier, we can have two different behaviors of time limit distribution with defining a proper reinforcement which comes from the role of parameters λ_js which in play. Indeed, parameters λ_js (being equal 0 or not) let the transformation ψ_js to make a different character of reinforcement (Pólya or Friedman respectively).

This is straightforward, to compute the following relationships that will be used frequently in this paper:

$$\mathbb{E}[Z_{n+1}(i)|\mathcal{F}_n] = (1 - 2\lambda_1 r_n^l - r_n^g)Z_n(i) + r_n^g(1 - 2\lambda_2)Z_n + \lambda_1 r_n^l + \lambda_2 r_n^g, \qquad (2)$$

and by averaging over i in $\{1, \dots, N\}$, we have

$$\mathbb{E}[Z_{n+1}|\mathcal{F}_n] = (1 - 2\lambda_1 r_n^l - 2\lambda_2 r_n^g)Z_n + \lambda_1 r_n^l + \lambda_2 r_n^g. \qquad (3)$$

Remark 3. In this paper the parameters λ_1, λ_2 are kept fixed. Cases where λ_j may converge to 0 depending on n, N are work in progress and will be considered in a forthecoming work [14].

3. Convergence and synchronization

In this section we study convergence of $(Z_n)_n$ and synchronization phenomenon. Indeed considering the different regime of γ_j, for $\frac{1}{2} < \gamma_j \leq 1$, we obtain different kind of limit (deterministic or random) of Z_n with different rate. Moreover synchronization happens with different rate according to γ_j's regimes. The following theorems describe them.

3.1. Case of a deterministic time-asymptotics

We call *deterministic*, the case when the limit is not random.

Theorem 1. *For any $\lambda_1 > 0$ and $\lambda_2 > 0$ and for $\frac{1}{2} < (\gamma_1, \gamma_2) \leq 1$, following results hold, where C denotes a generic constant:*

i) asymptotics of variances ($n \to \infty$):
$\mathbb{V}ar(Z_n) \sim \frac{C}{n^\gamma}$ *and* $\mathbb{V}ar(Z_n(i)) \sim \frac{C}{n^\gamma}$ *where* $\gamma := \min(\gamma_1, \gamma_2)$*;*

ii) behavior of the L^2 distance between Z_n and $Z_n(i)$ when $n \to \infty$:

a) if $\gamma_1 \leq \gamma_2$, then $\mathbb{E}[Z_n - Z_n(i)]^2 \sim \frac{C}{n^{\gamma_1}}$,

b) if $\gamma_2 < \gamma_1$, then $\mathbb{E}[Z_n - Z_n(i)]^2 \sim \frac{C}{n^{2\gamma_1 - \gamma_2}}$;

iii) almost sure convergence holds i.e.
$\forall i = 1, ..., N, \qquad \lim_{n\to+\infty} Z_n(i) = \lim_{n\to+\infty} Z_n = \frac{1}{2}$ *a.s.*

Two others cases lead to the following similar results.

Theorem 2. *In the following cases: either $\lambda_1 \neq 0$ and $\lambda_2 = 0$ or $\lambda_1 = 0$ and $\lambda_2 \neq 0$ and for $\frac{1}{2} < (\gamma_1, \gamma_2) \leq 1$, it holds $\lim_{n\to+\infty} Z_n(i) = \lim_{n\to+\infty} Z_n = \frac{1}{2}$ a.s. Moreover, the following table summarizes the L^2 speed of convergence with $\gamma := \min(\gamma_1, \gamma_2)$ and where C denotes a generic constant.*

	$\lambda_1 \neq 0,\ \lambda_2 = 0$	$\lambda_1 = 0,\ \lambda_2 \neq 0$
$\gamma_1 \leq \gamma_2$	$\mathbb{V}ar(Z_n) \sim \frac{C}{n^{\gamma_1}}$ $\mathbb{E}[Z_n - Z_n(i)]^2 \sim \frac{C}{n^{\gamma_1}}$	$\mathbb{V}ar(Z_n) \sim \frac{C}{n^{2\gamma_1 - \gamma_2}}$ $\mathbb{E}[Z_n - Z_n(i)]^2 \sim \frac{C}{n^{2\gamma_1 - \gamma_2}}$
$\gamma_2 < \gamma_1$	$\mathbb{V}ar(Z_n) \sim \frac{C}{n^{2\gamma_2 - \gamma_1}}$ $\mathbb{E}[Z_n - Z_n(i)]^2 \sim \frac{C}{n^{2\gamma_1 - \gamma_2}}$	$\mathbb{V}ar(Z_n) \sim \frac{C}{n^{\gamma_2}}$ $\mathbb{E}[Z_n - Z_n(i)]^2 \sim \frac{C}{n^{2\gamma_1 - \gamma_2}}$

3.2. Case of a common shared random time-asymptotics

Theorem 3. *For $\lambda_1 = \lambda_2 = 0$, with $\frac{1}{2} < (\gamma_1, \gamma_2) \leq 1$,*

(i) it holds $(n \to \infty)$ $\mathbb{V}ar(Z_n) - e^{-\zeta(2\gamma_2)} \sim \frac{C}{n^{2\gamma-1}}$, where $\gamma := \min(\gamma_1, \gamma_2)$ and C denotes a generic (known) constant, and $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ is Riemann's zeta function. In particular $(Z_n)_n$ converges a.s. to a non-degenerated random limit denoted by Z_∞ i.e. $\mathbb{V}ar(Z_\infty) > 0$.

(ii) The L^2-distance behaves as follows,

$$\mathbb{E}[Z_n(i) - Z_n]^2 \sim \frac{C}{n^{2\gamma_1 - \gamma_2}}$$

and synchronization holds almost surely in the sense $\lim_{n\to\infty} Z_n(i) = Z_\infty$ a.s.

4. Fluctuations

Theorem 4. *Let $\lambda_1 > 0, \lambda_2 > 0$; let $\gamma := \min(\gamma_1, \gamma_2)$. The following statements hold, where C is denoting a generic constant (depending on N and λ_j).*

i) For $\gamma_1 \leq 1,\ \gamma_2 \leq 1$, then

a) when $\gamma_1 \leq \gamma_2$, $n^{\frac{\gamma_1}{2}}(Z_n - Z_n(i)) \xrightarrow[n\to\infty]{stably} \mathcal{N}\Big(0, C\Big)$,

b) when $\gamma_2 < \gamma_1$, $n^{\gamma_1 - \frac{\gamma_2}{2}}(Z_n - Z_n(i)) \xrightarrow[n\to\infty]{stably} \mathcal{N}\Big(0, C\Big)$,

ii) When at least one of these conditions hold: $\gamma_1 < 1$ or $\gamma_2 < 1$, It holds $n^{\frac{\gamma}{2}}(Z_n - \frac{1}{2}) \xrightarrow[n\to\infty]{stably} \mathcal{N}\Big(0, C\Big)$.

iii) When both γ_1 and γ_2 are equal 1

a) for $(\lambda_1 + \lambda_2) > \frac{1}{4}$, $\sqrt{n}(Z_n - \frac{1}{2}) \xrightarrow[n\to\infty]{stably} \mathcal{N}\Big(0, C\Big)$.

b) for $(\lambda_1 + \lambda_2) = \frac{1}{4}$, $\frac{\sqrt{n}}{\sqrt{\ln n}}(Z_n - \frac{1}{2}) \xrightarrow[n\to\infty]{stably} \mathcal{N}\Big(0, C\Big)$.

Proposition 1. *Let $\lambda_1 > 0, \lambda_2 > 0$; when $\gamma_1 = \gamma_2 = 1$ following statement hold. When $(\lambda_1 + \lambda_2) < \frac{1}{4}$,*

$$\sqrt{n}(Z_n - \frac{1}{2}) \xrightarrow[n\to\infty]{a.s./\mathcal{L}^1} \tilde{X},$$

for some real random variable $\tilde{X}$ such that $\mathbb{P}(\tilde{X} \neq 0) > 0$.

Proposition 2. *In the following cases: $\lambda_1 \neq 0,\ \lambda_2 = 0$ and $\lambda_1 = 0,\ \lambda_2 \neq 0$*

	$\lambda_1 \neq 0,\ \lambda_2 = 0$	$\lambda_1 = 0,\ \lambda_2 \neq 0$
$\gamma_1 \leq \gamma_2$	$n^{\frac{\gamma_1}{2}}(Z_n - Z_n(i)) \xrightarrow{stably} \mathcal{N}\Big(0, C\Big)$ $n^{\frac{\gamma_1}{2}}(Z_n - \frac{1}{2}) \xrightarrow{stably} \mathcal{N}\Big(0, C\Big)$	$n^{\gamma_1 - \frac{\gamma_2}{2}}(Z_n - Z_n(i)) \xrightarrow{stably} \mathcal{N}\Big(0, C\Big)$ $n^{\gamma_1 - \frac{\gamma_2}{2}}(Z_n - \frac{1}{2}) \xrightarrow{stably} \mathcal{N}\Big(0, C\Big)$
$\gamma_2 < \gamma_1$	$n^{\gamma_1 - \frac{\gamma_2}{2}}(Z_n - Z_n(i)) \xrightarrow{stably} \mathcal{N}\Big(0, C\Big)$ $n^{\gamma_2 - \frac{\gamma_1}{2}}(Z_n - \frac{1}{2}) \xrightarrow{stably} \mathcal{N}\Big(0, C\Big)$	$n^{\gamma_1 - \frac{\gamma_2}{2}}(Z_n - Z_n(i)) \xrightarrow{stably} \mathcal{N}\Big(0, C\Big)$ $n^{\gamma_2}(Z_n - \frac{1}{2}) \xrightarrow{stably} \mathcal{N}\Big(0, C\Big)$

	$\lambda_i = 0,\ \lambda_j > \frac{1}{4}$	$\lambda_i = 0,\ \lambda_j = \frac{1}{4}$	$\lambda_i = 0,\ \lambda_j < \frac{1}{4}$
$\gamma_1 = \gamma_2 = \gamma$	$\sqrt{n}(Z_n - \frac{1}{2}) \xrightarrow{stably} \mathcal{N}\Big(0, C\Big)$	$\frac{\sqrt{n}}{\sqrt{\ln n}}(Z_n - \frac{1}{2}) \xrightarrow{stably} \mathcal{N}\Big(0, C\Big)$	$\sqrt{n}(Z_n - \frac{1}{2}) \xrightarrow{a.s./\mathcal{L}^1} \tilde{X}$

Theorem 5. *Let $\lambda_1 = \lambda_2 = 0$. Following statements hold.*

(i) For any $\frac{1}{2} < (\gamma_1, \gamma_2) \leq 1$, it holds

$$n^{\gamma_1 - \frac{\gamma_2}{2}}(Z_n - Z_n(i)) \xrightarrow[n\to\infty]{stably} \mathcal{N}\Big(0, C\ Z_\infty(1 - Z_\infty)\Big).$$

(ii) For $\frac{1}{2} < (\gamma_1, \gamma_2) \leq 1$, let $\gamma := \min(\gamma_1, \gamma_2)$, it holds

$$n^{\gamma - \frac{1}{2}}(Z_n - Z_\infty) \xrightarrow[n\to\infty]{stably} \mathcal{N}\Big(0, C_{N,\gamma} Z_\infty(1 - Z_\infty)\Big).$$

5. Proofs

5.1. First results about the variances

Using (2) and (3), we compute recursive equations satisfied by variance of $Z_n(i)$ and Z_n:

$$\begin{aligned}
&\mathbb{V}\mathrm{ar}(Z_{n+1}(i)|\mathcal{F}_n) = \mathbb{V}\mathrm{ar}\left[(1 - r_n^l - r_n^g)Z_n + r_n^l \xi_{n+1}^l(i) + r_n^g \xi_{n+1}^g \Big| \mathcal{F}_n\right] \\
&= (r_n^l)^2\, \mathbb{V}\mathrm{ar}(\xi_{n+1}^l(i)|\mathcal{F}_n) + (r_n^g)^2\, \mathbb{V}\mathrm{ar}(\xi_{n+1}^g|\mathcal{F}_n) \\
&= (r_n^l)^2 \left[(1-2\lambda_1)^2(Z_n(i) - Z_n(i)^2) + \lambda_1 - \lambda_1^2\right] + (r_n^g)^2 \left[(1-2\lambda_2)^2(Z_n - Z_n^2) + \lambda_2 - \lambda_2^2\right],
\end{aligned}$$

then using the usual decomposition $(*)$, we have

$$\begin{aligned}
&\mathbb{V}\mathrm{ar}(Z_{n+1}(i)) \overset{*}{=} \mathbb{E}[\mathbb{V}\mathrm{ar}(Z_{n+1}(i)|\mathcal{F}_n)] + \mathbb{V}\mathrm{ar}[\mathbb{E}(Z_{n+1}(i)|\mathcal{F}_n)] \\
&= (r_n^l)^2 \left[(1-2\lambda_1)^2(Z_n(i) - Z_n(i)^2) + \lambda_1 - \lambda_1^2\right] + (r_n^g)^2 \left[(1-2\lambda_2)^2(Z_n - Z_n^2) + \lambda_2 - \lambda_2^2\right] \\
&\quad + (1 - 2\lambda_1 r_n^l - r_n^g)^2\, \mathbb{V}\mathrm{ar}(Z_n(i)) + (r_n^g)^2(1-2\lambda_2)^2\, \mathbb{V}\mathrm{ar}(Z_n) + 2(1 - 2\lambda_1 r_n^l - r_n^g) r_n^g (1-2\lambda_2)\, \mathbb{V}\mathrm{ar}(Z_n).
\end{aligned} \tag{4}$$

Where in the last equation we used the fact that $\mathrm{Cov}(Z_n(i), Z_n) = \mathbb{V}\mathrm{ar}(Z_n)$. Indeed by symmetry

$$\mathbb{E}(Z_n(i)Z_n) = \frac{1}{N}\sum_{j=1}^N \mathbb{E}(Z_n(j)Z_n) = \mathbb{E}\left[\Big(\frac{1}{N}\sum_{j=1}^N (Z_n(j)\Big) Z_n\right] = \mathbb{E}(Z_n^2).$$

Thus,

$$\begin{aligned}
&\mathbb{V}\mathrm{ar}(Z_{n+1}|\mathcal{F}_n)) = \frac{(r_n^l)^2}{N^2}\sum_{i=1}^N \mathbb{V}\mathrm{ar}(\xi_{n+1}^l(i)|\mathcal{F}_n) + (r_n^g)^2\, \mathbb{V}\mathrm{ar}(\xi_{n+1}^g|\mathcal{F}_n) \\
&= \frac{(r_n^l)^2}{N}\left[(1-2\lambda_1)^2\Big(Z_n - \frac{1}{N}\sum_{i=1}^N Z_n(i)^2\Big) + \lambda_1 - \lambda_1^2\right] + (r_n^g)^2 \left[(1-2\lambda_2)^2(Z_n - Z_n^2) + \lambda_2 - \lambda_2^2\right]
\end{aligned}$$

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1-2\lambda_1 r_n^l - 2\lambda_1 r_n^g)^2\,\mathbb{V}\mathrm{ar}(Z_n) \\ + \frac{(r_n^l)^2}{N}\left[(1-2\lambda_1)^2\Big(\mathbb{E}(Z_n) - \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(Z_n(i)^2)\Big) + \lambda_1 - \lambda_1^2\right] \\ + (r_n^g)^2\left[(1-2\lambda_2)^2(\mathbb{E}(Z_n) - \mathbb{E}(Z_n^2)) + \lambda_2 - \lambda_2^2\right] \quad (5)\end{aligned}$$

Furthermore we can rewrite the (1) as

$$Z_{n+1} - Z_n = r_n^l\left(\frac{1}{N}\sum_{i=1}^{N}\xi_{n+1}^l(i) - Z_n\right) + r_n^g\left(\xi_{n+1}^g - Z_n\right), \quad (6)$$

also we can obtain easily

$$\mathbb{E}\left(\frac{1}{N}\sum \xi_{n+1}^l(i)\Big|\mathcal{F}_n\right) = (1-2\lambda_1)Z_n + \lambda_1,$$

$$\mathbb{E}(\xi_{n+1}^g|\mathcal{F}_n) = (1-2\lambda_2)Z_n + \lambda_2.$$

Finally, note that defining the following quantities

$$\Delta M_{n+1}^l := \frac{1}{N}\sum_{i=1}^{N}\xi_{n+1}^l(i) - \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}\xi_{n+1}^l(i)|\mathcal{F}_n\right)$$

$$\Delta M_{n+1}^g := \xi_{n+1}^g - \mathbb{E}(\xi_{n+1}^g|\mathcal{F}_n),$$

which means

$$\Delta M_{n+1}^l = \frac{1}{N}\sum_{i=1}^{N}\xi_{n+1}^l(i) - [(1-2\lambda_1)Z_n(i) + \lambda_1], \quad (7)$$

$$\Delta M_{n+1}^g = \xi_{n+1}^g - [(1-2\lambda_2)Z_n(i) + \lambda_2]. \quad (8)$$

Using (6), (7) and (8) it leads to

$$Z_{n+1} - \frac{1}{2} = \left(Z_n - \frac{1}{2}\right)\left[1 - 2r_n^l\lambda_1 - 2r_n^g\lambda_2\right] + r_n^l\Delta M_{n+1}^l + r_n^g\Delta M_{n+1}^g. \quad (9)$$

Remark 4. The assumption $\forall i \in \{1,..N\}\ Z_0(i) = \frac{1}{2}$ leads to fix point in the equation between expectations. Thus $\forall n \in \mathbb{N}$, $\mathbb{E}(Z_n) = \mathbb{E}(Z_n(i)) = \frac{1}{2}$. Indeed,

$$\mathbb{E}(Z_{n+1}(i)) = (1-2\lambda_1 r_n^l - r_n^g)\,\mathbb{E}(Z_n(i)) + (1-2\lambda_2)r_n^g\,\mathbb{E}(Z_n) + (\lambda_1 r_n^l + \lambda_2 r_n^g)$$

then, when $\lambda_1 \neq 0, \lambda_2 \neq 0$, the only unique point that causes $\mathbb{E}(Z_{n+1}(i)) = \mathbb{E}(Z_n(i))$ is $\frac{1}{2}$. Moreover when $\lambda_1 = \lambda_2 = 0$, $\mathbb{E}(Z_{n+1}(i)) = \mathbb{E}(Z_n(i))$ so due to $Z_0(i) = \frac{1}{2}$ and consequently $\mathbb{E}(Z_0(i)) = \frac{1}{2}$ yields by induction on n, $\mathbb{E}(Z_{n+1}(i)) = \mathbb{E}(Z_n(i)) = \frac{1}{2}$.

So from (5)

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}) &= \left[1-4\Big(\lambda_1 r_n^l+\lambda_2 r_n^g-2\lambda_1\lambda_2 r_n^l r_n^g-\lambda_1^2(r_n^l)^2-\lambda_2^2(r_n^g)^2+\frac{(r_n^g)^2}{4N}(1-2\lambda_2)\Big)\right]\mathbb{V}\mathrm{ar}(Z_n)\\ &+ \frac{(r_n^l)^2}{N}\left[(1-2\lambda_1)^2\Big(\frac{1}{2}-\frac{1}{N}\sum_{i=1}^N\mathbb{E}(Z_n(i)^2)\Big)+\lambda_1-\lambda_1^2\right]+\frac{(r_n^g)^2}{4}. \qquad (10)\end{aligned}$$

and from (4)

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}(i)) &= \Big[(1-2\lambda_1 r_n^l-r_n^g)^2-(r_n^l)^2(1-2\lambda_1)^2)\Big]\mathbb{V}\mathrm{ar}(Z_n(i))\\ &+ (r_n^l)^2\left[\frac{1}{2}-\lambda_1+\lambda_1^2-\frac{(1-2\lambda_1)^2}{4}\right]+(r_n^g)^2\left[\frac{1}{2}-\lambda_2+\lambda_2^2-\frac{(1-2\lambda_2)^2}{4}\right]\\ &+ 2(1-2\lambda_1 r_n^l-r_n^g)r_n^g(1-2\lambda_2)\,\mathbb{V}\mathrm{ar}(Z_n) \qquad (11)\end{aligned}$$

Remark 5. When $\gamma_1=\gamma_2=\gamma$,

$$\begin{aligned}\mathbb{E}(Z_{n+1}(i)|\mathcal{F}_n) &= (1-2r_n)Z_n(i)+r_n\,\mathbb{E}[\tilde{\xi}_{n+1}(i)|\mathcal{F}_n]\\ &= (1-(1+2\lambda_1)r_n)Z_n(i)+(1-2\lambda_2)r_nZ_n+(\lambda_1+\lambda_2)r_n\end{aligned}$$

because,

$$\begin{aligned}\mathbb{E}(\tilde{\xi}_{n+1}(i)|\mathcal{F}_n) &= \mathbb{P}(\xi_{n+1}^l(i)=1|\mathcal{F}_n)\times\mathbb{P}(\xi_{n+1}^g=0|\mathcal{F}_n)+\mathbb{P}(\xi_{n+1}^l(i)=0|\mathcal{F}_n)\times\mathbb{P}(\xi_{n+1}^g=1|\mathcal{F}_n)\\ &+ 2\,\mathbb{P}(\xi_{n+1}^l(i)=1|\mathcal{F}_n)\times\mathbb{P}(\xi_{n+1}^g=1|\mathcal{F}_n)\\ &= (1-2\lambda_1)Z_n(i)+(1-2\lambda_2)Z_n+\lambda_1+\lambda_2\end{aligned}$$

$$\begin{aligned}\mathbb{E}(Z_{n+1}|\mathcal{F}_n) &= (1-2r_n)Z_n+\frac{r_n}{N}\sum_{i=1}^n\mathbb{E}(\tilde{\xi}_{n+1}(i)|\mathcal{F}_n)\\ &= \Big(1-2(\lambda_1+\lambda_2)r_n\Big)Z_n+(\lambda_1+\lambda_2)r_n\end{aligned}$$

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}(i)|\mathcal{F}_n) &= \Big[(1-2\lambda_1)Z_n(i)+\lambda_1\Big]-\Big[(1-2\lambda_1)^2Z_n(i)^2+\lambda_1^2+2\lambda_1(1-2\lambda_1)Z_n(i)\Big]\\ &+ \Big[(1-2\lambda_2)Z_n+\lambda_2\Big]-\Big[(1-2\lambda_2)^2Z_n^2+\lambda_2^2+2\lambda_2(1-2\lambda_2)Z_n\Big]\end{aligned}$$

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}(i)) &\overset{*}{=} r_n^2[(1-2\lambda_1)^2(\mathbb{E}(Z_n(i))-\mathbb{E}(Z_n(i)^2))+\lambda_1-\lambda_1^2]+[(1-2\lambda_2)^2(\mathbb{E}(Z_n)-\mathbb{E}(Z_n^2))+\lambda_2-\lambda_2^2]\\ &+ (1-r_n(2\lambda_1+1))^2\,\mathbb{V}\mathrm{ar}(Z_n(i))+r_n^2(1-2\lambda_2)^2\,\mathbb{V}\mathrm{ar}(Z_n)+(1-r_n(2\lambda_1+1))(1-2\lambda_2)\,\mathbb{V}\mathrm{ar}(Z_n)\end{aligned}$$

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}(i)) &= r_n^2\left(\frac{1}{2}-\lambda_1+\lambda_2-\frac{(1-2\lambda_1)^2}{4}+\frac{1}{2}-\lambda_2+\lambda_2^2-\frac{(1-2\lambda_2)^2}{4}\right)\\ &+ [(1-r_n(2\lambda_1+1))^2-r_n(1-2\lambda_1)^2]\,\mathbb{V}\mathrm{ar}(Z_n(i))\\ &+ [(1-2\lambda_2)^2+r_n(1-r_n(2\lambda_1+1))(1-2\lambda_1)]\,\mathbb{V}\mathrm{ar}(Z_n) \qquad (12)\end{aligned}$$

$$\mathbb{V}\mathrm{ar}(Z_{n+1}|\mathcal{F}_n) \quad = \quad \frac{r_n^2}{N^2}\sum_{i=1}^{N}\mathbb{V}\mathrm{ar}(\tilde{\xi}_{n+1}(i)|\mathcal{F}_n) = r_n^2\left[\frac{1}{N^2}\sum_{i=1}^{N}\mathbb{V}\mathrm{ar}(\xi_{n+1}^l(i)|\mathcal{F}_n) + \mathbb{V}\mathrm{ar}(\xi_{n+1}^g|\mathcal{F}_n)\right]$$

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}) = \left(1 - 4[(\lambda_1+\lambda_2)r_n - (\lambda_1+\lambda_2)^2 r_n^2] - (1-2\lambda_2)^2 r_n^2\right)\mathbb{V}\mathrm{ar}(Z_n) \\ + \frac{r_n^2}{N}\left[(1-2\lambda_1)^2(\frac{1}{2} - \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(Z_n(i)^2)) + \lambda_1 - \lambda_1^2 + \frac{N}{4}\right] \quad (13)\end{aligned}$$

$$Z_{n+1} - \frac{1}{2} = \left(Z_n - \frac{1}{2}\right)[1 - 2r_n(\lambda_1+\lambda_2)] + r_n\Delta\tilde{M}_{n+1}(i), \quad (14)$$

where $\Delta\tilde{M}_{n+1}(i) = \Delta M_{n+1}^l(i) + \Delta M_{n+1}^g$.

Lemma 5.1. *Depending on the γ_j's relationship,*

i) there exists a constant $0 < C < \frac{1}{4}$ such that,

a) when $\gamma_1 < \gamma_2$, it holds $\mathbb{V}ar(Z_n) \leq \frac{C}{N}$;

b) when $\gamma_1 > \gamma_2$, it holds $\mathbb{V}ar(Z_n) \leq C$.

Moreover, in particular $\mathbb{E}(Z_n^2) < \frac{1}{2}$.

ii) When $\gamma_1 = \gamma_2$ there exists a constant $0 < C < \frac{1}{8}$ such that $\mathbb{V}ar(Z_n) < \frac{C(2+N)}{N}$.

Proof. Using the fact

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}) = \mathbb{V}\mathrm{ar}(Z_n) + \frac{(r_n^l)^2}{2N} + \frac{(r_n^g)^2}{4} \\ - 4(\lambda_1 r_n^l + \lambda_2 r_n^g - 2\lambda_1\lambda_2 r_n^l r_n^g - \lambda_1^2(r_n^l)^2 - \lambda_2^2(r_n^g)^2 + \frac{(r_n^g)^2}{4}(1-2\lambda_2))\,\mathbb{V}\mathrm{ar}(Z_n) \\ - (r_n^l)^2\left((1-2\lambda_1)^2\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(Z_n(i)^2) + \lambda_1 - \lambda_1^2\right).\end{aligned}$$

In particular,

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) \leq \mathbb{V}\mathrm{ar}(Z_n) + \frac{(r_n^l)^2}{2N} + \frac{(r_n^g)^2}{4}.$$

i)(a) When $\gamma_1 < \gamma_2$ and according to that we must take $r_n = \frac{1}{n+3}$ due to $Z_0 = \frac{1}{2}$ (we start with 2 balls),

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) \leq \mathbb{V}\mathrm{ar}(Z_n) + \frac{1}{2N}\frac{1}{(n+3)^{2\gamma_1}}$$

from which we obtain

$$\mathbb{V}\mathrm{ar}(Z_n) \leq \frac{1}{2N} \sum_{k=3}^{\infty} \frac{1}{k^{2\gamma_1}} < \frac{1}{4N}$$

i)(b) When $\gamma_2 < \gamma_1$, conclusion follows from

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) \leq \mathbb{V}\mathrm{ar}(Z_n) + \frac{(r_n^g)^2}{4}.$$

Moreover,

$$\mathbb{E}(Z_n^2) = \mathbb{V}\mathrm{ar}(Z_n) + \frac{1}{4} < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

(ii) Using

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) \leq \mathbb{V}\mathrm{ar}(Z_n) + r_n^2(\frac{1}{2N} + \frac{1}{4})$$

then

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) \leq \mathbb{V}\mathrm{ar}(Z_n) + (\frac{2+N}{4N})r_n^2$$

hence,

$$\mathbb{V}\mathrm{ar}(Z_n) \leq \frac{2+N}{4N} \sum_{k=3}^{\infty} \frac{1}{k^{2\gamma_1}} < \frac{(2+N)}{8N}.$$

□

5.2. Proofs of L^2 and a.s. convergence results

Proof. (first th.)(i) First considering the equation (10) which behaves like

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1 - \frac{4\lambda}{n^\gamma})\,\mathbb{V}\mathrm{ar}(Z_n) + \frac{K_n}{n^{2\gamma}} + \mathcal{O}(\frac{1}{n^{2\gamma}}),$$

where $\lambda = \begin{cases} \lambda_1 & \text{if } \gamma_1 < \gamma_2 \\ \lambda_2 & \text{if } \gamma_1 > \gamma_2 \end{cases}$.

If $\gamma_1 < \gamma_2$ then $C_{N,\lambda} = K_n/4\lambda_1$ where

$$K_n = \frac{1}{N}\left((1-2\lambda_1)^2\left(\frac{1}{2} - \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(Z_n(i)^2)\right) + \lambda_1 - \lambda_1^2\right)$$

is bounded and not equal zero. Indeed, according to $\mathbb{E}(Z_n^2) < \frac{1}{2}$;

$$\sum_{i=1}^{N}\mathbb{E}(Z_n(i)^2) = \mathbb{E}(Z_n^2(j)) + \sum_{i=1,\ i\neq j}^{N-1}\mathbb{E}(Z_n(i)^2) < \frac{1}{2} + \frac{N-1}{2} = \frac{N}{2}$$

so by Lemma A.1 $\mathbb{V}\mathrm{ar}(Z_n) \sim \frac{C_{N,\lambda}}{n^{\gamma_1}}$. Also if $\gamma_1 > \gamma_2$ then $C_{N,\lambda} = K_n/4\lambda_2$ where $K_n = \frac{1}{4}$ that is bounded clearly, so we obtain that $\mathbb{V}\mathrm{ar}(Z_n) \sim \frac{C_{N,\lambda}}{n^{\gamma_2}}$, hence in both cases $\lim_{n\to\infty} \mathbb{V}\mathrm{ar}(Z_n) = 0$, which means Z_n converges to constant.

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}(i)) = &\left[1-4\lambda_1 r_n^l - 2r_n^g + 4\lambda^2 (r_n^l)^2 + 4\lambda_1 r_n^l r_n^g - (r_n^l)^2(1-2\lambda_1)^2\right]\mathbb{V}\mathrm{ar}(Z_n(i))\\ &+ (r_n^l)^2\left[\frac{1}{2} - \lambda_1 + \lambda_1^2 - \frac{(1-2\lambda_1)^2}{4}\right] + (r_n^g)^2\left[\frac{1}{2} - \lambda_2 + \lambda_2^2 - \frac{(1-2\lambda_2)^2}{4}\right]\\ &+ [(1-2\lambda_2 r_n^g - 2\lambda_1(1-2\lambda_2) r_n^l r_n^g)(1-2\lambda_2) r_n^g]\,\mathbb{V}\mathrm{ar}(Z_n). \quad (15)\end{aligned}$$

When $\gamma_1 < \gamma_2$ we have $\mathbb{V}\mathrm{ar}(Z_n) \sim \frac{1}{n^{\gamma_1}}$ so,

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}(i)) &= \left[1 - 4\lambda_1 r_n^l + \mathcal{O}(r_n^l)\right]\mathbb{V}\mathrm{ar}(Z_n(i))\\ &+ (r_n^l)^2\left[\frac{1}{2} - \lambda_1 + \lambda_1^2 - \frac{(1-2\lambda_1)^2}{4}\right] + (1-2\lambda_2)\frac{1}{n^{\gamma_1+\gamma_2}}\mathcal{O}((r_n^l)^2) + \mathcal{O}(r_n^g),\end{aligned}$$

then $C_{N,\lambda} = K_n/4\lambda_1$ where $K_n = \left(\frac{1}{2} - \lambda_1 + \lambda_1^2 - \frac{(1-2\lambda_1)^2}{4}\right)$ which implies $\mathbb{V}\mathrm{ar}(Z_n(i)) \sim \frac{K_n/A}{n^{\gamma_1}}$.

When $\gamma_2 < \gamma_1$ we have $\mathbb{V}\mathrm{ar}(Z_n) \sim \frac{1}{n^{\gamma_2}}$ so,

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}(i)) &= [1 - 2r_n^g + \mathcal{O}(r_n^g)]\,\mathbb{V}\mathrm{ar}(Z_n(i))\\ &+ (r_n^g)^2\left[\frac{1}{2} - \lambda_2 + \lambda_2^2 - \frac{(1-2\lambda_2)^2}{4}\right] + (1-2\lambda_2)\frac{1}{n^{2\gamma_2}}\\ &\simeq (1-2r_n^g)\,\mathbb{V}\mathrm{ar}(Z_n(i)) + (r_n^g)^2\left[\frac{1}{2} - \lambda_2 + \lambda_2^2 - \frac{(1-2\lambda_2)^2}{4} + (1-2\lambda_2)\right]\end{aligned}$$

then $C_{N,\lambda} = K_n/2$ where $K_n = \left[\frac{1}{2} - \lambda_2 + \lambda_2^2 - \frac{(1-2\lambda_2)^2}{4} + (1-2\lambda_2)\right]$ which implies $\mathbb{V}\mathrm{ar}(Z_n(i)) \sim \frac{C_{N,\lambda}}{n^{\gamma_2}}$.

When $\gamma_1 = \gamma_2$, using (13) we have

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}) &= (1 - 2(\lambda_1+\lambda_2)r_n - N(1-2\lambda_2)^2 r_n^2)\,\mathbb{V}\mathrm{ar}(Z_n) + r_n^2 K_n\\ &\simeq \left[1 - 2(\lambda_1+\lambda_2)r_n\right]\mathbb{V}\mathrm{ar}(Z_n) + r_n^2 K_n,\end{aligned}$$

where $C_{N,\lambda_1,\lambda_2} = K_n/2(\lambda_1+\lambda_2)$ where

$$K_n := \frac{1}{N}\left((1-2\lambda_1)^2\left(\frac{1}{2} - \frac{1}{N}\sum_{i=1}^N \mathbb{E}(Z_n(i)^2)\right) + \lambda_1 - \lambda_1^2 + \frac{N}{4}\right),$$

which implies $\mathbb{V}\mathrm{ar}(Z_n) \sim \frac{C_{N,\lambda_1,\lambda_2}}{n^{\gamma}}$. Moreover, using (12) and fact $\mathbb{V}\mathrm{ar}(Z_n) \sim \frac{1}{n^{\gamma}}$,

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}(i)) &= \left[(1 - r_n(2\lambda_1+1))^2 - r_n(1-2\lambda_1)^2\right]\mathbb{V}\mathrm{ar}(Z_n(i))\\ &+ r_n^2 K_n\end{aligned}$$

then $C_{N,\lambda_1,\lambda_2}$ where

$$K_n := \left[\frac{1}{2} - \lambda_1 + \lambda_2 - \frac{(1-2\lambda_1)^2}{4} + \frac{1}{2} - \lambda_2 + \lambda_2^2 - \frac{(1-2\lambda_2)^2}{4}\right] + [(1-2\lambda_2)^2 + r_n(1-r_n(2\lambda_1+1))(1-2\lambda_1)]$$

which implies $\mathbb{V}\mathrm{ar}(Z_n(i)) \sim \frac{1}{n^\gamma}$.

(ii) Consider the following recursive equation satisfied, for any $i \in \{1,\dots,N\}$ by the L^2 distance between one component and the mean field

$$x_n := \mathbb{E}[(Z_n(i) - Z_n)^2] = \mathbb{V}\mathrm{ar}(Z_n(i) - Z_n),$$

$$\begin{aligned}
x_{n+1} &\overset{*}{=} \mathbb{E}\left(\mathbb{V}\mathrm{ar}\left[(1-r_n^l-r_n^g)(Z_n(i)-Z_n) + r_n^l(\xi_n^l(i) - \frac{1}{N}\sum_j \xi_n^l(j))\Big|\mathcal{F}_n\right]\right)\\
&+ \mathbb{V}\mathrm{ar}\left[Z_n(i) - 2\lambda_1 r_n^l Z_n(i) + r_n^g(Z_n(i) - (1-2\lambda_2)Z_n) - Z_n(1-2\lambda_1 r_n^l - r_n^g)\right]\\
&= (r_n^l)^2\,\mathbb{E}\left[\mathbb{V}\mathrm{ar}(\xi_n^l(i) - \frac{1}{N}\sum_i \xi_n^l(i)|\mathcal{F}_n)\right] + \mathbb{V}\mathrm{ar}\left[(1-2\lambda_1 r_n^l - r_n^g)(Z_n(i)-Z_n)\right]\\
&= (1-2\lambda_1 r_n^l - r_n^g)^2\,\mathbb{V}\mathrm{ar}(Z_n(i)-Z_n)\\
&+ (r_n^l)^2\Big((1-\frac{1}{N})^2 + (\frac{N-1}{N^2})\Big)\,\mathbb{E}\left[\mathbb{V}\mathrm{ar}(\xi_n^l(i)|\mathcal{F}_n)\right]\\
&= (1-2\lambda_1 r_n^l - r_n^g)^2 x_n\\
&+ \frac{N-1}{N}(r_n^l)^2\left[\left[(1-2\lambda_1)\,\mathbb{E}(Z_n(i)) + \lambda_1\right] - \left[\frac{(1-2\lambda_1)^2}{N}\,\mathbb{E}(Z_n(i)^2) + \lambda_1^2 + 2\lambda_1(1-2\lambda_1)\,\mathbb{E}(Z_n)\right]\right]
\end{aligned}$$

$$\begin{aligned}
x_{n+1} &= \Big(1 - 4\lambda_1 r_n^l - 2r_n^g + 4\lambda_1^2(r_n^l)^2 + (r_n^g)^2 + 2\lambda_1 r_n^l r_n^g\Big)x_n\\
&+ \frac{N-1}{N}(r_n^l)^2\left[\frac{1}{2} - \left[\frac{(1-2\lambda_1)^2}{N}\,\mathbb{E}(Z_n(i)^2) + \lambda_1 - \lambda_1^2\right]\right]
\end{aligned}$$

$$x_{n+1} = \Big(1 - 4\lambda_1 r_n^l - 2r_n^g + 4\lambda_1^2(r_n^l)^2 + (r_n^g)^2 + 2\lambda_1 r_n^l r_n^g\Big)x_n + (r_n^l)^2\varepsilon_n$$

where $\varepsilon_n = \frac{N-1}{N}\left(\frac{1}{2} - [\frac{(1-2\lambda_1)^2}{N}\,\mathbb{E}(Z_n(i)^2) + \lambda_1 - \lambda_1^2]\right)$ is bounded and not equal zero for $N > 1$.

(a) For $\gamma_1 < \gamma_2$ we have $x_{n+1} = [1 - 4\lambda_1 r_n^l - \mathcal{O}(r_n^l)]x_n + (r_n^l)^2\varepsilon_n$ can be written as $x_{n+1} = (1 - \frac{4\lambda_1}{n^{\gamma_1}})x_n + \frac{\varepsilon_n}{n^{2\gamma_1}}$ which implies $x_n \sim \frac{C_{N,\lambda_1}}{n^{\gamma_1}}$ where $C_{N,\lambda_1} = \varepsilon_n/4\lambda_1$.
Also for $\gamma_1 = \gamma_2$ we have

$$x_{n+1} = [(1 - r_n - 2\lambda_1 r_n)^2]x_n + r_n^2\varepsilon_n \simeq [1 - (2 + 4\lambda_1)r_n]x_n + r_n^2\varepsilon_n$$

which implies $x_n \sim \frac{C_{N,\lambda_1}}{n^\gamma}$ where $C_{N,\lambda_1} = \varepsilon_n/2 + 4\lambda_1$.

(b) When $\gamma_2 < \gamma_1$, $x_{n+1} = (1 - \frac{2}{n^{\gamma_2}} + \mathcal{O}(r_n^g))x_n + \frac{\varepsilon_n}{n^{2\gamma_1}}$ where $C_{N,\lambda_1} = \varepsilon_n/2$ implies $x_n \sim \frac{C_{N,\lambda_1}}{n^{2\gamma_1-\gamma_2}}$.

Thus in all cases, $\gamma_1 < \gamma_2$, $\gamma_2 < \gamma_1$ and $\gamma_1 = \gamma_2$, $\lim_{n\to\infty} \mathbb{E}[(Z_n(i) - Z_n)^2] = 0$ which means synchronization happens in L^2, but with different speed according to relation between γ_1 and γ_2.

(iii) Now to prove that the convergence's value is equal $\frac{1}{2}$ a.s, using (9) and let

$$
\begin{aligned}
&\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2|\mathcal{F}_n] \\
&= \left(Z_n - \frac{1}{2}\right)^2 [1 + 4(r_n^l)^2\lambda_1^2 + 4(r_n^g)^2\lambda_2^2 - 4r_n^l\lambda_1 - 4r_n^g\lambda_2 + 4r_n^l r_n^g\lambda_1\lambda_2] \\
&\quad + (r_n^l)^2\,\mathbb{E}[(\Delta M_{n+1}^l)^2|\mathcal{F}_n] + (r_n^g)^2\,\mathbb{E}[(\Delta M_{n+1}^g)^2|\mathcal{F}_n] \\
&= \left(Z_n - \frac{1}{2}\right)^2 \left[1 - 4r_n^l\lambda_1 - 4r_n^g\lambda_2 + \mathcal{O}(r_n^l) + \mathcal{O}(r_n^g)\right] \\
&\quad + (r_n^l)^2 \left[4\lambda_1^2(Z_n - \frac{1}{2})^2 + \mathbb{E}[(\Delta M_{n+1}^l)^2|\mathcal{F}_n]\right] + (r_n^g)^2 \left[4\lambda_2^2(Z_n - \frac{1}{2})^2 + \mathbb{E}[(\Delta M_{n+1}^g)^2|\mathcal{F}_n]\right]
\end{aligned}
$$

Thus, $\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2|\mathcal{F}_n] \le (Z_n - \frac{1}{2})^2 + (r_n^l)^2 W_n^l + (r_n^g)^2 W_n^g$ where

$$W_n^l := (r_n^l)^2 \left[4\lambda_1^2 \left(Z_n - \frac{1}{2}\right)^2 + \mathbb{E}[(\Delta M_{n+1}^l)^2|\mathcal{F}_n]\right],$$

$$W_n^g := (r_n^g)^2 \left[4\lambda_2^2 \left(Z_n - \frac{1}{2}\right)^2 + \mathbb{E}[(\Delta M_{n+1}^g)^2|\mathcal{F}_n]\right].$$

By assumption $\frac{1}{2} < \gamma_1 \le 1$ and $\frac{1}{2} < \gamma_2 \le 1$, so $(Z_{n+1} - \frac{1}{2})^2$ is a positive supermartingale and almost sure convergence holds.

It is enough to consider L^2 convergence in order to identify the (deterministic) limit.

$$
\begin{aligned}
\mathbb{E}\left(\mathbb{E}\left[\left(Z_{n+1} - \frac{1}{2}\right)^2|\mathcal{F}_n\right]\right) &= \mathbb{E}\left(Z_n - \frac{1}{2}\right)^2 \left[1 - 4r_n^l\lambda_1 - 4r_n^g\lambda_2 + 4r_n^l r_n^g\lambda_1\lambda_2\right] \\
&+ (r_n^l)^2 K_n^l + (r_n^g)^2 K_n^g.
\end{aligned}
$$

Let $y_n := \mathbb{E}(Z_n - \frac{1}{2})^2$, so

$$y_{n+1} = \left(1 - 4r_n^l\lambda_1 - 4r_n^g\lambda_2 + \lambda_1^2(r_n^l)^2 + \lambda_2^2(r_n^g)^2 + 4r_n^l r_n^g\lambda_1\lambda_2\right)x_n + (r_n^l)^2 K_{n+1}^l + (r_n^g)^2 K_{n+1}^g \tag{16}$$

where $0 \le K_{n+1}^l := \{\mathbb{E}[(\Delta M_{n+1}^l)^2]\} \le K$, $0 \le K_{n+1}^g := \{\mathbb{E}[(\Delta M_{n+1}^g)^2]\} \le K$, by $\lambda_i \le K$ and by lemma A.1 $\lim_{n\to\infty} x_n = 0$. So, $\lim \mathbb{E}(Z_n - \frac{1}{2})^2 = 0$. Using the fact that $(Z_n)_n$ converges almost surely, then $Z_n \xrightarrow{\text{a.s.}} \frac{1}{2}$.

The proof of the case when $\gamma_1 = \gamma_2$ is essentially the same as above using (14). Indeed,

$$\begin{aligned}\mathbb{E}[\Big(Z_{n+1} - \frac{1}{2}\Big)^2 |\mathcal{F}_n] &= \Big(Z_n - \frac{1}{2}\Big)^2 [1 - 2r_n(\lambda_1+\lambda_2)]^2 + r_n^2\, \mathbb{E}[\Delta\tilde{M}_{n+1}(i)^2|\mathcal{F}_n] \\ &+ 2\Big(Z_n - \frac{1}{2}\Big)[1 - 2r_n(\lambda_1+\lambda_2)] r_n\, \mathbb{E}[\Delta\tilde{M}_{n+1}(i)|\mathcal{F}_n]\end{aligned}$$

So $\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2|\mathcal{F}_n] \leq (Z_n - \frac{1}{2})^2 + r_n^2\tilde{W}_n$, where

$$\tilde{W}_n = r_n^2\left(4(\lambda_1+\lambda_2)^2(Z_n - \frac{1}{2})^2 + \mathbb{E}[(\Delta\tilde{M}_{n+1}(i))^2|\mathcal{F}_n]\right).$$

To prove the a.s. synchronization, since L^2 synchronization is hold, it is enough to show a.s convergence exists for $Z_n(i)$. We can obtain that $(Z_n(i))_n$ is quasi-martingale, *i.e.*

$$\sum_{n=0}^{+\infty} \mathbb{E}\,|\,\mathbb{E}[Z_{n+1}(i)|\mathcal{F}_n] - Z_n(i)| < +\infty$$

Indeed using the fact that

$$\mathbb{E}(Z_{n+1}(i) - Z_n(i)|\mathcal{F}_n) = \lambda_1 r_n^l\Big(1 - 2Z_n(i)\Big) + \lambda_2 r_n^g\Big(1 - 2Z_n\Big) + r_n^g\Big(Z_n - Z_n(i)\Big).$$

and corresponding to each part, $\mathbb{E}\,|1 - 2Z_n| = 2\,\mathbb{E}\,|Z_n - \frac{1}{2}|$ and $\mathbb{E}\,|1 - 2Z_n(i)| = 2\,\mathbb{E}\,|Z_n(i) - \frac{1}{2}|$, $Z_n(i), Z_n \xrightarrow{\mathrm{L}^2} \frac{1}{2}$, we obtain the convergence of the series of $\mathbb{E}\,|\,\mathbb{E}(Z_{n+1}(i) - Z_n(i)|\mathcal{F}_n)|$ using the following inequality and bounding the expectation with the second moment,

$$\sum_n \mathbb{E}\,|\,\mathbb{E}(Z_{n+1}(i) - Z_n(i)|\mathcal{F}_n)| \leq 2\lambda_1 r_n^l\,\mathbb{E}\left|Z_n(i) - \frac{1}{2}\right| + 2\lambda_2 r_n^g\,\mathbb{E}\left|Z_n - \frac{1}{2}\right| + r_n^g\,\mathbb{E}\,|Z_n - Z_n(i)|$$

□

Proof. (Th2) We shall consider to two different regimes of γ_j. First consider to the recursive equation of $\mathbb{V}\mathrm{ar}(Z_n)$ when $\gamma_1 < \gamma_2$,

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1 - \frac{4\lambda_1}{n^{\gamma_1}} + \mathcal{O}(r_n^l))\,\mathbb{V}\mathrm{ar}(Z_n) + \frac{K_n}{n^{2\gamma_1}},$$

where

$$K_n = \frac{1}{N}\left(\frac{1}{2} - \frac{(1-2\lambda_1)^2}{N}\sum_{i=1}^N \mathbb{E}(Z_n(i)^2) - (\lambda_1 - \lambda_1^2)\right),$$

where $C_{N,\lambda_1} = K_n/4\lambda_1$ which implies $\mathbb{V}\mathrm{ar}(Z_n) \sim \frac{C_{N,\lambda}}{n^{\gamma_1}}$, thus means Z_n converges to constant. Using similar argument in proof of Theorem 1(ii), we can proof that $\lim_{n\to\infty} Z_n = \frac{1}{2}$ a.s. To show it, using (9) and let $\lambda_2 = 0$, then the result is obtained. To study the synchronization, consider to the L^2-distance which behaves as follows

$$x_{n+1} = (1 - \frac{4\lambda_1}{n^{\gamma_1}} + \mathcal{O}(r_n^l))x_n + \frac{\varepsilon_n}{n^{2\gamma_1}},$$

where $\varepsilon_n = \frac{N-1}{N}\{\frac{1}{2} - [\frac{(1-2\lambda_1)^2}{N}\,\mathbb{E}(Z_n(i)^2) + \lambda_1 - \lambda_1^2]\}$ and where $C_{N,\lambda_1} = \varepsilon_n/4\lambda_1$ then can derive $x_n \sim \frac{C_{N,\lambda}}{n^{\gamma_1}}$ which implies $\lim_{n\to\infty} x_n = 0$ and using similar argument in Theorem 1(iv), synchronization holds a.s.

Case $\gamma_2 < \gamma_1$. Let us consider the recursive equation (10),

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1 - \frac{4\lambda_1}{n^{\gamma_1}})\,\mathbb{V}\mathrm{ar}(Z_n) + \frac{K_n}{n^{2\gamma_2}},$$

where $C_{N,\lambda_1} = K_n/4\lambda_1$ which implies $\mathbb{V}\mathrm{ar}(Z_n) \sim \frac{C_{N,\lambda_1}}{n^{2\gamma_2-\gamma_1}}$ which implies Z_n converges to constant then again by similar argument hence $\lim_{n\to\infty} Z_n = \frac{1}{2}$ a.s. Moreover considering the L^2-distance's behavior,

$$x_{n+1} = (1 - \frac{2}{n^{\gamma_2}} + \mathcal{O}(r_n^g))x_n + \frac{\varepsilon_n}{n^{2\gamma_1}},$$

where $C_{N,\lambda_1} = \varepsilon_n/2$ which implies $x_n \sim \frac{C_{N,\lambda_1}}{n^{2\gamma_1-\gamma_2}}$, synchronization holds a.s.

Case $\gamma_1 = \gamma_2$, so

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1 - \frac{4\lambda_1}{n^{\gamma}} + \mathcal{O}(r_n))\,\mathbb{V}\mathrm{ar}(Z_n) + \frac{K_n}{n^{2\gamma}},$$

where $C_{N,\lambda_1} = K_n/4\lambda_1$ which implies $\mathbb{V}\mathrm{ar}(Z_n) \sim \frac{C_{N,\lambda_1}}{n^{\gamma}}$. Using (14) and let $\lambda_2 = 0$ then the conclusion of *a.s.* is obtained.
To study the L^2-distance's behavior,

$$x_{n+1} = (1 - \frac{(1-2\lambda_1)}{n^{\gamma}} + \mathcal{O}(r_n))x_n + \frac{\varepsilon_n}{n^{2\gamma}},$$

where $C_{N,\lambda_1} = \varepsilon_n/(1-2\lambda_1)$ which implies $x_n \sim \frac{C_{N,\lambda_1}}{n^{\gamma}}$, synchronization holds a.s.
when $\gamma_1 < \gamma_2$,

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1 - \frac{4\lambda_2}{n^{\gamma_2}} + \mathcal{O}(r_n^g))\,\mathbb{V}\mathrm{ar}(Z_n) + \frac{K_n}{n^{2\gamma_1}},$$

where $C_{N,\lambda_2} = K_n/4\lambda_2$ and $K_n = \frac{1}{N}\{\frac{1}{2} - \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(Z_n(i)^2)\}$ which implies $\mathbb{V}\mathrm{ar}(Z_n) \sim \frac{C_{N,\lambda_2}}{n^{2\gamma_1-\gamma_2}}$, thus means Z_n converges to constant. Using similar argument in proof of Theorem 1(ii), we can prove that $\lim_{n\to\infty} Z_n = \frac{1}{2}$ a.s. To show it, using (9) and let $\lambda_1 = 0$, then the result is obtained. To study the synchronization, consider the L^2-distance which behaves as follows

$$x_{n+1} = (1 - \frac{2}{n^{\gamma_2}} + \mathcal{O}(r_n^g))x_n + \frac{\varepsilon_n}{n^{2\gamma_1}},$$

where $C_{N,\lambda_2} = \varepsilon_n/2$ which implies $x_n \sim \frac{C_{N,\lambda_2}}{n^{2\gamma_1-\gamma_2}}$, synchronization holds a.s.

Case $\gamma_2 < \gamma_1$. Let us consider the recursive equation (10),

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1 - \frac{4\lambda_2}{n^{\gamma_2}} + \mathcal{O}(r_n^g))\,\mathbb{V}\mathrm{ar}(Z_n) + \frac{K_n}{n^{2\gamma_2}},$$

where $C_{N,\lambda_2} = K_n/4\lambda_2$ which implies $\mathbb{V}\mathrm{ar}(Z_n) \sim \frac{C_{N,\lambda_2}}{n^{\gamma_2}}$ which implies Z_n converges to constant then again by similar argument Theorem 1(ii) hence $\lim_{n\to\infty} Z_n = \frac{1}{2}$ a.s. To study the synchronization, consider the L^2-distance which behaves as follows

$$x_{n+1} = (1 - \frac{2}{n^{\gamma_2}})x_n + \frac{\varepsilon_n}{n^{2\gamma_1}},$$

where $C_{N,\lambda_2} = \varepsilon_n/2$ which implies $x_n \sim \frac{C_{N,\lambda_2}}{n^{2\gamma_1-\gamma_2}}$, synchronization holds almost surely.

Case $\gamma_1 = \gamma_2$, so

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1 - \frac{4\lambda_2}{n^{\gamma}} + \mathcal{O}(r_n))\,\mathbb{V}\mathrm{ar}(Z_n) + \frac{K_n}{n^{2\gamma}},$$

where $C_{N,\lambda_2} = K_n/4\lambda_2$ which implies $\mathbb{V}\mathrm{ar}(Z_n) \sim \frac{C_{N,\lambda_2}}{n^{\gamma}}$. Using (14) and let $\lambda_1 = 0$ then the conclusion of *a.s.* is obtained.
To study the L^2-distance's behavior,

$$x_{n+1} = (1 - \frac{1}{n^{\gamma}} + \mathcal{O}(r_n))x_n + \frac{\varepsilon_n}{n^{2\gamma}},$$

which implies $x_n \sim \dfrac{C_{N,\lambda_2}}{n^{\gamma}}$, synchronization holds almost surely. □

Proof. (Th. 3)

(i)Using

$$\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = (1 - 2\lambda_1 r_n^l - 2\lambda_2 r_n^g)Z_n + r_n^l\lambda_1 + r_n^g\lambda_2,$$

If $\lambda_1 = \lambda_2 = 0$ then Z_n is a bounded martingale. Therefore, it converges a.s. to Z_∞. Let us consider

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1 - \frac{1}{4N\ n^{2\gamma_2}})\,\mathbb{V}\mathrm{ar}(Z_n) + \frac{K_n}{n^{2\min(\gamma_1,\gamma_2)}},$$

where

a) If $\gamma_1 < \gamma_2$ then $C_N = 4NK_n$ and $K_n = \frac{1}{N}\left[\left(\frac{1}{2} - \frac{1}{N}\sum_{i=1}^N \mathbb{E}(Z_n(i)^2)\right)\right]$. Using lemma A.1, the solution of the recursive equation would be as follows $\mathbb{V}\mathrm{ar}(Z_n) \sim \frac{C_N}{n^{2\gamma_1-1}} + e^{-\sum_{n=0}^{\infty}\frac{1}{n^{2\gamma_2}}}$.

b) $C_N = 4NK_n$ and $K_n = \frac{1}{4}$ when $\gamma_2 < \gamma_1$, so $\mathbb{V}\mathrm{ar}(Z_n) \sim \frac{C_N}{n^{2\gamma_2-1}} + e^{-\sum_{n=0}^{\infty}\frac{1}{n^{2\gamma_2}}}$. Also when $\gamma_1 = \gamma_2$,

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1 - \frac{1}{n^{2\gamma}})\,\mathbb{V}\mathrm{ar}(Z_n) + \frac{K_n}{n^{2\gamma}},$$

where $K_n = \left[(\frac{1}{2} - \frac{1}{N}\sum_{i=1}^N \mathbb{E}(Z_n(i)^2)) + \frac{N}{4}\right]$ which implies $\mathbb{V}\mathrm{ar}(Z_n) \sim \frac{C_N}{n^{2\gamma-1}} + e^{-\sum_{n=0}^{\infty}\frac{1}{n^{2\gamma}}}$ where $C_N = K_n$.

Hence, independently of the γ_j's relationship $\lim_{n\to\infty} \mathbb{V}\mathrm{ar}(Z_n) \neq 0$ which corresponds to Z_n converges to the random limit Z_∞ a.s. however convergence happens with

different rate according to γ_j's relation.

(ii) To study the synchronization phenomenon, consider the L^2-distance

$$x_{n+1} \quad = \quad [1-2r_n^g+(r_n^g)^2]x_n+(r_n^l)^2\varepsilon_n \simeq (1-\frac{2}{n^{\gamma_2}})x_n+\frac{\varepsilon_n}{n^{2\gamma_1}},$$

where $\varepsilon_n = \frac{N-1}{N}\{\mathbb{E}(Z_n)-[\frac{1}{N}\,\mathbb{E}(Z_n(i)^2)]\}$ and then $C_N=\varepsilon_n/2$. Thus, $x_n \sim \frac{C_N}{n^{2\gamma_1-\gamma_2}}$ goes to 0 in L^2 when $n\to\infty$. Moreover when $\gamma_1=\gamma_2$, $x_{n+1}=(1-\frac{2}{n^\gamma})x_n+\frac{\varepsilon_n}{n^{2\gamma}}$ which implies $x_n\sim\frac{C_N}{n^\gamma}$. To show that synchronization holds a.s., we shall show that the a.s. limit $Z_n(i)$ exists. we observe that $Z_n(i)$ is a quasi-martingale, *i.e.*

$$\sum_{n=0}^{+\infty}\mathbb{E}\,|\,\mathbb{E}[Z_{n+1}(i)|\mathcal{F}_n]-Z_n(i)|<+\infty.$$

Indeed, $\mathbb{E}(Z_{n+1}(i)-Z_n(i)|\mathcal{F}_n)=r_n^g\Big(Z_n-Z_n(i)\Big)$, we obtain

$$\begin{aligned}\sum_n \mathbb{E}\,|\,\mathbb{E}(Z_{n+1}(i)-Z_n(i)|\mathcal{F}_n)| \quad &= \quad \sum_n r_n^g\,\mathbb{E}\,|Z_n-Z_n(i)| \\ &\leq \quad \sum_n r_n^g(\mathbb{E}\,|Z_n-Z_n(i)|^2)^{1/2}<\infty\end{aligned}$$

Indeed, a bounded quasi-martingale has an almost sure limit. □

5.3. Proofs of the CLTs

Proof. (Th. i)
(Th. 1 F)
(i)(a) Define $X_k=Z_k-Z_k(i)$. (NOTE : DEPENDENT ON i)
Set $L_0=X_0$. Let us define

$$L_n := X_n-\sum_{k=0}^{n-1}(\mathbb{E}[X_{k+1}|\mathcal{F}_k]-X_k). \tag{17}$$

Then it holds:

$$X_{n+1}=[1-2\lambda_1 r_n^l-r_n^g]X_n+\Delta L_{n+1} \tag{18}$$

where $\Delta L_{n+1}=L_{n+1}-L_n$. Note that L_n is an $\mathcal{F}_n$-martingale by construction. Iterating the above relation, we can write

$$X_n=c_{1,n}X_1+\sum_{k=1}^{n-1}c_{k+1,n}\Delta L_{k+1} \tag{19}$$

where $c_{n,n}=1$ and $c_{k,n}=\prod_{h=k}^{n-1}[1-2\lambda_1 r_h^l-r_h^g]$ for $k<n$. It is easy to check that $\lim_{n\to\infty} n^{\frac{\gamma_1}{2}}\,c_{1,n}=0$.

$$\lim_{k\to\infty}\sup_{n\geq k}\Big|\frac{c_{k,n}}{\exp[\frac{-2\lambda_1}{1-\gamma_1}(n^{1-\gamma_1}-k^{1-\gamma_1})]}-1\Big|=0. \tag{20}$$

So it is enough to prove that $n^{\frac{\gamma_1}{2}} \sum_k c_{k+1,n} \Delta L_{n+1} \to \mathcal{N}(0, (1-1/N)/16\lambda_1)$. This can be proved by following conditions for $U_{n,k+1} = n^{\frac{\gamma_1}{2}} c_{k+1,n} \Delta L_{k+1}$.

(a) $\max_{1\leq k\leq n} |U_{n,k}| \to 0$.
(b) $\mathbb{E}[\max_{1\leq k\leq n} U_{n,k}^2]$ is bounded in n.
(c) $\sum_{k=1}^n U_{n,k}^2 \to (1-1/N)/16\lambda_1$ a.s.

To investigate these conditions, first consider to (a). Since $\Delta L_{n+1} = X_{n-1} - X_n + 2\lambda_1 X_n/n^{\gamma_1}$, $|\Delta L_{n+1}| \sim \mathcal{O}(n^{-\gamma_1})$.

For (b), use (12) and (a) to obtain

$$\begin{aligned}
\mathbb{E}[\max_{1\leq k\leq n} U_{n,k}^2] &\leq \mathbb{E}[\sum_{k=1}^n U_{n,k}^2] \\
&= n^{\gamma_1} \sum_{k=1}^n c_{k+1,n}^2 \, \mathbb{E}(\Delta L_{n+1})^2 \\
&\approx n^{\gamma_1} \sum_{k=1}^n e^{\frac{-4\lambda_1}{1-\gamma_1}(n^{1-\gamma_1} - k^{1-\gamma_1})} \mathcal{O}(n^{-2\gamma_1}) \\
&= n^{\gamma_1} e^{\frac{-4\lambda_1}{1-\gamma_1} n^{1-\gamma_1}} \sum_{k=1}^{n-1} k^{2\gamma_1} e^{\frac{-4\lambda_1}{1-\gamma_1} k^{1-\gamma_1}} \mathcal{O}(k^{-2\gamma_1}) + \frac{n^{2\gamma_1}\mathcal{O}(n^{-2\gamma_1})}{n^{\gamma_1}}.
\end{aligned}$$

Thus, $\mathbb{E}[\max_{1\leq k\leq n} U_{n,k}^2]$ is bounded in n. Let us now consider (c). We have

$$\sum_{k=1}^n U_{n,k}^2 = n^{\gamma_1} \sum_k c_{k+1,n}^2 (\Delta L_{n+1})^2 \approx n^{\gamma_1} \sum_{k=1}^n \frac{k^{-2\gamma_1} e^{\frac{4\lambda}{1-\gamma_1} k^{1-\gamma_1}}}{e^{\frac{4\lambda}{1-\gamma_1} n^{1-\gamma_1}}} (\Delta L_{k+1})^2 k^{2\gamma_1}.$$

From (a) we obtain

$$\begin{aligned}
(\Delta L_{k+1})^2 &= (X_{k+1} - X_k + 2\lambda r_k^l X_k)^2 \\
&= ((Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i)) + 2\lambda r_k^l (Z_k - Z_k(i)))^2 \\
&\approx [(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))]^2 + (r_k^l)^2 (Z_k - Z_k(i))^2 \\
&+ (r_k^l)(Z_k - Z_k(i))[(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))].
\end{aligned}$$

Since $Z_n \simeq Z_n(i)$ and $X_k^2 \sim \mathcal{O}(k^{-2\gamma_1})$ so,

$$\sum_{k=1}^n U_{n,k}^2 = n^{\gamma_1} \sum_{k=1}^n c_{k+1,n}^2 [(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))].$$

Let $V_k = k^{2\gamma_1}[(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$ and setting $b_n = \frac{1}{n^{\gamma_1}} e^{+4\lambda_1 \frac{n^{1-\gamma_1}}{1-\gamma_1}}$ and $a_k = \frac{k^{2\gamma_1}}{c_{1,n}^2} e^{-4\lambda_1 \frac{k^{1-\gamma_1}}{1-\gamma_1}}$. Hence, by Lemma A.2 given in Appendix is satisfied

$$\frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} \to \frac{1}{4\lambda_1}.$$

Indeed,

$$\begin{aligned}
\frac{1}{b_n}\sum_{k=1}^{n}\frac{1}{a_k} &= \frac{n^{\gamma_1}}{e^{\frac{4\lambda_1}{1-\gamma_1}}n^{1-\gamma_1}}\sum_{k=1}^{n}k^{-2\gamma_1}e^{\frac{4\lambda}{1-\gamma_1}k^{1-\gamma_1}} \\
&\approx n^{\gamma_1}e^{\frac{-4\lambda_1}{1-\gamma_1}n^{1-\gamma_1}}\int_1^n u^{-2\gamma_1}e^{4\lambda_1\frac{u^{1-\gamma_1}}{1-\gamma_1}}du \\
&= n^{\gamma_1}e^{\frac{-4\lambda_1}{1-\gamma_1}n^{1-\gamma_1}}\int_1^n \frac{u^{-\gamma_1}}{4\lambda_1}4\lambda_1 u^{-\gamma_1}e^{4\lambda_1\frac{u^{1-\gamma_1}}{1-\gamma_1}}du \\
&= n^{\gamma_1}e^{\frac{-4\lambda_1}{1-\gamma_1}n^{1-\gamma_1}}\Big[\Big(\frac{u^{-\gamma_1}}{4\lambda_1}e^{4\lambda_1\frac{u^{1-\gamma_1}}{1-\gamma_1}}\Big)_1^n+\frac{\gamma_1}{4\lambda_1}\int_1^n u^{-\gamma_1-1}e^{4\lambda_1\frac{u^{1-\gamma_1}}{1-\gamma_1}}du\Big] \\
&= \frac{1}{4\lambda_1}+\frac{\gamma_1 n^{\gamma_1}}{4\lambda_1}e^{-4\lambda_1}\frac{n^{1-\gamma_1}}{1-\gamma_1}\int_1^n\frac{1}{u^{1+\gamma_1}}e^{4\lambda_1\frac{u^{1-\gamma_1}}{1-\gamma_1}}du=\frac{1}{4\lambda_1}.
\end{aligned}$$

This implies that $\sum_{k=1}^n U_{n,k}^2$ converges to $\frac{V}{4\lambda_1}$ a.s., where V is deterministic such that $\mathbb{E}(V_{k+1}|\mathcal{F}_k)\longrightarrow V$. Indeed,

$$\begin{aligned}
&\mathbb{E}[k^{2\gamma_1}(Z_{k+1}(i)-Z_k(i))^2|\mathcal{F}_k]= \\
&k^{2\gamma_1}\Big[(r_k^l)^2\,\mathbb{E}\Big[(\xi_{k+1}^l(i)-Z_k(i))^2|\mathcal{F}_k\Big]+(r_k^g)^2\,\mathbb{E}\Big[(\xi_{k+1}^g-Z_k(i))^2|\mathcal{F}_k\Big]\Big] \\
&\quad+2r_k^lr_k^g\,\mathbb{E}\Big[(\xi_{k+1}^l(i)-Z_k(i))(\xi_{k+1}^g-Z_k(i))|\mathcal{F}_k\Big] \\
&=k^{2\gamma_1}\Big[(r_k^l)^2\Big(\mathbb{V}\mathrm{ar}[\xi_{k+1}^l(i)|\mathcal{F}_k]+\mathbb{E}^2(\xi_{k+1}^l(i)-Z_k(i)|\mathcal{F}_k)\Big)\Big]+(r_k^g)^2\Big(\mathbb{V}\mathrm{ar}[\xi_{k+1}^g|\mathcal{F}_k]+\mathbb{E}^2(\xi_{k+1}^g-Z_k(i)|\mathcal{F}_k)\Big)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[k^{2\gamma_1}(Z_{k+1}-Z_k)^2|\mathcal{F}_k] &= k^{2\gamma_1}\Big[(r_k^l)^2\,\mathbb{E}\Big[(\frac{1}{N}\sum\xi_{k+1}^l(i)-Z_k)^2|\mathcal{F}_k\Big]+(r_k^g)^2\,\mathbb{E}\Big[(\xi_{k+1}^g-Z_k)^2|\mathcal{F}_k\Big]\Big] \\
&+ 2r_k^lr_k^g\,\mathbb{E}\Big[(\frac{1}{N}\sum\xi_{k+1}^l(i)-Z_k)(\xi_{k+1}^g-Z_k)|\mathcal{F}_k\Big] \\
&= k^{2\gamma_1}\Big[(r_k^l)^2\Big(\mathbb{V}\mathrm{ar}[\frac{1}{N}\sum\xi_{k+1}^l(i)|\mathcal{F}_k]+\mathbb{E}^2(\frac{1}{N}\xi_{k+1}^l(i)-Z_k(i)|\mathcal{F}_k)\Big) \\
&+ (r_k^g)^2\Big(\mathbb{V}\mathrm{ar}[\xi_{k+1}^g|\mathcal{F}_k]+\mathbb{E}^2(\xi_{k+1}^g-Z_k|\mathcal{F}_k)\Big)\Big]
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}[k^{2\gamma_1}(Z_{k+1}-Z_k)(Z_{k+1}(i)-Z_k(i))|\mathcal{F}_k]= \\
&k^{2\gamma_1}\Big[(r_k^l)^2\,\mathbb{E}\Big[(\xi_{k+1}^l(i)-Z_k(i))(\frac{1}{N}\sum\xi_{k+1}^l(i)-Z_k)|\mathcal{F}_k\Big]+(r_k^g)^2\,\mathbb{E}\Big[(\xi_{k+1}^g-Z_k(i))(\xi_{k+1}^g-Z_k)|\mathcal{F}_k\Big]\Big] \\
&=k^{2\gamma_1}\Big[(r_k^l)^2\,\mathbb{E}\Big[(\xi_{k+1}^l(i)-Z_k(i))(\frac{1}{N}\sum\xi_{k+1}^l(i)-Z_k)|\mathcal{F}_k\Big]\Big]
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(V_{k+1}|\mathcal{F}_k)\approx k^{2\gamma_1}\Big[(r_k^l)^2\Big(&\mathbb{V}\mathrm{ar}[\xi_{k+1}^l(i)|\mathcal{F}_k]+\mathbb{V}\mathrm{ar}[\frac{1}{N}\sum\xi_{k+1}^l(i)|\mathcal{F}_k] \\
&-2\,\mathbb{E}\Big[(\xi_{k+1}^l(i)-Z_k(i))\Big(\frac{1}{N}\sum\xi_{k+1}^l(i)-Z_k\Big)\Big|\mathcal{F}_k\Big]\Big)\Big]\xrightarrow{a.s}\frac{1}{4}(1-\frac{1}{N}).
\end{aligned}$$

Thus, $V_k \xrightarrow{a.s} \frac{1}{4}(1-\frac{1}{N})$.
The proof of next parts and the other theorems follows along the same lines as above. We sketch the essential argument below.

Case $\gamma_1 = \gamma_2 = \gamma$. Let $Z_{k+1} - Z_{k+1}(i) = \Big(1-(1+2\lambda_1)r_n)(Z_k - Z_k(i)\Big)$ thus, $L_n = X_n + \sum_{k=0}^{n-1}(1+2\lambda_1)r_n X_k$ therefore, $X_{n+1} = [1-(1+2\lambda_1)r_n]X_n + \Delta L_{n+1}$ and $c_{1,n} \sim \exp[\frac{-(1+2\lambda_1)}{1-\gamma}n^{1-\gamma}]$. So

$$\lim_{k\to\infty}\sup_{n\geq k}\Big|\frac{c_{k,n}}{\exp[\frac{-(1+2\lambda_1)}{1-\gamma}(n^{1-\gamma}-k^{1-\gamma})]}-1\Big| = 0$$

and that (a), (b) (as in proof of theorem) hold. So it is enough to prove that $\sum_{k=1}^n U_{n,k}^2 \to (1-1/N)/4(1+2\lambda_1)$. $b_n = \frac{1}{n^\gamma}e^{\frac{2(1+2\lambda_1)}{1-\gamma}n^{1-\gamma}}$ and $a_k = \frac{k^{2\gamma}}{c_{1,n}^2}e^{-\frac{2(1+2\lambda_1)}{1-\gamma}k^{1-\gamma}}$ thus $\frac{1}{b_n}\sum_{k=1}^n \frac{1}{a_k} \to \frac{1}{2(1+2\lambda_1)}$.

$$\begin{aligned}\mathbb{E}(k^{2\gamma}(Z_{k+1}(i)-Z_k(i))^2|\mathcal{F}_k) = k^{2\gamma}r_k^{2\gamma}\,\mathbb{E}[(\tilde{\xi}_{k+1}(i)-Z_k(i))^2|\mathcal{F}_k] \xrightarrow{a.s}\\ \frac{(1-2\lambda_1)}{2}+\lambda_1(1-\lambda_1)-\frac{(1-2\lambda_1)^2}{4}-\lambda_1(1-2\lambda_1)\\ +\frac{(1-2\lambda_2)}{2}+\lambda_2(1-\lambda_2)-\frac{(1-2\lambda_2)^2}{4}-\lambda_2(1-2\lambda_2)=\frac{1}{2}.\end{aligned}$$

Similary, $\mathbb{E}(k^{2\gamma}(Z_{k+1}-Z_k)^2|\mathcal{F}_k) = k^{2\gamma}r_k^{2\gamma}\,\mathbb{E}[(\frac{1}{N}\sum_i \tilde{\xi}_{k+1}(i)-Z_k(i))^2|\mathcal{F}_k] \xrightarrow{a.s} \frac{1}{2N}$, and $\mathbb{E}(k^{2\gamma}(Z_{k+1}(i)-Z_k(i))(Z_{k+1}-Z_k)|\mathcal{F}_k) = k^{2\gamma}r_k^{2\gamma}\,\mathbb{E}[(\tilde{\xi}_{k+1}(i)-Z_k(i))(\frac{1}{N}\sum_i \tilde{\xi}_{k+1}(i)-Z_k)|\mathcal{F}_k] \xrightarrow{a.s} \frac{1}{2N}$, thus $V_k \xrightarrow{a.s} \frac{1}{2}(1-\frac{1}{N})$.

Case $\gamma_1 = \gamma_2 = 1$

$$\begin{aligned}L_n &= X_n - \sum_{k=0}^{n-1}([1-(1+2\lambda_1)r_n](Z_k - Z_k(i)) - (Z_k - Z_k(i)))\\ &= X_n + \sum_{k=0}^{n-1}(1+2\lambda_1)r_n X_k\end{aligned}$$

$L_{n+1} - L_n = X_{n+1} - [1-(1+2\lambda_1)r_n]X_n$ and

$$X_{n+1} = [1-(1+2\lambda_1)r_n]X_n + \Delta L_{n+1} \tag{21}$$

$c_{1,n} := \prod_h^n[1-(1+2\lambda_1)r_h] \sim n^{-(1+2\lambda_1)}$ Then $\sqrt{n}c_{1,n} \sim n^{-1-2\lambda_1+\frac{1}{2}} \to 0$. So it is enough to prove that $\sqrt{n}\sum_k c_{k+1,n}\Delta L_{n+1} \to \mathcal{N}(0,(1-1/N)/2(1+4\lambda_1))$. This can proved by following conditions for $U_{n,k+1} = \sqrt{n}\sum_k c_{k+1,n}\Delta L_{n+1}$.

(a) $\max_{1\leq k\leq n}|U_{n,k}| \to 0$.

(b) $\mathbb{E}[\max_{1\leq k\leq n} U_{n,k}^2]$ is bounded in n.

(c) $\sum_{k=1}^{n} U_{n,k}^2 \to (1-1/N)/2(1+4\lambda_1)$. To investigate these conditions, first consider to (a). Since $\Delta L_{n+1} = X_{n-1} - X_n + (1+2\lambda_1)X_n/n$, $|\Delta L_{n+1}| \sim \mathcal{O}(n^{-1})$.

For (b), use (12) and (a) to obtain

$$\mathbb{E}[\max_{1\le k\le n} U_{n,k}^2] \lessapprox \frac{1}{n^{1+4\lambda_1}} \sum_{k=1}^{n-1} \frac{k^2\mathcal{O}(k^{-2})}{k^{-4\lambda_1}} + \frac{n^2\mathcal{O}(n^{-2})}{n}$$

Thus, $\mathbb{E}[\max_{1\le k\le n} U_{n,k}^2]$ is bounded in n. Let us now consider (c). We have

$$\sum_{k=1}^{n} U_{n,k}^2 = n\sum_{k} c_{k+1,n}^2 (\Delta L_{n+1})^2 \approx \frac{1}{n^{1+4\lambda_1}} \sum_{k=1}^{n} \frac{k^2(\Delta L_{n+1})^2}{k^{-4\lambda_1}}.$$

From (a) we obtain

$$\begin{aligned}(\Delta L_{n+1})^2 \approx [(Z_{k+1}-Z_k) - (Z_{k+1}(i)-Z_k(i))]^2 + r_k^2(Z_k - Z_k(i))^2 \\ + r_k^2(Z_k - Z_k(i))[(Z_{k+1}-Z_k) - (Z_{k+1}(i)-Z_k(i))]\end{aligned}$$

Since $Z_n \xrightarrow{a.s} Z_n(i)$ and $X_k^2 \sim \mathcal{O}(k^{-2})$ so,

$$\sum_{k=1}^{n} U_{n,k}^2 \xrightarrow{a.s} n\sum_{k=1}^{n} c_{k+1,n}^2[(Z_{k+1}-Z_k)^2 + (Z_{k+1}(i)-Z_k(i))^2 - 2(Z_{k+1}-Z_k)(Z_{k+1}(i)-Z_k(i))]$$

we use Lemma A.2 in Appendix with $b_n = n^{1+4\lambda_1}$ and $a_k = k^{-4\lambda_1}$.
Let $U_k = k^2[(Z_{k+1}-Z_k)^2 + (Z_{k+1}(i)-Z_k(i))^2 - 2(Z_{k+1}-Z_k)(Z_{k+1}(i)-Z_k(i))]$. So $\frac{1}{b_n}\sum_{k=1}^{n} \frac{1}{a_k} \to \frac{1}{1+4\lambda_1}$. This implies that $\sum_{k=1}^{n} U_{n,k}^2$ converges to $\frac{U}{1+4\lambda_1}$ a.s., where V is deterministic such that $\mathbb{E}(V_{k+1}|\mathcal{F}_k) \longrightarrow V$. Indeed, $\mathbb{E}(k^2(Z_{k+1}(i)-Z_k(i))^2|\mathcal{F}_k) \xrightarrow{a.s} \frac{1}{2}$. Similarly, $\mathbb{E}(k^2(Z_{k+1}-Z_k)^2|\mathcal{F}_k) \xrightarrow{a.s} \frac{1}{2N}$, and $\mathbb{E}(k^2(Z_{k+1}(i)-Z_k(i))(Z_{k+1}-Z_k)|\mathcal{F}_k) \xrightarrow{a.s} \frac{1}{2N}$. Thus, $V_k \xrightarrow{a.s} \frac{1}{2}(1-\frac{1}{N})$.

(i)(b) Case $\gamma_2 < \gamma_1$. Let $X_k = Z_k - Z_k(i)$. Denote by L_n the martingale

$$L_0 = X_0, \quad L_n = X_n - \sum_{k=0}^{n-1}(\mathbb{E}[X_{k+1}|\mathcal{F}_n] - X_k).$$

Then $X_{n+1} = [1-r_n^g+\mathcal{O}(r_n^g)]X_n + \Delta L_{n+1}$. We can write $X_n = c_{1,n}X_1 + \sum_{k=1}^{n} c_{k+1,n}\Delta L_{n+1}$ Therefore $c_{1,n} \sim \exp[\frac{-1}{1-\gamma_2}n^{1-\gamma_2}]$. Then $n^{\gamma_1-\frac{\gamma_2}{2}}c_{1,n} \to 0$. Thus

$$\limsup_{k\to\infty} \sup_{n\ge k} \Big|\frac{c_{k,n}}{\exp[\frac{-1}{1-\gamma_2}(n^{1-\gamma_2}-k^{1-\gamma_2})]} - 1\Big| = 0$$

and that (a), (b) (as in proof of theorem) hold. So it is enough to prove that $\sum_{k=1}^{n} U_{n,k}^2 \to (1-1/N)/4$. We have

$$\begin{aligned}(\Delta L_{n+1})^2 &\approx [(Z_{k+1}-Z_k) - (Z_{k+1}(i)-Z_k(i))]^2 + (r_k^g)^2(Z_k - Z_k(i))^2 \\ &+ (r_k^g)^2(Z_k - Z_k(i))[(Z_{k+1}-Z_k) - (Z_{k+1}(i)-Z_k(i))]\end{aligned}$$

Since $Z_n \xrightarrow{a.s} Z_n(i)$ and $X_k^2 \sim \mathcal{O}(k^{-2\gamma_1+\gamma_2})$ so,

$$\sum_{k=1}^{n} U_{n,k}^2 \xrightarrow{a.s} n^{2\gamma_1-\gamma_2} \sum_{k=1}^{n} c_{k+1,n}^2[(Z_{k+1}-Z_k)^2+(Z_{k+1}(i)-Z_k(i))^2-2(Z_{k+1}-Z_k)(Z_{k+1}(i)-Z_k(i))]$$

we use Lemma A.2 in Appendix with $b_n = \frac{1}{n^{2\gamma_1-\gamma_2}} e^{\frac{2}{1-\gamma_2}} n^{1-\gamma_2}$ and $a_k = \frac{k^{2\gamma_1}}{c_{1,n}^2} e^{-\frac{2}{1-\gamma_2}} k^{1-\gamma_2}$ thus $\frac{1}{b_n}\sum_{k=1}^{n} \frac{1}{a_k} \to \frac{1}{2}$.
Let $V_k = k^{2\gamma_1}[(Z_{k+1}-Z_k)^2 + (Z_{k+1}(i)-Z_k(i))^2 - 2(Z_{k+1}-Z_k)(Z_{k+1}(i)-Z_k(i))]$. This implies that $\sum_{k=1}^{n} U_{n,k}^2$ converges to V a.s., where V is deterministic such that $\mathbb{E}(V_{k+1}|\mathcal{F}_k) \longrightarrow V$. Since we completed this computation in the proof of the previous part of theorem, we know that in this case $V_k \xrightarrow{a.s} \frac{1}{4}(1-\frac{1}{N})$.

(ii) When $\gamma_1 < \gamma_2$; Let $X_k = Z_k - \dfrac{1}{2}$ so,

$$L_n = X_n - \sum_{k=0}^{n-1} \Big(\mathbb{E}(Z_{k+1} - \frac{1}{2}|\mathcal{F}_k) - (Z_k - \frac{1}{2}) \Big) = X_n + 2(\lambda_1 r_n^l + \lambda_2 r_n^g) \sum_{k=0}^{n-1} X_k$$

and $X_{n+1} \simeq [1-2\lambda_1 r_n^l]X_n + \Delta L_{n+1}$. So $c_{1,n} \sim \exp[\frac{-2\lambda_1}{1-\gamma_1} n^{1-\gamma_1}]$. Then

$$\lim_{k\to\infty} \sup_{n\geq k} \Big| \frac{c_{k,n}}{\exp[\frac{-2\lambda_1}{1-\gamma_1}(n^{1-\gamma_1}-k^{1-\gamma_1})]} - 1 \Big| = 0.$$

Using Lemma A.2, $b_n = \frac{1}{n^{\gamma_1}} e^{\frac{2\lambda_1}{1-\gamma_1}} n^{1-\gamma_1}$ and $a_k = \frac{k^{2\gamma_1}}{c_{1,n}^2} e^{\frac{-2\lambda_1}{1-\gamma_1}} k^{1-\gamma_1}$. Therefore $\frac{1}{b_n}\sum_k \frac{1}{a_k} \to \frac{1}{2\lambda_1}$. Also $(\Delta L_{n+1})^2 = (Z_{k+1} - Z_k + 2\lambda_1 r_n^l (Z_k - \dfrac{1}{2}))^2 = (Z_{k+1}-Z_k)^2$. Then $k^{2\gamma_1}\,\mathbb{E}((Z_{k+1}-Z_k)^2|\mathcal{F}_k) = \frac{1}{4}$.
When $\gamma_2 < \gamma_1$; Set $X_k = Z_k - \dfrac{1}{2}$ then $X_n + 2(\lambda_1 r_n^l + \lambda_2 r_n^g)\sum_{k=0}^{n-1} X_k$. So $X_{n+1} \simeq [1-2\lambda_2 r_n^g]X_n + \Delta L_{n+1}$. Thus, $c_{1,n} \sim \exp[\frac{-2\lambda_2}{1-\gamma_2} n^{1-\gamma_2}]$. Then

$$\lim_{k\to\infty} \sup_{n\geq k} \Big| \frac{c_{k,n}}{\exp[\frac{-2\lambda_2}{1-\gamma_2}(n^{1-\gamma_2}-k^{1-\gamma_2})]} - 1 \Big| = 0.$$

Using Lemma A.2, $b_n = \frac{1}{n^{\gamma_2}} e^{\frac{2\lambda_2}{1-\gamma_2}} n^{1-\gamma_2}$ and $a_k = \frac{k^{2\gamma_2}}{c_{1,n}^2} e^{\frac{-2\lambda_2}{1-\gamma_2}} k^{1-\gamma_2}$. Therefore $\frac{1}{b_n}\sum_k \frac{1}{a_k} \to \frac{1}{2\lambda_2}$. Also

$$\begin{aligned}(\Delta L_{n+1})^2 &= (X_{n+1}X_n - 2\lambda_2 r_n^g X_n)^2 \\ &= (Z_{k+1} - Z_k + 2\lambda_2 r_n^g (Z_k - \frac{1}{2}))^2 \\ &= (Z_{k+1}-Z_k)^2.\end{aligned}$$

$k^{2\gamma_2}\,\mathbb{E}((Z_{k+1}-Z_k)^2|\mathcal{F}_k) = \frac{1}{4}$.

When $\gamma_1 = \gamma_2 = \gamma$, set $X_k = Z_k - \frac{1}{2}$ then $X_{n+1} = [1-2r_n(\lambda_1+\lambda_2)]X_n + \Delta L_{n+1}$ and $c_{1,n} \sim \exp[\frac{-2(\lambda_1+\lambda_2)}{1-\gamma} n^{1-\gamma}]$ Then

$$\lim_{k\to\infty} \sup_{n\geq k} \Big| \frac{c_{k,n}}{\exp[\frac{-2(\lambda_1+\lambda_2)}{1-\gamma}(n^{1-\gamma}-k^{1-\gamma})]} - 1 \Big| = 0.$$

Using Lemma A.2, $b_n = \frac{1}{n^\gamma} e^{\frac{2(\lambda_1+\lambda_2)}{1-\gamma} n^{1-\gamma_1}}$ and $a_k = \frac{k^{2\gamma}}{c_{1,n}^2} e^{\frac{-2(\lambda_1+\lambda_2)}{1-\gamma} k^{1-\gamma}}$. Therefore $\frac{1}{b_n}\sum_k \frac{1}{a_k} \to \frac{1}{2\lambda}$. Also $(\Delta L_{n+1})^2 = (Z_{k+1} - Z_k)^2$ so $k^{2\gamma}\,\mathbb{E}((Z_{k+1} - Z_k)^2|\mathcal{F}_k) = \frac{1}{4}$.

(iii) When $\gamma_1 = \gamma_2 = 1$, $c_{1,n} = \prod_h^n [1 - 2(\lambda_1 + \lambda_2) r_h] \sim \exp[\ln n^{-2(\lambda_1+\lambda_2)}] \sim n^{-2(\lambda_1+\lambda_2)}$ so $\sqrt{n}\; c_{1,n} = n^{-2(\lambda_1+\lambda_2)+\frac{1}{2}} \longrightarrow 0$ for $(\lambda_1 + \lambda_2) > \frac{1}{4}$. Then

$$\lim_{k\to\infty} \sup_{n\geq k} \left| \frac{c_{k,n}}{(\frac{k}{n})^{2(\lambda_1+\lambda_2)}} - 1 \right| = 0.$$

Moreover $\sum_k U_{k,n}^2 = n \sum_k (\frac{k}{n})^{4(\lambda_1+\lambda_2)} (\Delta L_{k+1})^2 k^2 k^{-2}$ $(\Delta L_{n+1})^2 = (Z_{k+1} - Z_k)^2$ then $k^2\,\mathbb{E}((Z_{k+1} - Z_k)^2|\mathcal{F}_k) = \frac{1}{4}$.

When $(\lambda_1+\lambda_2) = \frac{1}{4}$, $\sqrt{n}(\ln n)^{-\frac{1}{2}} c_{1,n} \longrightarrow 0$. So $c_{k,n} = (\frac{k}{n})^{\frac{1}{2}}$ and $U_{k,n} = \frac{\sqrt{n}}{\sqrt{\ln n}} c_{k+1,n} \Delta L_{k+1}$ so $\sum_k U_{k,n}^2 = \frac{n}{\ln n} \sum_k (\frac{k}{n})(\Delta L_{k+1})^2 = \frac{1}{\ln n} \sum_k \frac{1}{k} k^2 (\Delta L_{k+1})^2 = k^2 (\Delta L_{k+1})^2$ therefore $b_n = \ln n$ and $a_k = k$, so $\frac{1}{\ln n} \sum_k \frac{1}{k} \to 1$. □

Proof. (of Prop. 1)

Define $\sqrt{n}(Z_n - \frac{1}{2})$. $\mathbb{E}[\tilde{X}_n^2] < \infty$. It is therefore enough to show that $\tilde{X}_n$ is a quasimartingale. Indeed, we have

$$\begin{aligned}
\sum_k \mathbb{E}[|\,\mathbb{E}[\tilde{X}_{k+1}|\mathcal{F}_k] - \tilde{X}_k]|] &= \sum_k \mathbb{E}\left[\left|(k+1)^{\frac{1}{2}}(1 - (1+2\lambda_1) r_k) X_k - \tilde{X}_k\right|\right] \\
&= \sum_k \mathbb{E}\left[\left|(1+\frac{1}{k})^{\frac{1}{2}}(1 - 2(\lambda_1+\lambda_2) r_k)\tilde{X} - \tilde{X}_k\right|\right] \\
&= \sum_k \mathbb{E}\left[\left|[(1+\frac{1}{k})^{\frac{1}{2}}(1 - 2(\lambda_1+\lambda_2) r_k) - 1]\tilde{X}_k\right|\right] \\
&= \sum_k \left|(\frac{1}{k})^{\frac{1}{2}}(-2(\lambda_1+\lambda_2) r_k)\right| \mathbb{E}(|\tilde{X}_k|) \\
&= \sum_k \mathcal{O}(\frac{1}{k^2}) 2(\lambda_1+\lambda_2)\,\mathbb{E}(|\tilde{X}_k|) < +\infty
\end{aligned}$$

So $(\tilde{X}_n)_n$ is an $\mathcal{F}$-quasi-martingale. Moreover by theorem 3.2, $\mathbb{E}(\tilde{X}_n^2) < +\infty$ and so it converges $a.s$ and in mean to some real random variable $\tilde{X}$.
In order to prove that $\mathbb{P}(\tilde{X} \neq 0) > 0$, we will prove that $(\tilde{X}_n^2)_n$ is bounded in L^p for a suitable $p > 1$. Indeed this fact implies that $\tilde{X}_n^2$ converges in mean to $\tilde{X}^2$ and so, by Theorem 3.2, we obtain

$$\mathbb{E}(\tilde{X}^2) = \lim_n \mathbb{E}(\tilde{X}_n^2) = \lim_n n\,\mathbb{E}(\tilde{X}_n^2) > 0.$$

To this purpose, we set $p = 1 + \epsilon/2$, with $\epsilon > 0$ and $x_n = \mathbb{E}(|X_n|^{2+\epsilon})$. Using the following recursive equation:

$$X_{n+1} = (1 - 2r_n) Z_n + \frac{r_n}{N} \sum_{i=1}^{k} \tilde{\xi}_{k+1}(i) - \frac{1}{2}$$

$$x_{n+1} = \mathbb{E}(|X_n|^{2+\epsilon}) - (2+\epsilon)r_n 2Z_n\, \mathbb{E}(|X_n|^{1+\epsilon}) + (2+\epsilon)r_n\, \mathbb{E}\left[|X_n|^{1+\epsilon}\ \mathrm{sign}(X_n)\ (X_n)(\frac{1}{N}\sum_i \tilde{\xi}_{k+1}(i))\right] + R_n$$

where $R_n = \mathcal{O}(n^{-2})$. Now since $\mathbb{E}[\frac{1}{N}\sum_i \tilde{\xi}_{n+1}(i)|\mathcal{F}_n] = 2Z_n - 2(\lambda_1+\lambda_2)(Z_n - \frac{1}{2})$, we have

$$\begin{aligned} x_{n+1} &= \mathbb{E}(|X_n|^{2+\epsilon}) - 2(2+\epsilon)r_n Z_n\, \mathbb{E}(|X_n|^{1+\epsilon}) \\ &\quad + (2+\epsilon)r_n\, \mathbb{E}[|X_n|^{1+\epsilon}\ \mathrm{sign}(X_n)\ (2Z_n - 2(\lambda_1+\lambda_2))X_n] + R_n \\ &= \mathbb{E}(|X_n|^{2+\epsilon}) - (2+\epsilon)r_n 2(\lambda_1+\lambda_2)\, \mathbb{E}[|X_n|^{1+\epsilon}\ \mathrm{sign}(X_n)\ (X_n)X_n] + R_n \\ &= \mathbb{E}(|X_n|^{2+\epsilon}) - (2+\epsilon)r_n 2(\lambda_1+\lambda_2)\, \mathbb{E}\left(|X_n|^{2+\epsilon}\right) + R_n \\ &= \Big(1 - 2(\lambda_1+\lambda_2)(2+\epsilon)r_n\Big)x_n + g(n) \end{aligned}$$

with $g(n) = \mathcal{O}(n^{-2})$. Therefore, we have

$$x_{n+1} = \Big(1 - 2(\lambda_1+\lambda_2)(2+\epsilon)r_n\Big)x_n + g(n).$$

$$\begin{aligned} \prod_{k=0}^{n-1}\Big(1 - 2(\lambda_1+\lambda_2)(2+\epsilon)r_n\Big) &= \exp[\sum_{k=0}^{n-1}\ln\Big(1 - 2(\lambda_1+\lambda_2)(2+\epsilon)r_n\Big)] \\ &= \exp[-2(\lambda_1+\lambda_2)(2+\epsilon)\sum_{k=0}^{n-1} r_n] \\ &= n^{-2(\lambda_1+\lambda_2)(2+\epsilon)} \end{aligned}$$

So,

$$\mathbb{E}[|X_n|^{(2+\epsilon)}] = \mathcal{O}(\frac{1}{n^{2(\lambda_1+\lambda_2)(2+\epsilon)}})$$

and so it implies that $\tilde{X}^2$ is bounded in $\mathcal{L}^{1+\frac{\epsilon}{2}}$. □

Proof. (Th.5)
(i) When $\gamma_1 \neq \gamma_2$. Define $X_k = Z_k - Z_k(i)$. Set $L_0 = X_0$,

$$\begin{aligned} L_n &= X_n - \sum_{k=0}^{n-1}(\mathbb{E}[X_{k+1}|\mathcal{F}_n] - X_k) \\ &= X_n - \sum_{k=0}^{n-1}([1 - r_k^g](Z_k - Z_k(i)) - (Z_k - Z_k(i))) \\ &= X_n + \sum_{k=0}^{n-1} r_n^g X_k. \end{aligned}$$

Then $X_{n+1} = [1-r_n^g]X_n + \Delta L_{n+1}$. Note that L_n is an $\mathcal{F}_n$-martingale by construction. Iterating the above relation, we can write $X_n = c_{1,n}X_1 + \sum_{k=1}^n c_{k+1,n}\Delta L_{n+1}$ where $c_{n+1,n} = 1$ and $c_{k,n} = \prod_{h=k}^n [1-r_h^g]$ for $k \leq n$. $c_{1,n} = \prod_{h=1}^n [1-r_h^g] \sim \exp[\frac{-1}{1-\gamma_2} n^{1-\gamma_2}]$. Then $n^{\gamma_1 - \frac{\gamma_2}{2}} c_{1,n} \sim n^{\gamma_1 - \frac{\gamma_2}{2}} \exp[\frac{-1}{1-\gamma_2} n^{1-\gamma_2}] \to 0$.

$$\lim_{k\to\infty} \sup_{n\geq k} \Big| \frac{c_{k,n}}{\exp[\frac{-1}{1-\gamma_2}(n^{1-\gamma_2} - k^{1-\gamma_2})]} - 1 \Big| = 0.$$

So it is enough to prove that $n^{\gamma_1 - \frac{\gamma_2}{2}} \sum_k c_{k+1,n}\Delta L_{n+1} \to \mathcal{N}(0, (1-1/N)Z_\infty - Z_\infty^2/2)$. This can proved by following conditions for $U_{n,k+1} = n^{\gamma_1 - \frac{\gamma_2}{2}} \sum_k c_{k+1,n}\Delta L_{n+1}$.
(a) $\max_{1\leq k\leq n} |U_{n,k}| \to 0$. (b) $\mathbb{E}[\max_{1\leq k\leq n} U_{n,k}^2]$ is bounded in n. (c) $\sum_{k=1}^n U_{n,k}^2 \to (1-1/N)Z_\infty - Z_\infty^2/2$. It is easy to check the conditions (a) and (b). Let us now consider (c). We have

$$\sum_{k=1}^n U_{n,k}^2 = n^{2\gamma_1-\gamma_2} \sum_k c_{k+1,n}^2 (\Delta L_{n+1})^2 \approx n^{2\gamma_1-\gamma_2} \sum_{k=1}^n \frac{k^{-2\gamma_1} e^{\frac{1}{1-\gamma_2} k^{1-\gamma_2}}}{e^{\frac{1}{1-\gamma_2} n^{1-\gamma_2}}} (\Delta L_{n+1})^2 k^{2\gamma_1}$$

. From (a) we obtain

$$\begin{aligned} (\Delta L_{n+1})^2 &\approx [(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))]^2 + (r_k^g)^2 (Z_k - Z_k(i))^2 \\ &+ (r_k^g)^2 (Z_k - Z_k(i))[(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))] \end{aligned}$$

Since $Z_n \xrightarrow{a.s} Z_n(i)$ and $X_k^2 \sim \mathcal{O}(k^{-2\gamma_1+\gamma_2})$ so,

$$\sum_{k=1}^n U_{n,k}^2 \xrightarrow{a.s} n^{2\gamma_1-\gamma_2} \sum_{k=1}^n c_{k+1,n}^2 [(Z_{k+1}-Z_k)^2 + (Z_{k+1}(i)-Z_k(i))^2 - 2(Z_{k+1}-Z_k)(Z_{k+1}(i)-Z_k(i))]$$

where we use Lemma A.2 in Appendix with $b_n = \frac{1}{n^{2\gamma_1-\gamma_2}} e^{\frac{2}{1-\gamma_2} n^{1-\gamma_2}}$ and $a_k = \frac{k^{2\gamma_1}}{c_{1,n}^2} e^{-\frac{2}{1-\gamma_2} k^{1-\gamma_2}}$.
Let $V_k = k^{2\gamma_1}[(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$. So $\frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} \to \frac{1}{2}$ This implies that $\sum_{k=1}^n U_{n,k}^2$ converges to V a.s., where V is deterministic such that $\mathbb{E}(V_{k+1}|\mathcal{F}_k) \longrightarrow V$. Indeed,

$$\begin{aligned} \mathbb{E}(k^{2\gamma_1}(Z_{k+1}(i) - Z_k(i))^2 | \mathcal{F}_k) &= k^{2\gamma_1}(r_k^l)^2 \, \mathbb{E}[(\xi_{k+1}^l(i) - Z_k(i))^2 | \mathcal{F}_k] \\ &= k^{2\gamma_1}(r_k^l)^2 \, \mathbb{V}\mathrm{ar}[\xi_{k+1}^l(i) | \mathcal{F}_k] \\ &= k^{2\gamma_1}(r_k^l)^2 \{Z_k - Z_k^2\} \\ &\xrightarrow{a.s} Z_\infty - Z_\infty^2. \end{aligned}$$

Similary, $\mathbb{E}(k^{2\gamma_2}(Z_{k+1} - Z_k)^2 | \mathcal{F}_k) \xrightarrow{a.s} Z_\infty - Z_\infty^2$, and $\mathbb{E}(k^{2\gamma_2}(Z_{k+1}(i) - Z_k(i))(Z_{k+1} - Z_k) | \mathcal{F}_k) \xrightarrow{a.s} \frac{Z_\infty - Z_\infty^2}{N}$. Thus, $U_k \xrightarrow{a.s} 2(Z_\infty - Z_\infty^2)(1 - \frac{1}{N})$.

When $\gamma_1 = \gamma_2 = \gamma$; $L_n = X_n + \sum_{k=0}^{n-1} r_n X_k$. Then $L_{n+1} - L_n = X_{n+1} - [1-r_n]X_n$
So $X_{n+1} = [1-r_n]X_n + \Delta L_{n+1}$. Note that L_n is an $\mathcal{F}_n$-martingale by construction. Iterating the above relation, we can write $X_n = c_{1,n}X_1 + \sum_{k=1}^n c_{k+1,n}\Delta L_{n+1}$ where

$c_{n+1,n}=1$ and $c_{k,n}=\prod_{h=k}^{n}[1-r_h]$ for $k\leq n$. $c_{1,n}=\prod_{h=1}^{n}[1-r_h]\sim\exp[\frac{-1}{1-\gamma}n^{1-\gamma}]$. Then $n^{\frac{\gamma}{2}}c_{1,n}\sim n^{\frac{\gamma}{2}}\exp[\frac{-1}{1-\gamma}n^{1-\gamma}]\to 0$. Moreover $\frac{1}{b_n}\sum_k\frac{1}{a_k}\longrightarrow\frac{1}{2}$,

$$\begin{aligned}\mathbb{E}(k^{2\gamma}(Z_{k+1}(i)-Z_k(i))^2|\mathcal{F}_k) &= k^{2\gamma}r_k^{2\gamma}\,\mathbb{E}[(\tilde{\xi}_{k+1}(i)-Z_k(i))^2|\mathcal{F}_k]\\ &= k^{2\gamma}r_k^{2\gamma}\,\mathbb{V}\mathrm{ar}[\tilde{\xi}_{k+1}(i)|\mathcal{F}_k]\\ &\xrightarrow{a.s} 2(Z_\infty-Z_\infty^2).\end{aligned}$$

Similary, $\mathbb{E}(k^{2\gamma}(Z_{k+1}-Z_k)^2|\mathcal{F}_k)\xrightarrow{a.s}2(Z_\infty-Z_\infty^2)$, and
$\mathbb{E}(k^{2\gamma}(Z_{k+1}(i)-Z_k(i))(Z_{k+1}-Z_k)|\mathcal{F}_k)\xrightarrow{a.s}\frac{2(Z_\infty-Z_\infty^2)}{N}$
Thus, $U_k\xrightarrow{a.s}4(Z_\infty-Z_\infty^2)(1-\frac{1}{N})$.

When $\gamma_1=\gamma_2=1$; $L_n=X_n-\sum_{k=0}^{n-1}(Z_k-Z_k(i))(-r_n)=X_n+\sum_{k=0}^{n-1}r_nX_k$. Then $L_{n+1}-L_n=X_{n+1}-[1-r_n]X_n$ so, $X_{n+1}=[1-r_n]X_n+\Delta L_{n+1}$. Note that L_n is an $\mathcal{F}_n$-martingale by construction. Iterating the above relation, we can write $X_n=c_{1,n}X_1+\sum_{k=1}^{n}c_{k+1,n}\Delta L_{n+1}$ where $c_{n+1,n}=1$ and $c_{k,n}=\prod_{h=k}^{n}[1-r_h]$ for $k\leq n$. $c_{1,n}=\prod_{h=1}^{n}[1-r_h]\sim\exp[\ln n^{-1}]=n^{-1}$. Then $\sqrt{n}\;c_{1,n}\sim\sqrt{n}\;n^{-1}\to 0$. Choosing $b_n=n$ and $a_k=1$, $\frac{1}{b_n}\sum_k\frac{1}{a_k}\longrightarrow 1$. $\mathbb{E}(k^{2\gamma}(Z_{k+1}(i)-Z_k(i))^2|\mathcal{F}_k)\xrightarrow{a.s}2(Z_\infty-Z_\infty^2)$ Similary, $\mathbb{E}(k^{2\gamma}(Z_{k+1}-Z_k)^2|\mathcal{F}_k)\xrightarrow{a.s}2(Z_\infty-Z_\infty^2)$, and $\mathbb{E}(k^{2\gamma}(Z_{k+1}(i)-Z_k(i))(Z_{k+1}-Z_k)|\mathcal{F}_k)\xrightarrow{a.s}\frac{2(Z_\infty-Z_\infty^2)}{N}$. Thus, $U_k\xrightarrow{a.s}4(Z_\infty-Z_\infty^2)(1-\frac{1}{N})$.

(ii) Case $\gamma_1<\gamma_2$. Z_n is martingale. Indeed,

$$\mathbb{E}(Z_{n+1}|\mathcal{F}_n)=(1-2\lambda_1r_n^l-2\lambda_2r_n^g)Z_n+\lambda_1r_n^l+\lambda_2r_n^g,$$

when $\lambda_1,\lambda_2=0$ then $\mathbb{E}(Z_{n+1}|\mathcal{F}_n)=Z_n$, so Z_n is martingale and converges a.s.
To this purpose, satisfies the following two conditions:

1) $\mathbb{E}\left[\sup_k k^{\gamma_1-\frac{1}{2}}|Z_{k+1}-Z_k|\right]<+\infty$
2) $n^{2\gamma_1-1}\sum_{k\geq n}(Z_{k+1}-Z_k)^2\xrightarrow{a.s}\frac{1}{N(2\gamma_1-1)}(Z_\infty-Z_\infty^2)$

Indeed, the first condition immediately follows from

$$|Z_{k+1}-Z_k|=|r_n^l(\frac{1}{N}\sum_i\xi_{k+1}^l(i)-Z_k)+r_n^g(\xi_{k+1}^g-Z_k)|=\mathcal{O}(k^{-\gamma_1})$$

Regarding the second condition, we observe that

$$n^{2\gamma_1-1}\sum_{k\geq n}(Z_{k+1}-Z_k)^2=n^{2\gamma_1-1}\sum_{k\geq n}k^{-2\gamma_1}(r_k^l)^2(\frac{\sum_k\xi_{k+1}(i)}{N}-Z_k)^2k^{2\gamma_1}$$

and so the desired convergence follows by lemma with $a_k=k^{-2\gamma_1+2}$, $b_n=n^{2\gamma_1-1}$ and $U_k=k^{2\gamma_1}(r_k^l)^2(\frac{\sum_k\xi_{k+1}(i)}{N}-Z_k)^2$, $b_n\sum_{k\geq n}\frac{1}{a_kb_k^2}\to-\frac{1}{1-2\gamma_1}$

$$\mathbb{E}(\frac{\sum\xi_{k+1}(i)}{N}-Z_k)^2|\mathcal{F})=\mathbb{V}\mathrm{ar}(\frac{\sum\xi_{k+1}(i)}{N}|\mathcal{F}_n)=\frac{1}{N}(Z-Z^2).$$

Case $\gamma_2 < \gamma_1$. To this purpose, satisfies the following two conditions:

1) $\mathbb{E}\left[\sup_k k^{\gamma_2-\frac{1}{2}}|Z_{k+1}-Z_k|\right] < +\infty$
2) $n^{2\gamma_2-1}\sum_{k\geq n}(Z_{k+1}-Z_k)^2 \xrightarrow{a.s} \frac{1}{(2\gamma_2-1)}(Z_\infty - Z_\infty^2)$

Indeed, the first condition immediatly follows from

$$|Z_{k+1}-Z_k| = |r_n^l(\frac{1}{N}\sum_i \xi_{k+1}^l(i) - Z_k) + r_n^g(\xi_{k+1}^g - Z_k)| = \mathcal{O}(k^{-\gamma_2})$$

Regarding the second condition, we observe that

$$n^{2\gamma_2-1}\sum_{k\geq n}(Z_{k+1}-Z_k)^2 = n^{2\gamma_2-1}\sum_{k\geq n}(r_k^l)^2 k^{-2\gamma_2}(\frac{\sum_k \xi_{k+1}(i)}{N} - Z_k)^2 k^{2\gamma_2}$$

and so the desired convergence follows by lemma with $a_k = k^{2\gamma_2+2}$, $b_n = n^{2\gamma_2-1}$ and $U_k = k^{2\gamma_2}(r_k^g)^2(\xi_{k+1}^g - Z_k)^2$, $b_n\sum_{k\geq n}\frac{1}{a_k b_k^2} \to -\frac{1}{1-2\gamma_2}$ and $\mathbb{E}(\xi_{k+1}^g(i) - Z_k)^2|\mathcal{F}) = \mathbb{V}\mathrm{ar}(\xi_{k+1}^g|\mathcal{F}) = (Z_\infty - Z_\infty^2)$

Third, when $\gamma_1 = \gamma_2 = \gamma$. Z_n is matringle and converges a.s.. Indeed,

$$\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = (1-2r_n)Z_n + r_n\,\mathbb{E}(\frac{\sum_i \tilde{\xi}_{n+1}(i)}{N}|\mathcal{F}_n) = Z_n$$

To this purpose, satisfies the following two conditions:

1) $\mathbb{E}\left[\sup_k k^{\gamma-\frac{1}{2}}|Z_{k+1}-Z_k|\right] < +\infty$
2) $n^{2\gamma-1}\sum_{k\geq n}(Z_{k+1}-Z_k)^2 \xrightarrow{a.s} \frac{2}{N(2\gamma-1)}(Z_\infty - Z_\infty^2)$

Indeed, the first condition immediatly follows from

$$|Z_{k+1}-Z_k| = |r_n(\frac{1}{N}\sum_i \tilde{\xi}_{k+1}(i) - 2Z_k))| = \mathcal{O}(k^{-\gamma})$$

Regarding the second condition, we observe that

$$n^{2\gamma-1}\sum_{k\geq n}(Z_{k+1}-Z_k)^2 = n^{2\gamma-1}\sum_{k\geq n} r_k^2 k^{-2\gamma}(\frac{\sum_k \tilde{\xi}_{k+1}(i)}{N} - Z_k)^2 k^{2\gamma}$$

and so the desired convergence follows by lemma with $a_k = k^{-2\gamma+2}$, $b_n = n^{2\gamma-1}$ and $U_k = k^{2\gamma}r_k^2(\frac{\sum_k \tilde{\xi}_{k+1}(i)}{N} - 2Z_k)^2$,$b_n\sum_{k\geq n}\frac{1}{a_k b_k^2} \to -\frac{1}{1-2\gamma}$, $\mathbb{E}(\frac{\sum \tilde{\xi}_{k+1}(i)}{N} - 2Z_k)^2|\mathcal{F}) = \frac{2}{N}(Z_\infty - Z_\infty^2)$.

Case $\gamma_1 = \gamma_2 = 1$. To this purpose, satisfies the following two conditions:

1) $\mathbb{E}\left[\sup_k k^{\frac{1}{2}}|Z_{k+1}-Z_k|\right] < +\infty$;
2) $n\sum_{k\geq n}(Z_{k+1}-Z_k)^2 \xrightarrow{a.s} \frac{2}{N}(Z_\infty - Z_\infty^2)$.

Indeed, the first condition immediately follows from $|Z_{k+1}-Z_k| = |r_n(\frac{1}{N}\sum_i \tilde{\xi}_{k+1}(i) - 2Z_k))| = \mathcal{O}(k^{-1})$. The second condition, we observe that $n\sum_{k\geq n}(Z_{k+1}-Z_k)^2 = n\sum_{k\geq n} r_k^2 k^{-2}(\frac{\sum_k \tilde{\xi}_{k+1}(i)}{N} - Z_k)^2 k^2$ and so the desired convergence follows by lemma with $a_k = 1$, $b_n = n$ and $U_k = k^2 r_k^2(\frac{\sum_k \tilde{\xi}_{k+1}(i)}{N} - 2Z_k)^2$, $b_n\sum_{k\geq n}\frac{1}{a_k b_k^2} \to 1$. $\mathbb{E}(\frac{\sum \tilde{\xi}_{k+1}(i)}{N} - 2Z_k)^2|\mathcal{F}) = \frac{2}{N}(Z_\infty - Z_\infty^2)$. □

Appendix A. Appendix

Lemma A.1. *Let (x_n) be a sequence of positive number satisfying the following equation:*

$$x_{n+1} = (1-\frac{A}{n^{a\gamma_i}})x_n + \frac{K_n}{n^{b\gamma_j}}$$

where $n, a, b \in \mathbb{N}$, $0 \leq \gamma_i, \gamma_j \leq 1$, $A > 0$ and $0 < K_n \leq K$. Then

$$x_n \simeq \begin{cases} \frac{K_n/A}{n^{b\gamma_j-\gamma_i}} + e^{-\sum_{n=0}^{\infty}\frac{1}{n^{\gamma_i}}} & for\ a = 1; \\ \frac{K_n/A}{n^{b\gamma_j-1}} + e^{-\sum_{n=0}^{\infty}\frac{1}{n^{a\gamma_i}}} & for\ a > 1. \end{cases}$$

Proof. Let

$$x_{n+1} = (1-\varepsilon_n)x_n + \delta_n$$

where $\varepsilon_n = \frac{A}{n^{a\gamma_i}}$ and $\delta_n = \frac{K_n}{n^{b\gamma_j}}$. It holds

$$x_n = x_l\prod_{i=l}^{n-1}(1-\varepsilon_i) + \sum_{i=l}^{n-1}\delta_i\prod_{k=i+1}^{n-1}(1-\varepsilon_k). \tag{22}$$

If $\sum_n \varepsilon_n = +\infty$ implies $\prod_{u=l}^{n-1}(1-\varepsilon_u) \longrightarrow 0$. But $\sum_n \varepsilon_n < +\infty$ follows that $\prod_{u=l}^{n-1}(1-\varepsilon_u) = e^{-\sum_{u=l}^{n}\frac{1}{u^{a\gamma_i}}}$. Let $y_n := \sum_{i=l}^{n-1}\delta_i\prod_{k=i+1}^{n-1}(1-\varepsilon_k)$, so

$$\begin{aligned} y_n &= \sum_{i=l}^{n-1}\delta_i\prod_{k=i+1}^{n-1}(1-\varepsilon_k) \\ &\sim \int_l^n \delta_s\exp(-\int_s^n \varepsilon(u)du)ds \\ &= \int_l^n \frac{K_n}{s^{b\gamma_j}}\exp(-\int_s^n \frac{A}{u^{a\gamma_i}}du)ds \\ &= \int_l^n \frac{K_n}{s^{b\gamma_j}}\exp -[\frac{A}{(1-a\gamma_i)u^{a\gamma_i-1}}]_s^n ds \\ &= \int_l^n \frac{K_n}{s^{b\gamma_j}}\exp[\frac{A}{1-a\gamma_i}(\frac{1}{s^{a\gamma_i-1}} - \frac{1}{n^{a\gamma_i-1}})]ds \\ &= K_n\, e^{-\frac{A}{(1-a\gamma_i)n^{a\gamma_i-1}}}\int_l^n \frac{1}{s^{b\gamma_j}}e^{\frac{A}{(1-a\gamma_i)s^{a\gamma_i-1}}}ds \\ &= \frac{K_n}{n^{b\gamma_j}}\frac{\int_l^n s^{-b\gamma_j}e^{\frac{A}{(1-a\gamma_i)s^{a\gamma_i-1}}}ds}{n^{-b\gamma_j}e^{\frac{A}{(1-a\gamma_i)n^{a\gamma_i-1}}}} \end{aligned}$$

Letting $n \to \infty$, using de L'Hôpital rule,

$$\begin{aligned} &\sim \frac{K_n}{n^{b\gamma_j}} \frac{n^{-b\gamma_j} e^{\frac{A}{(1-a\gamma_i)n^{a\gamma_i-1}}}}{[(-b\gamma_j)n^{-b\gamma_j-1} + n^{-b\gamma_j} A n^{-a\gamma_i}] e^{\frac{A}{(1-a\gamma_i)s^{a\gamma_i-1}}}} \\ &= \frac{K_n}{n^{b\gamma_j}} \frac{1}{\frac{-b\gamma_j}{n} + \frac{A}{n^{a\gamma_i}}} \\ &= \frac{K_n}{A} \frac{1}{n^{b\gamma_j - a\gamma_i}} \left(\frac{1}{1 - \frac{b\gamma_j n^{a\gamma_i}}{nA}}\right) \end{aligned}$$

□

Lemma A.2. *Let $\mathcal{G}$ be an (increasing) filtration and (Y_k) be an $\mathcal{G}$-adapted sequence of real random variables such that $\mathbb{E}[Y_k|\mathcal{G}_{k-1}] \to Y$ a.s. for some real random variable Y. Moreover, let (a_k) and (b_k) be two sequences of strictly positive real numbers such that*

$$b_k \uparrow +\infty, \ \sum_{k=1}^{\infty} \frac{\mathbb{E}[Y_k^2]}{a_k^2 b_k^2} < +\infty.$$

Then we have:

a) If $\frac{1}{b_n}\sum_{k=1}^{n} \frac{1}{a_k} \to \sigma$ for some constant σ, then $\frac{1}{b_n}\sum_{k=1}^{n} \frac{Y_k}{a_k} \to \sigma Y$.

b) If $b_n \sum_{k\geq n} \frac{1}{a_k b_k^2} \to \sigma$ for some constant σ, then $b_n \sum_{k\geq n} \frac{Y_k}{a_k b_k^2} \to \sigma Y$.

Lemma A.3. *(Theorem 3.2 in [9])*
Let $\{S_{n,k}, \mathcal{F}_{n,k} : 1 \leq k \leq k_n, n \geq 1\}$ be a zero-mean, square-integrable martingale array with diffrences $Y_{n,k}$, and let η^2 be an a.s. finite random variable. Suppose that

1) $\max_{1\leq k\leq k_n} |Y_{n,k}| \xrightarrow{P} 0$;

2) $\mathbb{E}[\max_{1\leq k\leq k_n} Y_{n,k}^2]$ is bounded in n;

3) $\sum_{k=1}^{k_n} Y_{n,k}^2 \xrightarrow{P} \eta^2$

and the σ-fields are nested, i.e. $\mathcal{F}_{n,k} \subseteq \mathcal{F}_{n+1,k}$ for $1 \leq k \leq k_n$, $n \geq 1$. Then $S_{n,k_n} = \sum_{k=1}^{k_n} Y_{n,k}$ converges stably to a random variable with characteristic function $\varphi(u) = \mathbb{E}[\exp(-\eta^2 u^2/2)]$, i.e. to the Gaussian kernel $\mathcal{N}(0, \eta^2)$.

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