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| Georges Gras. Test of Vandiver's conjecture with Gauss sums – Heuristics. 2018. hal-01856083v4

HAL Id: hal-01856083

<https://hal.science/hal-01856083v4>

Preprint submitted on 25 Jun 2019

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Test of Vandiver’s conjecture with Gauss sums – Heuristics

Georges Gras

ABSTRACT. The link between Vandiver’s conjecture and Gauss sums is well known since the papers of Iwasawa (1975), Thaine (1995-1999) and Anglès–Nuccio (2010). This conjecture is required in many subjects and we shall give such examples of relevant references. In this paper, we recall our interpretation of Vandiver’s conjecture in terms of minus part of the torsion of the Galois group of the maximal abelian p -ramified pro- p -extension of the p th cyclotomic field (1984). Then we provide a specific use of Gauss sums of characters of order p of \mathbb{F}_ℓ^\times and prove new criteria for Vandiver’s conjecture to hold (Theorem 1.2 (a) using both the sets of exponents of p -irregularity and of p -primarity of suitable twists of the Gauss sums, and Theorem 1.2 (b) which does not need the knowledge of Bernoulli numbers or cyclotomic units). We propose in §5.2 new heuristics showing that any counterexample to the conjecture leads to excessive constraints modulo p on the above twists as ℓ varies and suggests analytical approaches to evidence. We perform numerical experiments to strengthen our arguments in direction of the very probable truth of Vandiver’s conjecture. All the calculations are given with their PARI/GP programs.

CONTENTS

1. Introduction	1
2. Pseudo-units – Notion of p -primarity	4
3. Abelian p -ramification	5
4. Twists of Gauss sums associated to primes $\ell \equiv 1 \pmod{p}$	8
5. Heuristics – Probability of a counterexample	19
6. Conclusion	28
References	29

1. Introduction

Let $K = \mathbb{Q}(\mu_p)$ be the field of p th roots of unity for a given prime $p > 2$ and let K_+ be its maximal real subfield. Put $G := \text{Gal}(K/\mathbb{Q})$.

We denote by \mathcal{C} and \mathcal{C}_+ the p -class groups of K and K_+ , then by \mathcal{C}_- the relative p -class group, so that $\mathcal{C} = \mathcal{C}_+ \oplus \mathcal{C}_-$.

Let E and E_+ be the groups of units of K and K_+ ; then $E = E_+ \oplus \mu_p$ (Kummer).

The conjecture of Vandiver (or Kummer–Vandiver) asserts that \mathcal{C}_+ is trivial. This statement is equivalent to say that the group of real cyclotomic units of K is of prime to p index in E_+ [52, Theorem 8.14]. One may refer to numerical tables using this property in [4, 9] (verifying the conjecture up to $2 \cdot 10^9$), and to more general results in [50, 51] where some relations

Date: June 25, 2019.

2010 Mathematics Subject Classification. 11R18, 11L05, 11R37, 11R29, 08-04.

Key words and phrases. Cyclotomic fields; Vandiver’s conjecture; Gauss sums; Jacobi sums; Kummer theory; Stickelberger’s theorem; Class field theory; p -ramification.

with Gauss and Jacobi sums are used to express the order of the isotypic components of \mathcal{C}_+ (e.g., [50, Theorem 4]).

Many heuristics are proposed about this conjecture; see Washington’s book [52, § 8.3, Corollary 8.19] for some history, criteria, and for probabilistic arguments, then [38] assuming Greenberg’s conjecture [22] for K_+ .

We have also given a probabilistic study in [15, II.5.4.9.2]. All these heuristics lead to the fact that the number of primes p less than x , giving a counterexample, can be of the form $O(1) \cdot \log(\log(x))$.

These reasonings, giving the possible existence of infinitely many counterexamples to Vandiver’s conjecture, are based on standard probabilities associated with the Borel–Cantelli heuristic, but many recent p -adic heuristics and conjectures (on class groups and units) may contradict such unfounded approaches.

In this paper, we shall work in another direction, in the framework of “abelian p -ramification”, using Gauss sums together with the “Main Theorem on abelian fields” restricted to \mathcal{C}_- , and giving the order of its isotypic components by means of generalized Bernoulli numbers (this aspect is related by Ribet in [40, 41] and we shall call it “Main Theorem” for short).

Such a link of Vandiver’s conjecture with Gauss sums and abelian p -ramification has been given first by Iwasawa [28], then by Anglès–Nuccio [1], and encountered by many authors in various directions (Iwasawa’s theory, Galois cohomology, Fermat curves, Galois representations,...), then often assuming Vandiver’s conjecture (e.g., [8, 23, 24, 26, 27, 31, 44, 45, 46, 47, 53, 54]).

This link does exist also in the context of the classical conjecture of Greenberg [22] considered as a generalization of Vandiver’s conjecture (e.g., [35], [16]). We propose, in Section 3.1, to explain the links with p -ramification and prove again the reflection theorem (Theorem 3.1 and Corollary 3.2).

Then we shall interpret a counterexample to Vandiver’s conjecture in terms of non-trivial “ p -primary pseudo-units” stemming from Gauss sums:

$$\tau(\psi) = - \sum_{x \in \mathbb{F}_\ell^\times} \psi(x) \xi_\ell^x,$$

for ψ of order p , ξ_ℓ of prime order $\ell \equiv 1 \pmod{p}$. Indeed, if $\#\mathcal{C}_+ \equiv 0 \pmod{p}$, there exists a class $\gamma = \mathcal{C}(\mathfrak{A}) \in \mathcal{C}_-$, of order p , such that $\mathfrak{A}^p = (\alpha)$, with α p -primary (to give the unramified extension $K(\sqrt[p]{\alpha})/K$, decomposed over K_+ into a cyclic unramified extension L_+/K_+ of degree p predicted by class field theory); the reciprocal being obvious.

Since α can be obtained explicitly by means of twists (giving products of Jacobi sums) of the above Gauss sums:

$$(1) \quad \mathfrak{g}_c(\ell) = \tau(\psi)^{c-\sigma_c} \in K,$$

with Artin automorphisms σ_c attached to a primitive root c modulo p , this will yield the main test verifying the validity of the conjecture at p ; this result is the object of the Theorem 4.7, Corollary 4.8 and Theorem 4.9, that we can summarize, in the Theorem 1.2 below, after the reminder of some notations and classical definitions.

Definition 1.1. (i) Let ζ_p be a primitive p th root of unity. We denote by ω the Teichmüller character of G (the p -adic character with values in $\mu_{p-1}(\mathbb{Q}_p)$ such that $\zeta_p^s = \zeta_p^{\omega(s)}$ for all $s \in G$).

The irreducible p -adic characters of G are the $\theta = \omega^m$, $1 \leq m \leq p-1$.

(ii) Let $e_\theta := \frac{1}{p-1} \sum_{s \in G} \theta(s^{-1}) s$ be the associated idempotents in $\mathbb{Z}_p[G]$.

(iii) Let $g_c(\ell)_\theta$ denotes the θ -component of the twist $g_c(\ell)$ defined by (1), as representative in K^\times of the class of $\overline{g_c(\ell)}^{e_\theta} \in K^\times/K^{\times p}$.

Theorem 1.2 (Main theorem). *For a prime $\ell \equiv 1 \pmod{p}$, let $\mathcal{E}_\ell(p)$ be the set of exponents of p -primarity of ℓ (even integers $n \in [2, p-3]$, such that $g_c(\ell)_{\omega^{p-n}} \equiv 1 \pmod{p}$). Then let $\mathcal{E}_0(p)$ be the set of exponents of p -irregularity of K (even integers $n \in [2, p-3]$, such that p divides the n th Bernoulli number B_n).*

(a) *Vandiver's conjecture holds for K if and only if there exists $\ell \equiv 1 \pmod{p}$ such that $\mathcal{E}_\ell(p) \cap \mathcal{E}_0(p) = \emptyset$.*

(b) *Vandiver's conjecture holds for K if and only if there exist $N \geq 1$ primes $\ell_i \equiv 1 \pmod{p}$ such that $\bigcap_{i=1}^N \mathcal{E}_{\ell_i}(p) = \emptyset$.*

Test (b) is numerically very frequent for $N = 1$ or N very small, and does not need the knowledge of Bernoulli's numbers; in fact, it does not need to know if p is irregular or not (see Theorem 4.9).

We show that some assumption of *independence*, of the congruential properties (mod p) of these twists, as ℓ varies, is an obstruction to any counterexample to Vandiver's conjecture.

This method is different from that needed to prove that some cyclotomic unit is not a global p th power, which does not give obvious probabilistic approach (nevertheless, see §5.2.4 for some complements).

Finally, we propose, in §§5.2, 5.3, new heuristics (to our knowledge) and give substantial numerical experiments confirming them.

Definition 1.3. (i) We denote by \mathcal{X}_+ the set of even characters $\theta \neq 1$ (i.e., $\theta = \omega^m$, $m \in [2, p-3]$ even), and by \mathcal{X}_- the set of odd characters distinct from ω (i.e., $\theta = \omega^m$, $m \in [3, p-2]$ odd).

If $\theta = \omega^m$, we put $\theta^* := \omega\theta^{-1} = \omega^{p-m}$. This defines an involution on the group of characters which applies \mathcal{X}_+ onto $\mathcal{X}_+^* = \mathcal{X}_-$.

(ii) For a finitely generated $\mathbb{Z}_p[G]$ -module M , we put $M_\theta := M^{e_\theta}$. The operation of the complex conjugation $s_{-1} \in G$ gives rise to the obvious definition of the components M_+ and M_- such that $M = M_+ \oplus M_-$.

(iii) We denote by $\text{rk}_p(A) := \dim_{\mathbb{F}_p}(A/A^p)$ the p -rank of any abelian group A .

(iv) For $\alpha \in K^\times$, prime to p , considered modulo $K^{\times p}$, we denote by α_θ a representative in K^\times of the class $\overline{\alpha}^{e_\theta} \in K^\times/K^{\times p}$ (e.g., $\alpha_\theta = \alpha^{e'_\theta}$ where $e'_\theta \in \mathbb{Z}[G]$ approximates $e_\theta \pmod{p}$).

(v) Let I be the group of prime to p ideals of K . For any $\mathfrak{A} \in I$ such that $\mathcal{d}(\mathfrak{A}) \in \mathcal{C}$, there exists an approximation $e'_\theta \in \mathbb{Z}[G]$ of e_θ modulo a sufficient high power of p such that $\mathfrak{A}_\theta := \mathfrak{A}^{e'_\theta}$ is defined up to a principal ideal of the form (x^p) , $x \in K^\times$.

(vi) We say that $\mathfrak{A} \in I$ is p -principal if it is principal in $I \otimes \mathbb{Z}_p$; thus $\mathfrak{A} = (\alpha)$, with $\alpha \in K^\times \otimes \mathbb{Z}_p$, defined up to the product by $\varepsilon \in E \otimes \mathbb{Z}_p$.¹

(vii) For $\chi =: \omega^n \in \mathcal{X}_+$, let $B_{1,(\chi^*)^{-1}} = B_{1,\omega^{n-1}} := \frac{1}{p} \sum_{a=1}^{p-1} (\chi^*)^{-1}(s_a) a$ be the generalized Bernoulli number of character $(\chi^*)^{-1}$ (where $s_a \in G$ is the Artin automorphism attached to a ; it is the restriction of the Artin automorphism σ_a defined above in larger extensions).

¹The distinction between $\mathfrak{A}^{e_\theta} \in I \otimes \mathbb{Z}_p$ and $\mathfrak{A}^{e'_\theta} \in I$ ($e'_\theta \equiv e_\theta \pmod{p^N \mathbb{Z}_p[G]}$, N large enough) has some importance in practice and programming, provided of a definition of $\mathfrak{A}^{e'_\theta}$ up to a principal ideal of the form (x^p) , for deciding, for instance in the writing $\mathfrak{A}^{e_\theta} =: (\alpha_\theta)$, of the " p -primarity" of $\alpha_\theta \in K^\times \otimes \mathbb{Z}_p$; whence $\mathfrak{A}^{e'_\theta} =: (\alpha')$ where $\alpha_\theta \cdot \alpha'^{-1} \in (K^\times \otimes \mathbb{Z}_p)^p \cdot E \otimes \mathbb{Z}_p$. This will be used for $\theta \in \mathcal{X}_+^*$ where θ -components of units do not intervene, giving $(\alpha_\theta) = (x)^p \Leftrightarrow \alpha_\theta \in K^{\times p}$.

The Bernoulli number $B_{1, \omega^{n-1}}$ is an element of \mathbb{Z}_p congruent modulo p to $\frac{B_n}{n}$, where B_n is the n th ordinary Bernoulli number; see [52, Proposition 4.1, Corollary 5.15].

(viii) We say that a finitely generated $\mathbb{Z}_p[G]$ -module M is monogenous if it is generated, over $\mathbb{Z}_p[G]$, by a single element; this is equivalent to $\text{rk}_p(M_\theta) \leq 1$ for all irreducible p -adic character θ of G .

The index of p -irregularity $i(p)$ is the number of even $n \in [2, p-3]$ such that $B_n \equiv 0 \pmod{p}$; thus $i(p) = \#\mathcal{E}_0(p)$. See [52, § 5.3 & Exercise 6.6] giving statistics and the heuristic $i(p) = O\left(\frac{\log(p)}{\log(\log(p))}\right)$.

For a general history of Bernoulli–Kummer–Herbrand–Ribet, then Mazur–Wiles–Thaine–Kolyvagin–Rubin–Greither works on cyclotomy see [13, 41, 52]; in this context, if for $\theta \in \mathcal{X}_-$, $B_{1, \theta^{-1}}$ is of p -valuation e , we shall have (Main Theorem):

$$\#\mathcal{C}_\theta = |B_{1, \theta^{-1}}|_p^{-1} = p^e.$$

2. Pseudo-units – Notion of p -primarity

Definition 2.1. (i) We call *pseudo-unit* any $\alpha \in K^\times$, prime to p , such that (α) is the p th power of an ideal of K .

(ii) We say that an arbitrary $\alpha \in K^\times$, prime to p , is *p -primary* if the Kummer extension $K(\sqrt[p]{\alpha})/K$ is unramified at the unique prime ideal \mathfrak{p} above p in K (but possibly ramified elsewhere).

Remark 2.2. (i) Let A be the group of pseudo-units of K . If $\alpha \in A$, there exists an ideal \mathfrak{a} such that $(\alpha) = \mathfrak{a}^p$; then if we associate with $\alpha K^{\times p}$ the class of \mathfrak{a} , we obtain the exact sequence, where ${}_p\mathcal{C} := \{\gamma \in \mathcal{C}, \gamma^p = 1\}$:

$$1 \longrightarrow E/E^p \longrightarrow AK^{\times p}/K^{\times p} \longrightarrow {}_p\mathcal{C} \longrightarrow 1,$$

giving $\dim_{\mathbb{F}_p}(AK^{\times p}/K^{\times p}) = \frac{p-1}{2} + \text{rk}_p(\mathcal{C})$. Thus the computation of $\dim_{\mathbb{F}_p}((AK^{\times p}/K^{\times p})_\theta)$ is immediate from the value of $\text{rk}_p(\mathcal{C}_\theta)$ and $\dim_{\mathbb{F}_p}((E/E^p)_\theta) = 1$ (resp. 0) if $\theta \in \mathcal{X}_+ \cup \{\omega\}$ (resp. $\theta \in \mathcal{X}_+^* \cup \{1\}$).

(ii) The general condition of p -primarity for any $\alpha \in K^\times$ (α prime to p but not necessarily a pseudo-unit) is “ α congruent to a p th power modulo $\mathfrak{p}^p = (p)\mathfrak{p}$ ” (e.g., [15, Ch. I, § 6, (b), Theorem 6.3]). Since in any case (replacing α by α^{p-1}) we can assume $\alpha \equiv 1 \pmod{\mathfrak{p}}$, the above condition is then equivalent to $\alpha \equiv 1 \pmod{\mathfrak{p}^p}$ (indeed, for any $x \equiv 1 \pmod{\mathfrak{p}}$ we get $x^p \equiv 1 \pmod{\mathfrak{p}^p}$).

For the pseudo-units of K , the p -primarity may be characterized as follows:

Proposition 2.3. *Let $\alpha \in K^\times$ be a pseudo-unit. Then α is p -primary if and only if it is a local p th power at \mathfrak{p} .*

Proof. One direction is trivial. Suppose that $K(\sqrt[p]{\alpha})/K$ is unramified at \mathfrak{p} ; since $\alpha = \mathfrak{a}^p$, this extension is unramified as a global extension and is contained in the p -Hilbert class field H of K . The Frobenius automorphism in H/K of the *principal ideal* $\mathfrak{p} = (\zeta_p - 1)$ is trivial; so \mathfrak{p} totally splits in H/K , thus in $K(\sqrt[p]{\alpha})/K$, proving the proposition. \square

When α is not necessarily a pseudo-unit, we have a similar result provided we only look at the p -primarity of α_θ for $\theta \neq 1, \omega$:

Proposition 2.4. *Let $\alpha \equiv 1 \pmod{\mathfrak{p}}$. Let $m \in [2, p-2]$ and let α_θ for $\theta = \omega^m$. Then $\alpha_\theta \equiv 1 \pmod{\mathfrak{p}^m}$; moreover α_θ is p -primary if and only if $\alpha_\theta \equiv 1 \pmod{p}$, in which case one gets $\alpha_\theta \equiv 1 \pmod{\mathfrak{p}^{m+p-1} = (p)\mathfrak{p}^m}$.*

Proof. Consider the Dwork uniformizing parameter ϖ in $\mathbb{Z}_p[\mu_p]$ which has the following properties:

- (i) $\varpi^{p-1} = -p$,
- (ii) $s(\varpi) = \omega(s) \cdot \varpi$, for all $s \in G$.

Put $\alpha_\theta = 1 + \varpi^k u$, where u is a unit of $\mathbb{Z}_p[\varpi]$ and $k \geq 1$; let $u_0 \in \mathbb{Z} \setminus p\mathbb{Z}$ be such that $u \equiv u_0 \pmod{\varpi}$ giving $\alpha_\theta \equiv 1 + \varpi^k u_0 \pmod{\varpi^{k+1}}$. Since $\alpha_\theta^s = \alpha_\theta^{\theta(s)}$ in $K^\times \otimes \mathbb{Z}_p$, we get, for all $s \in G$:

$$1 + s(\varpi^k) u_0 = 1 + \omega^k(s) \varpi^k u_0 \equiv (1 + \varpi^k u_0)^{\theta(s)} \equiv 1 + \omega^m(s) \varpi^k u_0 \pmod{\varpi^{k+1}},$$

which implies $k \equiv m \pmod{p-1}$ and $\alpha_\theta = 1 + \varpi^k u$, $k \in \{m, m+p-1, \dots\}$; whence the first claim. The p -primarity condition for α_θ is $\alpha_\theta \equiv 1 \pmod{\varpi^p}$ giving the obvious direction “ α_θ p -primary $\Rightarrow \alpha_\theta \equiv 1 \pmod{p}$ ” since $(\varpi^p) = (p\varpi)$.

Suppose $\alpha_\theta \equiv 1 \pmod{\varpi^{p-1}}$; so $k = m$ does not work in the writing $\alpha_\theta = 1 + \varpi^k u$ since $m \leq p-2$, and necessarily k is at least $m+p-1 \geq p+1$, because $m \geq 2$ (which is also the local p th power condition). \square

3. Abelian p -ramification

Let's give an overview of the theory of abelian p -ramification, which is not our main purpose, but the natural framework for Vandiver's conjecture and Gauss sums.

3.1. Vandiver's conjecture and abelian p -ramification. Let U be the group of principal local units at p of K and let \overline{E} be the closure of the image of E in U . Let \mathcal{T} be the torsion group of the Galois group of the maximal abelian p -ramified (i.e., unramified outside p) pro- p -extension H^{pr} of K . This extension contains the p -Hilbert class field H and the compositum \tilde{K} of the \mathbb{Z}_p -extensions of K . In the case of $K = \mathbb{Q}(\mu_p)$, the theory is summarized by the following exact sequences (since Leopoldt's conjecture holds for abelian fields):

$$\begin{aligned} 1 &\longrightarrow \text{tor}_{\mathbb{Z}_p}(U/\overline{E}) \longrightarrow \mathcal{T} \longrightarrow \tilde{\mathcal{C}} \longrightarrow 1 \\ 1 &\longrightarrow \text{tor}_{\mathbb{Z}_p}(U)/\text{tor}_{\mathbb{Z}_p}(\overline{E}) = 1 \longrightarrow \text{tor}_{\mathbb{Z}_p}(U/\overline{E}) \xrightarrow{\log} \mathcal{R} \longrightarrow 0, \end{aligned}$$

where $\tilde{\mathcal{C}} \subseteq \mathcal{C}$ corresponds, by class field theory, to the subgroup $\text{Gal}(H/H \cap \tilde{K})$, and where $\mathcal{R} := \text{tor}_{\mathbb{Z}_p}(\log(U)/\log(\overline{E}))$ is the normalized p -adic regulator [19, Proposition 5.2]. Taking the θ -components, we obtain the exact sequences (where $\mathcal{R}_\theta = 1$ for all odd θ):

$$1 \longrightarrow \mathcal{R}_\theta \longrightarrow \mathcal{T}_\theta \longrightarrow \tilde{\mathcal{C}}_\theta \longrightarrow 1.$$

For more information, see [15, 17, 19]. We then have $\text{Gal}(H^{\text{pr}}/K) \simeq \Gamma \oplus \mathcal{T} \simeq \mathbb{Z}_p^{\frac{p+1}{2}} \oplus \mathcal{T}$ where $\Gamma := \text{Gal}(\tilde{K}/K)$ is such that $\Gamma_+ = \Gamma_1 \simeq \mathbb{Z}_p$ and $\Gamma_- \simeq \mathbb{Z}_p[G]_-$ giving $\Gamma_\theta \simeq \mathbb{Z}_p$ for all odd θ .

Write $\mathcal{T} = \mathcal{T}_+ \oplus \mathcal{T}_-$ and define $H_-^{\text{pr}} \subseteq H^{\text{pr}}$ (fixed by $\text{Gal}(H^{\text{pr}}/K)_+$), then $H_+^{\text{pr}} \subseteq H^{\text{pr}}$ (fixed by $\text{Gal}(H^{\text{pr}}/K)_-$). Thus $\text{Gal}(H_+^{\text{pr}}/K) \simeq \mathbb{Z}_p \oplus \mathcal{T}_+$ and $\text{Gal}(H_-^{\text{pr}}/K) \simeq \mathbb{Z}_p^{\frac{p-1}{2}} \oplus \mathcal{T}_-$.

One defines in the same way the fields H_θ^{pr} for which $\text{Gal}(H_\theta^{\text{pr}}/K) \simeq \Gamma_\theta \oplus \mathcal{T}_\theta$ (reduced to \mathcal{T}_θ , finite, for all $\theta \in \mathcal{X}_+$). We have $H_\theta \subset H_\theta^{\text{pr}}$ in terms of components of H .

Note that H_+^{pr}/K is decomposed over K_+ to give the maximal abelian p -ramified pro- p -extension of K_+ .

Theorem 3.1. *For all irreducible p -adic character θ of K , we have $\text{rk}_p(\mathcal{T}_{\theta^*}) = \text{rk}_p(\mathcal{C}_\theta)$.*

Proof. We will give an outline of this famous reflection result as follows from classical Kummer duality between radicals and Galois groups (see, e.g., [15, Theorem I.6.2 & Corollary I.6.2.1]), using the fact that $K(\sqrt[p]{\beta})/K$, $\beta \in K^\times$, is p -ramified if and only if $(\beta) = \mathfrak{p}^e \cdot \mathfrak{A}^p$, $e \geq 0$, $\mathfrak{A} \in I$. We shall have to take the θ or θ^* -components for each object considered in $K^\times \otimes \mathbb{Z}_p$, $I \otimes \mathbb{Z}_p \dots$, modulo p th powers:

Let θ be even. The Kummer radical of the compositum of the cyclic extensions of degree p of K , contained in $H_{\theta^*}^{\text{pr}}$, is generated (modulo $K^{\times p}$) by the part E_θ of real units, giving a p -rank 1 for $\theta \neq 1$ (and 0 for $\theta = 1$), by p (of character 1), and by the pseudo-units α_θ coming from the elements of order p of \mathcal{C}_θ , which gives a radical of p -rank $1 + \text{rk}_p(\mathcal{C}_\theta)$. Since $\text{rk}_p(\text{Gal}(H_{\theta^*}^{\text{pr}}/K)) = 1 + \text{rk}_p(\mathcal{T}_{\theta^*})$, we get $\text{rk}_p(\mathcal{T}_{\theta^*}) = \text{rk}_p(\mathcal{C}_\theta)$. Similarly, we have $\text{rk}_p(\mathcal{T}_\theta) = \text{rk}_p(\mathcal{C}_{\theta^*})$. \square

Corollary 3.2. *One has $\mathcal{T}_1 = \mathcal{T}_\omega = \mathcal{C}_\omega = \mathcal{C}_1 = 1$ and for all $\chi \in \mathcal{X}_+$, we have $\mathcal{R}_{\chi^*} = 1$ and $\mathcal{T}_{\chi^*} = \tilde{\mathcal{C}}_{\chi^*} \subseteq \mathcal{C}_{\chi^*}$, which establishes the Hecke reflection theorem or Leopoldt spiegelungssatz $\text{rk}_p(\mathcal{C}_{\chi^*}) = \text{rk}_p(\mathcal{C}_\chi) + \delta_\chi$, $\delta_\chi \in \{0, 1\}$ since $\Gamma_{\chi^*} \simeq \mathbb{Z}_p$ (particular case of [15, Theorem II.5.4.5, 5.4.9.2]).*

Remark 3.3. (i) One says that K is p -rational if $\mathcal{T} = 1$ (same definition for any number field fulfilling the Leopoldt conjecture at p ; see [17, 21] for more details and programs testing the p -rationality of any number field). For the p th cyclotomic field K this is equivalent to its “ p -regularity” in the more general context of “regular kernel” given in [12, Théorème 4.1] ($\mathcal{T}_- = 1$ may be interpreted as the conjectural “relative p -rationality” of K).

(ii) As we have seen, at each unramified cyclic extension L_+ of degree p of K_+ is associated a p -primary pseudo-unit $\alpha \in (K^\times/K^{\times p})_-$ such that $L_+K = K(\sqrt[p]{\alpha})$. Put $(\alpha) = \mathfrak{A}^p$, where $\mathcal{C}(\mathfrak{A}) \in \mathcal{C}_-$; moreover \mathfrak{A} is not principal, otherwise α should be, up to a p th power factor, a unit ε such that $\varepsilon^{1+s-1} = 1$, which gives $\varepsilon \in \mu_p$ (absurd). In the same way, if G operates via χ on $\text{Gal}(L_+/K_+)$ then by Kummer duality G operates via χ^* on $\langle \alpha \rangle K^{\times p}/K^{\times p}$.

(iii) As explained in the Introduction, we shall prove in Section 4 that such pseudo-units α may be found by means of twists $\mathfrak{g}_c(\ell) := \tau(\psi)^{c-\sigma_c}$ associated to primes $\ell \equiv 1 \pmod{p}$ and Artin automorphisms σ_c .

3.2. Vandiver’s conjecture and Gauss sums. Recall the formula [15, Corollary III.2.6.1]:

$$\#\mathcal{T}_- = \frac{\#\mathcal{C}_-}{\#(\mathbb{Z}_p \log(I) / \mathbb{Z}_p \log(U))_-},$$

where I is the group of prime to p ideals of K and $U = 1 + \varpi \mathbb{Z}_p[\varpi]$. For any $\mathfrak{A} \in I$, let $m \geq 1$ be such that $\mathfrak{A}^m = (\alpha)$, then $\log(\mathfrak{A}) := \frac{1}{m} \log(\alpha)$ where \log is the p -adic logarithm; taking the minus parts, $\log(\mathfrak{A})$ becomes well-defined since $\mathbb{Q}_p \log(E)_- = 0$. We obtain:

$$(2) \quad \#\mathcal{T}_{\chi^*} = \frac{\#\mathcal{C}_{\chi^*}}{\#(\mathbb{Z}_p \log(I) / \mathbb{Z}_p \log(U))_{\chi^*}}, \text{ for all } \chi =: \omega^n \in \mathcal{X}_+$$

The following reasoning (from [14, § 3]) gives another interpretation of the result of Iwasawa [28]. Consider the Stickelberger element $S := \frac{1}{p} \sum_{a=1}^{p-1} a s_a^{-1} \in \mathbb{Q}[G]$; it is such that $S \cdot e_{\chi^*} = B_{1, (\chi^*)^{-1}} \cdot e_{\chi^*} \in \mathbb{Z}_p[G]$ for all $\chi \in \mathcal{X}_+$; then $\chi^* = \omega^{p-n}$ for which $\#\mathcal{C}_{\chi^*}$ corresponds to the ordinary Berrnoulli numbers B_n giving the “exponents of p -irregularity” n for $B_n \equiv 0 \pmod{p}$ (see Definitions 1.3 (vii)).

Let ℓ be a prime number totally split in K (thus $\ell \equiv 1 \pmod{p}$). Let ψ be a character of order p of \mathbb{F}_ℓ^\times . We define the Gauss sum (where ξ_ℓ is a primitive ℓ th root of unity):

$$(3) \quad \tau(\psi) := - \sum_{x \in \mathbb{F}_\ell^\times} \psi(x) \xi_\ell^x \in \mathbb{Z}[\mu_{p\ell}].$$

Lemma 3.4. *We have $\tau(\psi)^{\sigma_a} = \psi(a)^{-a} \tau(\psi^a)$, where σ_a is the Artin automorphism attached to a in $\text{Gal}(\mathbb{Q}(\mu_{p\ell})/\mathbb{Q})$, and $\tau(\psi)^p \in \mathbb{Z}[\zeta_p]$; then $\tau(\psi) \equiv 1 \pmod{\mathfrak{p}\mathbb{Z}[\mu_{p\ell}]}$.*

Proof. By definition of σ_a , one has $\tau(\psi)^{\sigma_a} = - \sum_{x \in \mathbb{F}_\ell^\times} \psi(x)^a \xi_\ell^{ax} = -\psi^a(a^{-1}) \sum_{y \in \mathbb{F}_\ell^\times} \psi^a(y) \xi_\ell^y$;

whence the second claim taking $\sigma_a \in \text{Gal}(\mathbb{Q}(\mu_{p\ell})/K)$ (i.e., $a \equiv 1 \pmod{p}$).

Then $\tau(\psi) \equiv - \sum_{x \in \mathbb{F}_\ell^\times} \xi_\ell^x \pmod{\mathfrak{p}\mathbb{Z}[\mu_{p\ell}]}$; since ℓ is prime, $\sum_{x \in \mathbb{F}_\ell^\times} \xi_\ell^x = -1$. \square

We then have the fundamental classical relation in K (see [52, §§ 6.1, 6.2, 15.1]):

$$(4) \quad \mathfrak{L}^{pS} = \tau(\psi)^p \mathbb{Z}[\zeta_p],$$

for $\mathfrak{L} \mid \ell$ such that ψ is defined on the multiplicative group of $\mathbb{Z}[\zeta_p]/\mathfrak{L} \simeq \mathbb{F}_\ell$.

Remark 3.5. (i) Since various choices of $\mathfrak{L} \mid \ell$, ξ_ℓ and ψ , from a given ℓ , correspond to Galois conjugations and/or products by a p th root of unity, we denote simply $\tau(\psi)$ such a Gauss sum, where ψ is for instance the canonical character of order p ; for convenience, we shall have in mind that ℓ defines such a $\tau(\psi)$ (and some other objects) in an obvious way. One verifies that the forthcoming properties (p -primaryities, Kummer radicals...) do not depend on these choices especially because of the action of the θ -components.

(ii) If we consider $\alpha := \tau(\psi)^p \in K^\times$ as the Kummer radical of the cyclic extension $M_\ell := K(\tau(\psi))$ of K , we have $\alpha^{c^{-sc}} =: g_c(\ell)^p$, where $g_c(\ell) := \tau(\psi)^{c-\sigma_c} \in K^\times$; which gives $M_\ell = K(\sqrt[p]{\alpha}) = F_\ell K$, where F_ℓ is the subfield of $\mathbb{Q}(\mu_\ell)$ of degree p (the character of $\langle \alpha \rangle K^{\times p}/K^{\times p}$ is ω and that of $\text{Gal}(M_\ell/K)$ is 1). Thus p is unramified in M_ℓ/K (which is coherent with $\tau(\psi) \equiv 1 \pmod{\mathfrak{p}\mathbb{Z}_p[\mu_{p\ell}]}$ implying $\tau(\psi)^p \equiv 1 \pmod{\mathfrak{p}^p}$); it splits if and only if $\tau(\psi)^p \equiv 1 \pmod{\mathfrak{p}^{p+1}}$.

Taking the logarithms in (4), we obtain, for all $\chi \in \mathcal{X}_+$:

$$(S \cdot e_{\chi^*}) \cdot \log(\mathfrak{L}) = B_{1, (\chi^*)^{-1}} \cdot \log(\mathfrak{L}) \cdot e_{\chi^*} = \log(\tau(\psi)) \cdot e_{\chi^*},$$

where $\log(\tau(\psi)) := \frac{1}{p} \log(\tau(\psi)^p) \in \mathbb{Z}_p[\varpi]$. Put $B_{1, (\chi^*)^{-1}} \sim p^e$, $e \geq 1$, where \sim means equality up to a p -adic unit. Then $p^e \mathbb{Z}_p \log(\mathfrak{L}) \cdot e_{\chi^*} = \mathbb{Z}_p \log(\tau(\psi)) \cdot e_{\chi^*}$, thus, from (2), since I/P may be represented by prime ideals of degree 1:

$$(5) \quad \#\mathcal{T}_{\chi^*} = \frac{p^e}{\#(\mathbb{Z}_p \log(\mathcal{G})/p^e \log(U))_{\chi^*}},$$

where \mathcal{G} is the group generated by all the previous Gauss sums.

So, the ‘‘Vandiver conjecture at $\chi \in \mathcal{X}_+$ ’’ is equivalent to $(\mathbb{Z}_p \log(\mathcal{G})/\log(U))_{\chi^*} = 1$, and is, as expected, obviously fulfilled if $e = 0$. The whole Vandiver conjecture is equivalent to the fact that the images of the Gauss sums in U generate the minus part of this \mathbb{Z}_p -module giving again Iwasawa’s result [28].

We shall from now make the following working hypothesis which corresponds to the more subtle case for testing Vandiver’s conjecture with Theorems 4.7, 4.9 (or Theorem 1.2), the case where some \mathcal{A}_{χ^*} are not cyclic being obvious for all the forthcoming statements, as soon as one knows that $B_{1, (\chi^*)^{-1}} \sim p^e$ gives the order of \mathcal{A}_{χ^*} thus its annihilation and identities of the form $\mathfrak{a}^{p^e} = (\beta^p)$, $\beta \in K^\times$. So, this will give $\mathcal{E}_\ell(p) \cap \mathcal{E}_0(p) \neq \emptyset$ for all $\ell \equiv 1 \pmod{p}$ (see § 4.2):

Hypothesis 3.6 (Cyclicity hypothesis). *We assume that, for all $\chi \in \mathcal{X}_+$, the component \mathcal{C}_{χ^*} of the p -class group is cyclic (which implies the cyclicity of \mathcal{C}_χ); in other words, we restrict ourselves to the case where \mathcal{C} is $\mathbb{Z}_p[G]$ -monogenous (cf. Definition 1.3 (viii)), giving $\text{rk}_p(\mathcal{C}_-) = i(p)$.*

3.3. Vandiver's conjecture and ray class group modulo (p). Assume the Hypothesis 3.6 and let $\chi = \omega^n \in \mathcal{X}_+$ be such that $B_{1,(\chi^*)^{-1}} \sim p^e$, $e \geq 1$ (i.e., $\mathcal{C}_{\chi^*} \simeq \mathbb{Z}/p^e\mathbb{Z}$); thus, from (5), we have $\mathcal{T}_{\chi^*} = 1$ (i.e., $\mathcal{C}_\chi = 1$) if and only if there exists a prime number $\ell \equiv 1 \pmod{p}$ such that the corresponding $\log(\tau(\psi)_{\chi^*})$ generates $\log(U_{\chi^*}) = \log(1 + \varpi^{p-n}\mathbb{Z}_p[\varpi]) = \varpi^{p-n}\mathbb{Z}_p[\varpi]$ (Proposition 2.4), which indicates analytically the non- p -primarity of $\tau(\psi)_{\chi^*}$ in $\mathbb{Z}[\zeta_p]$ since $n > 1$.

There is also the fact that the Gauss sums (or the $g_c(\ell)$), considered modulo p th powers and computed modulo p , are indexed by infinitely many ℓ ; in other words there are some non-obvious large periodicities in the results as ℓ varies since numerical data are finite in number.

This may be explained as follows (giving also an interesting criterion which will imply new heuristics):

Theorem 3.7. *Let $\mathcal{C}^{(p)}$ be the p -subgroup of the ray class group $I/\{(x), x \equiv 1 \pmod{p}\}$ of modulus $p\mathbb{Z}[\zeta_p]$. Then for any $\chi \in \mathcal{X}_+$, we have (under the Hypothesis 3.6) the following properties:*

(i) $\#\mathcal{C}_{\chi^*}^{(p)} = p \cdot \#\mathcal{C}_{\chi^*}$.

(ii) *The condition $\mathcal{C}_\chi = 1$ is equivalent to the cyclicity of $\mathcal{C}_{\chi^*}^{(p)}$.*

Proof. Let $V := \{x \in K^\times, x \equiv 1 \pmod{p}\}$ and $W := \{x \in K^\times, x \equiv 1 \pmod{p}\}$. Since $E_{\chi^*} = 1$, we have the exact sequence (using Proposition 2.4):

$$1 \rightarrow (V/W)_{\chi^*} \simeq \mathbb{F}_p \longrightarrow \mathcal{C}_{\chi^*}^{(p)} \longrightarrow \mathcal{C}_{\chi^*} \rightarrow 1,$$

giving (i). The statement (ii) is obvious if $\mathcal{C}_{\chi^*} = 1$. Suppose $\#\mathcal{C}_{\chi^*} = p^e$, with $e \geq 1$.

Then $\mathcal{C}_\chi = 1$ implies $\mathcal{T}_{\chi^*} = 1$ (from Theorem 3.1) which implies $\mathcal{C}_{\chi^*}^{(p)} \simeq \mathbb{Z}/p^{e+1}\mathbb{Z}$: indeed, the χ^* -part $H_{\chi^*}^{\text{pr}}/K$ of the pro- p -extension H^{pr}/K is a \mathbb{Z}_p -extension, thus the p -ray class field corresponding to $\mathcal{C}_{\chi^*}^{(p)}$, contained in $H_{\chi^*}^{\text{pr}}$, is a cyclic extension of K .

Reciprocally, if $\mathcal{C}_{\chi^*}^{(p)} \simeq \mathbb{Z}/p^{e+1}\mathbb{Z}$, $e \geq 1$ (thus $\mathcal{C}_{\chi^*} \simeq \mathbb{Z}/p^e\mathbb{Z}$), there exists \mathfrak{A} (whose class generates $\mathcal{C}_{\chi^*}^{(p)}$) such that $\mathfrak{A}_{\chi^*}^{p^e} = (\alpha_{\chi^*})$ (where α_{χ^*} is unique up to a p th power since $E_{\chi^*} = 1$) with $\alpha_{\chi^*} \equiv 1 \pmod{p^{p-n}}$ ($\chi =: \omega^n$, $n \in [2, p-3]$ even), but $\alpha_{\chi^*} \not\equiv 1 \pmod{p}$. Note that $\text{rk}_p(\mathcal{T}_\chi) = \text{rk}_p(\mathcal{C}_{\chi^*}) = 1$. Thus α_{χ^*} defines the radical of the unique p -ramified (but not unramified) cyclic extension of degree p of K decomposed over K_+ into L_+/K_+ and contained in H_χ^{pr} (its Galois group is a quotient of order p of the *cyclic group* \mathcal{T}_χ since $\Gamma_\chi = 1$ for an even $\chi \neq 1$); thus $\mathcal{C}_\chi = 1$. \square

4. Twists of Gauss sums associated to primes $\ell \equiv 1 \pmod{p}$

Let \mathcal{L}_p be the set of primes ℓ totally split in K (namely, $\ell \equiv 1 \pmod{p}$). For $\ell \in \mathcal{L}_p$, let $\psi : \mathbb{F}_\ell^\times \rightarrow \mu_p$ be a multiplicative character of order p ; if g is a primitive root modulo ℓ , we put $\psi(g \pmod{\ell}) = \zeta_p$. Let ξ_ℓ be a primitive ℓ -th root of unity; then the Gauss sum associated to ℓ may be written in $\mathbb{Z}[\mu_p\ell]$:

$$(6) \quad \tau(\psi) := - \sum_{x \in \mathbb{F}_\ell^\times} \psi(x) \cdot \xi_\ell^x = - \sum_{k=0}^{\ell-2} \zeta_p^k \cdot \xi_\ell^{g^k}.$$

4.1. Computation and properties of the twists $g_c(\ell) := \tau(\psi)^{c-\sigma_c}$. Let $c \in [2, p-2]$ be a primitive root modulo p ; to get an integer of K (a PARI/GP program in $\mathbb{Z}[\mu_p]$ overflows as ℓ increases, even if $\tau(\psi)_{\chi^*} = \tau(\psi)^{e'_{\chi^*}}$ makes sense in $\mathbb{Z}[\zeta_p]$, a posteriori), one uses the twist $\tau(\psi)^{c-\sigma_c}$, where σ_c is the Artin automorphism attached to c in $\text{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$. We define for $\ell \in \mathcal{L}_p$ (cf. Lemma 3.4):

$$(7) \quad g_c(\ell) := \tau(\psi)^{c-\sigma_c} \in \mathbb{Z}[\zeta_p] \quad (\text{see formulas (3), (4) and Remark 3.5}).$$

giving for all $\chi \in \mathcal{X}_+$, up to $K^{\times p}$ for the generators of ideals:

$$\mathfrak{L}^{S_c} = g_c(\ell) \mathbb{Z}[\zeta_p] \quad \& \quad \mathfrak{L}_{\chi^*}^{(c-\chi^*(s_c)) \cdot B_{1,(\chi^*)^{-1}}} = g_c(\ell)_{\chi^*} \mathbb{Z}[\zeta_p]$$

(see Definitions 1.3), where $\mathfrak{L} \mid \ell$ in K , $S_c := (c - s_c) \cdot S \in \mathbb{Z}[G]$ is the corresponding twist of the Stickelberger element and where $g_c(\ell) \in \mathbb{Z}[\zeta_p]$. Put:

$$(8) \quad b_c(\chi^*) := (c - \chi^*(s_c)) \cdot B_{1,(\chi^*)^{-1}} \sim B_{1,(\chi^*)^{-1}}, \quad \text{for all } \chi \in \mathcal{X}_+.$$

Then we obtain the main relation that will be of a constant use:

$$(9) \quad \mathfrak{L}_{\chi^*}^{b_c(\chi^*)} = g_c(\ell)_{\chi^*} \mathbb{Z}[\zeta_p].$$

Remark 4.1. (i) In the above definition (7) of $g_c(\ell)$, $\tau(\psi)^{\sigma_c} = \tau(\psi^c) \cdot \psi^{-c}(c)$ (Lemma 3.4); but for all $\chi \neq 1$, $\mu_p^{e_{\chi^*}} = 1$, defining $g_c(\ell)_{\chi^*}$ without ambiguity up to $K^{\times p}$, which does not change the p -primarity properties. But in some sense the best definition of the twists should be $\psi^{-c}(c) \cdot g_c(\ell) = \psi^{-c}(c) \cdot \tau(\psi)^{c-\sigma_c}$.

(ii) Note that, since $\tau(\psi)^{1+s-1} = \ell$, this yields $g_c(\ell)_{\chi} \in K^{\times p}$ for all $\chi \in \mathcal{X}_+$.

Lemma 4.2. *Let $\ell \in \mathcal{L}_p$ be given. Then $\psi^{-c}(c) \cdot g_c(\ell)$ is a product of Jacobi sums and $\psi^{-c}(c) \cdot g_c(\ell) \equiv g_c(\ell) \equiv 1 \pmod{\mathfrak{p}}$.*

Proof. The classical formula [52, § 6.1] for Jacobi sums (with $\psi \psi' \neq 1$) is:

$$J(\psi, \psi') := \tau(\psi) \cdot \tau(\psi') \cdot \tau(\psi \psi')^{-1} = - \sum_{x \in \mathbb{F}_\ell \setminus \{0,1\}} \psi(x) \cdot \psi'(1-x).$$

Whence $\tau(\psi)^c = J_1 \cdots J_{c-1} \cdot \tau(\psi^c)$, where $J_i = - \sum_{x \in \mathbb{F}_\ell \setminus \{0,1\}} \psi^i(x) \cdot \psi(1-x)$, thus:

$$\tau(\psi)^{c-\sigma_c} = J_1 \cdots J_{c-1} \cdot \tau(\psi^c) \tau(\psi)^{-\sigma_c} = J_1 \cdots J_{c-1} \cdot \psi^c(c),$$

from Lemma 3.4; then $\tau(\psi) \equiv 1 \pmod{\mathfrak{p} \mathbb{Z}[\mu_p]}$ implies the result for $g_c(\ell)$. \square

Thus, in the numerical computations, we shall use the relation:

$$(10) \quad g_c(\ell)_{\chi^*} = (J_1 \cdots J_{c-1})_{\chi^*} \text{ for any } \chi \in \mathcal{X}_+.$$

The following definitions will be of constant use in the paper:

Definition 4.3 (exponents of p -primarity and p -irregularity). (i) We call set of exponents of p -primarity, of a prime $\ell \in \mathcal{L}_p$, the set $\mathcal{E}_\ell(p)$ of even integers $n \in [2, p-3]$ such that $g_c(\ell)_{\omega^{p-n}}$ is p -primary, thus $g_c(\ell)_{\omega^{p-n}} \equiv 1 \pmod{p}$ (Definition 2.1 (ii), Proposition 2.4).

(ii) We call set of exponents of p -irregularity, the set $\mathcal{E}_0(p)$ of even integers $n \in [2, p-3]$ such that $B_n \equiv 0 \pmod{p}$, thus, $B_{1,\omega^{n-1}} \equiv 0 \pmod{p}$ (see Definitions 1.3 (vii)).

Remark 4.4. Let $\chi =: \omega^n \in \mathcal{X}_+$ and $\ell \in \mathcal{L}_p$. If $g_c(\ell)_{\chi^*}$ is p -primary ($n \in \mathcal{E}_\ell(p)$) this does not give necessarily a counterexample to Vandiver's conjecture for the two following possible reasons considering $S_c e_{\chi^*} = b_c(\chi^*) e_{\chi^*}$; recall that from (8),

$$b_c(\chi^*) = (c - \chi^*(s_c)) \cdot B_{1,(\chi^*)^{-1}} \sim B_{1,(\chi^*)^{-1}} = B_{1,\omega^{n-1}}.$$

(i) The number $b_c(\chi^*)$ is a p -adic unit ($n \notin \mathcal{E}_0(p)$), so the radical $\mathfrak{g}_c(\ell)_{\chi^*}$ is not the p th power of an ideal (thus not a pseudo-unit, even if Proposition 2.4 applies) and leads to a cyclic ℓ -ramified Kummer extension of degree p of K_+ .

For instance, for $p = 11$ ($c = 2$), $\ell = 23$, the exponent of 11-primarity is $n = 2$ so that $\alpha := \mathfrak{g}_c(\ell)_{\chi^*}$ is the integer (where $x = \zeta_{11}$):

```
-8491773970656065727678427465045288222*x^9-1963231019856677733688722439078492228*x^8
+11757523232198873159205810348854526320*x^7-5860674150310922200348907606983566648*x^6
-644088006192816851608142123579276962*x^5-611074014289231284308386817199658010*x^4
+2673005955545675004066087284224877298*x^3+15023028737838809151251842166615658188*x^2
+1520229819300797188419125563036321734*x+17836238554732163868933693789025679469
```

for which $K(\sqrt[11]{\alpha})/K$ is decomposed over K_+ into L_+/K_+ , ℓ -ramified; then (α) is a product of prime ideals above ℓ ($s = s_2$): $(\alpha) = \mathfrak{L}^{1+2s+2^2s^2+2^3s^3+2^4s^4+2^5s^5+2^6s^6+2^7s^7+2^8s^8+2^9s^9}$, up to the 11th power of an ℓ -ideal. We get $N_{K/\mathbb{Q}}(\alpha) = \ell^{275}$ and $N_{K/\mathbb{Q}}(\alpha - 1) \sim 11^{13}$. In fact the program gives $(\alpha) = \mathfrak{L}_1^{25} \cdot \mathfrak{L}_2^{27} \cdot \mathfrak{L}_3^{31} \cdot \mathfrak{L}_4^{24} \cdot \mathfrak{L}_5^{28} \cdot \mathfrak{L}_6^{15} \cdot \mathfrak{L}_7^{30} \cdot \mathfrak{L}_8^{23} \cdot \mathfrak{L}_9^{32} \cdot \mathfrak{L}_{10}^{40}$ and one must discover the significance given above ! Here $b_c(\chi^*) \equiv 1 \pmod{11}$.

(ii) The number $b_c(\chi^*)$ is divisible by p , but the ideal \mathfrak{L}_{χ^*} is p -principal and then $\mathfrak{g}_c(\ell)_{\chi^*}$ is a p th power in K^\times (numerical examples in § 4.4.1).

4.2. First main theorem. So, from the previous Remark 4.4, a *sufficient condition for the existence of a counterexample* to Vandiver's conjecture is the existence of $\chi \in \mathcal{X}_+$ and $\ell \in \mathcal{L}_p$ such that the three following conditions are fulfilled:

- (a) $b_c(\chi^*) \equiv 0 \pmod{p}$,
- (b) $\mathfrak{g}_c(\ell)_{\chi^*}$ is p -primary,
- (c) $\mathfrak{g}_c(\ell)_{\chi^*}$ is not a global p th power.

We make here a fundamental remark:

Remark 4.5. If $\text{rk}_p(\mathcal{C}_{\chi_0^*}) \geq 2$ for $\chi_0 = \omega^{n_0} \in \mathcal{X}_+$ (giving a counterexample to Vandiver's conjecture), we get, from the "Main Theorem", $\#\mathcal{C}_{\chi_0^*} \sim b_c(\chi_0^*)$; then the p -part of $b_c(\chi_0^*)$ is strictly larger than the exponent of $\mathcal{C}_{\chi_0^*}$ so that, in any relation $\mathfrak{L}_{\chi_0^*}^{b_c(\chi_0^*)} = (\mathfrak{g}_c(\ell)_{\chi_0^*})$ where $\mathfrak{L}_{\chi_0^*}$ define a generating class of $\mathcal{C}_{\chi_0^*}$, necessarily $\mathfrak{g}_c(\ell)_{\chi_0^*}$ is a global p th power (condition (c) is never fulfilled), whence the property $n_0 \in \mathcal{E}_\ell(p) \cap \mathcal{E}_0(p) \neq \emptyset$ for all $\ell \in \mathcal{L}_p$; thus Theorems 4.7 and 4.9 will apply for trivial reasons and we can go back to the cases $\text{rk}_p(\mathcal{C}_{\chi^*}) < 2$ (Hypothesis 3.6) for the reciprocal.

Lemma 4.6. *Let $\chi \in \mathcal{X}_+$ such that $\mathcal{C}_\chi \neq 1$. There exists a totally split prime ideal \mathfrak{L} such that \mathfrak{L}_{χ^*} represents a generator of \mathcal{C}_{χ^*} . Then $\mathfrak{L}^{S_{c \cdot e_{\chi^*}}} = \mathfrak{L}_{\chi^*}^{b_c(\chi^*)} = (\alpha_{\chi^*})$, where α_{χ^*} is unique (up to a p th power), thus equal to $\mathfrak{g}_c(\ell)_{\chi^*}$ which is p -primary and not a global p th power.*

Proof. From the Chebotarev density theorem in H/\mathbb{Q} , there exists a prime ℓ and $\overline{\mathfrak{L}} \mid \ell$ in H such that the Frobenius $(\frac{H/\mathbb{Q}}{\overline{\mathfrak{L}}})$ generates the subgroup of $\text{Gal}(H/K)$ corresponding to \mathcal{C}_{χ^*} by class field theory. So ℓ splits completely in K/\mathbb{Q} ($\ell \in \mathcal{L}_p$) and the ideal \mathfrak{L} of K under $\overline{\mathfrak{L}}$ is (as \mathfrak{L}_{χ^*}) a representative of a *generator* of $\mathcal{C}_{\chi^*} \simeq \mathbb{Z}_p/b_c(\chi^*)\mathbb{Z}_p$. Then $\mathfrak{L}_{\chi^*}^{b_c(\chi^*)} = (\alpha_{\chi^*})$ where $\alpha_{\chi^*} \notin K^{\times p}$; α_{χ^*} is unique since $E_{\chi^*} = 1$ for $\chi^* \neq \omega$. In terms of Gauss sums, $\mathfrak{L}_{\chi^*}^{b_c(\chi^*)} = (\mathfrak{g}_c(\ell)_{\chi^*})$, thus $\alpha_{\chi^*} = \mathfrak{g}_c(\ell)_{\chi^*}$. The p -primarity of α_{χ^*} is necessary to obtain the *unique* (still thanks to Hypothesis 3.6) unramified Kummer extension $K(\sqrt[p]{\alpha_{\chi^*}})/K$ of degree p , decomposed over K_+ into the unramified extension L_+/K_+ of degree p in H_χ , associated to $\mathcal{C}_\chi/\mathcal{C}_\chi^p$ by class field theory, whence the p -primarity of $\mathfrak{g}_c(\ell)_{\chi^*}$. \square

Drawing the consequences of the above, we get, unconditionally, the main test for Vandiver's conjecture stated in the Introduction (Theorem 1.2(a)). We refer to the relations (7), (8), (9) and the Definition 4.3.

Theorem 4.7. *Vandiver's conjecture holds for $K = \mathbb{Q}(\mu_p)$ if and only if there exists $\ell \equiv 1 \pmod{p}$ such that $\mathcal{E}_\ell(p) \cap \mathcal{E}_0(p) = \emptyset$.*

Proof. As explained in the Remark 4.5, we may assume the cyclicity Hypothesis 3.6.

Suppose $\mathcal{E}_\ell(p) \cap \mathcal{E}_0(p) = \emptyset$ and consider, for $\chi =: \omega^n \in \mathcal{X}_+$, and $\chi^* = \omega^{p-n}$, the relation $\mathfrak{L}_{\chi^*}^{b_c(\chi^*)} = (\text{g}_c(\ell)_{\chi^*})$ for the prime ℓ under consideration, and examine the two possibilities:

(i) If n is not an exponent of p -irregularity (namely, $b_c(\chi^*) \not\equiv 0 \pmod{p}$ or $B_n \not\equiv 0 \pmod{p}$), then $\mathcal{C}_{\chi^*} = 1$ and $\mathcal{C}_\chi = 1$ from reflection theorem (Corollary 3.2).

(ii) If n is an exponent of p -irregularity, then $b_c(\chi^*) \sim p^e$, $e \geq 1$, giving, for some p -adic unit u , $\mathfrak{L}_{\chi^*}^{p^e u} = (\text{g}_c(\ell)_{\chi^*})$ (Lemma 4.6); if $\mathfrak{L}_{\chi^*}^{p^{e-1}u}$ is p -principal, then $\text{g}_c(\ell)_{\chi^*}$ is a global p th power, hence p -primary (absurd by assumption). So \mathfrak{L}_{χ^*} defines a class of order p^e in \mathcal{C}_{χ^*} for which the pseudo-unit $\text{g}_c(\ell)_{\chi^*}$ is not p -primary by assumption; since $\text{Gal}(H_\chi^{\text{pr}}/K_+) = \mathcal{T}_\chi$ is cyclic, from relation (3.1), by Kummer duality $K(\sqrt[p]{\text{g}_c(\ell)_{\chi^*}})$ is the unique extension cyclic of degree p , decomposed over K_+ and contained in H_χ^{pr} . Since it is ramified at p and since H_χ^{pr} contains the χ -component of the p -Hilbert class field of K_+ , this implies $\mathcal{C}_\chi = 1$.

Reciprocally, if Vandiver's conjecture holds, then $\mathcal{C} = \mathcal{C}_-$ is $\mathbb{Z}_p[G]$ -monogenous, thus the direct sum of non-trivial cyclic isotypic components generated by some p -classes $\gamma^{(n_i)} = \mathcal{C}(\mathfrak{L}_{\omega^{p-n_i}}^{(n_i)}) \in \mathcal{C}_{\omega^{p-n_i}}$ ($n_i \in \mathcal{E}_0(p)$) related to non- p -primary $\text{g}_c(\ell^{(n_i)})_{\omega^{p-n_i}}$; thus there exists, from density theorem, $\ell \in \mathcal{L}_p$ and $\mathfrak{L} \mid \ell$ such that $\mathcal{C}(\mathfrak{L})_{\omega^{p-n_i}} = \gamma^{(n_i)}$ for all i (e.g., $\mathfrak{L} = (z) \cdot \prod_i \mathfrak{L}_{\omega^{p-n_i}}^{(n_i)}$). So each $\text{g}_c(\ell)_{\omega^{p-n_i}} = \text{g}_c(\ell^{(n_i)})_{\omega^{p-n_i}}$ (up to a p th power) is non- p -primary, whence $\mathcal{E}_\ell(p) \cap \mathcal{E}_0(p) = \emptyset$ for this prime ℓ . \square

Corollary 4.8. *Let $\ell \in \mathcal{L}_p$. If, for all $\chi \in \mathcal{X}_+$, the numbers $\text{g}_c(\ell)_{\chi^*}$ are not p -primary (i.e., $\mathcal{E}_\ell(p) = \emptyset$), then the Vandiver conjecture holds for p .*

4.2.1. Program computing $\mathcal{E}_\ell(p)$. For $p \in [3, 199]$ and for the least $\ell \in \mathcal{L}_p$, the program computes $\text{g}_c(\ell)$ in $\text{Mod}(\mathbb{J}, \mathbb{P})$, with $\mathbb{P} = \text{polcyclo}(p)$, where $\mathbb{J} = \mathbb{J}_1 \cdots \mathbb{J}_{c-1}$ is written in $\mathbb{Z}[x]$ modulo $p\mathbb{Z}[x]$; c is a primitive root \pmod{p} (see the relation (10)).

For the computation of \mathbb{J}_i we use the discrete logarithm `znlog` to interpret the $1 - g^k$ in $g^{\mathbb{Z}/(\ell-1)\mathbb{Z}}$. We put $\chi = \omega^n$ & $\chi^* = \omega^{1-n}$, taking $n = 2 * m$ for m in $[1, (p-3)/2]$.

The program takes into account the relation $J^{1+s-1} \equiv 1 \pmod{p}$ in the action of the idempotents and drops the coefficient $\frac{1}{p-1}$ in e_{χ^*} (in which $\chi^*(s_a^{-1})$ is replaced by the residue of a^{n-1} modulo p), thus computes in reality $\text{g}_c(\ell)^{-1/2}$ up to p th powers. Then the polynomials \mathbb{J}_j give, in the list `LJ`, the powers \mathbb{J}^j modulo p , $j = 1, \dots, p-1$.

The result is given in

$$\text{Sn} = \prod_{a=1}^{(p-1)/2} s_a(\mathbb{J}^{an}), \quad \text{from} \quad \text{g}_c(\ell)_{\chi^*}^{-1/2} = \prod_{a=1}^{(p-1)/2} s_a(\text{g}_c(\ell)^{\omega^{n-1}(a)})$$

(up to a p th power), where $\omega^{n-1}(a) \equiv a^{n-1} \pmod{p}$ is computed in `an` and \mathbb{J}^{an} is given by `component(LJ, an)`. The conjugate $s_a(\mathbb{J}^{an})$ is computed in `sJan` via the conjugation $x \mapsto x^a$ in \mathbb{J}^{an} , whence the product in `Sn` (the exponents of p -primarity are denoted `expp`):

Note: To copy and past the programs in verbatim text, one must perhaps replace the symbol of power (in `a^b`) by the PARI/GP symbol (= that of the keyboard); otherwise the program does not work (this is due to the character font used by some Journals).

```
{forprime(p=3,200,c=lift(znprimroot(p));P=polcyclo(p)+Mod(0,p);
X=Mod(x,P);el=1;while(isprime(el)==0,el=el+2*p);g=znprimroot(el);
print("p=",p," el=",el," c=",c," g=",g);J=1;for(i=1,c-1, Ji=0;
for(k=1,el-2, kk=znlog(1-g^k,g);e=lift(Mod(kk+i*k,p)); Ji=Ji-X^e); J=J*Ji);
LJ=List; Jj=1;for(j=1,p-1, Jj=lift(Jj*J));listinsert(LJ,Jj,j));
for(m=1,(p-3)/2,n=2*m; Sn=Mod(1,P);for(a=1,(p-1)/2,
an=lift(Mod(a,p)^(n-1)); Jan=component(LJ,an); sJan=Mod(0,P);
for(j=0,p-2, aj=lift(Mod(a*j,p)); sJan=sJan+x^(aj)*component(Jan,1+j));
Sn=Sn*sJan);if(Sn==1,print(" exponents of p-primarity: ",n))}}}
```

p=3	el=7	c=2	g=3		p=97	el=389	c=5	g=2	expp:26
p=5	el=11	c=2	g=2		p=101	el=607	c=2	g=3	expp:10
p=7	el=29	c=2	g=2		p=103	el=619	c=5	g=3	
p=11	el=23	c=3	g=5	expp:2	p=107	el=643	c=2	g=11	
p=13	el=53	c=2	g=2		p=109	el=1091	c=6	g=2	expp:14,86
p=17	el=103	c=3	g=5		p=113	el=227	c=3	g=2	
p=19	el=191	c=4	g=19		p=127	el=509	c=3	g=2	
p=23	el=47	c=2	g=5		p=131	el=263	c=2	g=5	expp:16
p=29	el=59	c=2	g=2	expp:2	p=137	el=823	c=3	g=3	expp:78
p=31	el=311	c=7	g=17		p=139	el=557	c=2	g=2	
p=37	el=149	c=2	g=2		p=149	el=1193	c=2	g=3	
p=41	el=83	c=6	g=2		p=151	el=907	c=6	g=2	
p=43	el=173	c=9	g=2	expp:26	p=157	el=1571	c=5	g=2	expp:94
p=47	el=283	c=2	g=3		p=163	el=653	c=2	g=2	expp:42
p=53	el=107	c=2	g=2	expp:34,10	p=167	el=2339	c=5	g=2	expp:122
p=59	el=709	c=3	g=2		p=173	el=347	c=2	g=2	
p=61	el=367	c=2	g=6		p=179	el=359	c=2	g=7	expp:138
p=67	el=269	c=4	g=2		p=181	el=1087	c=2	g=3	expp:114,164
p=71	el=569	c=2	g=3		p=191	el=383	c=19	g=5	
p=73	el=293	c=5	g=2		p=193	el=773	c=5	g=2	expp:108,172
p=79	el=317	c=2	g=2		p=197	el=3547	c=2	g=2	expp:62
p=83	el=167	c=3	g=5		p=199	el=797	c=3	g=2	
p=89	el=179	c=3	g=2						

4.2.2. Minimal prime $\ell \in \mathcal{L}_p$ such that $\mathcal{E}_\ell(p) = \emptyset$. The following program examines, for each p , the successive prime numbers $\ell_i \in \mathcal{L}_p$, $i \geq 1$, and returns the first one, ℓ_N (in \mathbb{L} with its index N), such that $\mathcal{E}_{\ell_N}(p) = \emptyset$. Its existence is of course a strong conjecture, but the numerical results are extremely favorable to the existence of such primes; which strengthens the conjecture of Vandiver. Moreover, since the integer $i(p) = \#\mathcal{E}_0(p)$ is rather small regarding p , as doubtless for $\#\mathcal{E}_\ell(p)$, and can be both in $O\left(\frac{\log(p)}{\log(\log(p))}\right)$, the intersection $\mathcal{E}_\ell(p) \cap \mathcal{E}_0(p)$ may be easily empty *if these sets are independent*. The experiments give the impression that the sets $\mathcal{E}_\ell(p)$ are random when ℓ varies and have no link with $\mathcal{E}_0(p)$.

```
{forprime(p=3,100,c=lift(znprimroot(p));P=polcyclo(p)+Mod(0,p);
N=0;T=1;el=1;while(T==1,el=el+2*p;if(isprime(el)==1,N=N+1;g=znprimroot(el);
J=Mod(1,P);for(i=1,c-1, Ji=0;for(k=1,el-2, kk=znlog(1-g^k,g);
e=lift(Mod(kk+i*k,p)); Ji=Ji-x^e); J=J*Ji);LJ=List; Jj=1;
for(j=1,p-1, Jj=lift(Jj*J));listinsert(LJ,Jj,j));T=0;for(m=1,(p-3)/2,n=2*m;
Sn=Mod(1,P);for(a=1,(p-1)/2, an=lift(Mod(a,p)^(n-1)); Jan=component(LJ,an);
sJan=0;for(j=0,p-2, aj=lift(Mod(a*j,p)); sJan=sJan+x^(aj)*component(Jan,1+j));
Sn=Sn*sJan);if(Sn==1,T=1;break);if(T==0,print(p," ",el," ",N);break))}}}
```

For $p < 400$, we only write the primes p, ℓ_N for which $N > 1$, then a complete list for $p \in [409, 683]$:

p	el	N	p	el	N	p	el	N	p	el	N	p	el	N
11	67	2	197	4729	2	409	4091	2	499	1997	1	601	25243	5

29	233	2	211	10973	4	419	839	1	503	3019	1	607	20639	3
43	431	2	223	6691	2	421	4211	1	509	4073	2	613	6131	1
53	743	2	227	5903	2	431	863	1	521	16673	1	617	30851	3
97	971	2	229	5039	2	433	5197	2	523	6277	2	619	17333	3
101	809	2	233	1399	2	439	4391	1	541	9739	1	631	6311	1
109	2399	2	251	4519	2	443	887	1	547	5471	1	641	1283	1
131	1049	3	277	4987	3	449	3593	1	557	24509	3	643	10289	2
137	1097	2	337	6067	3	457	21023	3	563	7883	1	647	9059	1
157	7537	5	349	8377	2	461	9221	2	569	6829	1	653	1307	1
163	5869	3	367	3671	2	463	5557	1	571	5711	1	659	1319	1
167	7349	3	383	16087	4	467	2803	1	577	3463	2	661	14543	3
179	1433	2	389	14783	2	479	3833	1	587	8219	1	673	2693	1
181	1811	2	397	6353	2	487	1949	1	593	1187	1	677	5417	1
193	1931	2	401	10427	4	491	983	1	599	4793	1	683	4099	2

The comparison with the table of exponents of p -irregularity does not show any relation.

4.3. Second main theorem. Let n_0 be an exponent of p -irregularity; put $\chi_0 = \omega^{n_0}$ and let $b_c(\chi_0^*) \sim p^e$, $e \geq 1$. If $\mathcal{C}_{\chi_0^*}$ is not cyclic, Remark 4.5 implies $n_0 \in \bigcap_{\ell \in \mathcal{L}_p} \mathcal{E}_\ell(p) \neq \emptyset$ and Theorem 4.9 will hold. Then we may assume $\mathcal{C}_{\chi_0^*} \simeq \mathbb{Z}/p^e\mathbb{Z}$. We shall examine what happens when $\ell \in \mathcal{L}_p$ varies.

Let $\ell \in \mathcal{L}_p$ and let $\mathfrak{L}_{\chi_0^*}$ with $\mathfrak{L} \mid \ell$. There are two cases as we have seen previously in the monogenous case:

(i) $\mathfrak{L}_{\chi_0^*}^{p^{e-1}}$ is p -principal. Since $b_c(\chi_0^*) \sim p^e$, $e \geq 1$, $g_c(\ell)_{\chi_0^*}$ is a global p th power in K^\times , whence $g_c(\ell)_{\chi_0^*}$ is p -primary and $n_0 \in \mathcal{E}_\ell(p)$, but this does not lead to an unramified cyclic extension of degree p of K_+ of character χ_0 ;

(ii) $\mathfrak{L}_{\chi_0^*}^{p^{e-1}}$ is not p -principal (from density theorem, such primes ℓ always exist). Thus it defines a generator of $\mathcal{C}_{\chi_0^*}$ and Vandiver's conjecture "holds for $\chi_0 = \omega^{n_0}$ " if and only if $g_c(\ell)_{\chi_0^*}$ is not p -primary (Theorem 4.7).

If $g_c(\ell)_{\chi_0^*} \equiv 1 \pmod{p}$ (counterexample to Vandiver's conjecture), we fix this ℓ once for all, and whatever the ideal $\mathfrak{L}' \mid \ell'$, $\ell' \in \mathcal{L}_p$, we have $\mathfrak{L}'_{\chi_0^*} = (z) \cdot \mathfrak{L}_{\chi_0^*}^r$, with $z \in K^\times$ and $r \in [0, p^e - 1]$, so:

$$\mathfrak{L}'_{\chi_0^*}{}^{p^e u} = (z^{p^e u}) \cdot \mathfrak{L}_{\chi_0^*}{}^{r p^e u} \quad \& \quad g_c(\ell')_{\chi_0^*} \equiv g_c(\ell)_{\chi_0^*}^r \equiv 1 \pmod{p}.$$

Whence, the exponent n_0 of p -irregularity is a common exponent of p -primarity for all $\ell \in \mathcal{L}_p$, giving $n_0 \in \mathcal{E}_0(p) \cap (\bigcap_{\ell \in \mathcal{L}_p} \mathcal{E}_\ell(p)) \neq \emptyset$. In other words, the existence of an empty intersection $\mathcal{E}_{\ell_1}(p) \cap \cdots \cap \mathcal{E}_{\ell_N}(p)$ implies Vandiver's conjecture. We shall also prove the reciprocal, that gives the new criterion:

Theorem 4.9. *Vandiver's conjecture holds if and only if there exist $N \geq 1$ and $\ell_1, \dots, \ell_N \in \mathcal{L}_p$ such that $\mathcal{E}_{\ell_1}(p) \cap \cdots \cap \mathcal{E}_{\ell_N}(p) = \emptyset$.*

Proof. It remains to prove that Vandiver's conjecture implies such an empty intersection. Assume, on the contrary, that for all $N \geq 1$ and all sets $\{\ell_1, \dots, \ell_N\} \subset \mathcal{L}_p$, one has $\mathcal{E}_{\ell_1}(p) \cap \cdots \cap \mathcal{E}_{\ell_N}(p) \neq \emptyset$.

Since \mathcal{X}_+ is finite, there exists such an n_0 in $\bigcap_{\ell \in \mathcal{L}_p} \mathcal{E}_\ell(p)$ (if $\bigcap_{\ell \in \mathcal{L}_p} \mathcal{E}_\ell(p) = \emptyset$ then for all even $n \in [2, p-3]$ there exists $\ell(n)$ such that $n \notin \mathcal{E}_{\ell(n)}(p)$ whence $\bigcap_{n \in [2, p-3] \text{ even}} \mathcal{E}_{\ell(n)}(p) = \emptyset$ (absurd)). This means that for the fixed character $\chi_0 := \omega^{n_0}$, we have the property:

$$g_c(\ell)_{\chi_0^*} \equiv 1 \pmod{p}, \quad \text{for all } \ell \in \mathcal{L}_p.$$

To simplify, put $\alpha(\ell) := g_c(\ell)_{\chi_0^*}$ and consider the extensions $K(\sqrt[p]{\alpha(\ell)})/K$; these extensions, with Galois groups of character χ_0 , are decomposed over K_+ into cyclic extensions $L_+(\ell)/K_+$

(possibly trivial), and are ℓ -ramified since $(\alpha(\ell)) = \mathfrak{L}_{\chi_0^*}^{b_c(\chi_0^*)}$ with $\alpha(\ell) \equiv 1 \pmod{p}$ (non-ramification at p). Examine the two possibilities about $b_c(\chi_0^*)$:

(i) $b_c(\chi_0^*) \equiv 0 \pmod{p}$. Then $\alpha(\ell)$ is, for all ℓ , a p -primary pseudo-unit, and choosing ℓ such that $\mathfrak{L}_{\chi_0^*}$ generates $\mathcal{C}_{\chi_0^*}$ (which is cyclic since $\mathcal{C}_{\chi_0} = 1$), the extension $L_+(\ell)/K_+$ is unramified of degree p (absurd).

(ii) $b_c(\chi_0^*) \not\equiv 0 \pmod{p}$. Then $L_+(\ell)/K_+$ is, for all ℓ , a ℓ -ramified degree p cyclic extension of character χ_0 . We restrict ourselves to primes $\ell \not\equiv 1 \pmod{p^2}$ and consider the p -ray class fields, $H_+(\ell)$ over K_+ , of modulus (ℓ) ; we have $L_+(\ell) \subseteq H_+(\ell)$. Since $\mathcal{C}_+ = 1$, $\text{Gal}(H_+(\ell)/K_+) \simeq (P_+/P_+(\ell)) \otimes \mathbb{Z}_p$, where P_+ is the group of principal ideals prime to ℓ of K_+ and $P_+(\ell)$ the subgroup of P_+ of ideals generated by an element $x \equiv 1 \pmod{\ell}$.

From the G -modules exact sequence $1 \rightarrow E_+/E_+(\ell) \rightarrow \bigoplus_{\mathfrak{L}_+|\ell} \mathbb{F}_\ell^\times \rightarrow P_+/P_+(\ell) \rightarrow 1$, where $E_+(\ell) := \{\varepsilon \in E_+, \varepsilon \equiv 1 \pmod{\ell}\}$, we get (for $\ell \not\equiv 1 \pmod{p^2}$):

$$1 \rightarrow (E_+/E_+(\ell))_{\chi_0} \rightarrow (\mathbb{Z}/p\mathbb{Z})_{\chi_0} \rightarrow \text{Gal}(H_+(\ell)/K_+)_{\chi_0} \rightarrow 1.$$

Since $\text{Gal}(H_+(\ell)/K_+)_{\chi_0}$ is, at least, of order p , the generating χ_0 -unit, $\varepsilon_{\chi_0} =: \varepsilon$, is in $E_+(\ell)_{\chi_0}$, thus locally a p th power at ℓ , for all $\ell \in \mathcal{L}_p$, $\ell \not\equiv 1 \pmod{p^2}$. Thus ℓ totally splits in $K(\sqrt[p]{\varepsilon})/K$. Let M be the compositum $K(\sqrt[p]{\varepsilon}) \cdot K_1$, where $K_1 = \mathbb{Q}(\mu_{p^2})$; this Galois field M only depends on p and χ_0 and the primes $\ell \not\equiv 1 \pmod{p^2}$ are inert in K_1/K . Then choose ℓ such that the decomposition group of $\mathcal{L} | \ell$ does not fix $K(\sqrt[p]{\varepsilon})$ (since $\text{Gal}(M/K) \simeq (\mathbb{Z}/p\mathbb{Z})^2$, this allows $p-1$ possibilities). Thus ℓ is inert in $K(\sqrt[p]{\varepsilon})/K$ (contradiction).

Whence the reciprocal. \square

Remark 4.10. This theorem suggests that if the sets $\mathcal{E}_\ell(p)$ are random when ℓ varies and independent, the (conjectural) triviality of \mathcal{C}_+ is a *consequence* of a natural p -adic property of Gauss sums and the statement does exist with $N = 1$.

On the contrary, the structure of \mathcal{C}_- is independent of the Gauss sums because the *even components* $g_c(\ell)_\chi$ are global p th powers for all $\ell \in \mathcal{L}_p$ (Remark 4.1 (ii)) and do not yield any obstruction ! Thus the cases of non-triviality of \mathcal{C}_- may follow standard probabilities under the monogenous case.

4.4. Study of the case $p = 37$. So it is fundamental to see if the sets $\mathcal{E}_\ell(p)$ are independent (or not) of the choice of $\ell \in \mathcal{L}_p$ for $\mathcal{E}_0(p) \neq \emptyset$. We analyse the case of $p = 37$ ($n_0 = 32$) giving $\#\mathcal{C}_{\omega^5} = 37$ and compute (in `expp`) the sets $\mathcal{E}_\ell(37)$ when $\ell \in \mathcal{L}_{37}$ varies. If $n_0 \in \mathcal{E}_\ell(37)$, this means that \mathfrak{L}_{χ^*} is necessarily 37-principal and then $g_c(\ell)_{\omega^5} \in K^{\times 37}$:

```
{p=37;c=lift(znprimroot(p));P=polcyclo(p)+Mod(0,p);X=Mod(x,P);
for(i=1,100,e1=1+2*i*p;if(isprime(e1)!=1,next);g=znprimroot(e1);
print("e1=",e1," g=",lift(g));J=1;for(i=1,c-1,Ji=0;for(k=1,e1-2,kk=znlog(1-g^k,g);
e=lift(Mod(kk+i*k,p));Ji=Ji-X^e);J=J*Ji);LJ=List;Jj=1;for(j=1,p-1,Jj=lift(Jj*J));
listinsert(LJ,Jj,j)};for(m=1,(p-3)/2,n=2*m;Sn=Mod(1,P);for(a=1,(p-1)/2,
an=lift(Mod(a,p)^(n-1));Jan=component(LJ,an);sJan=Mod(0,P);
for(j=0,p-2,aj=lift(Mod(a*j,p));sJan=sJan+x^(aj)*component(Jan,1+j));
Sn=Sn*sJan);if(Sn==1,print("      exponent of p-primarity: ",n))}}
```

e1=149	g=2		e1=3331	g=3	expp: 22
e1=223	g=3		e1=3701	g=2	
e1=593	g=3		e1=3923	g=2	
e1=1259	g=2		e1=4219	g=2	expp: 18,16
e1=1481	g=3	expp: 30	e1=4441	g=21	
e1=1777	g=5		e1=4663	g=3	
e1=1999	g=3		e1=5107	g=2	
e1=2221	g=2		e1=5477	g=2	

e1=2591	g=7	expp: 34	e1=6143	g=5	expp: 28
e1=2887	g=5		e1=6217	g=5	
e1=3109	g=6		e1=6661	g=6	
e1=3257	g=3		e1=6883	g=2	

e1=742073	g=3	expp: 12	e1=768343	g=11	expp: 18
e1=742369	g=7		e1=768491	g=10	
e1=742591	g=3		e1=768787	g=2	expp: 20
e1=743849	g=3		e1=769231	g=11	expp: 24
e1=743923	g=3	expp: 16	e1=769453	g=2	expp: 30
e1=744071	g=22		e1=772339	g=3	
e1=744811	g=10		e1=773153	g=3	expp: 14
e1=744959	g=13	expp: 10	e1=774337	g=5	expp: 28
e1=745033	g=10	expp: 16	e1=774929	g=3	expp: 18
e1=745181	g=2		e1=775669	g=10	expp: 18
e1=745477	g=2		e1=776483	g=2	
e1=745699	g=2		e1=776557	g=2	expp: 20
e1=746069	g=2		e1=777001	g=31	expp: 18,28
e1=746957	g=2		e1=778111	g=11	
e1=747401	g=3		e1=778333	g=2	expp: 28
e1=747919	g=3		e1=778777	g=5	
e1=748807	g=6	expp: 22	e1=779221	g=2	
e1=749843	g=2	expp: 34	e1=779591	g=7	
e1=750287	g=5		e1=779887	g=10	expp: 18
e1=750509	g=2	expp: 14,22	e1=780257	g=3	expp: 8
e1=751027	g=3		e1=780553	g=10	
e1=751841	g=3	expp: 14,16,24	e1=781367	g=5	expp: 34
e1=752137	g=10	expp: 8	*e1=781589	g=2	expp: 32
e1=752359	g=3	expp: 18	e1=782107	g=2	
e1=752581	g=2	expp: 16	e1=782329	g=13	expp: 18
e1=752803	g=2	expp: 22,32	e1=782921	g=3	expp: 20
e1=753617	g=3		e1=783143	g=5	
e1=753691	g=11	expp: 16	e1=783661	g=2	
e1=753839	g=7	expp: 4,22	e1=784327	g=3	
e1=754283	g=2		e1=784697	g=3	
e1=755171	g=6		e1=784919	g=7	
e1=755393	g=3	expp: 22	e1=785363	g=2	
e1=756281	g=3	expp: 2	e1=786251	g=2	
e1=756799	g=15	expp: 18	e1=786547	g=2	
e1=757243	g=2		e1=787139	g=2	expp: 20
e1=757909	g=2	expp: 16	e1=787361	g=6	
e1=758279	g=7		e1=787879	g=6	expp: 10,18,20
e1=758501	g=2	expp: 18	e1=788027	g=2	expp: 34
e1=759019	g=2		e1=789137	g=3	expp: 24
e1=759167	g=5	expp: 12	e1=790099	g=2	
e1=759463	g=3		e1=791209	g=7	
e1=759833	g=3	expp: 4	e1=791431	g=12	
e1=760129	g=11		e1=791801	g=3	
e1=760499	g=2		*e1=792023	g=5	expp: 32
e1=762053	g=2		e1=792689	g=3	
e1=762571	g=10		e1=793207	g=5	
e1=763237	g=2		e1=795427	g=2	
e1=764051	g=2		*e1=795649	g=22	expp: 2,32
e1=764273	g=3		e1=795797	g=2	
e1=764717	g=2	expp: 2	e1=795871	g=3	
e1=765383	g=5		e1=796759	g=3	
e1=765827	g=2	expp: 34	e1=796981	g=7	
e1=766049	g=3	expp: 22	e1=797647	g=3	
e1=766937	g=3	expp: 34	e1=797869	g=10	

e1=767381	g=2	expp: 18	e1=798461	g=2	
e1=767603	g=5	expp: 34	e1=798757	g=2	
e1=767677	g=5		e1=800089	g=7	expp: 20

For $\ell = 149, 223, 593, 1259, 1777, \dots$, $\mathcal{E}_\ell(37) = \emptyset$, which proves the Vandiver conjecture for $p = 37$ a great lot of times. For $\ell = 1481$ one finds a p -primarity for $\chi^* = \omega^7$ ($\chi = \omega^{30} \neq \omega^{32}$). Theorem 4.9 applies at will.

It remains to give statistics about the p -principality (or not) of the $\mathfrak{L}_{\chi_0^*}$ when $\ell \in \mathcal{L}_p$ varies. For $p = 37$, $\mathfrak{L}_{\chi_0^*}$ is 37-principal if and only if \mathfrak{L} is principal since the class number of K is $h = 37$.

4.4.1. Table of the classes of \mathfrak{L} for $p = 37$. We give a table with a generator of \mathfrak{L} in the principal cases (indicated by *). Otherwise, the class of \mathfrak{L} is of order 37 in K . We only write the cases $\mathcal{E}_\ell(37) \neq \emptyset$:

```
{p=37;c=lift(znprimroot(p));P=polcyclo(p);K=bnfinit(P,1);P=P+Mod(0,p);
X=Mod(x,P);Lsplit=List;N=0;for(i=1,2000,e1=1+2*i*p;if(isprime(e1)!=1,next);
N=N+1;listinsert(Lsplit,e1,N));for(j=1,N,e1=component(Lsplit,j));
F=bnfisintnorm(K,e1);if(F!=[],print("e1=",e1," ",component(F,1)));
g=znprimroot(e1);J=1;for(i=1,c-1, Ji=0;for(k=1,e1-2, kk=znlog(1-g^k,g);
e=lift(Mod(kk+i*k,p)); Ji=Ji-X^e); J=J*Ji); LJ=List;
Jj=1;for(j=1,p-1, Jj=lift(Jj*J));listinsert(LJ,Jj,j));for(m=1,(p-3)/2,
n=2*m; Sn=Mod(1,P);for(a=1,(p-1)/2, an=lift(Mod(a,p)^(n-1));
Jan=component(LJ,an);sJan=Mod(0,P);for(j=0,p-2, aj=lift(Mod(a*j,p));
sJan=sJan+x^(aj)*component(Jan,1+j)); Sn=Sn*sJan);
if(Sn==1,print("e1=",e1," expp:",n))})
```

e1=1481	expp: 30	e1=56167	expp: 10, 14, 26
e1=2591	expp: 34	e1=57203	expp: 34
e1=3331	expp: 22	e1=58313	expp: 28
e1=4219	expp: 16, 18	e1=58757	expp: 16, 18
e1=6143	expp: 28	e1=58831	expp: 24, 30
e1=7993	expp: 16, 20	e1=59497	expp: 28
e1=8363	expp: 8	e1=61051	expp: 10
e1=9769	expp: 20	e1=62383	expp: 2
e1=10657	expp: 4, 18, 26	e1=62753	expp: 2
e1=12433	expp: 20	e1=63493	expp: 2
e1=13099	expp: 28	e1=64381*	expp: 6, 32 [x^20+x^9+x]
e1=14431	expp: 4, 14, 22	e1=66749	expp: 30
e1=17021	expp: 6	e1=67489*	expp: 30, 32 [x^24-x^3-x^2]
e1=17909	expp: 30	e1=67933	expp: 6
e1=18131	expp: 22	e1=68821*	expp: 32 [x^15-x^9+x^4]
e1=19463	expp: 6	e1=69931	expp: 12
e1=20129	expp: 6	e1=71411	expp: 4
e1=21017	expp: 2, 4	e1=72817	expp: 28
e1=21313	expp: 18	e1=74149	expp: 2
e1=21757	expp: 8	e1=75407	expp: 10
e1=22349	expp: 8	e1=75629	expp: 12, 20
e1=23459	expp: 6	e1=76961	expp: 14
e1=23977	expp: 26	e1=78737	expp: 28
e1=25087	expp: 26	e1=79181	expp: 10
e1=25457	expp: 30	e1=80513	expp: 16, 26
e1=29009	expp: 8, 24	e1=81031	expp: 18, 34
e1=30859	expp: 2	e1=82067	expp: 34
e1=32783*	expp: 32 [x^11+x^3+x]	e1=83621	expp: 34
e1=33301	expp: 30	e1=83843	expp: 2
e1=33967	expp: 26	e1=84731	expp: 6
e1=36187	expp: 8	e1=85027	expp: 26

e1=37889	expp: 16	e1=86729	expp: 22
e1=38629	expp: 22	e1=86951	expp: 8
e1=40627	expp: 30	e1=87691	expp: 24
e1=40849	expp: 6	e1=91243	expp: 22, 34
e1=42773	expp: 4	e1=91909	expp: 30
e1=45289	expp: 8	e1=94351	expp: 10
e1=45659	expp: 26	e1=94573	expp: 18
e1=48619	expp: 8	e1=95239	expp: 18, 28
e1=48989	expp: 20	e1=96497	expp: 10
e1=51283	expp: 14, 16	e1=98347	expp: 28
e1=51431	expp: 20	e1=98939	expp: 30
e1=53281	expp: 16	e1=99679	expp: 10, 22
e1=55057	expp: 20	e1=100049	expp: 14

Give some examples (\mathcal{L}^{1+s-1} is always principal giving an easy characterization):

(ii) Non-principal case $\mathcal{L} \mid 149$. The instruction `bnfisintnorm(K, 149k)`:

```
{P=polcyclo(37);K=bnfinit(P,1);for(k=1,2,print(bnfisintnorm(K,149^k)))}
```

yields an empty set for $k = 1$ (since \mathcal{L} is not principal) and, for $k = 2$, it gives (with $x = \zeta_{37}$) the 18 conjugates of the real integer:

$$-2x^{35}-2x^{34}-x^{32}-2x^{31}+x^{29}-x^{28}-2x^{27}-2x^{24}-x^{23}+x^{22}-2x^{20}-x^{19}-x^{17}-2x^{16}+x^{14}-x^{13}-2x^{12}-2x^9-x^8+x^7-2x^5-x^4-2x^2-2x$$

(i) Principal case $\mathcal{L} \mid 32783$. The principal \mathcal{L} are rare (which comes from density theorems); the first one is $\mathcal{L} = (\zeta_{37}^{11} + \zeta_{37}^3 + \zeta_{37})$ where $\ell = 32783$. Thus in that case, in the relation $\mathcal{L}_{\chi_0^*}^{b_c(\chi_0^*)} = (g_c(\ell)_{\chi_0^*})$, the number $g_c(\ell)_{\chi_0^*}$ must be a global 37th power (which explains that one shall find the exponent of 37-primarity $n_0 = 32$ equal to that of 37-irregularity in the table); unfortunately, the data are too large to be given.

Nevertheless, the reader can easily compute

```
factor(norm(Sn)) = 3278337·16·9 and use K = bnfinit(P, 1); idealfactor(K, Sn),
```

which gives the 37th power of $\mathcal{L} \mid 32783$.

We obtain the following excerpts of the table (up to 10^6) of principal cases:

e1=32783	expp:32	e1=64381	expp:6,32	e1=67489	expp:30,32
e1=68821	expp:32	e1=108929	expp:32	e1=132313	expp:32
e1=325379	expp:10,32	e1=332039	expp:6,10,14,32	e1=351797	expp:32
e1=364451	expp:28,32	e1=387169	expp:32	e1=396937	expp:32
e1=960151	expp:32	e1=973397	expp:32	e1=983239	expp:32
e1=1000777	expp:32	e1=1002109	expp:2,32	e1=1040959	expp:20,32

4.4.2. Densities of the exponents of p -primarity. The following program intends to show that all exponents of p -primarity are obtained, with (perhaps) some specific densities, taking sufficiently many $\ell \in \mathcal{L}_p$ (each even $n \in [2, p - 3]$, such that $g_c(\ell)_{\omega^{p-n}}$ is p -primary for some new ℓ , is counted in the $(n/2)$ th component of the list Eel).

```
{p=37;c=lift(znprimroot(p));P=polcyclo(p)+Mod(0,p);X=Mod(x,P);Nel=0;Npp=0;Eel=List;
for(j=1,(p-3)/2,listput(Eel,0,j));for(i=1,1000,el=1+2*i*p;if(isprime(el)!=1,next);
g=znprimroot(el);Nel=Nel+1;J=1;for(i=1,c-1,Ji=0;for(k=1,el-2,kk=znlog(1-g^k,g);
e=lift(Mod(kk+i*k,p));Ji=Ji-X^e);J=J*Ji);LJ=List;Jj=1;for(j=1,p-1,Jj=lift(Jj*J);
listinsert(LJ,Jj,j));for(m=1,(p-3)/2,n=2*m;Sn=Mod(1,P);for(a=1,(p-1)/2,
an=lift(Mod(a,p)^(n-1));Jan=component(LJ,an);sJan=Mod(0,P);for(j=0,p-2,
aj=lift(Mod(a*j,p));sJan=sJan+x^(aj)*component(Jan,1+j));Sn=Sn*sJan);
if(Sn==1,Npp=Npp+1;listput(Eel,1+component(Eel,n/2),n/2);
print(Nel," ",Npp," ",el," ",Eel))})}
```

In the first column, one shall find the index i (in Nel) of the prime ℓ_i considered; if some index i is missing, this means that $\mathcal{E}_{\ell_i}(p) = \emptyset$. The second integer gives the whole number

of exponents of p -primarity obtained at this step (in Npp); then the third one is ℓ_i (in el). In some cases, a prime ℓ gives rise to several exponents of p -primarity

(i) Results for $p = 37$. The end of the table for the selected interval is:

Ne1	Npp	el
3015	1426	1414067 [83,95,84,91,80,80,86,83,92,83,97,76,83,78,85,74,76]
3015	1427	1414067 [83,95,84,91,80,80,86,83,92,83,97,76,83,78,86,74,76]
3027	1428	1419839 [83,95,84,91,80,80,86,83,92,83,98,76,83,78,86,74,76]
3030	1429	1420949 [83,95,84,91,80,80,86,83,92,83,98,76,83,78,86,75,76]
3032	1430	1421911 [83,95,85,91,80,80,86,83,92,83,98,76,83,78,86,75,76]
3033	1431	1422133 [83,95,86,91,80,80,86,83,92,83,98,76,83,78,86,75,76]
3042	1432	1428127 [83,96,86,91,80,80,86,83,92,83,98,76,83,78,86,75,76]
3889	1819	1863913[106,114,108,113,99,111,115,100,117,113,116,93,103,97,108,103,103]
3894	1820	1865911[106,114,108,114,99,111,115,100,117,113,116,93,103,97,108,103,103]
3898	1821	1868501[106,114,108,114,100,111,115,100,117,113,116,93,103,97,108,103,103]
3900	1822	1869389[106,114,108,114,100,112,115,100,117,113,116,93,103,97,108,103,103]
3900	1823	1869389[106,114,108,114,100,112,115,101,117,113,116,93,103,97,108,103,103]
3900	1824	1869389[106,114,108,114,100,112,115,101,117,113,116,93,104,97,108,103,103]

The penultimate column corresponds to the exponent of 37-irregularity $n_0 = 32$; since there is no counterexamples to Vandiver's conjecture, when this component increases, this means that the new ℓ gives rise to a principal \mathfrak{L} for which $g_c(\ell)_{\omega^5}$ is a 37th power.

(ii) Results for $p = 157$. For $p = 157$ (exponents of p -irregularity 62, 110), one finds the partial analogous information after 590 distinct primes $\ell \in \mathcal{L}_p$ tested (proving also Vandiver's conjecture for a lot of times):

Ne1	Npp	el
590	309	1161487 [9,3,2,6,8,3,1,4,5,10,3,1,3,1,6,3,4,4,2,2,1,2,5, 5,3,2,2,1,5,7,6,2,2,1,5,5,5,4,4,3,3,4,5,4,5,5,5,5,3,6,1,6,3,5,4,5, 0,2,3,5,7,3,3,3,2,4,4,7,6,6,5,6,1,7,4,7]
590	310	1161487 [9,3,2,6,8,3,1,4,5,10,3,1,3,1,6,3,4,4,2,2,1,2,5, 5,3,2,2,1,5,7,6,2,2,1,5,5,5,4,4,3,3,4,5,4,5,6,5,5,3,6,1,6,3,5,4,5, 0,2,3,5,7,3,3,3,2,4,4,7,6,6,5,6,1,7,4,7]
590	311	1161487 [9,3,2,6,8,3,1,4,5,10,3,1,3,1,6,3,4,4,2,2,1,2,5, 5,3,2,2,1,5,7,6,2,2,1,5,5,5,4,4,3,3,4,5,4,5,6,5,5,3,6,1,6,3,5,4,5, 0,2,3,5,7,3,3,3,2,4,5,7,6,6,5,6,1,7,4,7]

The remaining column of zeros (for $n/2 = 58$) stops at the following lines:

602	318	1185979 [9,3,2,6,8,3,2,4,6,10,3,1,3,1,6,4,4,4,2,2,1,2,5, 5,3,2,2,1,5,7,6,3,2,1,5,5,5,4,4,3,3,4,5,4,5,6,5,5,3,6,1,6,4,5,4,6, 0,2,3,5,7,3,3,3,3,4,5,7,6,6,5,6,1,7,4,7]
602	319	1185979 [9,3,2,6,8,3,2,4,6,10,3,1,3,1,6,4,4,4,2,2,1,2,5, 5,3,2,2,1,5,7,6,3,2,1,5,5,5,4,4,3,3,4,5,4,5,6,5,5,3,6,1,6,4,5,4,6, 1,2,3,5,7,3,3,3,3,4,5,7,6,6,5,6,1,7,4,7]
602	320	1185979 [9,3,2,6,8,3,2,4,6,10,3,1,3,1,6,4,4,4,2,2,1,2,5, 5,3,2,2,1,5,7,6,3,2,1,5,5,5,4,4,3,3,4,5,4,5,6,5,5,3,6,1,6,4,5,4,6, 1,2,4,5,7,3,3,3,3,4,5,7,6,6,5,6,1,7,4,7]

These numbers may depend on the orders of ω^n and/or ω^{p-n} , but this needs to be clarified taking much $\ell \in \mathcal{L}_p$.

4.4.3. Vandiver's conjecture and p -adic regulator of K_+ . We return to the case $p = 37$ and $n_0 = 32$. We see that ω^{32} is a character of order 9, hence a character of the real subfield k_9 of degree 9, which is such that $\mathcal{T}_{k_9} \neq 1$ from the reflection relation (3.1); so, k_9 admits a cyclic 37-ramified extension of degree 37 which is not unramified. To verify, we use [17, Program I], for real fields, which indeed gives $\#\mathcal{T}_{k_9} = 37$ (nt must verify $p^{\text{nt}} > p^t$, the exponent of \mathcal{T}):

```
{p=37;n=32;d=(p-1)/gcd(p-1,n);P=polsubcyclo(p,d);K=bnfinit(P,1);nt=6;
Kpn=bnrinit(K,p^nt);Hpn=component(component(Kpn,5),2);L=List;
e=component(matsize(Hpn),2);R=0;for(k=1,e-1,c=component(Hpn,e-k+1);
if(Mod(c,p)==0,R=R+1;listinsert(L,p^valuation(c,p),1));if(R>0,
print("rk(T)=",R," K is not ",p,"-rational ",L));
if(R==0,print("rk(T)=",R," K is ",p,"-rational"))}
```

```
rk(T)=1 K is not 37-rational List([37])
```

We find here another interpretation of the reflection theorem since we have the typical formula $\#\mathcal{T}_+ = \#\mathcal{C}_+ \cdot \#\mathcal{R}_+$, where the p -group \mathcal{R}_+ is the normalized p -adic regulator of K_+ [19, Proposition 5.2]. Whence $\#\mathcal{T}_\chi = \#\mathcal{C}_\chi \cdot \#\mathcal{R}_\chi$ and $\mathcal{R}_{\chi^*} = 1$, for all $\chi \in \mathcal{X}_+$; but we have $\#\mathcal{T}_{\chi^*} = \#\tilde{\mathcal{C}}_{\chi^*}$ for the subgroup $\tilde{\mathcal{C}}_{\chi^*}$ of \mathcal{C}_{χ^*} . The above data shows that the relation $\#\mathcal{T}_{\chi_0} = 37$ comes from $\#\mathcal{R}_{\chi_0} = 37$, which is not surprising:

Remark 4.11. We have the analytic formula $\#\mathcal{C}_{\chi_0} = \#(E_{\chi_0}/\langle \eta_{\chi_0} \rangle)$, where η is a suitable cyclotomic unit; so a classical method (explained in [52, Corollary 8.19], applied in [4, 9] and developed in [50, 51]) consists in finding $\ell \in \mathcal{L}_p$ such that η_{χ_0} is not a local p th power at ℓ proving Vandiver's conjecture at χ_0 ; so when we find that $\mathcal{R}_{\chi_0} \neq 1$ (with $\mathcal{C}_{\chi_0} = 1$), this means that η_{χ_0} generates E_{χ_0} and is a local p th power at p by p -primarity, so that $K(\sqrt[p]{\eta_{\chi_0}})$ is contained in the χ_0^* -component of the p -Hilbert class field of K .

We shall give in §5.2.4 some insights in this direction to state new heuristics for the probability of p -primarity of $g_c(\ell)_{\chi_0^*}$ to be at most $\frac{O(1)}{p^2}$.

5. Heuristics – Probability of a counterexample

5.1. Use of classical standard probabilities. We may suppose in a first approximation that, for a given p , the sets $\mathcal{E}_\ell(p)$ of exponents of p -primarity of primes $\ell \in \mathcal{L}_p$, are random with the same behavior as for the set $\mathcal{E}_0(p)$ of exponents of p -irregularity. More precisely, assume, as in Washington's book (see in [52], the Theorem 5.17 and some statistical computations), that for given primes p and $\ell \in \mathcal{L}_p$, the probabilities of a cardinality k is $\binom{N}{j} \cdot \left(1 - \frac{1}{p}\right)^{N-j} \cdot \left(\frac{1}{p}\right)^j$ (where $N := \frac{p-3}{2}$). This would imply that, for p given, $\mathcal{E}_\ell(p) \neq \emptyset$ for many $\ell \in \mathcal{L}_p$, but that $\mathcal{E}_\ell(p) = \emptyset$ in a proportion close to $e^{-\frac{1}{2}}$, which is in accordance with previous tables. Then the probability, for p and ℓ given, of $\mathcal{E}_0(p) \cap \mathcal{E}_\ell(p) \neq \emptyset$ with cardinalities $j \in [0, N]$ and $k \in [0, N]$ fixed, is:

$$1 - \frac{(N-k)! \cdot (N-j)!}{N! \cdot (N-k-j)!}.$$

So, an approximation of the whole probability of $\mathcal{E}_0(p) \cap \mathcal{E}_\ell(p) \neq \emptyset$ is:

$$(11) \quad \sum_{j,k \geq 0} \binom{N}{j} \binom{N}{k} \cdot \left(1 - \frac{1}{p}\right)^{2N-j-k} \cdot \left(\frac{1}{p}\right)^{j+k} \cdot \left(1 - \frac{(N-k)! \cdot (N-j)!}{N! \cdot (N-k-j)!}\right).$$

The computations show that this expression is around $\frac{1}{2p}$, which does not allow to conclude easily for a single ℓ , but *this does not take into account* the “infiniteness” of \mathcal{L}_p giving, a priori, *independent informations*, but limited by the Theorem 3.7 on periodicities due to the density theorem (see the Weil interpretation of Jacobi sums defining Hecke Größencharacters [55, Theorem, p. 489] where a module of definition of our Jacobi sums is p^2 , which may give an order of magnitude of the cardinality of this “infiniteness”). So this must be put in relation with the Theorem 4.7 to characterize “non-Vandiver”.

5.2. New heuristics and probabilities. Many reasons imply that the generic probability $\frac{1}{p}$ must be replaced by a much lower one:

5.2.1. Results from K-theory. For some characters $\chi \in \mathcal{X}_+$, of the form $\chi =: \omega^{p-(1+h)}$, for small $h = 2, 4, \dots$, one may prove that $\mathcal{C}_{\omega^{p-(1+h)}} = 1$, as the case of $\mathcal{C}_{\omega^{p-3}} = 1$ proved unconditionally by Kurihara [30]; then Soulé proved in [48] that for $n \in [2, p-3]$ even, $\mathcal{C}_{\omega^{p-n}} = 1$ for all p large enough (see also [11, 49, 3] among other references applying the same approach via K-theory). Unfortunately these bounds are not usable in practice, but demonstrate the existence of a fundamental general principle.

5.2.2. Archimedean aspects. At the opposite, for $\chi \in \mathcal{X}_+$ of small order, \mathcal{C}_χ may be trivial because of the ‘‘archimedean’’ order of magnitude of the whole class number of the subfield of K_+ fixed by $\text{Ker}(\chi)$ (which is proved for the quadratic case when $p \equiv 1 \pmod{4}$, the cubic case when $p \equiv 1 \pmod{3}$); see the tables of Schoof [42] for serious arguments about the order of magnitude of the whole class number. Moreover, we have the p -rank ϵ -conjecture for p -class groups [10] that we state for the real abelian fields k_d of constant degree d , of discriminant $D = p^{d-1}$, when $p \equiv 1 \pmod{d}$ increases:

For all $\epsilon > 0$ there exists $C_{p,\epsilon}$ such that $\log(\#(\mathcal{C}_{k_d}/\mathcal{C}_{k_d}^p)) \leq \log(C_{p,\epsilon}) + \epsilon \cdot \log(p)$,

which would give $\mathcal{C}_{k_d} = 1$ for $\log(p) > \frac{\log(C_{p,\epsilon})}{1-\epsilon}$ and any $\epsilon < 1$. But this does not apply for any p with ‘‘small’’ d and the constant $C_{p,\epsilon}$ is not effective.

5.2.3. Heuristics about Gauss sums. The standard probabilities (11) assume that when $\ell \in \mathcal{L}_p$ varies, the sets $\mathcal{E}_\ell(p)$ are *random and independent*, giving probabilities close to 0, which is not the case when p is irregular at some $\chi_0^* = \omega^{p-n_0}$ with $\mathcal{C}_{\chi_0^*} \simeq \mathbb{Z}/p^e\mathbb{Z}$, $e \geq 1$, and when $\text{g}_c(\ell)_{\chi_0^*}$ is a global p th power because $\mathfrak{L}_{\chi_0^*}^{p^{e-1}}$ is p -principal.

Fix $\ell \in \mathcal{L}_p$ such that $\mathfrak{L}_{\chi_0^*}$ generates $\mathcal{C}_{\chi_0^*} \simeq \mathbb{Z}/p^e\mathbb{Z}$ (thus $\text{g}_c(\ell)_{\chi_0^*}$ is not a global p th power); put (Proposition 2.4) $\text{g}_c(\ell)_{\chi_0^*} = 1 + \beta_0(\ell) \cdot \varpi^{p-n_0}$, $\beta_0(\ell) \in \mathbb{Z}_p[\varpi]$, where $\beta_0(\ell)$ is invertible modulo ϖ if and only if $\text{g}_c(\ell)_{\chi_0^*}$ is not p -primary.

Whatever $\ell' \in \mathcal{L}_p$ and $\mathfrak{L}' \mid \ell'$, one has, from §4.3 (ii), $\text{g}_c(\ell')_{\chi_0^*} \equiv \text{g}_c(\ell)_{\chi_0^*}^r \pmod{p}$, with $r \in [0, p^e - 1]$ ($r = 0$ if $\mathfrak{L}'_{\chi_0^*}$ is p -principal, thus $\text{g}_c(\ell')_{\chi_0^*} \in K^{\times p}$), giving:

$$(12) \quad \text{g}_c(\ell')_{\chi_0^*} =: 1 + \beta_0(\ell') \cdot \varpi^{p-n_0}, \quad \beta_0(\ell') \equiv r \cdot \beta_0(\ell) \pmod{\varpi}.$$

Contrary to the classical idea that $\beta_0(\ell) \pmod{\varpi}$ follow standard probabilities $\frac{O(1)}{p}$ (even under the condition $\text{g}_c(\ell)_{\chi_0^*} \notin K^{\times p}$), we propose the following heuristic:

For each $\chi \in \mathcal{X}_+$, the mod p values, at $\chi^* = \omega \chi^{-1}$, of the Gauss sums (more precisely of the $\psi^{-c}(c) \cdot \text{g}_c(\ell) = J_1 \cdots J_{c-1}$), are uniformly distributed (or at least with explicit non-trivial densities), when $\ell \in \mathcal{L}_p$ varies.

Because of the density theorems on the ideal classes when ℓ varies in \mathcal{L}_p and χ in \mathcal{X}_+ , we must examine two cases concerning the χ -components of \mathcal{C} when there exists $\chi_0 = \omega^{n_0} \in \mathcal{X}_+$ such that $\mathcal{C}_{\chi_0^*} \simeq \mathbb{Z}/p^e\mathbb{Z}$, $e \geq 1$:

(a) $\chi \neq \chi_0$ and $\mathcal{C}_{\chi^*} = 1$. The numerical experiments show that when $\ell \in \mathcal{L}_p$ varies, $\text{g}_c(\ell)_{\chi^*} = 1 + \beta(\ell) \cdot \varpi^{p-n}$, with random $\beta(\ell) \pmod{\varpi}$ (probabilities $\frac{O(1)}{p}$).

(b) $\chi = \chi_0$ (and $\mathcal{C}_{\chi_0^*} \neq 1$). If $\text{g}_c(\ell)_{\chi_0^*}$ is p -primary for some given generator $\mathfrak{L}_{\chi_0^*}$, then from (12) all the $\text{g}_c(\ell')_{\chi_0^*}$ are p -primary, whatever the class of $\mathfrak{L}'_{\chi_0^*}$ (p^e possibilities) because $\beta_0(\ell') \equiv 0 \pmod{\varpi}$. So, n_0 is always in $\mathcal{E}_\ell(p)$ and $\mathcal{E}_0(p) \cap \mathcal{E}_\ell(p) \neq \emptyset$ for all $\ell \in \mathcal{L}_p$, which corresponds to $\mathcal{C}_{\chi_0} \neq 1$ and the non-cyclicity of $\mathcal{C}_{\chi_0^*}^{(p)}$ (Theorem 3.7).

Thus, to have analogous densities of p -primarity on \mathcal{L}_p (as for the p -principal case (a)), $\beta_0(\ell) \equiv 0 \pmod{\varpi}$ (under the condition $\text{g}_c(\ell)_{\chi_0^*} \notin K^{\times p}$) must occur at least p times less,

giving a probability in $\frac{O(1)}{p^2}$ instead of $\frac{O(1)}{p}$; it is even possible that such a circumstance be of probability 0 depending on more accurate properties of Gauss or Jacobi sums; for this, the computation of $\beta(\ell)$ should be very interesting (see [51] where, for any $\ell \equiv 1 \pmod{p}$, the coefficients $d_{i,k}$ of $J_i := \sum_{k=0}^{p-1} d_{i,k} \zeta_p^k$, with $\sum_{k=0}^{p-1} d_{i,k} = 1$, are studied and the starting point of future investigations, in relation with the other heuristics).

5.2.4. Use of p th power residue symbols and cyclotomic units. We refer to [52, § 8.3] for the classical p -adic interpretation of the numbers $\#\mathcal{C}_\chi$, for $\chi \in \mathcal{X}_+$, as indices of the form $(E_\chi : F_\chi)$, where F is the group of cyclotomic units.

We need the following p th power criterion (from [15, II.6.3.8]):

Lemma 5.1. *Let $\alpha \in K^\times$ be a pseudo-unit (namely, α is prime to p and $(\alpha) = \mathfrak{a}^p$). Let any set \mathcal{S} of places \mathfrak{q} of K such that $\langle \mathcal{C}(\mathcal{S}) \rangle_{\mathbb{Z}} = \mathcal{C}$ (or such that $\langle \mathcal{C}(\mathcal{S}) \rangle_{\mathbb{Z}[G]} = \mathcal{C}$ if $K(\sqrt[p]{\alpha})/\mathbb{Q}$ is Galois).*

Then $\alpha \in K^{\times p}$ if and only if α is p -primary and locally a p th power at \mathcal{S} (i.e., $\alpha \in K_{\mathfrak{q}}^{\times p}$ for all $\mathfrak{q} \in \mathcal{S}$ where $K_{\mathfrak{q}}$ is the \mathfrak{q} -completion of K).

Proof. Consider the non-trivial direction, in the Galois case, assuming that α is p -primary and such that $\alpha \in K_{\mathfrak{q}}^{\times p}$ for all $\mathfrak{q} \in \mathcal{S}$. So $K(\sqrt[p]{\alpha})/K$ is unramified and \mathcal{S} -split; thus, due to the Galois condition, all the conjugates of $\mathfrak{q} \in \mathcal{S}$ split and the Galois group of $K(\sqrt[p]{\alpha})/K$ corresponds, by class field theory, to a quotient of $\mathcal{C}/\langle \mathcal{C}(\mathcal{S}) \rangle_{\mathbb{Z}[G]}$, trivial by assumption. \square

Theorem 5.2. *Let $\chi_0 = \omega^{n_0} \in \mathcal{X}_+$ with $n_0 \in \mathcal{E}_0(p)$ and $\mathcal{C}_{\chi_0^*} \simeq \mathbb{Z}/p^e\mathbb{Z}$, $e \geq 1$ (i.e., $b_c(\chi_0^*) \sim p^e$). Let $\eta := \zeta_p^{\frac{1-c}{2}} \frac{1-\zeta_p^c}{1-\zeta_p}$ be a generating real cyclotomic unit, where c is a primitive root modulo p (cf. [52, Proposition 8.11]).*

(i) *There exists an infinite subset $\mathcal{L}_p(\chi_0) \subseteq \mathcal{L}_p$ of primes ℓ such that the G -module generated by the p -class of $\mathfrak{L} \mid \ell$ is $\mathcal{C}_{\chi_0} \oplus \mathcal{C}_{\chi_0^*}$.*

(ii) *$\mathcal{C}_{\chi_0} \neq 1$ if and only if $g_c(\ell)_{\chi_0^*}$ is locally a p th power at \mathfrak{p} but not at \mathfrak{L} ($\ell \in \mathcal{L}_p(\chi_0)$).*

(iii) *$\mathcal{C}_{\chi_0} \neq 1$ if and only if η_{χ_0} is locally a p th power at \mathfrak{p} and at \mathfrak{L} ($\ell \in \mathcal{L}_p(\chi_0)$).*

Proof. (i) In the $\mathbb{Z}_p[G]$ -monogenous case, the ideals \mathfrak{L} are of the form $(z) \cdot \mathfrak{A} \cdot \mathfrak{A}^*$, $z \in K^\times$, where $\mathcal{C}(\mathfrak{A})$ generates \mathcal{C}_{χ_0} and $\mathcal{C}(\mathfrak{A}^*)$ generates $\mathcal{C}_{\chi_0^*}$.²

(ii) & (iii) Define the p th power residue symbol $\left(\frac{\alpha}{\mathfrak{L}}\right) := \alpha^{\frac{\ell-1}{p}} \pmod{\mathfrak{L}}$ for $\mathfrak{L} \mid \ell \in \mathcal{L}_p$ when $\alpha \in K^\times$ is prime to \mathfrak{L} . By abuse of notation, we shall write $\left(\frac{\alpha}{p}\right) = 1$ if α is p -primary and $\left(\frac{\alpha}{\mathfrak{L}}\right) = 1$ if $\alpha \in K_{\mathfrak{L}}^{\times p}$ is not prime to \mathfrak{L} . Take $\ell \in \mathcal{L}_p(\chi_0)$:

(a) Consider $\alpha = g_c(\ell)_{\chi_0^*}$, where $(g_c(\ell)_{\chi_0^*}) = \mathfrak{L}_{\chi_0^*}^{b_c(\chi_0^*)}$. This gives rise to a counterexample to Vandiver's conjecture at χ_0 if and only if α is p -primary since $\mathcal{C}(\mathfrak{L}_{\chi_0^*})$ is a generator of $\mathcal{C}_{\chi_0^*}$; it follows that $\left(\frac{\alpha}{\mathfrak{L}}\right) \neq 1$, otherwise, from Lemma 5.1 applied in H_{χ_0} , $\alpha = g_c(\ell)_{\chi_0^*}$ should be a global p th power (contradiction).

²If, for instance, $\mathcal{C}_{\chi_0} \simeq \mathcal{C}_{\chi_0^*} \simeq \mathbb{Z}/p\mathbb{Z}$, these prime ideals \mathfrak{L} have density $\left(1 - \frac{1}{p}\right)^2$; otherwise, if $\mathcal{C}_{\chi_0} = 1$ and $\mathcal{C}_{\chi_0^*} \simeq \mathbb{Z}/p\mathbb{Z}$, the density is $1 - \frac{1}{p}$.

(b) Consider $\alpha = \eta_{\chi_0}$. Thus $b_c(\chi_0^*) \equiv 0 \pmod{p}$ is equivalent to the p -primarity of η_{χ_0} ; so a counterexample to Vandiver's conjecture at χ_0 , equivalent to $\eta_{\chi_0} \in E_{\chi_0}^p$, is equivalent to $\left(\frac{\eta_{\chi_0}}{\mathfrak{L}}\right) = 1$ since $\left(\frac{\eta_{\chi_0}}{p}\right) = 1$. Whence, with a prime $\mathfrak{L} \mid \ell \in \mathcal{L}_p(\chi_0)$:

$$\mathcal{C}_{\chi_0} \neq 1 \Leftrightarrow \left(\frac{g_c(\ell)_{\chi_0^*}}{\mathfrak{L}}\right) \neq 1 \ \& \ \left(\frac{g_c(\ell)_{\chi_0^*}}{p}\right) = 1 \Leftrightarrow \left(\frac{\eta_{\chi_0}}{\mathfrak{L}}\right) = \left(\frac{\eta_{\chi_0}}{p}\right) = 1. \quad \square$$

Let $\chi \in \mathcal{X}_+$ and $\ell \in \mathcal{L}_p(\chi)$ fixed. If $\text{Prob}\left(\left(\frac{g_c(\ell)_{\chi^*}}{\mathfrak{L}}\right) \neq 1\right)$ is close to 1, this suggests a probability around $\frac{O(1)}{p^2}$ for the p -primarity of $g_c(\ell)_{\chi^*}$ if the two above symbols of η_χ are independent with probabilities $\frac{O(1)}{p}$ for a single ℓ .

So it is necessary to compute the symbol $\left(\frac{g_c(\ell)_{\chi_0^*}}{\mathfrak{L}}\right)$ since $g_c(\ell)_{\chi_0^*}$ and \mathfrak{L} are non-independent data. For $\chi_0 = \omega^{n_0}$, $n_0 \in \mathcal{E}_0(p)$, the primes ℓ of the theorem are not effective, but experiments with random ℓ seem sufficient for statistics. Then a first condition for $\left(\frac{g_c(\ell)_{\chi_0^*}}{\mathfrak{L}}\right) = 1$ is that $g_c(\ell)_{\chi_0^*}$ be the p th power of an ℓ -ideal, which is fulfilled since $b_c(\chi_0^*) \equiv 0 \pmod{p}$.

Then, using the general program computing $g_c(\ell)_{\chi_0^*}$ in $\mathbf{S}n \in \mathbb{Z}[\zeta_p]$ (in other words *not reduced modulo* p), we divide this integer by the maximal power ℓ^v , so that there exists a prime ideal $\mathfrak{L} \mid \ell$ which does not divide this new integer (still denoted $\mathbf{S}n$ and p th power of an ℓ -ideal); the computation reduces to R , prime to \mathfrak{L} and most likely random, whose symbol $\left(\frac{R}{\mathfrak{L}}\right) = R^{\frac{\ell-1}{p}} \pmod{\mathfrak{L}}$, computed in \mathbf{u} , is immediate and gives the statistics:

```
{p=37;n=32;print("p=",p," n=",n);c=lift(znprimroot(p));P=polcyclo(p);X=Mod(x,P);
for(i=1,100,el=1+2*i*p;if(isprime(el)!=1,next);g=znprimroot(el);M=(el-1)/p;J=1;
for(i=1,c-1,Ji=0;for(k=1,el-2,kk=znlog(1-g^k,g);e=lift(Mod(kk+i*k,p));
Ji=Ji-X^e);J=J*Ji);LJ=List;Jj=1;for(j=1,p-1,Jj=lift(Jj*J);listinsert(LJ,Jj,j));
Sn=1;for(a=1,p-1,an=lift(Mod(a,p)^(n-1));Jan=component(LJ,an);sJan=Mod(0,P);
for(j=0,p-2,aj=lift(Mod(a*j,p));sJan=sJan+X^(aj)*component(Jan,1+j));Sn=Sn*sJan);
Sn=lift(Sn);s=valuation(Sn-1,p);v=valuation(Sn,el);Sn=Sn/el^v;ro=g^M;
for(b=1,p-1,r=lift(ro^b);R=0;for(k=0,p-2,R=R+component(Sn,k+1)*r^k);
if(valuation(R,el)==0,y=R;break);u=lift(Mod(y,el)^M);
print("p=",p," el=",el," v=",v," u=",u);if(s!=0,print("Sn local pth power at P"));
if(Mod(v,p)==0 & u==1,print("Sn local pth power at L"));
if(Mod(v,p)!=0 || u!=1,print("Sn NON local pth power at L"));
if(Mod(v,p)==0 & u==1 & s!=0,print("Sn GLOBAL pth power"))}
p=37 n=32
```

el=149	v=259	u=102	Sn NON local pth power at L
el=223	v=259	u=132	Sn NON local pth power at L
el=6883	v=259	u=6850	Sn NON local pth power at L
el=7253	v=259	u=4947	Sn NON local pth power at L
el=32783	v=259	u=1	Sn local pth power at P
el=32783	v=259	u=1	Sn local pth power at L
el=32783	v=259	u=1	Sn GLOBAL pth power

We found $u = \left(\frac{g_c(\ell)_{\chi_0^*}}{\mathfrak{L}}\right) = 1$ for the following ℓ (including the underlined numbers corresponding to primes $\ell \notin \mathcal{L}_p(\chi_0)$ such that $g_c(\ell)_{\chi_0^*} \in K^{\times p}$, i.e., \mathcal{L} p -principal):

$\ell \in \{22571;32783;46103;53503;57943; \underline{64381};\underline{67489};\underline{68821};79847;83177;96497; 98939;104933;\underline{108929};$
 $117883;\underline{132313};146521;\underline{146891};151553;151849;158657; 158731;\underline{167759}; \underline{172717};197359;\underline{198839};\underline{207497}\}$

confirming existence and rarity of primes ℓ in the interval $[149; 207497]$ such that $u = 1$ by accident ($g_c(\ell)_{\chi_0^*} \notin K^{\times p}$, i.e., \mathcal{L} non- p -principal).

For $n = 22 \notin \mathcal{E}_0(37)$, we found $u = 1$ for the few examples (up to $2 \cdot 10^5$):

$\ell \in \{2221; 2887; 3923; 49211; 51283; 69709; 147779; 164503; 170497; 179969; 192697; 197803\}$

but $g_c(\ell)_{\chi^*}$ is not the p th power of an ℓ -ideal, whence it is never in $K_{\mathfrak{L}}^{\times p}$. One finds an exponent of p -primarity 22 for $\ell = 3331$, then 14, 16 for $\ell = 51283$, 10 for $\ell = 147779$, and 28 for $\ell = 164503$. In the exceptional case $\ell = 3331$, $g_c(\ell)_{\chi^*}$ is p -primary.

This confirms the expected properties of the symbol $\left(\frac{g_c(\ell)_{\chi_0^*}}{\mathfrak{L}}\right)$. A similar program computing the two symbols of η_{χ_0} gives all expected results.

5.2.5. Classical heuristics on class groups. A first important reason for a very rare occurrence of the non-cyclic case for $\mathcal{C}_{\chi^*}^{(p)}$ may come from classical heuristics on p -class groups, assuming that they can be applied to ray class groups as $\mathcal{C}_{\chi^*}^{(p)}$ when it is, for instance, of order p^2 .

Whatever the (numerous) references concerning this subject and independently of some improvements or questions on the relevance of the formulas giving $\text{Prob}(\text{rk}_p(C) = r)$ for such a p -group C , we observe that the quotient of the two probabilities for $r = 2$ and $r = 1$ (for instance under the condition $\#C = p^2$) is at most $\frac{O(1)}{p}$ giving probabilities in $\frac{O(1)}{p^2}$ to have $\mathcal{C}_{\chi^*}^{(p)} \simeq (\mathbb{Z}/p\mathbb{Z})^2$. Since $\text{rk}_p(\mathcal{C}_{\chi}) = 1$ splits in the two cases of the reflection theorem, $\text{rk}_p(\mathcal{C}_{\chi} \oplus \mathcal{C}_{\chi^*}) = 2$ or $\text{rk}_p(\mathcal{C}_{\chi^*}) = 2$, the above applies. As Nguyen Quang Do pointed out, this may come from the relation $H^2(\mathcal{C}_{\chi^*}, (V/W)_{\chi^*}) \simeq \mathbb{F}_p$, assuming the uniform randomness of the exact sequences $1 \rightarrow (V/W)_{\chi^*} \simeq \mathbb{F}_p \rightarrow \mathcal{C}_{\chi^*}^{(p)} \rightarrow \mathcal{C}_{\chi^*} \simeq \mathbb{F}_p \rightarrow 1$ (proof of Theorem 3.7), the non-cyclic case corresponding to the single cohomology class 0.

5.2.6. Heuristics from p -ramification theory. Another investigation is about the groups \mathcal{T}_{χ} , $\chi \in \mathcal{X}_+$, and the formula $\#\mathcal{T}_{\chi} = \#\mathcal{C}_{\chi} \cdot \#\mathcal{R}_{\chi}$ with the equivalence (3.1) of reflection, $\mathcal{C}_{\chi^*} \neq 1$ if and only if $\mathcal{T}_{\chi} \neq 1$ (illustrated in § 4.4.3).

Indeed, it is interesting to estimate in what proportions the relation $\#\mathcal{C}_{\chi} \cdot \#\mathcal{R}_{\chi} \neq 1$ is due to \mathcal{C}_{χ} or \mathcal{R}_{χ} . Of course, it is impossible to experiment with the cyclotomic fields K ; so, since this problem must be considered as general and may result from some insights in p -ramification theory as done in a number of our articles (see [20] and its bibliography), we give some examples with quadratic and cyclic cubic fields.

(a) **Real quadratic fields and $p \geq 3$ fixed.** For each of the ND real quadratic field of discriminant $D \in [\text{bD}, \text{BD}]$, for which $\mathcal{T} \neq 1$ (counted in Nt), we compute the proportions of cases for which this is due to $\#\mathcal{C}$ or $\#\mathcal{R}$; we privilege the case $\mathcal{C} \neq 1$ (counted in Nh) even if the two groups \mathcal{C} and \mathcal{R} are both non-trivial; this may give a slightly larger proportion for $\frac{\text{Nh}}{\text{Nt}}$ but a much faster program:

```
{p=3; bD=10^6; BD=10^6+5*10^4; ND=0; Nh=0; Nt=0; for (D=bD, BD, e=valuation(D, 2); M=D/2^e;
if (core(M) != M, next); if ((e==1 || e>3) || (e==0&Mod(M, 4) != 1) || (e==2&Mod(M, 4) == 1), next);
m=D; if (e!=0, m=D/4); ND=ND+1; P=x^2-m; K=bnfinit(P, 1); Kpn=bnrinit(K, p^2);
C5=component(Kpn, 5); Hpn0=component(C5, 1); Hpn=component(C5, 2);
Hpn1=component(Hpn, 1); vptor=valuation(Hpn0/Hpn1, p);
if (vptor>=1, Nt=Nt+1; C8=component(K, 8); h=component(component(C8, 1), 1);
vph= valuation(h, p); if (vph>=1, Nh=Nh+1)); print("[", bD, ", ", " ", BD, "]"); print
("p=", p, " ND=", ND, " Nt=", Nt, " Nh=", Nh, " Nh/Nt=", Nh/Nt, ", Nh/Nt+0., " 1/p=", 1./p)}
```

[bD, BD]=[1000000, 1050000]

p	ND	Nt	Nh	Nh/Nt	1/p
3	15204	7308	2050	0.28051450	0.33333333
5	15204	3522	634	0.18001135	0.20000000
7	15204	2464	331	0.13433441	0.14285714
11	15204	1497	97	0.06479625	0.09090909

[bD, BD]=[10000000, 10050000]

```
p=3  ND=15198  Nt=7516  Nh=2161  Nh/Nt=0.28751995  1/p=0.33333333
p=5  ND=15198  Nt=3597  Nh=720  Nh/Nt=0.20016680  1/p=0.20000000
p=7  ND=15198  Nt=2443  Nh=347  Nh/Nt=0.14203847  1/p=0.14285714
p=11 ND=15198  Nt=1512  Nh=122  Nh/Nt=0.08068783  1/p=0.09090909
```

```
[bD, BD]=[100000000, 100100000]
p=3  N=30410  Nt=15133  Nh=4456  Nh/Nt=0.29445582  1/p=0.33333333
```

The proportion Nh/Nt becomes close to $\frac{1}{p}$ for intervals with large discriminants.

(b) **Cyclic cubic fields and $p \equiv 1 \pmod{3}$ fixed.** We obtain analogous results with the same rough calculation (e.g., we may have $\mathcal{C}_{\chi_1} \neq 1$ and $\mathcal{R}_{\chi_1} \neq 1$ or $\mathcal{R}_{\chi_2} \neq 1$), but this does not affect the statistics ($f \in [bf, Bf]$ denotes the conductor):

```
{p=7;bf=10^5;Bf=5*10^5;Nf=0.0;Nh=0;Nt=0;for(f=bf,Bf,e=valuation(f,3);
if(e!=0 & e!=2,next);F=f/3^e;if(Mod(F,3)!=1||core(F)!=F,next);F=factor(F);
D=component(F,1);d=component(matsize(F),1);for(j=1,d-1,l=component(D,j);
if(Mod(1,3)!=1,break));for(b=1,sqrt(4*f/27),if(e==2 & Mod(b,3)==0,next);
A=4*f-27*b^2;if(issquare(A,&a)==1,if(e==0,if(Mod(a,3)==1,a=-a);
P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);if(e==2,if(Mod(a,9)==3,a=-a);
P=x^3-f/3*x-f*a/27);Nf=Nf+1;K=bnfinit(P,1);Kpn=bnrinit(K,p^2);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
if(vptor>=1,Nt=Nt+1;C8=component(K,8);h=component(component(C8,1),1);
vph=valuation(h,p);if(vph>=1,Nh=Nh+1)))));print("[",bf," ",",",Bf,""]");print
("p=",p," Nf=",Nf," Nt=",Nt," Nh=",Nh," Nh/Nt=",Nh/Nt," 1/p=",1./p)}
```

```
[bf, Bf]=[50000, 100000]
p=7  Nf=7928  Nt=2302  Nh=344  Nh/Nt=0.14943527  1/p=0.14285714
[bf, Bf]=[100000, 500000]
p=7  Nf=63427  Nt=18533  Nh=2690  Nh/Nt=0.14514649  1/p=0.14285714
[bf, Bf]=[100000, 500000]
p=13 Nf=63427  Nt=9979  Nh=754  Nh/Nt=0.07555867  1/p=0.07692307
[bf, Bf]=[100000, 500000]
p=19 Nf=63427  Nt=6850  Nh=389  Nh/Nt=0.05678832  1/p=0.05263157
[bf, Bf]=[100000, 500000]
p=31 Nf=63427  Nt=4316  Nh=139  Nh/Nt=0.03220574  1/p=0.03225806
```

The fact that \mathcal{R}_χ is much often non-trivial than \mathcal{C}_χ , in a computable proportion, is a positive argument for Vandiver's conjecture. We suggest that for totally real fields (like K_+), abelian p -ramification is essentially governed by the normalized p -adic regulator and that the p -class group is in some sense a "secondary" invariant.

5.2.7. Folk heuristic. Consider the Gauss sum $\tau(\psi) = -\sum_{k=0}^{\ell-2} \zeta_p^k \cdot \xi_\ell^{g^k}$ (where g is a primitive root modulo ℓ , $\zeta_p := \psi(g)$, see (6)), and put $k = ap + b$, $0 \leq a \leq \frac{\ell-1}{p} - 1$, $0 \leq b \leq p - 1$. Whence:

$$(13) \quad \tau(\psi) = -\sum_{b=0}^{p-1} \zeta_p^b \cdot [\mathrm{Tr}_{\mathbb{Q}(\xi_\ell)/F_\ell}(\xi_\ell)]^{\sigma(b)},$$

where F_ℓ is the cyclic subextension of degree p of $\mathbb{Q}(\xi_\ell)$, where $\sigma(b)$ is the automorphism acting trivially on ζ_p and such that $\xi_\ell \mapsto \xi_\ell^{g^b}$, giving an exact system of representatives for $\mathrm{Gal}(F_\ell/\mathbb{Q})$ independent of the choice of g .

From Remark 3.5 (ii), we know that F_ℓ is obtained as the decomposition over \mathbb{Q} of the extension $K(\sqrt[p]{\alpha})/K$, with $\alpha = \tau(\psi)^p \in \mathbb{Z}[\zeta_p]$, and it is immediate to see that the p -class group of

F_ℓ is trivial because of Chevalley's formula on invariant classes giving here $\#\mathcal{C}_{F_\ell}^{\text{Gal}(F_\ell/\mathbb{Q})} = 1$ since ℓ is the unique ramified prime in F_ℓ/\mathbb{Q} .

(i) The first observation is that the p -class group of F_ℓ does not depend on that of K as ℓ varies! Indeed, this context is neither more nor less than class field theory over \mathbb{Q} giving the existence of a unique cyclic extension F_ℓ of conductor $\ell \equiv 1 \pmod{p}$, for which one considers the set of conjugates of the relative trace of ξ_ℓ which moreover defines a normal basis of F_ℓ ; then the unique link with the arithmetic of K is the linear combination (13) involving the traces to built α , but the character of $\langle \alpha \rangle_{\mathbb{Z}[G]} K^{\times p}/K^{\times p}$ is ω which gives, as we know, a "poor" information on the arithmetic of K .

Thus, the relationship of $\alpha = \tau(\psi)^p$ (whence of $\tau(\psi)$) with class field theory over K (namely, with p -classes and units of K) is tenuous, possibly empty; which is quite the opposite for the twists $g_c(\ell)$ because of the relations $\alpha^{c-s_c} = g_c(\ell)^p$ and the fact that the $g_c(\ell)_{\chi^*}$ are radicals defining non-trivial (arithmetically) cyclic extensions of degree p of K_+ for any even character χ .

(ii) In another direction, suggested by the work of Lecouturier [33] generalizing results of Calegari–Emerton and Iimura, consider the non-Galois extension $\widetilde{F}_\ell := \mathbb{Q}(\sqrt[p]{\widetilde{\alpha}})$, where $\widetilde{\alpha} := \ell$; of course, $K(\sqrt[p]{\widetilde{\alpha}})/K$ is a cyclic extension of degree p (undecomposed over a strict subfield of K), ramified at the $p-1$ primes $\mathfrak{L} \mid \ell$ and at p if and only if $\ell \not\equiv 1 \pmod{p^2}$. On the contrary, as shown by many results of [33], the p -class group of \widetilde{F}_ℓ strongly depends on the arithmetic of K while the radical $\widetilde{\alpha}$ does not.

This second observation comes from the fact that, for $\widetilde{M} := K(\sqrt[p]{\widetilde{\alpha}})$:

$$\#\mathcal{C}_{\widetilde{M}}^{\text{Gal}(\widetilde{M}/K)} = \#\mathcal{C}_K \cdot \frac{p^{p-2+\delta}}{(E_K : E_K \cap N_{\widetilde{M}/K}(\widetilde{M}^\times))} \leq \#\mathcal{C}_K \cdot p^{\frac{p-1}{2}},$$

where $\delta = 1$ or 0 according as p ramifies or not and where ζ_p is norm for $\delta = 0$; but the non-abelian Galois structure yields various non-trivial p -class groups for \widetilde{F}_ℓ as ℓ varies, and genera theory implies $\text{rk}_p(\mathcal{C}_{\widetilde{F}_\ell}) \geq 1$ for all ℓ (for the metabelian genera theory, see [29]). However, for $M = K(\sqrt[p]{\alpha}) = F_\ell K$:

$$\#\mathcal{C}_M^{\text{Gal}(M/K)} = \#\mathcal{C}_K \cdot \frac{p^{p-2}}{(E_K : E_K \cap N_{M/K}(M^\times))} \leq \#\mathcal{C}_K \cdot p^{\frac{p-1}{2}},$$

and we left to the reader the computation of the (non-trivial) order of the minus part; but M/K decomposes into F_ℓ/\mathbb{Q} and only the isotopic component for the unit character is concerned, which gives in fact a trivial part of the above Chevalley's formula (contrary to the metabelian case \widetilde{M}/\mathbb{Q}). So the "folk heuristic" should be:

Because of F_ℓ defined by the radical $\alpha = \tau(\psi)^p$, the p -adic properties of the Gauss sums are independent of the arithmetic of K as ℓ varies (despite the apparent complexity of the radical $\alpha = \tau(\psi)^p$), while the properties of \widetilde{F}_ℓ are strongly dependent (despite the obvious simplicity of the radical $\widetilde{\alpha} = \ell$).

In other words we have probably some "dualities" about the arithmetic complexity of Kummer theory in the comparison "radicals versus extensions".

5.3. Additive p -adic statistics. Of course, we are only concerned with the multiplicative p -adic properties of the Gauss sums $\tau(\psi)$ and mainly of the twists $g_c(\ell)$, and these are given by their χ^* -components for $\chi \in \mathcal{X}_+$. Nevertheless, the additive properties seem to follow more explicit rules, which is an interesting information about the numerical repartition and the independence as ℓ varies, and this probably has an impact on the multiplicative properties regarding the results of § 4.3. We shall examine the case of the twists $g_c(\ell)$ (more precisely

of $\psi^{-c}(c)g_c(\ell)$, then the case of the original Gauss sums $\tau(\psi)$ from the arithmetic of the fields F_ℓ .

5.3.1. \mathbb{Z} -rank of the family $(\psi^{-c}(c)g_c(\ell))_{\ell \in \mathcal{L}_p}$. Put, for p and c fixed:

$$(14) \quad \mathbf{J}(\ell) := \psi^{-c}(c)g_c(\ell) = \psi^{-c}(c)\tau(\psi)^{c-\sigma_c} = J_1 \cdots J_{c-1} \text{ (see (10))}$$

written on the basis $\{1, \zeta_p, \dots, \zeta_p^{p-2}\}$, under the form $\mathbf{J}(\ell) = \sum_{k=0}^{p-2} a_k(\ell) \zeta_p^k$, the integers $a_k(\ell)$ being considered modulo p . A first information, about the p -adic repartition of the $\mathbf{J}(\ell)$ as ℓ varies, is to compute the \mathbb{F}_p -rank of the \mathbb{F}_p -matrix $(a_k(\ell))_{k,\ell}$. The following program gives *systematically*:

$$\text{Rank}_{\mathbb{F}_p} [(a_k(\ell))_{\ell,k}] = p - 4,$$

for all the primes $p \geq 7$ tested (rank 1 for $p = 3$ and rank 2 for $p = 5$), *despite the fact that the lines are not canonical* (up to circular permutations of their elements since $\mathbf{J}(\ell)$ is defined up to conjugation). We have verified it up to $p \leq 331$, an interval which contains 16 irregular primes. The program gives p , the \mathbb{F}_p -rank of the matrix (in rank) and the least ℓ_p (in elp) for which the sub-matrix built from $\{\ell \in \mathcal{L}_p, \ell \leq \ell_p\}$ has rank $p - 4$:

```
{forprime(p=7,500,c=lift(znprimroot(p));P=polcyclo(p)+Mod(0,p);M=matrix(0,p-1);
r=0;for(i=1,10^8,el=1+2*i*p;if(isprime(el)!=1,next);g=znprimroot(el);J=1;
for(i=1,c-1,Ji=0;for(k=1,el-2,kk=znlog(1-g^k,g);e=lift(Mod(kk+i*k,p));
Ji=Ji-x^e);J=J*Ji);J=lift(Mod(J,P));V=vector(p-1,j,component(J,j));
A=concat(M,V);rr=matrank(A);if(rr==r,next);r=rr;M=A;
if(r==p-4,print("p=",p," r=",r," elp=",el);break))}
```

p	rank	elp	p	rank	elp	p	rank	elp	p	rank	elp
7	3	113	11	7	397	13	9	599	17	13	1259
(...)											
71	67	42743	73	69	48473	79	75	50087	83	79	65239
151	147	247943	157	153	273181	163	159	294053	167	163	305611

We have $\mathbf{J}(\ell) \equiv 1 \pmod{p}$, in other words $\sum_{k=0}^{p-2} a_k(\ell) \equiv 1 \pmod{p}$, and we can write $\mathbf{J}(\ell) = 1 + \sum_{k=1}^{p-2} a_k(\ell) (\zeta_p^k - 1)$ depending on $p - 2$ parameters; then, due to the relations $\mathbf{J}(\ell)^{1+s-1} \equiv 1 \pmod{p}$ and $\mathbf{J}(\ell)^{e_\omega} \in K^{\times p}$ (because $\omega(c - s_c) \equiv 0 \pmod{p}$), this yields the three relations of “derivation” (for $p \geq 7$) $\sum_{k=1}^{p-2} k^\delta \cdot a_k(\ell) \equiv 0 \pmod{p}$, $\delta \in \{1, 2, 4\}$, for any $\ell \in \mathcal{L}_p$. Whence a \mathbb{F}_p -rank at most $p - 4$, but we have no proof of the equality.

The order of magnitude of ℓ_p seems to be $O(1)p^2 \log(p^2)$, which is in agreement with a “conductor” p^2 for these Hecke Grössencharacters [55, Theorem, p.489], but the program slows down very much, as p increases, to be more accurate.

Moreover, the number of consecutive primes ℓ needed to reach the rank $p - 4$ is equal to $p - 4$, except probably for finitely many cases, which confirms the above order. Give now the end of the above table with an estimation of the $O(1)$:

p	elp	0(1)	p	elp	0(1)	p	elp	0(1)	p	elp	0(1)
211	517373	1.0856	223	628861	1.1693	227	604729	1.0816	229	631583	1.1082
233	642149	1.0849	239	695491	1.1116	241	684923	1.0750	251	784627	1.1269
257	862493	1.1766	263	819509	1.0631	269	928051	1.1461	271	906767	1.1019
277	925181	1.0719	281	1055437	1.1853	283	979747	1.0834	293	988583	1.0136
307	1174583	1.0881	311	1214767	1.0941	313	1203799	1.0692	317	1276243	1.1026

The \mathbb{F}_p -rank $r_p(\ell)$ of the $p - 1$ conjugates of $\mathbf{J}(\ell) \pmod{p}$, $\ell \in \mathcal{L}_p$, is close to $p - 4$ (e.g., for $p = 37$, $r_{37}(\ell) \in \{33, 32, 31, 30\}$ in similar proportions, and we only have the local minimum $(r_{37}(\ell), \ell) = (29, 2591)$ for ℓ up to 37000.

5.3.2. Repartition of the conjugates of the traces $\text{Tr}_{\mathbb{Q}(\xi_\ell)/F_\ell}(\xi_\ell)$. Let Z_{F_ℓ} be the ring of integers of F_ℓ and let Z_{F_ℓ}/pZ_{F_ℓ} be the residue ring modulo p . These residue rings are isomorphic to \mathbb{F}_{p^p} or to \mathbb{F}_p^p , but there is no canonical map between them as $\ell \in \mathcal{L}_p$ varies.

Thus, in the expression (13) giving $\tau(\psi) = -\sum_{b=0}^{p-1} \zeta_p^b \cdot [\text{Tr}_{\mathbb{Q}(\xi_\ell)/F_\ell}(\xi_\ell)]^{\sigma(b)}$, the images in Z_{F_ℓ}/pZ_{F_ℓ} of the conjugates of the relative traces $\text{Tr}(\xi_\ell) := \text{Tr}_{\mathbb{Q}(\xi_\ell)/F_\ell}(\xi_\ell)$ may be easily analysed and compared, for $\ell \in \mathcal{L}_p$, by means of the image R_ℓ in $\mathbb{F}_p[x]$ of the polynomial $Q_\ell = \prod_{\bar{\sigma} \in \text{Gal}(F_\ell/\mathbb{Q})} (x - \text{Tr}(\xi_\ell)^{\bar{\sigma}}) \in \mathbb{Z}[x]$.

Proposition 5.3. *Let $\ell_1, \ell_2 \in \mathcal{L}_p$ and let $\tau(\psi_1), \tau(\psi_2)$ be the corresponding Gauss sums normalized via $\psi_1(g_1) = \psi_2(g_2) = \zeta_p$. Let $F = F_{\ell_1}F_{\ell_2}$. If $R_{\ell_1} \neq R_{\ell_2}$, then for all $\sigma \in \text{Gal}(FK/\mathbb{Q})$, $\tau(\psi_2) \not\equiv \tau(\psi_1)^\sigma \pmod{\mathfrak{p}^p Z_{FK}}$.*

Proof. Suppose there exists $\sigma \in \text{Gal}(FK/\mathbb{Q})$ such that $\tau(\psi_2) \equiv \tau(\psi_1)^\sigma \pmod{\mathfrak{p}^p Z_{FK}}$; recall that $\tau(\psi_1)^\sigma = \zeta_\sigma \tau(\psi_1^e)$, $\zeta_\sigma \in \mu_p$, $e \in (\mathbb{Z}/p\mathbb{Z})^\times$. Then:

$$\tau(\psi_2) = -\sum_{b=0}^{p-1} \zeta_p^b \cdot \text{Tr}(\xi_{\ell_2})^{\sigma_2(b)} \quad \text{and} \quad \tau(\psi_1)^\sigma = -\sum_{b=0}^{p-1} \zeta_p^b \cdot \text{Tr}(\xi_{\ell_1})^{\pi(\sigma_1(b))},$$

where π is a permutation of the $\sigma_1(b)$. Using $\text{Tr}_{\mathbb{Q}(\xi_{\ell_i})/\mathbb{Q}}(\xi_{\ell_i}) = -1$, we get:

$$\tau(\psi_2) = 1 - \sum_{b=1}^{p-1} (\zeta_p^b - 1) \cdot \text{Tr}(\xi_{\ell_2})^{\sigma_2(b)}, \quad \tau(\psi_1)^\sigma = 1 - \sum_{b=1}^{p-1} (\zeta_p^b - 1) \cdot \text{Tr}(\xi_{\ell_1})^{\pi(\sigma_1(b))}, \quad \text{whence:}$$

$$\tau(\psi_1)^\sigma - \tau(\psi_2) = \sum_{b=1}^{p-1} (\zeta_p^b - 1) \cdot (\text{Tr}(\xi_{\ell_2})^{\sigma_2(b)} - \text{Tr}(\xi_{\ell_1})^{\pi(\sigma_1(b))}) \equiv 0 \pmod{\mathfrak{p}^p Z_{FK}}.$$

Since the $\zeta_p^b - 1$, $b \in [1, p-1]$, define a \mathbb{Z} -basis of $\mathfrak{p}Z_K$, then a Z_F -basis of $\mathfrak{p}Z_{FK}$, this relation implies $\text{Tr}(\xi_{\ell_2})^{\sigma_2(b)} \equiv \text{Tr}(\xi_{\ell_1})^{\pi'(\sigma_1(b))} \pmod{p}$ for all b , which yields $R_{\ell_1} = R_{\ell_2}$ in $\mathbb{F}_p[x]$ (contradiction). \square

Since $\tau(\psi_2) \not\equiv \tau(\psi_1)^\sigma \pmod{\mathfrak{p}^p}$ for all σ implies $g_c(\ell_2) \not\equiv g_c(\ell_2)^\sigma \pmod{\mathfrak{p}^p}$ for all σ (except for the ω -components because $\omega(c - \sigma_c) \equiv 0 \pmod{p}$), we can say that the number of distinct polynomials R_ℓ , $\ell \in \mathcal{L}_p$, gives a partial idea of the repartition modulo p of the sets $\mathcal{E}_\ell(p)$ as ℓ varies. As p increases, the number of distinct R_ℓ seems to be $O(p^2 \cdot \log(p^2))$.

The following program, computing the monic polynomial $R = R_\ell \in \mathbb{F}_p[x]$ returns: $e1 = \ell$, the residue degree $f = f$ of p in F_ℓ/\mathbb{Q} , and R .

```
{p=7;B=5*10^3;e1=1;while(e1<B,e1=e1+2*p;if(isprime(e1)!=1,next);g=znprimroot(e1);
h=g^p;g=lift(g);h=lift(h);P=polcyclo(e1);z=Mod(x,P);Q=1;e=1;for(k=1,p,Tr=0;e=e*g;
for(j=1,(e1-1)/p,e=e*h;e=lift(Mod(e,e1));Tr=Tr+z^e;Q=Q*(T-Tr));
Q=component(lift(Q),1);R=0; for(i=0,p,C=component(Q,i+1);C=lift(Mod(C,p));
R=R+x^i*C);F=znorder(Mod(p,e1));f=1;v=valuation(F,p);w=valuation(e1-1,p);
if(w==v,f=p);print("e1=",e1," f=",f," R=",R)}
```

```
e1=29    f=7      R=x^7 + x^6 + 2*x^5 + 5*x + 1
e1=43    f=1      R=x^7 + x^6 + 3*x^5 + 3*x^3 + 6*x^2
e1=71    f=7      R=x^7 + x^6 + 5*x^5 + 3*x^4 + 2*x^3 + 6*x^2 + 4
e1=4943  f=7      R=x^7 + x^6 + 3*x^5 + x^4 + x^3 + 3*x + 5
e1=4957  f=7      R=x^7 + x^6 + 4*x^5 + 2*x^4 + 5*x^3 + 3*x^2 + 2*x + 1
e1=4999  f=7      R=x^7 + x^6 + 4*x^3 + 5*x^2 + 2*x + 6
```

It is hopeless to write wide lists of polynomials R_ℓ for large p , but any experiment suggests a random distribution of the (non-independent) coefficients (except that of x^{p-1} since $\text{Tr}_{\mathbb{Q}(\xi_\ell)/\mathbb{Q}}(\xi_\ell) = -1$). For $p = 3$ the six possible polynomials are of the form R_ℓ . For $p = 5$ (resp. $p = 7$) there are 150 (resp. 17192) possible polynomials.

(i) For $p = 5$, we obtain the following end of the calculations (two days of computer; it seems that only 35 distinct polynomials R_ℓ are available):

e1=5591	f=5	R=x^5 + x^4 + 4*x^3 + x^2 + 4*x + 2
e1=6211	f=1	R=x^5 + x^4 + x^3 + x^2 + x
e1= 6271	f=1	R=x^5 + x^4 + 2*x^3 + 4*x^2 + 3*x + 4
e1=1345	f=1	R=x^5 + x^4

(ii) For $p = 7$, ℓ up to 17977, we get painfully a little more than 250 distinct R_ℓ , but the exact number is unknown.

Remark 5.4. It is clear that a large number of polynomials R_ℓ strengthens Vandiver's conjecture since the corresponding $\mathbf{J}(\ell) = \psi^{-c}(c) g_c(\ell)$ cover sufficiently possibilities modulo p , especially since we know that the \mathbb{F}_p -rank associated to the family of $(\mathbf{J}(\ell))_{\ell \in \mathcal{L}_p}$ is probably always $p - 4$, but these informations are not "equivalent". Moreover, an assumption about the order of magnitude of $\mathcal{N}_p := \#\{R_\ell, \ell \in \mathcal{L}_p\}$ is *not necessary* to obtain Vandiver's conjecture for p ; indeed, a *single* suitable ℓ may ensure a positive test for Vandiver's conjecture as shown by the table given in § 4.2.2.

We propose the following heuristic, about the sets $\mathcal{E}_\ell(p)$ of exponents of p -primarity for which the reference [25] may be useful:

The probability of $\mathcal{E}_\ell(p) = \emptyset$, for a single $\ell \in \mathcal{L}_p$, is $(1 + o(1)) \cdot e^{-\frac{1}{2}}$; that of at least a counterexample to Vandiver's conjecture is of the form $O(1) (1 - e^{-\frac{1}{2}})^{\mathcal{N}_p}$, where $\mathcal{N}_p := \#\{R_\ell, \ell \in \mathcal{L}_p\}$, with the polynomial $R_\ell = \prod_{\bar{\sigma} \in \text{Gal}(F_\ell/\mathbb{Q})} (x - \text{Tr}_{\mathbb{Q}(\xi_\ell)/F_\ell}(\xi_\ell)^{\bar{\sigma}})$ seen in $\mathbb{F}_p[x]$.

6. Conclusion

Under these experiments and heuristics, the existence of sets $\mathcal{E}_\ell(p)$, disjoint from $\mathcal{E}_0(p)$, or probably the existence of primes $\ell \in \mathcal{L}_p$ such that $\mathcal{E}_\ell(p) = \emptyset$, may occur conjecturally for all p . Possibly, our computations in § (4.2.2) show the existence of general properties of the sets $\mathcal{E}_\ell(p)$ coming from the fact that *all* $\ell \in \mathcal{L}_p$ *intervene* (and that these primes are probably independent), which is a new argument compared with classical ones. This is strengthened by the computation of the conjugates of the traces $\text{Tr}_{\mathbb{Q}(\xi_\ell)/F_\ell}(\xi_\ell)$, as $\ell \in \mathcal{L}_p$ varies (coefficients of the Gauss sums), the fields $\mathbb{Q}(\mu_\ell)$ being, a priori, independent of the arithmetic of K .

Remark 6.1. There are two constraints, for the Gauss and Jacobi sums that we have considered, but they only concern the auxiliary prime numbers $\ell \in \mathcal{L}_p$:

(i) The p -classes of ideals $\mathfrak{L} \mid \ell$, $\ell \in \mathcal{L}_p$, are all represented with standard densities.

(ii) The ideal factorization of $\tau(\psi)^p$ is related to *congruences modulo the conjugates of a prime ideal $\mathfrak{L} \mid \ell$ and is canonical* (this yields Stickelberger's theorem and its consequences [52, § 15.1], [6, 55], and [43] for the annihilation of $\mathcal{O}_{\chi_0}^{(p)}$ with generalizations of the Stickelberger ideal). A similar context is that of the ℓ -adic Γ -function of Morita.

However, since we consider characters ψ of order p , the *p -adic congruential properties* of Gauss sums (or Jacobi sums) do not follow any known law (in our opinion, the classical literature being mute about this).

These fundamental p -adic properties of Gauss sums may have crucial consequences in various domains since Vandiver's conjecture is often required; for instance:

In [8] about the Galois cohomology of Fermat curves, in [47] for the root numbers of the Jacobian varieties of Fermat curves, then in several papers on Galois p -ramification theory as in [36, 44, 45, 46], or [53, 54] in relation with modular forms, then in numerous papers

and books on the theory of deformations of Galois representations as in [2, 37], Iwasawa's theory context and cyclotomy, as in [7] on Ihara series, [5] for μ -invariants in Hida families, [32] for the main conjecture of the Iwasawa theory).

Then it may be legitimate to think that all these numerous basic congruential aspects are (logically) governing principles of a wide part of algebraic number theory, as follows, beyond the case of the p th cyclotomic field (not to mention all the geometrical aspects as the theory of elliptic curves where some analogies can be found, and all the generalizations of the present abelian case over a number field $k \neq \mathbb{Q}$):

Gauss and Jacobi sums \longrightarrow *Hecke Grössencharacters* \longrightarrow *Stickelberger element*
 \longrightarrow *p -adic L -functions* \longrightarrow *Herbrand–Ribet theorem* \longrightarrow *Main Theorem on abelian fields*
 \longrightarrow *annihilation of the p -torsion group \mathcal{T} of real abelian fields* \longrightarrow *universal isomorphism $\mathcal{T} \simeq H^2(G_{S_p}, \mathbb{Z}_p)^*$* \longrightarrow *p -rationality of fields ($\mathcal{T} = 1$)* \longrightarrow *cohomological obstructions in Galois theory* \longrightarrow \dots

Which gives again an example of a *basic p -adic problem*, analogous to those we have analysed about deep conjectures: Greenberg's conjectures (on Iwasawa theory over totally real fields [22] and on representation theory [24]), p -rationalities of a number field as $p \rightarrow \infty$, generalizations of the conjecture of Ankeny–Artin–Chowla from the conjectural existence of a p -adic Brauer–Siegel theorem [20]...

As shown by the evidences given in §5.2, Vandiver's conjecture may be justified, for $p \gg 0$, by the Borel–Cantelli heuristic, on exceptional features of Gauss sums; but this point of view allows cases of failure of the conjecture, which is not satisfactory for the theoretical foundations of the above quoted fundamental subjects.

To be optimistic (but not very rigorous), one can say that Vandiver's conjecture is true because it holds for sufficiently many prime numbers [4, 9] since probabilities may be in $\frac{O(1)}{p^{\lambda(p)}}$, $\lambda(p) \rightarrow \infty$. In a more serious claim, we can say that Vandiver's conjecture holds for almost all prime numbers; the accurate cardinality of the finite set of counterexamples (\emptyset or not) is (in our opinion) not of algebraic nature nor enlightened by class field theory, Galois cohomology or Iwasawa's theory, but is perhaps accessible by the way of analytical/geometrical techniques or depends on a more general hypothetic “complexity theory” in number theory.

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