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Friedrich Wehrung

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RANGES OF FUNCTORS IN ALGEBRA

FRIEDRICH WEHRUNG

ABSTRACT. The problem of determining the range of a given functor arises in various parts of mathematics. We present a sample of such problems, with focus on various functors, arising in the contexts of nonstable K_0 -theory of rings, congruence lattices of universal algebras, spectral spaces of ring-like objects. We also sketch some of the ideas involved in the solutions of those problems.

1. INTRODUCTION

The present short survey paper deals with the following kind of problem. We are given classes \mathcal{A} and \mathcal{B} , together with a map $\Psi: \mathcal{A} \rightarrow \mathcal{B}$. We would like to find a way to recognize the members of the range $\Psi[\mathcal{A}] \stackrel{\text{def}}{=} \{\Psi(A) \mid A \in \mathcal{A}\}$ of Ψ . That is, we would like to find simple criteria for a given $B \in \mathcal{B}$ to be $\Psi(A)$ for some $A \in \mathcal{A}$.

Our main line of argument is that in many situations, if there exists a solution A , then there exists a “nice” solution A_{nice} , where “nice” means that part of the structure of B can be read on the construction of A_{nice} .

In most of those situations, \mathcal{A} and \mathcal{B} are classes of *structures*: groups, rings, modules, lattices (which are all particular instances of so-called *universal algebras*), but also topological spaces. In all those instances, both \mathcal{A} and \mathcal{B} afford a notion of *isomorphism*, which enables us to state that two structures, although not identical in the set-theoretical sense, share all the properties that matter (e.g., isomorphic groups, homeomorphic topological spaces).

The relevant context, for this kind of problem, is thus *category theory*. Now \mathcal{A} and \mathcal{B} are both categories and Ψ is a *functor* from \mathcal{A} to \mathcal{B} . Moreover, instead of asking for $\Psi(A) = B$, we only ask for $\Psi(A) \cong B$ (say that A *lifts* B *with respect to* Ψ), and the “range” of Ψ is the class of all B that are isomorphic to $\Psi(A)$ for some object A of \mathcal{A} : in formula,

$$\Psi[\mathcal{A}] \stackrel{\text{def}}{=} \{B \in \mathcal{B} \mid (\exists A \in \mathcal{A})(\Psi(A) \cong B)\}. \quad (1.1)$$

Observe right away that (1.1) already involves a quite common abuse of notation, used there for simplicity’s sake: “ $B \in \mathcal{B}$ ” should be “ B is an object of \mathcal{B} ”, and, similarly, “ $A \in \mathcal{A}$ ” should be “ A is an object of \mathcal{A} ”. Calling the right hand side of (1.1) the “range” of Ψ is a similar abuse of terminology, as the $\Psi[\mathcal{A}]$ of (1.1) usually properly contains the set-theoretical range of Ψ .

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“Niceness” of the solution A means that if B is obtained as a colimit of (usually simpler) structures B_i , then A is also the colimit of suitably chosen structures A_i with each $\Psi(A_i) \cong B_i$. Thinking of the B_i as all the “finitely generated” substructures of B , we would like to lift (with respect to Ψ) not only the *object* B , but the *diagram* formed by all the B_i and the inclusion maps between them. We are thus trying to lift the *slice category* $\mathcal{B} \downarrow B$ of all arrows from some object of \mathcal{B} to B .

Now it is often the case that we are only trying to lift a *subdiagram* of $\mathcal{B} \downarrow B$. A “diagram” in \mathcal{B} is, really, a functor $\Phi: \mathcal{J} \rightarrow \mathcal{B}$, for some category \mathcal{J} . Our initial problem can thus be recast as follows: we are given categories $\mathcal{J}, \mathcal{A}, \mathcal{B}$ together with functors $\Phi: \mathcal{J} \rightarrow \mathcal{B}$ and $\Psi: \mathcal{A} \rightarrow \mathcal{B}$. We are asking whether there exists a functor $\Gamma: \mathcal{J} \rightarrow \mathcal{A}$ such that the functors Φ and $\Psi \circ \Gamma$ are isomorphic, in notation $\Phi \cong \Psi \circ \Gamma$ (two functors are *isomorphic* if there exists a natural transformation from one to the other all of whose components are isomorphisms; the relation $\Phi \cong \Psi \circ \Gamma$ could be paraphrased as “ $\Phi = \Psi \circ \Gamma$ up to isomorphism”).

The monograph Gillibert and Wehrung [12] deals extensively with this kind of problem, in situations arising from algebra. It is articulated around a technical statement called there the *Condensate Lifting Lemma* (CLL for short, Lemma 3.4.2 in [12]). This statement makes it possible, in many situations, to reduce the liftability of a *diagram* \vec{B} of \mathcal{B} to the liftability of a suitable *object* B of \mathcal{B} , called a *condensate* of \vec{B} . Even for diagrams indexed by a finite partially ordered set (*poset* for short) P , the construction of a condensate can be a difficult matter, relying on objects called *lifters* of P and whose existence follows from statements of infinite combinatorics based on *Kuratowski’s Free Set Theorem* (cf. Kuratowski [21], Erdős *et al.* [10, Theorem 46.1]). The “size” (often defined as cardinality) of a condensate may be (strictly) larger than the size of the diagram \vec{B} : even in case \vec{B} is a diagram of finite structures, indexed by a finite poset P , the condensate B may have transfinite cardinality, often \aleph_{n-1} where n is nothing else than the order-dimension of the poset P (cf. Gillibert and Wehrung [13]). In addition, this construction works if P is a finite lattice, but may fail for general finite posets.

Many readers would object that this is sounding a bit abstract. Hence, let us now inch away from category theory and present some more specific situations.

2. SPECTRAL SPACES: STONE DUALITY AND HOCHSTER’S THEOREM

For more details and references about the present section, we refer the reader to Grätzer [15, § 2.5], Hochster [17], Johnstone [18, § II.3 and Ch. V], Stone [27].

2.1. Hochster’s Theorem. The *Zariski spectrum* of a commutative unital ring A , is a topological space, fundamental in algebraic geometry. Let us recall the definition of that space. An ideal P of A is *prime* if it is proper (i.e., $P \neq A$, equivalently $1 \notin P$) and the quotient ring A/P is a domain (equivalently, $xy \in P$ implies that either $x \in P$ or $y \in P$, whenever $x, y \in A$). We denote by $\text{Spec } A$ the set of all prime ideals of A , endowed with the topology whose *closed* sets are exactly those of the form

$$\text{Spec}(A, X) \stackrel{\text{def}}{=} \{P \in \text{Spec } A \mid X \subseteq P\}, \quad (2.1)$$

for $X \subseteq A$. This is a so-called *hull-kernel topology*.

The assignment $A \mapsto \text{Spec } A$ can be extended to a (contravariant) *functor*, by sending any homomorphism $f: A \rightarrow B$ of commutative unital rings to the map $\text{Spec } f: \text{Spec } B \rightarrow \text{Spec } A$, $Q \mapsto f^{-1}[Q] \stackrel{\text{def}}{=} \{x \in A \mid f(x) \in Q\}$. This map is easily

seen to be *continuous* (this is not the best possible guess, but it will do for now). Hence, Spec defines a contravariant functor, from the category of all commutative unital rings, with unital ring homomorphisms, to the category of all topological spaces, with continuous maps.

What kind of topological space is $\text{Spec } A$? Easy examples show that it may not be Hausdorff. However, other easy examples show that not every topological space is the Zariski spectrum of a commutative unital ring, even in the finite case. Let us now prepare for the definition of the relevant topological spaces.

Definition 2.1. A nonempty closed set F in a topological space X is *irreducible* if $F = A \cup B$ implies that either $F = A$ or $F = B$, for all *closed* sets A and B . We say that X is *sober* if every irreducible closed set is the closure of a unique singleton¹.

Definition 2.1 can be paraphrased by saying that a topological space is sober if it has as few points as possible with respect to the lattice structure of its collection of open sets.

For a topological space X , we shall denote by $\overset{\circ}{\mathcal{K}}(X)$ the set of all open and compact² subsets of X , partially ordered under set inclusion. In general, for arbitrary $U, V \in \overset{\circ}{\mathcal{K}}(X)$, the union $U \cup V$ always belongs to $\overset{\circ}{\mathcal{K}}(X)$. However, the intersection $U \cap V$ may not belong to $\overset{\circ}{\mathcal{K}}(X)$.

Definition 2.2. A topological space X is *spectral* if it is sober and $\overset{\circ}{\mathcal{K}}(X)$ is a basis of the topology of X , closed under finite intersection.

Defining the empty intersection as the whole space X , this shows that a spectral space is always compact.

It is well known, and easy to verify, that the Zariski spectrum of any commutative unital ring is a spectral space. The converse is given by the following theorem of Hochster [17].

Theorem 2.3 (Hochster). *Every spectral space is homeomorphic to the Zariski spectrum of some commutative unital ring.*

Hochster's construction, that assigns, to any spectral topological space X , a commutative unital ring A_X such that $X \cong \text{Spec } A_X$, is *functorial*, from the category of all commutative unital rings with unital ring *embeddings*, to spectral spaces with *spectral maps*. By definition, for spectral spaces X and Y , a map $\varphi: X \rightarrow Y$ is *spectral* if $\varphi^{-1}[V]$ is compact open whenever V is a compact open subset of Y ; every spectral map is continuous, but not every continuous map is spectral.

In particular, Hochster's Theorem implies that the range of the Zariski spectrum functor is completely described: it is the class of all spectral topological spaces.

2.2. Stone duality. The construction of the Zariski spectrum, for commutative unital rings, can be extended to bounded distributive lattices, in the following way. Recall that a *lattice* is a structure (L, \vee, \wedge) , where \vee and \wedge are both binary operations on a set L such that there is a partial ordering \leq for which $x \vee y = \sup(x, y)$ (the *join* of $\{x, y\}$) and $x \wedge y = \inf(x, y)$ (the *meet* of $\{x, y\}$) whenever

¹Due to the uniqueness, every sober space is T_0 (not all references assume this).

²Throughout the paper, "compact" means what some other references call "quasicompact" (i.e., every open cover has a finite subcover); in particular, it does not imply Hausdorff.

$x, y \in L$. Necessarily, $x \leq y \Leftrightarrow x \vee y = y \Leftrightarrow x \wedge y = x$ whenever $x, y \in L$, so the partial ordering \leq is uniquely determined by either \vee or \wedge . We say that L is

- *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ whenever $x, y, z \in L$;
- *bounded* if \leq has a smallest element (then denoted by 0) and a largest element (then denoted by 1).

An ideal P , in a bounded distributive lattice D , is *prime* if it is proper (i.e., $P \neq D$, equivalently $1 \notin P$) and $x \wedge y \in P$ implies that either $x \in P$ or $y \in P$, whenever $x, y \in D$.

The remainder of the definition of the spectrum, of a bounded distributive lattice, follows the one for commutative unital rings. We denote by $\text{Spec } D$ the set of all prime ideals of D , endowed with the topology whose closed sets are exactly those of the form

$$\text{Spec}(D, X) \stackrel{\text{def}}{=} \{P \in \text{Spec } D \mid X \subseteq P\},$$

for $X \subseteq D$. It can be easily seen that $\text{Spec } D$ is a spectral space.

The assignment $D \mapsto \text{Spec } D$ can be extended to a (contravariant) *functor*, by sending any 0,1-lattice homomorphism $f: D \rightarrow E$ to the map $\text{Spec } f: \text{Spec } E \rightarrow \text{Spec } D$, $Q \mapsto f^{-1}[Q]$. This map is easily seen to be *continuous*, and even *spectral* (cf. Section 2.1). Hence, Spec defines a contravariant functor, from the category of all bounded distributive lattices, with 0,1-lattice homomorphisms, to spectral spaces, with spectral maps.

Stone duality [27] is contained in the following theorem and the subsequent comments.

Theorem 2.4 (Stone). *The pair $(\text{Spec}, \overset{\circ}{\mathcal{K}})$ induces a (categorical) duality, between bounded distributive lattices with 0,1-lattice homomorphisms and spectral spaces with spectral maps.*

The dual of a bounded distributive lattice D is its spectrum $\text{Spec } D$, and the dual of a spectral space X is the lattice $\overset{\circ}{\mathcal{K}}(X)$ of all its compact open subsets. The dual of a spectral map $\varphi: X \rightarrow Y$, between spectral spaces, is the 0,1-lattice homomorphism $\overset{\circ}{\mathcal{K}}(\varphi): \overset{\circ}{\mathcal{K}}(Y) \rightarrow \overset{\circ}{\mathcal{K}}(X)$, $V \mapsto \varphi^{-1}[V]$. For a bounded distributive lattice D , the assignment $a \mapsto \{P \in \text{Spec } D \mid a \notin P\}$ defines a lattice isomorphism $\varepsilon_D: D \rightarrow \overset{\circ}{\mathcal{K}}(\text{Spec } D)$, and the assignment $D \mapsto \varepsilon_D$ defines a natural transformation from the identity functor, on bounded distributive lattices, to the functor $\overset{\circ}{\mathcal{K}} \circ \text{Spec}$. For a spectral space X , the assignment $x \mapsto \{U \in \overset{\circ}{\mathcal{K}}(X) \mid x \notin U\}$ defines a homeomorphism $\eta_X: X \rightarrow \text{Spec } \overset{\circ}{\mathcal{K}}(X)$, and the assignment $X \mapsto \eta_X$ is a natural transformation from the identity functor, on spectral spaces, to the functor $\text{Spec} \circ \overset{\circ}{\mathcal{K}}$.

The duality described above restricts to the classical Stone duality between Boolean algebras and zero-dimensional compact Hausdorff spaces. It can also be extended, *mutatis mutandis*, to what should be called Stone duality between distributive lattices with zero, with cofinal³ zero-preserving lattice homomorphisms, and so-called “generalized spectral spaces” with spectral maps (cf. Rump and Yang [26, page 63], Johnstone [18, § II.3], Grätzer [15, § II.5]).

³For posets P and Q , a map $f: P \rightarrow Q$ is *cofinal* if for every $q \in Q$ there exists $p \in P$ such that $q \leq f(p)$.

While Stone’s Theorem and Hochster’s Theorem both characterize spectral spaces as spectra of commutative rings and bounded distributive lattices, respectively, the former result achieves one more feature: since we are dealing with a *duality*, every spectral space is the spectrum of a *unique* (up to isomorphism) bounded distributive lattice, while a commutative unital ring is usually not determined by its Zariski spectrum (e.g., the Zariski spectrum of any field is the one-point topological space).

Something can still be said about the interaction between Stone duality and Hochster’s Theorem. For a subset X in a commutative unital ring A , the Zariski-closed set $\text{Spec}(A, X)$ introduced in (2.1) depends only on the ideal I generated by X : namely, $\text{Spec}(A, X) = \text{Spec}(A, I)$. This can be even made more precise, by observing that $\text{Spec}(A, I) = \text{Spec}(A, \sqrt{I})$, where \sqrt{I} , the *radical* of I , is defined as $\{x \in A \mid x^n \in I \text{ for some positive integer } n\}$. Now this is as precise as it can get, as by Krull’s Theorem, every radical ideal (an ideal I is *radical* if $I = \sqrt{I}$) is the intersection of all the prime ideals containing it. Building on that observation, it is easy to verify that the Stone dual of $\text{Spec } A$ is isomorphic to the (bounded, distributive) lattice $\text{Id}_c^r A$ of all radicals of finitely generated ideals⁴ (beware the ambiguity about the “finitely generated radical ideal” terminology!) of A . Hence, by applying Stone duality to Theorem 2.3, we obtain the following.

Corollary 2.5. *Every bounded distributive lattice with zero is isomorphic to $\text{Id}_c^r A$ for some commutative unital ring A .*

3. CONGRUENCE LATTICES

For more detail and references about this section, we refer the reader to Wehrung [36, 37, 38].

3.1. The Congruence Lattice Problem. A *congruence* of a lattice (L, \vee, \wedge) is an equivalence relation θ on L such that $x_1 \equiv_\theta y_1$ and $x_2 \equiv_\theta y_2$ implies both $x_1 \vee x_2 \equiv_\theta y_1 \vee y_2$ and $x_1 \wedge x_2 \equiv_\theta y_1 \wedge y_2$, whenever $x_1, x_2, y_1, y_2 \in L$ (we say that θ is *compatible* with the operations \vee and \wedge). Here, $x \equiv_\theta y$ is short for $(x, y) \in \theta$.

The set $\text{Con } L$ of all congruences of a lattice L , partially ordered under \subseteq , is a *complete lattice*, in which

$$\bigwedge_{i \in I} \theta_i = \bigcap_{i \in I} \theta_i,$$

$$\bigvee_{i \in I} \theta_i = \text{congruence generated by } \bigcup_{i \in I} \theta_i,$$

for every collection $\{\theta_i \mid i \in I\}$ of congruences of L .

A congruence θ is *finitely generated* if it is the least one such that $x_1 \equiv_\theta y_1$ and \dots and $x_n \equiv_\theta y_n$, for some $x_i, y_i \in L$ ($1 \leq i \leq n$). A congruence θ is finitely generated iff it is a *compact* element of $\text{Con } L$, that is, whenever $\theta \subseteq \bigvee_{i \in I} \theta_i$, there exists a finite subset J of I such that $\theta \subseteq \bigvee_{i \in J} \theta_i$.

The lattice $\text{Con } L$ is *algebraic*, that is, it is complete and every congruence can be written in the form $\bigvee_{i \in I} \theta_i$ with all θ_i *compact*.

Up to this point, there is nothing special about the structure of lattice and the concept of congruence can be extended to any “universal algebra” (i.e., nonempty set A with a [possibly infinite] collection of operations $A^n \rightarrow A$ for various n). For

⁴The letter “r” stands for “radical” while the letter “c” stands for “compact”, which is the lattice-theoretical counterpart of “finitely generated”.

example, the congruences of a *group* G are in one-to-one correspondence with the *normal subgroups* of G . However, the congruences of a lattice L are, usually, *not* in any natural one-to-one correspondence with subsets of L .

The congruence lattice of any universal algebra is an algebraic lattice. The converse, stating that every algebraic lattice arises as the congruence lattice of some universal algebra, is a deep theorem of Grätzer and Schmidt [16].

A large part of what makes lattices so special is the following fundamental result, established in Funayama and Nakayama [11]. We show a proof for convenience.

Theorem 3.1 (Funayama and Nakayama). *The congruence lattice of every lattice is distributive.*

Proof. Define the *lower median* operation on a lattice L by setting $m(x, y, z) \stackrel{\text{def}}{=} (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$, whenever $x, y, z \in L$. The map m is a *majority operation*, that is, $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$ whenever $x, y \in L$. Now let $\alpha, \beta, \gamma \in \text{Con } L$ and let $(x, y) \in \alpha \cap (\beta \vee \gamma)$. Since (x, y) belongs to $\beta \vee \gamma$, which is the transitive closure of $\beta \cup \gamma$, there exists a finite sequence (z_0, \dots, z_n) of elements of L such that $z_0 = x$, $z_n = y$, and each $(z_i, z_{i+1}) \in \beta \cup \gamma$. Since m is a composition of the fundamental operations \vee and \wedge , every congruence of L is compatible with m . Setting $t_i \stackrel{\text{def}}{=} m(x, y, z_i)$ for each i , it follows that each $(t_i, t_{i+1}) \in \beta \cup \gamma$. Moreover, since m is a majority operation, $t_0 = x$ and $t_n = y$. From $x \equiv_\alpha y$ it follows that each $t_i \equiv_\alpha m(x, x, z_i) = x$, thus each $(t_i, t_{i+1}) \in \alpha \cap (\beta \cup \gamma) = (\alpha \cap \beta) \cup (\alpha \cap \gamma)$. Therefore, $(x, y) \in (\alpha \cap \beta) \vee (\alpha \cap \gamma)$, thus completing the proof that $\alpha \cap (\beta \vee \gamma) \subseteq (\alpha \cap \beta) \vee (\alpha \cap \gamma)$. The converse containment is trivial. \square

Theorem 3.1 is quite lattice-specific, in the sense that it does *not* extend to groups, modules, rings... For example, usually $A \cap (B + C) \neq (A \cap B) + (A \cap C)$ for submodules A, B, C of a given module.

In the 1940's, Dilworth proved that conversely, every *finite* distributive lattice is the congruence lattice of a (finite) lattice. Then he asked whether this could be extended to the infinite case:

The Congruence Lattice Problem (CLP), ~ 1940 . *Is every distributive algebraic lattice the congruence lattice of a lattice?*

CLP initiated a considerable amount of work, leading to a host of *positive* results. All those results are more conveniently stated in terms of the set $\text{Con}_c L$ of all compact (i.e., finitely generated) congruences of L , partially ordered under set inclusion. It should be noted that $\text{Con}_c L$ is *not* a lattice as a rule: for compact congruences α and β , the join $\alpha \vee \beta$ is compact, but the meet $\alpha \cap \beta$ may not be compact. Hence, $\text{Con}_c L$ is a $(\vee, 0)$ -*semilattice*⁵. It is *distributive*, that is, whenever $\alpha \subseteq \beta_1 \vee \beta_2$ in $\text{Con}_c L$, there are $\alpha_i \subseteq \beta_i$ in $\text{Con}_c L$ such that $\alpha = \alpha_1 \vee \alpha_2$. Moreover, one can go naturally from $\text{Con } L$ to $\text{Con}_c L$ (the latter is the semilattice of all compact elements of the former) and back (the former is isomorphic to the ideal lattice of the latter), and this *functorially*. We thus reach the following

Semilattice formulation of CLP. *Is every distributive $(\vee, 0)$ -semilattice representable, that is, isomorphic to $\text{Con}_c L$ for some lattice L ?*

⁵Or, equivalently, a commutative, idempotent monoid, endowed with the partial ordering \leq given by $x \leq y \Leftrightarrow x \vee y = y$.

Some known positive instances of CLP are given by the following result:

Theorem 3.2. *Let S be a distributive $(\vee, 0)$ -semilattice. In each of the following cases, S is representable:*

- (i) S is countable (Bauer ~ 1980);
- (ii) $\text{card } S \leq \aleph_1$ (Huhn 1989);
- (iii) S is a lattice (Schmidt 1981);
- (iv) $S = \varinjlim_{n < \omega} S_n$, with all transition maps $S_n \rightarrow S_{n+1}$ $(\vee, 0)$ -homomorphisms and all S_n distributive lattices (Wehrung 2003).

In any of those cases, a representing lattice L (such that $\text{Con}_c L \cong S$) can be taken *sectionally complemented* (a lattice L with zero is sectionally complemented if whenever $a \leq b$ in L , there exists $x \in L$ such that $a \vee x = b$ and $a \wedge x = 0$). The following result was established in Wehrung [30, 31].

Theorem 3.3. *For every cardinal number $\kappa \geq \aleph_2$, there exists a distributive $(\vee, 0, 1)$ -semilattice S_κ , of cardinality κ , not isomorphic to $\text{Con}_c L$ for any sectionally complemented lattice L .*

The task of removing “sectionally complemented” from the statement of Theorem 3.3 was completed nearly ten years later in Wehrung [32], thus yielding a negative solution of CLP, using the class of counterexamples constructed for Theorem 3.3 (with a far more elaborate proof):

Theorem 3.4. *The distributive $(\vee, 0, 1)$ -semilattice $S_{\aleph_{\omega+1}}$ is not representable.*

The optimal cardinality bound was subsequently obtained by Růžička in [24]:

Theorem 3.5 (Růžička). *The distributive $(\vee, 0, 1)$ -semilattice S_{\aleph_2} is not representable.*

Those results, together with the proof of Theorem 3.1, shift the original Congruence Lattice Problem to the following (still unsolved) problem: *is every distributive algebraic lattice the congruence lattice of a majority algebra?* By definition, a *majority algebra* is a pair (A, m) , where $m: A^3 \rightarrow A$ is a majority operation. The proof of Theorem 3.1 shows that the congruence lattice of any majority algebra is distributive.

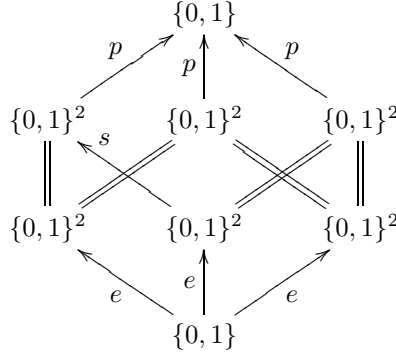
3.2. A heavy cube, and congruence-permutable algebras. Referring to the terminology “lifting” (with respect to a functor) from Section 1, “heavy” is intended to mean “hard to lift”.

We consider the diagram \mathcal{D}_c of $(\vee, 0)$ -semilattices and $(\vee, 0)$ -homomorphisms represented in Figure 3.1, where $e(x) = (x, x)$, $p(x, y) = x \vee y$, and $s(x, y) = (y, x)$ whenever $x, y \in \{0, 1\}$. This diagram is obviously commutative.

The following result was established in Tůma and Wehrung [28].

Theorem 3.6. *The cube \mathcal{D}_c cannot be lifted (with respect to the functor Con_c), by any cube of sectionally complemented lattices and lattice homomorphisms.*

In fact, it turns out that Theorem 3.6 can be extended to a much broader algebraic context; in particular, it is not lattice-specific. For binary relations α and β on a set A , we set $\alpha \circ \beta \stackrel{\text{def}}{=} \{(x, y) \in A \times A \mid (\exists z \in A)((x, z) \in \alpha \text{ and } (z, y) \in \beta)\}$. We say that an algebra A is *congruence-permutable* if $\alpha \circ \beta = \beta \circ \alpha$ for all congruences α and β of A . For example, *groups, modules, rings* are all congruence-permutable

FIGURE 3.1. The heavy cube \mathcal{D}_c

(e.g., $HK = KH$ for normal subgroups in a group). However, *not every lattice is congruence-permutable* (e.g., consider the three-element chain). The following result was established in Růžička, Tůma, and Wehrung [25].

Theorem 3.7. *The cube \mathcal{D}_c cannot be lifted (with respect to the functor Con_c), by any cube of congruence-permutable (universal) algebras. In particular, it cannot be lifted by groups, rings, or modules.*

It is still unknown whether every diagram of finite Boolean $(\vee, 0)$ -semilattices and $(\vee, 0)$ -homomorphisms, indexed by a finite lattice, can be lifted with respect to the functor Con_c on lattices.

The proof of Theorem 3.7 (i.e., non-liftability of \mathcal{D}_c by congruence-permutable algebras), yielded the construction of a distributive $(\vee, 0)$ -semilattice that cannot be lifted by congruence-permutable algebras, in Růžička, Tůma, and Wehrung [25].

Theorem 3.8. *For every cardinal number $\kappa \geq \aleph_2$, the distributive $(\vee, 0, 1)$ -semilattice S_κ is not isomorphic to $\text{Con}_c A$ for any congruence-permutable algebra A . In particular, S_{\aleph_2} is not isomorphic to $\text{Con}_c A$ whenever A is a sectionally complemented lattice, a group, a module, or a ring. Moreover, in the case of sectionally complemented lattices, groups, modules, rings, the cardinality bound \aleph_2 is optimal.*

Since congruences of a module are identified with submodules, it follows, for example, that S_{\aleph_2} is not isomorphic to the submodule lattice of any module. Similarly, it is not isomorphic to the normal subgroup lattice of any group.

In all the results described in this section, the unliftability statements about cubes (of order-dimension $n = 3$) always parallel unliftability statements about objects of cardinality $\aleph_{n-1} = \aleph_2$. General principles explaining this correspondence are developed in the monograph Gillibert and Wehrung [12]. Nonetheless, the constructions described in the present section predate the monograph.

4. NONSTABLE K_0 -THEORY

For further details and references about this section, see Ara [2], Goodearl [14], Wehrung [33].

Two idempotent matrices a and b over a (not necessarily commutative or unital) ring R are *Murray - von Neumann equivalent*, in symbol $a \sim b$, if there are matrices x and y such that $a = xy$ and $b = yx$. For square matrices x and y over R ,

we set $x \oplus y \stackrel{\text{def}}{=} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. If $x_1 \sim y_1$ and $x_2 \sim y_2$, then $x_1 \oplus x_2 \sim y_1 \oplus y_2$, for all square matrices (not necessarily of the same dimension) x_1, x_2, y_1, y_2 . Hence, Murray - von Neumann equivalence classes $[a] \stackrel{\text{def}}{=} \{x \mid a \sim x\}$, for idempotent matrices a over R , can be added, *via* the rule $[a] + [b] \stackrel{\text{def}}{=} [a \oplus b]$. The monoid $V(R) \stackrel{\text{def}}{=} \{[a] \mid a \text{ idempotent matrix on } R\}$ is *commutative* ($x + y = y + x$) and *conical* ($x + y = 0 \Rightarrow x = y = 0$). It encodes the *nonstable* K_0 -theory of R .

If R is *unital*, then $V(R)$ is isomorphic to the monoid of isomorphism classes of all finitely generated projective right (or, equivalently, left) R -modules, and the enveloping group (also called *Grothendieck group*) of $V(R)$ is the group usually denoted by $K_0(R)$.

For example, if R is a field, or more generally a division ring, then the finitely generated right R -modules are exactly the finite-dimensional right R -vector spaces, which are classified by dimension. Hence, $V(R)$ is isomorphic to the monoid $\mathbb{N}_0 \stackrel{\text{def}}{=} \{0, 1, 2, \dots\}$ of nonnegative integers.

The following result was established in Bergman [4], Bergman and Dicks [5]:

Theorem 4.1 (Bergman and Dicks). *Every commutative conical monoid M is isomorphic to $V(R_M)$ for some ring R_M .*

To my knowledge, the functoriality of the assignment $M \mapsto R_M$ has not yet been studied.

Further restrictions on the ring R are usually met with restrictions on the nonstable K_0 -theory. An important class of rings for which this occurs is the following. It was introduced by Warfield [29] in the unital case, Ara [1] in the general case.

Definition 4.2. A ring R is an *exchange ring* if for all $x \in R$, there are an idempotent $e \in R$ and $r, s \in R$ such that $e = rx = x + s - sx$.

It can be established (and this is not trivial) that this condition is left-right symmetric. Every *von Neumann regular ring* (i.e., satisfying $(\forall x)(\exists y)(xyx = x)$) is an exchange ring, and a C^* -algebra is an exchange ring iff it has *real rank zero* (Ara *et al.* [3], Ara [1]).

Extending earlier results about von Neumann regular rings and C^* -algebras of real rank zero, Ara established in [1] the following result, in full generality.

Theorem 4.3 (Ara). *Let R be an exchange ring. Then $V(R)$ is a refinement monoid, that is, for all $a_0, a_1, b_0, b_1 \in V(R)$ such that $a_0 + a_1 = b_0 + b_1$, there are $c_{i,j} \in V(R)$, for $i, j \in \{0, 1\}$, such that each $a_i = c_{i,0} + c_{i,1}$ and $b_i = c_{0,i} + c_{1,i}$.*

It is not known whether every conical refinement monoid M arises in this way, that is, $M \cong V(R)$ for some exchange ring R . Nevertheless, it is known since Wehrung [30] that the answer to that problem is negative for von Neumann regular rings (with counterexamples of cardinality \aleph_2), and it is known since Wehrung [33] that the answer is negative for C^* -algebras of real rank zero (with counterexamples of cardinality \aleph_3 , using the CLL tool from Gillibert and Wehrung [12]).

Say that a monoid is *simplicial* if it is isomorphic to a finite power of the additive monoid \mathbb{N}_0 of all nonnegative integers. The following result was established in Wehrung [33]:

Theorem 4.4. *There is a commutative cube, of simplicial monoids and monoid homomorphisms, that can be lifted, with respect to the functor V , by exchange rings*

and by C^* -algebras of real rank 1, but not by semiprimitive exchange rings, thus neither by von Neumann regular rings nor by C^* -algebras of real rank 0.

The cube of Theorem 4.4 is obtained from the one of Figure 3.1 by replacing $\{0, 1\}$ by \mathbb{N}_0 and setting $e(x) = (x, x)$, $p(x, y) = x + y$, and $s(x, y) = (y, x)$ whenever $x, y \in \mathbb{N}_0$.

Theorem 4.4 could be paraphrased by saying that *at diagram level*, the nonstable K_0 -theory of exchange rings *properly* contains both the one of von Neumann regular rings and the one of C^* -algebras of real rank zero. By using again the tools of Gillibert and Wehrung [12], this result is extended to *objects* in [33].

Theorem 4.5. *There exists a unital exchange ring of cardinality \aleph_3 (resp., an \aleph_3 -separable unital C^* -algebra of real rank 1) R such that $V(R)$ is not isomorphic to $V(B)$ for any ring B which is either a C^* -algebra of real rank 0 or a von Neumann regular ring.*

5. BACK TO SPECTRAL SPACES

For more details and references about the present section, we refer the reader to Delzell and Madden [8], Grätzer [15, § 2.5], Johnstone [18, § II.3 and Ch. V], Keimel [20], Coste and Roy [6], Dickmann [9, Ch. 6], Wehrung [34, 35].

The Zariski spectrum construction can be extended to various contexts, such as *Abelian ℓ -groups* (yielding the *ℓ -spectrum*) and *partially ordered, commutative unital rings* (yielding the *real spectrum*). Tailoring the methods above (*in particular, CLL*) to that new context, further results can be obtained on ℓ -spectra and real spectra. Let us give a short summary of such results.

5.1. The ℓ -spectrum of an Abelian ℓ -group. An *ℓ -group* is a group G endowed with a translation-invariant lattice ordering. An *ℓ -ideal* of G is an order-convex normal subgroup of G , closed under the lattice operations. An ℓ -ideal P of an Abelian ℓ -group G is *prime* if it is proper (i.e., $P \neq G$) and $x \wedge y \in P$ implies that either $x \in P$ or $y \in P$, whenever $x, y \in G$. The *ℓ -spectrum* of G , denoted $\text{Spec}_\ell G$, is the set of all prime ℓ -ideals of G , endowed with the topology whose closed sets are exactly those of the form

$$\text{Spec}_\ell(G, X) \stackrel{\text{def}}{=} \{P \in \text{Spec}_\ell G \mid X \subseteq P\}, \quad (5.1)$$

for $X \subseteq G$. An *order-unit* of G is an element e of the positive cone G^+ of G such that every element of G^+ lies below an integer multiple of e . To ease the presentation, let us deal only with Abelian ℓ -groups with order-unit. For those ℓ -groups, $\text{Spec}_\ell G$ is a spectral space. (If G has no order-unit, then $\text{Spec}_\ell G$ is only a *generalized spectral space*.)

A spectral space X is *completely normal* if for all $z \in X$ and all x, y in the closure of $\{z\}$, either x belongs to the closure of $\{y\}$ or y belongs to the closure of $\{x\}$. It is known since Keimel [19] that the ℓ -spectrum of any Abelian ℓ -group with unit is completely normal. The characterization problem of all ℓ -spectra of Abelian ℓ -groups with unit has been open since then, sometimes under equivalent names (mostly Mundici's *MV-spectrum problem*, see [23]). The countable case was recently solved in Wehrung [34]:

Theorem 5.1. *Every second countable, completely normal spectral space is homeomorphic to the ℓ -spectrum of an Abelian ℓ -group with order-unit.*

Due to a counterexample by Delzell and Madden [7], Theorem 5.1 cannot be extended to spectral spaces that fail to be second countable. In fact, due to the results of [34], the class of all Stone duals of ℓ -spectra of Abelian ℓ -groups with unit cannot be characterized by any class of $\mathcal{L}_{\infty, \omega}$ -formulas (thus, in particular, it is not first-order).

5.2. The real spectrum of a commutative unital ring. Let A be a commutative unital ring. A subset C of A is a *cone* if it is both an additive and a multiplicative submonoid of A , containing all squares in A . A cone P of A is *prime* if $A = P \cup (-P)$ and the *support* $P \cap (-P)$ is a prime ideal of A . For a prime cone P , $-1 \notin P$ (otherwise $1 \in P \cap (-P)$, thus $P \cap (-P) = A$, a contradiction). We denote by $\text{Spec}_r A$ the set of all prime cones of A , endowed with the topology generated by all subsets of the form $\{P \in \text{Spec}_r A \mid a \notin P\}$, for $a \in A$, and we call $\text{Spec}_r A$ the *real spectrum of A* . The real spectrum of A is always a completely normal spectral space. Delzell and Madden [7] established that not every completely normal spectral space arises in this way. Mellor and Tressl [22] extended that result by proving that *for any infinite cardinal number λ , the class of Stone duals of all real spectra cannot be defined by any class of $\mathcal{L}_{\infty, \lambda}$ -formulas.*

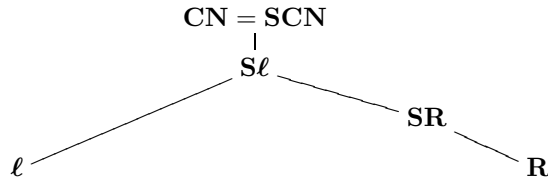
It is not known whether the result of Theorem 5.1 can be extended to real spectra: *is every second countable completely normal spectral space the real spectrum of some commutative unital ring?*

It is still possible to compare ℓ -spectra, real spectra, and their spectral subspaces, with respect to inclusion. Let

$$\begin{aligned} \mathbf{CN} &=_{\text{def}} \{\text{completely normal spectral spaces}\}, \\ \ell &=_{\text{def}} \{\ell\text{-spectra of Abelian } \ell\text{-groups with unit}\}, \\ \mathbf{R} &=_{\text{def}} \{\text{real spectra of commutative unital rings}\}, \\ \mathbf{SX} &=_{\text{def}} \{\text{spectral subspaces of members of } \mathbf{X}\}, \end{aligned}$$

for any class \mathbf{X} of spectral spaces. (By definition, a spectral space X is a *spectral subspace* of a spectral space Y if $X \subseteq Y$ and the inclusion map from X into Y is a spectral map.) The following result was established in Wehrung [35].

Theorem 5.2. *All containments and non-containments between the classes \mathbf{CN} , ℓ , $\mathbf{S}\ell$, \mathbf{R} , \mathbf{SR} can be read on the following diagram:*



Moreover, the topological counterexamples proving the various non-containments in that diagram all have bases of cardinality \aleph_1 , except for the one proving $\mathbf{S}\ell \subsetneq \mathbf{CN}$, which has a basis of cardinality \aleph_2 .

The Stone duals of the topological counterexamples proving the relations $\ell \subsetneq \mathbf{S}\ell$, $\mathbf{R} \subsetneq \mathbf{SR}$, $\mathbf{R} \not\subseteq \ell$ are all constructed as condensates of one-arrow diagrams. The counterexample illustrating $\ell \not\subseteq \mathbf{SR}$ is obtained *via* a direct construction. The

counterexample illustrating $\mathbf{S\ell} \not\subseteq \mathbf{CN}$ is obtained *via* a “free construction” similar, in spirit, to the one introduced in Wehrung [30].

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LMNO, CNRS UMR 6139, DÉPARTEMENT DE MATHÉMATIQUES, BP 5186, UNIVERSITÉ DE CAEN, CAMPUS 2, 14032 CAEN CEDEX, FRANCE

E-mail address: friedrich.wehrung01@unicaen.fr

URL: <http://wehrungf.users.lmno.cnrs.fr>