



HAL
open science

ON A MIXTURE OF BRENIER AND STRASSEN THEOREMS

Nathael Gozlan, Nicolas Juillet

► **To cite this version:**

Nathael Gozlan, Nicolas Juillet. ON A MIXTURE OF BRENIER AND STRASSEN THEOREMS. 2018. hal-01855463v1

HAL Id: hal-01855463

<https://hal.science/hal-01855463v1>

Preprint submitted on 8 Aug 2018 (v1), last revised 21 Dec 2018 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON A MIXTURE OF BRENIER AND STRASSEN THEOREMS

NATHAEL GOZLAN AND NICOLAS JUILLET

ABSTRACT. We give a characterization of optimal transport plans for a variant of the usual quadratic transport cost introduced in [31]. Optimal plans are composition of a deterministic transport given by the gradient of a continuously differentiable convex function followed by a martingale coupling.

1. INTRODUCTION

Given two probability measures μ, ν on \mathbb{R}^d , we recall that a coupling between μ and ν is a probability measure π on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$\pi(A \times \mathbb{R}^d) = \mu(A) \quad \text{and} \quad \pi(\mathbb{R}^d \times B) = \nu(B),$$

for all Borel sets A, B of \mathbb{R}^d . The set of all couplings between μ and ν will be denoted by $C(\mu, \nu)$ in all the paper. In the sequel, it will be convenient to represent a coupling $\pi \in C(\mu, \nu)$ in the following disintegrated form

$$(1) \quad d\pi(x, y) = d\mu(x)dp_x(y),$$

where $x \mapsto p_x$ is a (μ -almost surely unique) probability kernel. By a slight abuse of vocabulary, a couple (X, Y) of random variables such that $X \sim \mu$ and $Y \sim \nu$ will be also called a coupling of μ and ν .

1.1. Existence of structured couplings between probability measures. The question to construct couplings between probability measures having some nice structures or properties is natural both in probability theory and analysis. Let us recall two famous results guarantying the existence of couplings with rigid but very different structures, namely the theorems of Brenier and Strassen.

A fundamental result by Brenier [11, 12] (see [54, §1.3] for a discussion of the bibliography around this result) shows that if μ is say absolutely continuous with respect to Lebesgue measure, then there exists some convex function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\nabla\varphi$ pushes forward μ onto ν . In other words, there exists a *deterministic* coupling π^* of the form

$$d\pi^*(x, y) = d\mu(x)\delta_{\nabla\varphi(x)}(y).$$

Moreover, assuming in addition that μ and ν have finite second moments, this π^* is the unique optimal coupling in the Monge-Kantorovich transport problem for the quadratic cost:

$$\iint |x - y|^2 d\pi^*(x, y) = \inf_{\pi \in C(\mu, \nu)} \iint |x - y|^2 d\pi(x, y),$$

denoting by $|\cdot|$ the standard Euclidean norm on \mathbb{R}^d . This result has then been extended to various cost functions [24] and state spaces [53, 26, 3, 22, 21, 19, 7, 27] and has had numerous applications in PDE, geometric analysis or probability theory, see [2, 4, 69, 70] and the references therein.

On the other hand, from a probabilistic point of view it is natural to investigate the existence of *martingale couplings*, that is to say to look for couplings $\pi \in C(\mu, \nu)$ which

Date: August 8, 2018.

1991 Mathematics Subject Classification. 60A10, 49J55, 60G42.

Key words and phrases. Optimal transport, Martingales, Brenier Theorem, Strassen Theorem.

correspond to the law of a martingale $(X_t)_{t \in \{0;1\}}$. This martingale requirement means that the kernel p appearing in (1) satisfies

$$\int y dp_x(y) = x, \quad \text{for } \mu \text{ almost every } x \in \mathbb{R}^d.$$

As for the Brenier Theorem, it turns out that the existence of a martingale coupling between μ and ν is not automatic. Assuming that μ and ν have finite first moments, a necessary and sufficient condition is given by Strassen Theorem [66]: there exists a martingale coupling between μ and ν if and only if μ is dominated by ν in the convex order. Recall that given two probability measures with finite first moments μ and ν , one says that μ is dominated by ν in the convex order if $\int f d\mu \leq \int f d\nu$ for all convex functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$. This is denoted as follows in the sequel: $\mu \preceq_c \nu$. Note that Strassen's result has been generalized at the time continuous process level by Kellerer [44, 45] (see also [5]): if a family $(\mu_t)_{t \geq 0}$ of probability measures on \mathbb{R} is increasing for the convex order (a so-called *peacocks* using the terminology of [35]), then there exists a process $(X_t)_{t \geq 0}$ which is together *Markovian* and a martingale such that $X_t \sim \mu_t$ for all $t \geq 0$. See the monograph [36] on peacocks, [49, 38] for the approach by Lowther and [42, 37, 9] for extensions.

1.2. In between Brenier and Strassen. Given two probability measures μ, ν on \mathbb{R}^d having their first moments finite, it is not always possible to go from μ to ν using a deterministic mapping given by the gradient of a convex function, nor to go from μ to ν using a martingale coupling. In this paper we will be interested in couplings obtained by composition of these two classical transport methods, which as we will see below always exist.

More precisely, we will look for couplings (X, Y) of μ and ν such that there exists a convex function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ of class \mathcal{C}^1 such that

$$(2) \quad \mathbb{E}[Y|X] = \nabla\varphi(X), \quad \text{a.s.}$$

Introducing $X' = \nabla\varphi(X)$, we can see (X, Y) as the initial and final states of a (time inhomogeneous) Markov chain (X, X', Y) where the first transition step is deterministic and given by the gradient of a convex function and the second transition step is a martingale. As mention above, such couplings always exist: simply choosing X and Y independent gives a trivial example.

The purpose of our main result (Theorem 1.2 below) is to distinguish some special couplings (X, Y) among those satisfying (2) by showing that they are solutions of an optimal transport problem that we shall now present.

Let $\theta : \mathbb{R}^d \rightarrow \mathbb{R}^+$ be some convex function; using the terminology of [31] the barycentric optimal transport cost between μ and ν is defined by

$$(3) \quad \overline{\mathcal{T}}_\theta(\nu|\mu) = \inf_{\pi \in C(\mu, \nu)} \int \theta \left(x - \int y dp_x(y) \right) d\mu(x).$$

In probabilistic notations, this optimal transport cost can be expressed as follows:

$$\overline{\mathcal{T}}_\theta(\nu|\mu) = \inf \mathbb{E} [\theta(X - \mathbb{E}[Y|X])]$$

where the infimum runs over the set of random vectors (X, Y) such that $X \sim \mu$ and $Y \sim \nu$. If $d\pi(x, y) = d\mu(x)dp_x(y) \in C(\mu, \nu)$ is such that $\overline{\mathcal{T}}_\theta(\nu|\mu) = \int \theta(x - \int y dp_x(y)) d\mu(x)$, we call it an optimal transport plan (or coupling) from μ to ν (for the cost $\overline{\mathcal{T}}_\theta$). These barycentric costs are actually part of a more general family of transport costs, called ‘weak transport costs’, introduced by the first author together with Roberto, Samson and Tetali in [31] (see Section 1.4 below for a more complete presentation).

In this paper, we will mainly focus on the barycentric transport cost, denoted by $\overline{\mathcal{T}}_2$ in the sequel, associated to the function $\theta(u) = |u|^2$, $u \in \mathbb{R}^d$: for all $\mu, \nu \in \mathbb{R}^d$ having finite

first moments,

$$\overline{\mathcal{T}}_2(\nu|\mu) = \inf_{\pi \in C(\mu, \nu)} \int \left| x - \int y dp_x(y) \right|^2 d\mu(x).$$

In [31, Theorem 2.11], the following Kantorovich type duality formula has been obtained:

Theorem 1.1. *If μ, ν have finite first moments then*

$$\overline{\mathcal{T}}_\theta(\nu|\mu) = \sup_f \left\{ \int Q_\theta f d\mu - \int f d\nu \right\},$$

where

$$Q_\theta f(x) = \inf_{y \in \mathbb{R}^d} \{f(y) + \theta(x - y)\}, \quad x \in \mathbb{R}^d,$$

and where the infimum runs over the set of all functions f which are convex, Lipschitz and bounded from below.

Several applications of this duality formula were already investigated in [31], mainly in connection with transport-entropy inequalities and deviation inequalities for convex functions. In particular, let us mention that Strassen Theorem can be derived from this duality theorem, see [31, Section 3]. Here we shall use this duality result to describe optimal transport plans for $\overline{\mathcal{T}}_2$.

Before stating our main result, we need some preparation. First recall that the transport distance W_2 is defined for all probability measures μ, ν having finite second moments by

$$W_2^2(\mu, \nu) = \inf_{\pi \in C(\mu, \nu)} \iint |x - y|^2 d\pi(x, y).$$

Given a probability $\nu \in \mathcal{P}(\mathbb{R}^d)$ (the set of all Borel probability measures on \mathbb{R}^d), we denote by

$$C_\nu = \{\eta \in \mathcal{P}(\mathbb{R}^d) : \eta \preceq_c \nu\}$$

the set of all probability measures which are dominated by ν in the convex order. This set C_ν is easily seen to be convex in the usual sense and the proof of Proposition 1.1 will show that (generalized) W_2 -geodesics with endpoints in C_ν are contained in C_ν .

Proposition 1.1. *Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ be probability measures with finite second moment. There exists a unique probability measure $\mu^* \in C_\nu$ such that*

$$W_2(\mu^*, \mu) = \inf_{\eta \in C_\nu} W_2(\eta, \mu) = \overline{\mathcal{T}}_2(\nu|\mu).$$

We call μ^* the projection of μ on C_ν .

We recall that if $h : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is some convex function, the Legendre transform h^* of h is defined by

$$h^*(y) = \sup_{x \in \mathbb{R}^d} \{x \cdot y - h(x)\}, \quad y \in \mathbb{R}^d.$$

With these notions in hand, we can now state the main result of this paper which describes the set of all optimal couplings for $\overline{\mathcal{T}}_2$.

Theorem 1.2. *Let μ and ν be probability measures with compact support on \mathbb{R}^d .*

(a) *There exists some convex function $\bar{f} : \mathbb{R}^d \rightarrow \mathbb{R}$ such that*

$$(4) \quad \overline{\mathcal{T}}_2(\nu|\mu) = \int Q_2 \bar{f} d\mu - \int \bar{f} d\nu,$$

where, for any function $g : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$Q_2 g(x) = \inf_{y \in \mathbb{R}^d} \{g(y) + |x - y|^2\}, \quad x \in \mathbb{R}^d.$$

(b) Let h and φ be the convex functions defined by

$$h(x) = \frac{\bar{f}(x) + |x|^2}{2}, \quad x \in \mathbb{R}^d \quad \text{and} \quad \varphi(y) = h^*(y), \quad y \in \mathbb{R}^d.$$

The function φ is \mathcal{C}^1 -smooth on \mathbb{R}^d and the map $\nabla\varphi$ is 1-Lipschitz on \mathbb{R}^d . The projection μ^* of μ on C_ν is such that

$$\mu^* = \nabla\varphi\#\mu.$$

(c) There exists random vectors X, X^*, Y with $X \sim \mu$, $X^* \sim \mu^*$ and $Y \sim \nu$ and such that $X^* = \nabla\varphi(X)$ and (X^*, Y) is a martingale. For any such random vectors, it holds

$$\bar{\mathcal{T}}_2(\nu|\mu) = \mathbb{E} \left[|X - \mathbb{E}[Y|X]|^2 \right].$$

Conversely, if (X', Y') is a coupling of μ, ν such that $\bar{\mathcal{T}}_2(\nu|\mu) = \mathbb{E}[|X' - \mathbb{E}[Y'|X']|^2]$, then $\mathbb{E}[Y'|X']$ has law μ^* , $\mathbb{E}[Y'|X'] = \nabla\varphi(X')$ almost surely and $(\mathbb{E}[Y'|X'], Y')$ is a martingale (which is always true).

Remark 1.1.

- We underline the fact that there is no assumption on the measure μ (as for instance the condition that it does not give mass to small sets, which is classical for Brenier transport, see [26] for a minimal condition).
- It follows also from Item (b), that the optimal transport between μ and its projection μ^* onto the set C_ν is always Lipschitz continuous (without again any assumption on μ).
- The identity

$$\bar{\mathcal{T}}_2(\nu|\mu) = \inf_{\eta \in C_\nu} W_2^2(\eta, \mu)$$

of Proposition 1.1 was already observed in [29, Proposition 3.1] in dimension 1 when μ has no atom and then generalized to higher dimensions in [63, Proposition 4.1] when μ is absolutely continuous with respect to Lebesgue.

Before presenting the organization of the paper, let us mention that the methods and results developed in this work can be easily generalized to study other barycentric (or even more general) optimal transport problems. We focus on the particular case of $\bar{\mathcal{T}}_2$ to avoid unnecessary generality and make the reading of this paper more fluid.

1.3. Other results and organization of the paper. Let us now describe the content of the paper.

Section 2 addresses the question of characterizing equality cases between W_2^2 and $\bar{\mathcal{T}}_2$. We show in Theorem 2.1, that $W_2^2(\nu, \mu) = \bar{\mathcal{T}}_2(\nu|\mu)$ if and only if there exists some continuously differentiable convex function φ such that $\nabla\varphi$ is 1-Lipschitz and $\nu = \nabla\varphi\#\mu$. In other words, this result shows that the Brenier map from μ to ν is a contraction if and only if $\mu^* = \nu$. This observation is then used in connection with the classical Caffarelli Theorem [13] showing that probability measures with a log-concave density with respect to the standard Gaussian are contractions of it.

In Section 3, we generalize the notion of c -monotonicity to the case of the transport cost $\bar{\mathcal{T}}_2$. We illustrate the power of this notion, by giving a new proof of Strassen Theorem for submartingale in dimension 1.

In Section 4, we study a concrete example and describe the function $\nabla\varphi$ and the projected measure μ^* when μ is some arbitrary probability measure on \mathbb{R}^d (with finite second moment) and ν is a discrete probability measure concentrated on a simplex of \mathbb{R}^d .

Section 5 recalls some other examples appearing in the recent literature.

Finally, Section 6 contains the proof of Proposition 1.1 and Theorem 1.2 (and some auxiliary lemmas).

1.4. More about weak optimal transport costs. As mentioned above, the barycentric transport costs (3) enter a more general family of transport costs, that we will now recall.

Let (X, d) be a polish space and $\mathcal{P}(X)$ be the set of Borel probability measures on X . Given a cost function $c : X \times \mathcal{P}(X) \rightarrow \mathbb{R}^+$ and probability measures μ, ν on X the optimal weak transport cost $\mathcal{T}_c(\nu|\mu)$ from μ to ν is defined by

$$\mathcal{T}_c(\nu|\mu) = \inf_{\pi \in C(\mu, \nu)} \int c(x, p_x) d\mu(x),$$

where π and p are related together by (1). Note that this definition makes sense under mild regularity assumptions on c , that we will not detail in this exposition.

When $c(x, p) = \int \rho(x, y) dp(y)$ for some measurable non-negative function ρ on X^2 , one recovers the usual Monge-Kantorovich optimal transport cost

$$\mathcal{T}_\rho(\mu, \nu) = \inf_{\pi \in C(\mu, \nu)} \int \rho(x, y) d\pi(x, y).$$

On the other hand, the optimal barycentric transport costs presented above corresponds to cost functions of the form $c(x, p) = \theta(x - \int y dp(y))$.

The first appearance of this form of transport cost goes back to the works of Marton [52, 51] and Talagrand [67, 68] in connections with the concentration of measure phenomenon via the so-called transport-entropy inequalities (see [46] for an introduction to concentration of measure phenomenon and [28] for a survey on transport-entropy inequalities). For instance, Marton considers in [51] the following simple cost function:

$$c_{\text{Marton}}(x, p) = \left(\int \mathbf{1}_{x \neq y} dp(y) \right)^2, \quad x \in X,$$

and recovers a fundamental concentration of measure result for product probability measures of Talagrand [67]. After Marton, some other universal concentration of measures results by Talagrand were recovered or improved by Dembo [18] and Samson [60, 61, 62] using variants of Marton's cost c_{Marton} . Motivated by questions related to concentration and curvature properties of discrete measures, the paper [31] introduces the general definition given above and studies some of the properties of this new class of transport costs. In particular Kantorovich type duality formulas are obtained [31, Theorem 9.6] under the assumption that c is convex with respect to the p variable (and some additional mild regularity conditions). We refer to [30, 29, 64, 65, 20] for works directly connected to [31] and to [63] for an up-to-date survey of applications of weak transport costs to concentration of measure.

Besides their many applications in the field of functional inequalities and concentration of measure, it turns out that weak transport costs are also interesting in themselves as a natural generalization of the transportation problem. Indeed, the main interest of this definition is that it enables the introduction of additional constraints to the transfers of mass. For instance, a cost function of the form

$$c(x, p) = \begin{cases} \int \rho(x, y) dp(y) & \text{if } \int y dp(y) = x \\ +\infty & \text{otherwise} \end{cases}$$

where $\rho : \mathbb{R}^d \times \mathbb{R}^d$ is some non-negative measurable function gives back the notion of optimal transport with martingale constraint: if $\mu \preceq_c \nu$ (say compactly supported to avoid definition problem), then

$$\mathcal{T}_c(\nu|\mu) = \inf \{ \mathbb{E}[\rho(X, Y)] : X \sim \mu, Y \sim \nu, (X, Y) \text{ is a martingale} \}.$$

This optimal transport problem with martingale constraint has been thoroughly studied in [6] for the dimension 1. There, the martingale transport problem is studied for particular families of costs satisfying the cross derivative condition $\partial_y^2 \partial_x \rho < 0$ giving rise to the left-curtain coupling (on this coupling, see also [33, 43, 41]) and for the cost functions

$\rho : (x, y) \mapsto \pm|y - x|$, generalizing results by Hobson and coauthors [40, 39]. The supermartingale problem in dimension 1 with cross-derivative condition is studied in [56]. In higher dimension $\rho : (x, y) \mapsto \pm\|y - x\|$ is studied in [25, 48], more general costs are considered in [50]. We refer to [1] for existence results in the dual optimal transport problem with martingale constraints making use of the formulation in terms of weak cost as above.

As observed by Conforti [14] and Conforti and Ripani [15] the class of weak transport costs also entails the notion of entropic costs related to the Schrödinger problem (see the nice survey by C. Léonard [47]). We refer to the recent articles by Alibert, Bouchitté and Champion [1] (see also Section 5) and Bowles and Ghoussoub [10] for further developments and examples.

2. LINK WITH CAFFARELLI'S CONTRACTION THEOREM

In this section we investigate the question of determining in which case

$$W_2^2(\nu, \mu) = \overline{\mathcal{T}}_2(\nu|\mu)$$

and we show how this question is related (when μ is the standard Gaussian) to Caffarelli's contraction theorem of [13].

Theorem 2.1. *Let μ, ν be two compactly supported probability measures on \mathbb{R}^d equipped with the standard Euclidean norm. The following statements are equivalent:*

- (a) *There exists a convex function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ of class \mathcal{C}^1 such that $\nabla\varphi$ is 1-Lipschitz such that $\nu = \nabla\varphi\#\mu$.*
- (b) *The projection μ^* of μ on the set $C_\nu = \{\eta \in \mathcal{P}(\mathbb{R}^d) : \eta \preceq_c \nu\}$ is equal to ν .*
- (c) *It holds $W_2^2(\nu, \mu) = \overline{\mathcal{T}}_2(\nu|\mu)$.*

Let us denote by γ_d the standard Gaussian measure on \mathbb{R}^d . We recall a celebrated result by Caffarelli [13]:

Theorem 2.2 (Caffarelli [13]). *If ν is a probability measure with a density with respect to γ_d of the form e^{-V} , with $V : \mathbb{R}^d \rightarrow \mathbb{R}$ a convex function, then there exists a convex function φ of class \mathcal{C}^1 on \mathbb{R}^d such that the Brenier transport map $\nabla\varphi$ from γ_d to ν is 1-Lipschitz.*

This important result has numerous applications in the field of functional inequalities [32, 17, 55].

Actually, the conclusion of Theorem 2.1 can be extended to probability measures with finite moments of order 2. This will be developed in a forthcoming version of the present paper. The following result then immediately follows from Theorems 2.1 and 2.2.

Corollary 2.1. *If ν is a probability measure with a density with respect to γ_d of the form e^{-V} , with $V : \mathbb{R}^d \rightarrow \mathbb{R}$ a convex function, then the projection γ_d^* of γ_d on C_ν is equal to ν .*

Remark 2.1. *Let $d\nu = e^{-V} d\gamma_d$ with V convex on \mathbb{R}^d .*

- *According to Theorem 2.1 the conclusion of Caffarelli Theorem 2.2 is logically equivalent to the statement $\gamma_d^* = \nu$. It would be very interesting to prove directly that $\gamma_d^* = \nu$ since this would give an alternative proof of Caffarelli's result.*
- *To complete the picture, let us mention a nice result of Hargé [32] that shows that if in addition $\int x d\nu(x) = 0$, then $\nu \preceq_c \gamma_d$.*

Now let us turn to the proof of Theorem 2.1. We will need the following lemma whose proof is postponed to Section 6.

Lemma 2.1. *Let $g : \mathbb{R}^d \rightarrow \mathbb{R}$ be a function and let $g^*(y) = \sup_{x \in \mathbb{R}^d} \{x \cdot y - g(x)\}$, $y \in \mathbb{R}^d$. The following are equivalent:*

- (a) *The function $x \mapsto g(x) - \frac{|x|^2}{2}$ is convex on \mathbb{R}^d .*

- (b) The function $y \mapsto \frac{|y|^2}{2} - g^*(y)$ is convex on \mathbb{R}^d .
(c) The function g^* is of class \mathcal{C}^1 and ∇g^* is 1-Lipschitz on \mathbb{R}^d .

Now let us prove Theorem 2.1.

Proof of Theorem 2.1. Let us show that (a) implies (b). Define $h = \varphi^*$ and $f = 2\varphi^* - |x|^2$. According to Lemma 2.1 (applied with $g = \varphi^*$) we have that f is convex. An easy calculation shows that $Q_2 f(x) = |x|^2 - 2\varphi(x)$, $x \in \mathbb{R}^d$, and moreover it holds $\varphi^*(\nabla\varphi(x)) = x \cdot \nabla\varphi(x) - \varphi(x)$, $x \in \mathbb{R}^d$. This together with the fact that $\nabla\varphi$ sends μ onto ν yields that

$$\begin{aligned} W_2^2(\nu, \mu) &\leq \int |x - \nabla\varphi(x)|^2 d\mu(x) \\ &= \int |x|^2 d\mu(x) + \int |y|^2 d\nu(y) - 2 \int x \cdot \nabla\varphi(x) d\mu(x) \\ &= \int |x|^2 d\mu(x) + \int |y|^2 d\nu(y) - 2 \int \varphi(x) + \varphi^*(\nabla\varphi(x)) d\mu(x) \\ &= \int Q_2 f d\mu - \int f d\nu \\ &\leq \overline{\mathcal{T}}_2(\nu|\mu) \\ &\leq W_2^2(\nu, \mu), \end{aligned}$$

where the second inequality comes from the convexity of f and the duality formula for $\overline{\mathcal{T}}_2$. Therefore, the function f is optimal for the dual problem and so, according to Theorem 1.2, $\nu = \nabla h_{\#}^* \mu = \nabla \varphi_{\#} \mu = \mu^*$, which shows (b). Now according to Theorem 1.2, $W_2^2(\mu^*, \mu) = \overline{\mathcal{T}}_2(\nu|\mu)$ so (b) implies (c). On the other hand, if $W_2^2(\nu, \mu) = \overline{\mathcal{T}}_2(\nu|\mu)$, then according to Proposition 1.1 and Theorem 1.2, it holds $\mu^* = \mu$, so (c) implies (b) as well. Finally, according to Theorem 1.2, there always exist a convex function φ of class \mathcal{C}^1 on \mathbb{R}^d such that $\nabla\varphi$ is 1-Lipschitz and $\mu^* = \nabla\varphi_{\#} \mu$. So (b) implies (a), which completes the proof. \square

3. A MONOTONICITY THEOREM

In this section, we generalize the notion of c -monotonicity, which plays an important role in optimal transport [59, 24], to cost functions c defined on $\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$ and we use it to formulate a necessary condition of optimality for transport plans. Then we give a new proof of Strassen theorem on the existence of submartingale couplings with given marginals in dimension 1 using this c -monotonicity criterium.

3.1. A necessary condition of optimality. The following definition generalizes the notion of c -monotonicity ; we refer to [58, 69, 70] and the references therein for a complete account on the subject. In what follows, we denote by $\mathcal{P}_1(\mathbb{R}^d)$ the set of probability measures on \mathbb{R}^d having finite first moment.

Definition 3.1. Let $c : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \rightarrow \mathbb{R}^+$ be a cost function. We will say that a set $\Gamma \subset \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d)$ is c -monotone if for any $N \geq 1$ and points $(x_i, p_i) \in \Gamma$ and probability measures $q_i \in \mathcal{P}_1(\mathbb{R}^d)$, $i \in \{1, \dots, N\}$ such that $\sum_{i=1}^N p_i = \sum_{i=1}^N q_i$, it holds

$$\sum_{i=1}^N c(x_i, p_i) \leq \sum_{i=1}^N c(x_i, q_i).$$

We now formulate a necessary condition for optimality of a coupling for the transport cost $\overline{\mathcal{T}}_2$ which we recall is associated to the cost function c_2 defined by

$$c_2(x, p) = \left| x - \int y dp(y) \right|^2, \quad x \in \mathbb{R}^d, \quad p \in \mathcal{P}_1(\mathbb{R}^d).$$

Theorem 3.1. *Let μ, ν be two compactly supported probability measures on \mathbb{R}^d . There exists a set $\Gamma \subset \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d)$ which is monotone with respect to the cost function c_2 and such that if $\pi(dx dy) = d\mu(x) dp_x(y)$ is an optimal coupling for the cost $\bar{\mathcal{T}}_2(\nu|\mu)$, it holds*

$$\mu\left(\{x \in \mathbb{R}^d : (x, p_x) \in \Gamma\}\right) = 1.$$

Proof. Recall the duality formula

$$\bar{\mathcal{T}}_2(\nu|\mu) = \sup_f \left\{ \int Q_2 f(x) d\mu(x) - \int f(y) d\nu(y) \right\},$$

where the infimum is running over the set of convex functions bounded from below and $Q_2 f(x) = \inf_{y \in \mathbb{R}^d} \{f(y) + |x - y|^2\}$, $x \in \mathbb{R}^d$. Note that if f is convex, then

$$Q_2 f(x) = \inf_{p \in \mathcal{P}(\mathbb{R}^d)} \left\{ \int f dp + c_2(x, p) \right\}, \quad x \in \mathbb{R}^d.$$

Indeed, the inequality \geq is always true and since f is convex, it holds

$$\int f(y) dp(y) + c_2(x, p) \geq f\left(\int y dp(y)\right) + \left|x - \int y dp(y)\right|^2 \geq Q_2 f(x)$$

so taking the infimum over p gives the converse inequality. Let \bar{f} be such that $\bar{\mathcal{T}}_2(\nu|\mu) = \int Q_2 \bar{f}(x) d\mu(x) - \int \bar{f}(y) d\nu(y)$ (which exists according to Theorem 1.2). Then, it holds

$$\begin{aligned} \bar{\mathcal{T}}_2(\nu|\mu) &= \int Q_2 \bar{f}(x) d\mu(x) - \int \bar{f}(y) d\nu(y) \\ &\leq \int \left(\int \bar{f}(y) dp_x(y) + c_2(x, p_x) \right) d\mu(x) - \int \bar{f}(y) d\nu(y) \\ &= \int c_2(x, p_x) d\mu(x) = \bar{\mathcal{T}}_2(\nu|\mu). \end{aligned}$$

Therefore, $Q_2 \bar{f}(x) = \int \bar{f}(y) dp_x(y) + c_2(x, p_x)$ for μ -almost all $x \in \mathbb{R}^d$. So denoting by Γ the set of couples (x, p) so that $Q_2 f(x) = \int f dp + c_2(x, p)$, we have that $\mu(\{x \in \mathbb{R}^d : (x, p_x) \in \Gamma\}) = 1$. Now, let us check that Γ is monotone with respect to the cost function c_2 . Take a family of points $(x_i, p_i) \in \Gamma$ and probability measures $q_i \in \mathcal{P}(\mathbb{R}^d)$, $i \in \{1, \dots, N\}$ such that $\sum_{i=1}^N p_i = \sum_{i=1}^N q_i$. Then, it holds

$$Q_2 f(x_i) = \int f dp_i + c_2(x_i, p_i), \quad \forall i \in \{1, \dots, N\}$$

and on the other hand

$$Q_2 f(x_i) \leq \int f dq_i + c_2(x_i, q_i), \quad \forall i \in \{1, \dots, N\}.$$

Summing these inequalities, and using the fact that $\sum_{i=1}^N p_i = \sum_{i=1}^N q_i$ gives immediately $\sum_{i=1}^N c(x_i, p_i) \leq \sum_{i=1}^N c(x_i, q_i)$. \square

Remark 3.1. *With the notation of Theorem 1.2, let us show how one can recover the fact that $\nabla\varphi$ is 1-Lipschitz on the support of μ with the help of Theorem 3.1. Let $d\pi(x, y) = d\mu(x) dp_x(y)$ be an optimal transport plan from μ to ν and Γ be a c_2 -monotone set such that $(x, p_x) \in \Gamma$ for μ almost every $x \in \mathbb{R}^d$. If (x, p_x) and (y, p_y) are elements of Γ , then comparing this pair with $(x, (1 - \varepsilon)p_x + \varepsilon p_y)$ and $(y, (1 - \varepsilon)p_y + \varepsilon p_x)$ where $\varepsilon \in [0, 1]$ yields*

$$|x - x^*|^2 + |y - y^*|^2 \leq |x - [(1 - \varepsilon)x^* + \varepsilon y^*]|^2 + |y - [(1 - \varepsilon)y^* + \varepsilon x^*]|^2,$$

where $x^* = \int z dp_x(z)$ and $y^* = \int z dp_y(z)$. Noting that this is an equality for $\varepsilon = 0$ and taking the right derivative at $\varepsilon = 0_+$, we thus obtain

$$2\langle x^* - x, y^* - x^* \rangle - 2\langle y^* - y, y^* - x^* \rangle = 2\langle y - x, y^* - x^* \rangle - 2|y^* - x^*|^2 \geq 0,$$

hence $|y^* - x^*| \leq |y - x|$. Since $x^* = \nabla\varphi(x)$ and $y^* = \nabla\varphi(y)$ this proves that $\nabla\varphi$ is 1-Lipschitz on the support of μ .

3.2. An example related to the increasing convex order.

Definition 3.2. Let μ, ν be two probability measures on \mathbb{R} ; μ is dominated by ν for the stochastic order, denoted by $\mu \preceq_{sto} \nu$, if $\int f d\mu \leq \int f d\nu$ for all bounded increasing functions $f : \mathbb{R} \rightarrow \mathbb{R}$. Assuming that μ, ν have finite first moments, μ is dominated by ν for the increasing convex order, denoted by $\mu \preceq_{c,sto} \nu$, if $\int f d\mu \leq \int f d\nu$ for every increasing convex function $f : \mathbb{R} \rightarrow \mathbb{R}$.

Notice that if $\int x d\mu(x) = \int y d\nu(y)$ and $\mu \preceq_{c,sto} \nu$, then it is easy to see that the inequality $\int f d\mu \leq \int f d\nu$ can be extended first to all convex functions f with finite asymptotic slopes at $\pm\infty$ and second, by using the monotone convergence theorem, to all convex functions. Therefore, for probability measures with finite first moments, $\mu \preceq_c \nu$ is equivalent to $\mu \preceq_{c,sto} \nu$ and $\int x d\mu(x) = \int x d\nu(x)$.

In this section we prove that on \mathbb{R} under the assumption $\mu \preceq_{c,sto} \nu$, our Theorem 1.2 on the quadratic barycentric transport problem can be completed with new informations.

Theorem 3.2. Let μ and ν be probability measures on \mathbb{R} , with compact support and assume $\mu \preceq_{c,sto} \nu$. Then, with the notations of Proposition 1.1 and Theorem 1.2, the probability measure μ^* satisfies $\mu \preceq_{sto} \mu^*$ and any optimal coupling (X, Y) satisfies $X \leq \mathbb{E}(Y|X)$ almost surely.

In the terminology of martingale transport, (X, Y) is a submartingale coupling and its law a submartingale transport plan.

Conversely, if (X, Y) is a coupling of μ and ν such that $\mathbb{E}(Y|X) \geq X$ almost surely, then for every increasing and convex function $f : \mathbb{R} \rightarrow \mathbb{R}$, it holds

$$\mathbb{E}(f(Y)) \geq \mathbb{E}[\mathbb{E}(f(Y)|X)] \geq \mathbb{E}(f(\mathbb{E}(Y|X))) \geq \mathbb{E}(f(X)).$$

and so $\mu \preceq_{c,sto} \nu$. We therefore recover the classical result by Strassen [66, Theorem 9] on submartingale couplings (under the additional assumption that the measures are compactly supported): $\mu \preceq_{c,sto} \nu$ if and only if there exists $X \sim \mu$ and $Y \sim \nu$ in the same probability space with $\mathbb{E}(Y|X) \geq X$ almost surely.

Remark 3.2. Theorem 3.2 also gives back Strassen Theorem in the very specific case of the real line (and for measures of compact support). The fact that $\mathbb{E}(Y|X) = X$ implies $\mu \preceq_c \nu$ is an obvious consequence of the conditional Jensen inequality. Conversely if $\mu \preceq_c \nu$ we have also $\mu \preceq_{c,sto} \nu$. Therefore there exists $X \sim \mu$ and $Y \sim \nu$ in the same probability space with $\mathbb{E}(Y|X) \geq X$. Moreover, $\mu \preceq_c \nu$ implies $\int x d\mu(x) = \int x d\nu(x)$, i.e. $\mathbb{E}(X) = \mathbb{E}(Y)$. Hence $\mathbb{E}(Y|X)$ and X have the same expectation. They are therefore almost surely equal, which proves the converse implication of Strassen Theorem. For another elementary proof, see [6, Section 2]

Proof of Theorem 3.2. We denote a solution of the quadratic barycentric problem by $d\pi(x, y) = d\mu(x)dp_x(y)$. Let (X, Y) such that $(X, Y) \sim \pi$ and denote $\mathbb{E}(Y|X)$ by X^* . According to Theorem 1.2, π may not be uniquely determined but the law of X^* is uniquely determined and denoted by μ^* . The inequality $X \leq \mathbb{E}[Y|X]$ almost surely immediately implies that $\mu \preceq_{sto} \mu^*$. Now to show that $X \leq \mathbb{E}[Y|X]$ almost surely amounts to show that $\int y dp_x(y) \geq x$ for μ almost all $x \in \mathbb{R}$. The rest of the proof is devoted to this question.

According to Theorem 3.1, there exists $\Gamma \subset \mathbb{R} \times \mathcal{P}_1(\mathbb{R})$ such that, $\mu(\{x \in \mathbb{R} : (x, p_x) \in \Gamma\}) = 1$ and such that if $(x, p) \in \Gamma$ and $(x', p') \in \Gamma$, then for all probability measures q, q' such that $p + p' = q + q'$, it holds

$$(5) \quad \left| x - \int y dp(y) \right|^2 + \left| x' - \int y dp'(y) \right|^2 \leq \left| x - \int y dq(y) \right|^2 + \left| x' - \int y dq'(y) \right|^2.$$

Let us show that if $x, x' \in \mathbb{R}$ are such that $(x, p_x) \in \Gamma$ and $(x', p_{x'}) \in \Gamma$, then for all $a \in \text{Support}(p_x)$ and $b \in \text{Support}(p_{x'})$, it holds

$$(6) \quad \left[\left(\int y dp_x(y) - x \right) - \left(\int y dp_{x'}(y) - x' \right) \right] (b - a) \geq 0.$$

Let $a \in \text{Support}(p_x)$ and $b \in \text{Support}(p_{x'})$ and $\varepsilon > 0$ and define

$$r_a = p_x(\cdot | [a - \varepsilon, a + \varepsilon]) \quad \text{and} \quad r_b = p_{x'}(\cdot | [b - \varepsilon, b + \varepsilon]).$$

For all $t > 0$, define

$$q_x^t = p_x + t(r_b - r_a) \quad \text{and} \quad q_{x'}^t = p_{x'} + t(r_a - r_b)$$

and note that if t is small enough, q_x^t and $q_{x'}^t$ are probability measures such that $q_x^t + q_{x'}^t = p_x + p_{x'}$. Applying (5) and letting $t \rightarrow 0$ gives

$$\left[\left(\int y dp_x(y) - x \right) - \left(\int y dp_{x'}(y) - x' \right) \right] \left(\int y dr_b(y) - \int y dr_a(y) \right) \geq 0.$$

Finally, letting $\varepsilon \rightarrow 0$ yields to (6).

Define $\Gamma^1 = \{x \in \mathbb{R} : (x, p_x) \in \Gamma\}$ and for any interval $I \subset \mathbb{R}$, let A_I the set of points $x \in \Gamma^1$ such that $\int y dp_x(y) - x \in I$. We aim at proving that $\mu(A_{]-\infty, 0[}) = 0$.

Striving for a contradiction suppose that $\mu(A_{]-\infty, 0[}) > 0$. Our assumption $\mu \preceq_{c, \text{sto}} \nu$ implies $\int x d\mu(x) \leq \int y d\nu(y)$ so that $\int_{\Gamma^1} [(\int y dp_x(y)) - x] d\mu(x) \geq 0$. It follows that $\mu(A_{[0, +\infty[}) > 0$. Henceforth $A_{]-\infty, 0[}$ and $A_{[0, +\infty[}$ are not empty. Several configurations may be considered.

- In the first case we assume that there exists $x' \in A_{]-\infty, 0[}$ and $x \in A_{[0, +\infty[}$ such that $x' \leq x$. One thus has $\int y dp_{x'}(y) < x' \leq x \leq \int y dp_x(y)$. So in this case, one can find $a \in \text{Support}(p_x)$ and $b \in \text{Support}(p_{x'})$ such that $a > b$. Applying (6) with these a, b provides a contradiction.
- We can now assume that $A_{[0, +\infty[}$ has a supremum smaller than or equal to the infimum of $A_{]-\infty, 0[}$. If there exists $a \in \text{Support}(p_x)$ and $b \in \text{Support}(p_{x'})$ such that $a > b$, we get the same contradiction as before.
- We are finally reduced to the case where $A_{[0, +\infty[}$ has a supremum smaller than or equal to the infimum of $A_{]-\infty, 0[}$ and moreover every p_x with $x \in A_{[0, +\infty[}$ has the essential supremum smaller than or equal to the essential infimum of every $p_{x'}$ with $x' \in A_{]-\infty, 0[}$. Let $m \in \mathbb{R}$ be between $\sup_{x \in A_{[0, +\infty[}} [\sup \text{Support}(p_x)]$ and $\inf_{x' \in A_{]-\infty, 0[}} [\inf \text{Support}(p_{x'})]$. Integration of $x \mapsto (x - m)_+$ provides a contradiction to $\mu \preceq_{c, \text{sto}} \nu$. Indeed, on the one hand, it holds

$$\int [x - m]_+ d\mu(x) \geq \int [x - m]_+ \mathbf{1}_{A_{]-\infty, 0[}}(x) d\mu(x)$$

(with actually equality). On the other,

$$\begin{aligned} \int [y - m]_+ d\nu(y) &= \int \left(\int [y - m]_+ dp_x(y) \right) d\mu(x) \\ &\stackrel{(i)}{=} \int \left(\int [y - m]_+ dp_x(y) \right) \mathbf{1}_{A_{]-\infty, 0[}}(x) d\mu(x) \\ &\stackrel{(ii)}{=} \int \left(\int y - m dp_x(y) \right) \mathbf{1}_{A_{]-\infty, 0[}}(x) d\mu(x) \\ &\stackrel{(iii)}{<} \int (x - m) \mathbf{1}_{A_{]-\infty, 0[}}(x) d\mu(x) \\ &\stackrel{(iv)}{=} \int [x - m]_+ \mathbf{1}_{A_{]-\infty, 0[}}(x) d\mu(x), \end{aligned}$$

where

- (i) comes from the fact that for any $x \in A_{[0, +\infty[}$, $\text{Support}(p_x) \subset] - \infty, m]$,

- (ii) comes from the fact that for any $x \in A_{]-\infty, 0[}$, $\text{Support}(p_x) \subset [m, +\infty[$,
- (iii) comes from the definition of $A_{]-\infty, 0[}$ and the fact that if $f < g$ on a set A such that $\mu(A) > 0$, then $\int_A f d\mu < \int_A g d\mu$,
- (iv) follows from the inequality $x > \int y dp_x(y) \geq m$, for all $x \in A_{]-\infty, 0[}$.

In any case there is a contradiction to the fact that $\mu(A_{]-\infty, 0[}) > 0$ and this completes the proof. \square

Remark 3.3.

- (1) *Optimal transport plans for the weak transport problem are not uniquely determined. As stated in Theorem 1.2, μ^* and the transport from μ to μ^* are uniquely determined but the martingale transport plan from μ^* to ν is completely free. It is precisely the goal of Martingale Optimal Transport to determine special martingale transport plans from μ^* to ν .*
- (2) *This remark holds in particular for the dimension 1. Namely, in Theorem 3.2 we did not defined a submartingale coupling but only the non-decreasing/increasing part of its Doob decomposition. In relation with this decomposition, the paper [56] proposes two supermartingale transport problems and describes their solutions, extending the theory on curtain couplings initiated in [6]. In particular the two optimal supermartingale couplings respectively corresponding to the two problems coincide with the curtain coupling when $\mu \preceq_c \nu$. However, transforming the supermartingale problem in submartingale problem, we stress that $X^* = \mathbb{E}(Y|X)$ and its coupling with X is different from ours as we show in this example: if X is the uniform measure on $[-1, 0]$ and Y is uniform on $[0, 3]$, our coupling is $\mathbb{E}[Y|X] = X^* = X + 2$ (it is a translation see the second example). The first supermartingale coupling by Nutz and Stebegg (reversed in order to be a submartingale) gives $Y = \mathbb{E}[Y|X] = 3(X + 1)$. The second one provides $Y = \mathbb{E}[Y|X] = -3X$.*

4. EXAMPLE OF ν CONCENTRATED ON THE VERTICES OF A SIMPLEX

In this section we prove the following theorem, mainly by geometric means.

Theorem 4.1. *Let μ be a compactly supported probability measure on \mathbb{R}^d and ν an atomic measure whose support is a simplex $\{y_0, \dots, y_k\}$ of \mathbb{R}^d with $k \leq d$. Denote the convex hull of $\{y_0, \dots, y_k\}$ by Δ . Then there exists $v \in \mathbb{R}^d$ such that the map T defined by*

$$T : \mathbb{R}^d \rightarrow \mathbb{R}^d : x \mapsto T(x) = \text{proj}_\Delta(x + v)$$

is such that $\mu^ = T\#\mu$ (with the notations of Proposition 1.1 and Theorem 1.2), where proj_Δ denotes the orthogonal projection on the closed convex set Δ .*

Remark 4.1.

- *To be consistent with Theorem 1.2, note that the map T given above can be written as $T = \nabla\varphi$, where $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ is the convex function of class \mathcal{C}^1 defined by*

$$\varphi(x) = \frac{|x + v|^2}{2} - \frac{1}{2}d(x + v, \Delta)^2, \quad x \in \mathbb{R}^d,$$

where $d(z, \Delta) = \inf_{y \in \Delta} |z - y|$.

- *The assumption that μ is compactly supported could be easily relaxed into the assumption that μ admits a moment of order 2 finite.*

Proof. Let μ be a probability with compact support and ν an atomic measure with support the simplex $\{y_0, \dots, y_k\}$ of \mathbb{R}^d with $k \leq d$. We denote by μ^* the projection of μ on $C_\nu = \{\eta \in \mathcal{P}(\mathbb{R}^d) : \eta \preceq_c \nu\}$ given by Proposition 1.1 and Theorem 1.2. The assumption that the support of ν is a simplex is in order for the following property: for any point y in the convex hull $\Delta = \text{Conv}(y_0, \dots, y_k)$ the barycentric coordinates $(\lambda_0, \dots, \lambda_k)$, defined by $\sum_{i=0}^k \lambda_i = 1$ and $\sum_{i=0}^k \lambda_i y_i = y$, are uniquely determined. Moreover, notice that all the

coordinates are non-negative. If $v \in \mathbb{R}^d$, we will denote by μ_v the translation of μ by the vector v , i.e. $\mu_v = \text{Law}(X + v)$, with $X \sim \mu$. The four following properties will permit us to determine μ^* and find the optimal coupling between μ and μ^* .

- (a) Let η be a probability measure with a finite moment of order 2. A coupling (X, Y) is an optimal coupling for W_2 between μ and η if and only if $(X + v, Y)$ is an optimal coupling for W_2 between μ_v and η . Moreover,

$$W_2^2(\mu_v, \eta) = W_2^2(\mu, \eta) + |v|^2 + 2 \left\langle \int x d\mu - \int x d\eta, v \right\rangle,$$

- (b) As $\mu^* \preceq_c \nu$ the measure μ^* is concentrated on Δ and has the same barycenter as ν . Conversely any measure η concentrated on Δ and with the same barycenter as ν satisfies $\eta \preceq_c \nu$.
- (c) Among the measures concentrated on Δ , the one that minimises the quadratic cost W_2 with respect to a given probability measure μ' having a finite moment of order 2 is $(\text{proj}_\Delta)_\# \mu'$.
- (d) There exists $v \in \mathbb{R}^d$ such that $(\text{proj}_\Delta)_\# \mu_v$ has the same barycenter as ν .

Before we prove the different four points, let us finish the proof. According to Item (b), all the elements of C_ν have the same barycenter as ν . Therefore, applying Item (a), we have for any $\eta \in C_\nu$

$$W_2^2(\mu_v, \eta) = W_2^2(\mu, \eta) + |v|^2 + 2 \left\langle \int x d\mu - \int x d\nu, v \right\rangle.$$

One easily concludes from this identity that $(\mu_v)^* = \mu^*$ for any $v \in \mathbb{R}^d$. According to Item (d), there exists $v \in \mathbb{R}^d$ such that $\eta_v := (\text{proj}_\Delta)_\# \mu_v$ has the same barycenter as ν . Since η_v is also concentrated on Δ , it follows from Item (b) that η_v belongs to C_ν . Therefore, according to Item (c),

$$\inf_{\eta \in C_\nu} W_2^2(\mu_v, \eta) \geq \inf_{\eta(\Delta)=1} W_2^2(\mu_v, \eta) = W_2^2(\mu_v, \eta_v)$$

and so $\eta_v = (\mu_v)^* = \mu^*$. Finally if $X \sim \mu$, it follows from Item (a) that $(X, \text{proj}_\Delta(X + v))$ is an optimal coupling between μ and μ^* .

The four points above can be proved as follows:

- (a) This assumption simply comes from

$$\mathbb{E}[|(X + v) - X^*|^2] = \mathbb{E}[|X - X^*|^2] + \underbrace{|v|^2 + 2\langle \mathbb{E}[X - X^*], v \rangle}_{\text{Depends only on Law}(X), \text{Law}(X^*)}.$$

- (b) The first implication is obvious. For the second implication, assume η is concentrated on Δ and has the same barycenter as ν . For every $x \in \Delta$, let p_x be the unique probability measure concentrated on $\{y_0, \dots, y_k\}$ with barycenter x and let $\nu' = \int p_x d\eta(x)$. The probability measure ν' is concentrated on the same set as ν . Its barycenter permits to determine it uniquely. The barycenter of ν' is $\int x d\eta(x)$, the barycenter of η that is also the one of ν . Therefore $\nu' = \nu$. We have proved that there exists a martingale having η on ν as marginals. Therefore $\eta \preceq_c \nu$.
- (c) For every $x \in \mathbb{R}^d$, the point $\text{proj}_\Delta(x)$ is by definition the closest point in Δ . So if $X' \sim \mu'$ and Y takes values in Δ almost surely, one gets

$$\mathbb{E}[|X' - Y|^2] \geq \mathbb{E}[|X' - \text{proj}_\Delta(X')|^2]$$

and so $W_2^2(\mu', \eta) \geq \mathbb{E}[|X' - \text{proj}_\Delta(X')|^2]$ for any η concentrated in Δ . In particular, $W_2^2(\mu', (\text{proj}_\Delta)_\# \mu') = \mathbb{E}[|X' - \text{proj}_\Delta(X')|^2]$ which proves the claim.

- (d) Without loss of generality we can assume $k = d$ for the following reason: if Δ has positive codimension, the measure $(\text{proj}_\Delta)_\# \mu'$ is exactly $(\text{proj}_\Delta)_\# \circ (\text{proj}_A)_\# \mu'$ where A is the affine space spanned by Δ . As a consequence, in our investigation

we can replace μ by $(\text{proj}_A)_\# \mu$, consider only translations in A and see Δ as a simplex with full dimension.

Now, let y_ν be the barycenter of ν and let us prove the existence of $v \in \mathbb{R}^d$ such that the barycenter of $(\text{proj}_\Delta)_\# \mu_\nu$ is equal to y_ν . This point can be proved using the notion of topological degree coming from algebraic topology (we refer to [57, Chapter IV] for an introduction). If Ω is a bounded open set of \mathbb{R}^d and $f : \overline{\Omega} \rightarrow \mathbb{R}^d$ is a continuous mapping, the degree of f is a \mathbb{Z} -valued quantity denoted by $d(f, \Omega, a)$ defined for every $a \in \mathbb{R}^d \setminus f(\partial\Omega)$. We will use the following basic properties:

- if $f = \text{Id}$, then $d(\text{Id}, \Omega, a) = 1$, for all $a \in \mathbb{R}^d \setminus \partial\Omega$;
- if $d(f, \Omega, a) \neq 0$, then the equation $f(x) = a$, $x \in \Omega$, admits at least one solution;
- if $F : \overline{\Omega} \times [0, 1] \rightarrow \mathbb{R}^d$ is a continuous function and $a \in \mathbb{R}^d$ is such that for all $t \in [0, 1]$, $a \in \mathbb{R}^d \setminus F(t, \partial\Omega)$, then $d(F(\cdot, 0), \Omega, a) = d(F(\cdot, 1), \Omega, a)$ (invariance by homotopy).

Consider X a random variable with law μ and $R > 0$ a positive number to be fixed later such that the open ball of center 0 and radius R denoted by $\mathcal{B}(0, R)$ contains Δ . The sphere of center 0 and radius R will be denoted by $\mathcal{S}(0, R)$ in the sequel. We are interested in the map

$$(7) \quad \Phi : \overline{\mathcal{B}}(0, R) \times [0, 1] \rightarrow \Delta \subset \mathbb{R}^d,$$

defined so that $\Phi(v, t)$ is the barycenter of $(\text{proj}_\Delta)_\#(\text{Law}(v + tX))$. Note that this map is continuous so that it can be seen as an homotopy between proj_Δ and $v \mapsto (\text{proj}_\Delta)_\# \mu_\nu$. Another homotopy is possible between Id and proj_Δ

$$\Psi : \overline{\mathcal{B}}(0, R) \times [0, 1] \rightarrow \mathbb{R}^d$$

defined by $\Psi(x, t) = (1 - t)x + t\text{proj}_\Delta(x)$. Note that since y_ν lies in the interior of Δ , it does not belong to $\Psi(\mathcal{S}(0, R) \times [0, 1])$. Therefore, by invariance by homotopy,

$$1 = d(\text{Id}, \mathcal{B}(0, R), y_\nu) = d(\Psi(\cdot, 0), \mathcal{B}(0, R), y_\nu) = d(\Psi(\cdot, 1), \mathcal{B}(0, R), y_\nu),$$

from which we infer that $d(\Phi(\cdot, 0), \mathcal{B}(0, R), y_\nu) = 1$. According to Lemma 4.1 below, if R is large enough then $\Phi(\mathcal{S}(0, R) \times [0, 1]) \subset \partial\Delta$. Since y_ν lies in the interior of Δ , we conclude that $\Phi(\mathcal{S}(0, R) \times [0, 1])$ does not meet y_ν , and so using the homotopy invariance again, we get that $d(\Phi(\cdot, 1), \mathcal{B}(0, R), y_\nu) = 1$ and so the equation $\Phi(v, 1) = \int \text{proj}_\Delta(x) d\mu_\nu(x) = y_\nu$ admits at least one solution $v \in \mathcal{B}(0, R)$ which completes the proof. \square

Lemma 4.1. *Under the preceding assumptions, if R is large enough, the map Φ defined in (7) is such that $\Phi(\mathcal{S}(0, R) \times [0, 1]) \subset \partial\Delta$.*

Proof. Recall that

$$\Delta = \left\{ \sum_{i=0}^d \lambda_i y_i : \sum_{i=0}^d \lambda_i = 1, \lambda_0, \dots, \lambda_d \geq 0 \right\}.$$

For J a subset of $\{0, \dots, d\}$ we denote by Δ_J the set

$$\Delta_J = \left\{ \sum_{i=0}^d \lambda_i y_i : \forall i \in J, \lambda_i = 0 \right\} \subset \Delta.$$

We denote by u_j the outward normal unit vector of the facet $\Delta_{\{j\}}$. Recall that the relative interior of a subset is the interior of this set in the topology induced by its affine span. Every point of \mathbb{R}^d whose projection belongs to the relative interior of Δ_J can be written in the form $\sum_{i \notin J} \lambda_i y_i + \sum_{i \in J} \xi_i u_i$ where the coefficients $(\lambda_i)_{i \notin J}$ are positive and satisfy $\sum_{i \notin J} \lambda_i = 1$ and the coefficients $(\xi_i)_{i \in J}$ are non-negative.

We prove now that for every $r > 0$, there exists $R_0 > 0$ such that for all $R \geq R_0$ it holds

- Δ is contained in $\mathcal{B}(0, R - r + 1)$,
- for every $v \in \mathcal{S}(0, R)$ there exists $j \in \{0, \dots, d\}$ such that $\text{proj}_\Delta(\mathcal{B}(v, r)) \subset \Delta_{\{j\}}$.

Therefore if $X \sim \mu$ with μ such that $\mu(\mathcal{B}(0, r)) = 1$ and $v \in \mathcal{S}(0, R)$, there exists j such that the barycenter of $(\text{proj}_\Delta)_\#(\text{Law}(v + tX))$ is in $\Delta_{\{j\}} \subset \partial\Delta$, which proves the claim.

Let us consider

$$g : x \in \mathbb{R}^d \setminus \Delta \mapsto (x - \text{proj}_\Delta(x))/|x - \text{proj}_\Delta(x)| \in \mathbb{S}^{d-1}.$$

For our proof it is enough to find R such that for every $x \in \mathcal{S}(0, R)$ the range $g(\mathcal{B}(x, r))$ only contains vectors of \mathbb{S}^{d-1} that are in the cones $(C_J)_{J \ni j, \#J \leq d}$ for some j , where

$$C_J = \left\{ u \in \mathbb{R}^d : u = \sum_{i \in J} \xi_i u_i, \xi_i \geq 0, \forall i \in J \right\}.$$

In particular it does not intersect $C_{\{0, \dots, d\} \setminus \{j\}}$. Striving for a contradiction we assume that there exists an increasing sequence $R_n \rightarrow \infty$ and $x_n \in \mathcal{S}(0, R_n)$ such that the property is not satisfied. Therefore, for every n the sets $g(\mathcal{B}(x_n, r)) \subset \mathbb{S}^{d-1}$ intersects for every j a cone C_J with $j \in J$. However, the diameter of $g(\mathcal{B}(x_n, r))$ tends to zero. Up to selecting a subsequence, the sequence converges in the Hausdorff topology to a compact set of diameter zero, i.e a point $\{u_\infty\} \subset \mathbb{S}^{d-1}$. We have $u_\infty \in C_{J_\infty}$ for some $J_\infty \subset \{0, \dots, d\}$. Let j_∞ be in J_∞ . We have also $g(x_n) \rightarrow u_\infty$ (up to a subsequence) and the fact that g is locally Lipschitz with a constant that tends to zero at infinity tells us that the j_∞ coordinate is not zero in the cone coordinate $u = \sum_j \xi_j u_j$ for all the points u of $g(\mathcal{B}(x_n, r)) \subset \mathbb{S}^{d-1}$ when n is great enough - a contradiction. \square

5. OTHER EXAMPLES AND DISCUSSION OF THE LITERATURE

In this section, we briefly present and discuss other examples of optimal transport problems involving weak cost functions.

- (1) In [29, Theorem 1.5], it is shown that if μ, ν are probability measures on \mathbb{R} having finite first moments and if $\mu^* \in C_\nu$ denotes the projection of μ on C_ν as defined in Proposition 1.1, then for any even convex cost function $\theta : \mathbb{R} \rightarrow \mathbb{R}_+$, it holds

$$\overline{\mathcal{T}}_\theta(\nu|\mu) = \inf_{\pi \in C(\mu, \mu^*)} \iint \theta(x - y) d\pi(x, y),$$

where we recall that the barycentric optimal transport cost $\overline{\mathcal{T}}_\theta(\nu|\mu)$ is defined by (3).

- (2) In the recent paper [1] (see in particular the section 5.2 of [1]), the authors consider the family of cost functions $(c_\lambda)_{\lambda \geq 0}$ defined for all $x \in \mathbb{R}^d$ and $p \in \mathcal{P}(\mathbb{R}^d)$ having a finite first moment by

$$c_\lambda(x, p) = (\lambda - 1) \int |x - y|^2 dp(y) + \left| x - \int y dp(y) \right|^2$$

and study the associated optimal transport problem:

$$\overline{\mathcal{T}}_{c_\lambda}(\nu|\mu) = \inf_{\pi \in C(\mu, \nu)} \int c_\lambda(x, p_x) d\mu(x)$$

(this quantity is denoted by $F_\lambda(\mu, \nu)$ in [1]). It turns out that our Theorem 1.2 yields a full description of optimal couplings for these costs for $\lambda > 0$ (which completes the somehow implicit characterization of [1, Theorem 5.6]). Namely, an easy calculation reveals that if $d\pi(x, y) = d\mu(x)dp_x(y) \in C(\mu, \nu)$ where μ, ν are compactly supported, then

$$\int c_\lambda(x, p_x) d\mu(x) = C(\lambda) + \int \left| x - \int y dp_{x/\lambda}(y) \right|^2 d\mu_\lambda(x),$$

where $C(\lambda) = -\lambda(\lambda - 1) \int |x|^2 d\mu(x) + (\lambda - 1) \int |y|^2 d\nu(y)$ and μ_λ is the image of μ under the map $x \mapsto \lambda x$. Since the kernel $q_x = p_{x/\lambda}$, $x \in \mathbb{R}^d$, is such that $\nu(\cdot) = \int q_x(\cdot) d\mu_\lambda(x)$, we conclude that

$$\overline{\mathcal{T}}_{c_\lambda}(\nu|\mu) = C(\lambda) + \overline{\mathcal{T}}_2(\nu|\mu_\lambda),$$

and p is optimal in $\overline{\mathcal{T}}_{c_\lambda}(\nu|\mu)$ if and only if q is optimal in $\overline{\mathcal{T}}_2(\nu|\mu_\lambda)$. Moreover, Theorem 1.2 yields that optimal kernels p are all of the form

$$p_x = r_{\nabla\varphi_\lambda(\lambda x)}, \quad \text{with } r \text{ s.t. } \int y dr_u(y) = u, \text{ for } (\mu_\lambda)^* \text{ almost all } u \in \mathbb{R}^d,$$

where φ_λ is the \mathcal{C}^1 smooth convex function associated to the transport from μ_λ to the projection $(\mu_\lambda)^*$ of μ_λ on C_ν . In terms of random vectors, $\pi_\lambda = \text{Law}(X_\lambda, Y_\lambda) \in C(\mu, \nu)$ is optimal for c_λ if and only if $\mathbb{E}[Y_\lambda|X_\lambda] = \nabla\varphi_\lambda(\lambda X_\lambda)$ (which has distribution $(\mu_\lambda)^*$).

- (3) The case $\lambda = 0$ is also interesting since, in this case,

$$\overline{\mathcal{T}}_{c_0}(\nu|\mu) = - \sup_{\pi \in C(\mu, \nu)} \int \text{Var}(p_x) d\mu(x),$$

where as usual $d\pi(x, y) = d\mu(x)dp_x(y)$ and for all $p \in \mathcal{P}(\mathbb{R}^d)$ having finite first moment, $\text{Var}(p) = \int |y|^2 dp(y) - |\int y dp(y)|^2$ (note that this is a concave function of p). In this case, as observed in [1], the unique optimal coupling π^* is the product one : $\pi^* = \mu \otimes \nu$.

- (4) As proved in [5, Proposition 5.2], the so-called *shadow couplings* are solutions to a weak optimal transport problem. Shadow couplings from μ to ν are martingale transport parametrised by a measure $\hat{\mu}$ with marginals λ and μ . Let $(\lambda_x)_{x \in [0,1]}$ be a disintegration of $\hat{\mu}$ with respect to μ . Then

$$c^{\hat{\mu}}(x, p_x) = \inf \int (1 - u) \sqrt{1 + y^2} d\alpha(u, y)$$

where the inf goes among all α with first marginal λ_x and second marginal p_x and such that we have $\int f(u)(y - x) d\alpha(u, y) = 0$ for every bounded f .

- (5) Last but not least, let us mention that after the completion of this work, we learned that a recent remarkable economics paper on optimal mechanisms for the multiple-good monopoly problem shows interesting similarities with our context. In [16], the authors study the maximization problem among convex coordinate-wise nondecreasing 1-Lipschitz potential functions defined on some d -dimensional rectangle. In their Theorem 2, they prove the strong duality between this problem and “strong” dual problem (for us a primal problem). This problem is a transport problem with ℓ_1 -norm as cost function and the possibility to replace μ and ν by μ' and ν' with $\mu \preceq_{c, \text{sto}} \mu'$ and $\nu' \preceq_{c, \text{sto}} \nu$.

6. PROOFS

This section contains the proofs of Proposition 1.1 and Theorem 1.2 and of the technical Lemma 2.1.

Proof of Proposition 1.1. In this proof, we consider the set $\mathcal{P}_2(\mathbb{R}^d)$ of all probability measures having a finite moment of order 2 and we equip it with the topology generated by the metric W_2 . We recall (see *e.g.* [69, Theorem 7.12]) that if $(\eta_k)_{k \geq 1}$ is a sequence of elements of $\mathcal{P}_2(\mathbb{R}^d)$ and $\eta \in \mathcal{P}_2(\mathbb{R}^d)$, then $W_2(\eta_k, \eta) \rightarrow 0$ if and only if $\int f d\eta_k \rightarrow \int f d\eta$ for any continuous function f satisfying $|f|(x) \leq a + b|x|^2$, $x \in \mathbb{R}^d$, for some $a, b \geq 0$.

Let us show that the set C_ν is compact for this topology. By assumption, $\int |x|^2 d\nu < +\infty$, therefore, according to the de la Vallée-Poussin Theorem (see *e.g.* [8, Theorem 4.5.9]), there exists some increasing convex function $\theta : [0, \infty) \rightarrow [0, \infty)$ such that $\theta(r)/r \rightarrow +\infty$

as $r \rightarrow \infty$ and $\int \theta(|x|^2) d\nu(x) < +\infty$. Since θ is convex and increasing, the function $x \mapsto \theta(|x|^2)$ is convex on \mathbb{R}^d . Therefore, by definition of C_ν , for any $\eta \in C_\nu$, $\int \theta(|x|^2) d\eta \leq \int \theta(|x|^2) d\nu := D$. In particular, applying Markov's inequality yields that for all $R > 0$,

$$\sup_{\eta \in C_\nu} \int |x|^2 \mathbf{1}_{|x| > R} d\eta(x) \leq D\varepsilon(R)$$

where $\varepsilon(R) := \sup_{r \geq R} r/\theta(r) \rightarrow 0$ as $R \rightarrow \infty$. According to *e.g.* [31, Theorem 9.10], this shows that C_ν is precompact for the W_2 -topology. Now, let us show that C_ν is also closed. Namely, C_ν can be written as follows

$$C_\nu = \left\{ \eta \in \mathcal{P}_2(\mathbb{R}^d) : \int f d\eta \leq \int f d\nu, \forall f : \mathbb{R}^d \rightarrow \mathbb{R} \text{ convex and Lipschitz} \right\}$$

(the fact that one can restrict to convex and Lipschitz functions in the definition of the convex order is classical ; one can see this by noting that if f is convex then $f_n = \inf_{x \in \mathbb{R}^d} \{f(y) + n|x - y|\}$, $x \in \mathbb{R}^d$, is a sequence of n -Lipschitz convex functions converging to f monotonically). Since functionals $\eta \mapsto \int f d\eta$ with f convex and Lipschitz are continuous for the W_2 topology, it follows that C_ν is closed and thus C_ν is compact. The map $\eta \mapsto W_2(\eta, \mu)$ therefore reaches its minimum on C_ν .

Now let us show that the minimizer is unique. This will follow from the strict convexity of $W_2^2(\mu, \cdot)$ along generalized geodesics with base point μ and from the convexity of the set C_ν along those generalized geodesics. More precisely, suppose that η_0, η_1 are in C_ν and let (X, Y_0, Y_1) be a random vector such that $X \sim \mu$, $Y_0 \sim \eta_0$, $Y_1 \sim \eta_1$ and so that $\text{Law}(X, Y_i) \in C(\mu, \eta_i)$ and $W_2^2(\mu, \eta_i) = \mathbb{E}[|X - Y_i|^2]$, $i = 0, 1$. Then define $\eta_t = \text{Law}((1-t)Y_0 + tY_1)$ for all $t \in [0, 1]$. First note that $\eta_t \in C_\nu$ for all $t \in [0, 1]$. Indeed, if f is a convex function on \mathbb{R}^d , it holds

$$\int f d\eta_t = \mathbb{E}[f((1-t)Y_0 + tY_1)] \leq (1-t)\mathbb{E}[f(Y_0)] + t\mathbb{E}[f(Y_1)] \leq \int f d\nu.$$

Moreover, according to Lemma 9.2.1 of [2], it holds

$$W_2^2(\mu, \eta_t) \leq (1-t)W_2^2(\mu, \eta_0) + tW_2^2(\mu, \eta_1) - t(1-t)W_2^2(\eta_0, \eta_1), \quad \forall t \in [0, 1]$$

So if $W_2^2(\mu, \eta_0) = W_2^2(\mu, \eta_1) = \inf_{\eta \in C_\nu} W_2^2(\mu, \eta)$, then necessarily $\eta_0 = \eta_1$, which shows uniqueness.

Finally, let us show that $\overline{\mathcal{T}}_2(\nu|\mu) = W_2^2(\mu^*, \mu)$. First, let (X, Y) be a coupling of μ and ν and set $X' = \mathbb{E}[Y|X]$. Since (X', Y) is a martingale, it follows that $\text{Law}(X') \in C_\nu$. Therefore $\mathbb{E}[|X - X'|^2] \geq W_2^2(\mu, \mu^*)$ and so optimizing gives $\overline{\mathcal{T}}_2(\nu|\mu) \geq W_2^2(\mu, \mu^*)$. On the other hand, let $\pi \in C(\mu, \eta)$ be a coupling between μ and some $\eta \in C_\nu$. Since $\eta \preceq_c \nu$, one can construct a Markov chain (X, X', Y) such that $(X, X') \sim \pi$ and (X', Y) is a martingale. Then it holds,

$$\begin{aligned} \mathbb{E}[|X - X'|^2] &= \mathbb{E}[|X - \mathbb{E}[Y|X, X']|^2] \\ &= \mathbb{E} \left[\mathbb{E}[|X - \mathbb{E}[Y|X, X']|^2 | X] \right] \\ &\geq \mathbb{E}[|X - \mathbb{E}[Y|X]|^2] \geq \overline{\mathcal{T}}_2(\nu|\mu), \end{aligned}$$

where the inequality comes from Jensen inequality for conditional expectation. Optimizing over X' and η gives that $\overline{\mathcal{T}}_2(\nu|\mu) \leq W_2^2(\mu, \mu^*)$ and completes the proof. \square

Remark 6.1. Note that it is easy to conclude using similar arguments that there always exists a deterministic map T transporting μ on μ^* such that $W_2^2(\mu^*, \mu) = \int |x - T(x)|^2 d\mu(x)$. Indeed, suppose that (X, X^*) is an optimal coupling for $W_2^2(\mu^*, \mu)$ and consider $X' = \mathbb{E}[X^*|X]$. Then, Jensen inequality gives

$$(8) \quad \mathbb{E}[|X - X'|^2] = \mathbb{E}[|X - \mathbb{E}[X^*|X]|^2] \leq \mathbb{E}[|X - X^*|^2] = W_2^2(\mu^*, \mu).$$

Let $\eta' := \text{Law}(X')$. Then $\eta' \preceq_c \mu^*$ and $\mu^* \preceq_c \nu$ so $\eta' \in C_\nu$. Therefore, $\mathbb{E}[|X - X'|^2] \geq \inf_{\eta \in C_\nu} W_2^2(\eta, \mu) = W_2^2(\mu^*, \mu)$ and there is equality in (8). So $X' = \mathbb{E}[X^*|X] \sim \mu^*$ and (X, X') is an optimal coupling. Finally, by definition of conditional expectation, there exists a measurable $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $\mathbb{E}[X^*|X] = T(X)$ almost surely, which proves the claim.

Our proof of Theorem 1.2 follows closely the scheme developed by Gangbo [23] in his alternative proof of Brenier Theorem: first we show the dual attainment, and then we obtain the existence of the transport map $\nabla\varphi$ by doing a first variation around the minimizer \bar{f} .

Proof of Item (a) of Theorem 1.2. We will assume that μ and ν are supported in a closed ball B of radius $R > 0$ centered at 0. Let α be the convex function on $[0, \infty)$ defined by

$$\alpha(t) = t^2 \text{ if } t \in [0, 2R] \quad \text{and} \quad \alpha(t) = 4Rt - 4R^2 \text{ if } t \geq 2R.$$

and let $\theta(u) = \alpha(|u|)$, for all $u \in \mathbb{R}^d$. Since μ and ν are supported in B it is easily seen that

$$\bar{\mathcal{T}}_2(\nu|\mu) = \bar{\mathcal{T}}_\theta(\nu|\mu) = \sup_h \left\{ \int Q_\theta h d\mu - \int h d\nu \right\},$$

where the supremum runs over the set of convex functions h bounded from below.

Step 1 - Preparation. Let P_θ be the operator acting on functions defined as follows

$$P_\theta g(y) = \sup_{x \in \mathbb{R}^d} \{g(x) - \theta(x - y)\}, \quad y \in \mathbb{R}^d.$$

First, let us show that

$$\bar{\mathcal{T}}_2(\nu|\mu) = \sup_{f \in \mathcal{F}} \left\{ \int Q_\theta f d\mu - \int f d\nu \right\},$$

where Q_θ is defined as above and \mathcal{F} is the class of functions f which are convex bounded from below with $f(0) = 0$ and such that $f = P_\theta(Q_\theta f)$. Indeed, let h be a convex function bounded from below on \mathbb{R}^d . Then, by definition of Q_θ , it holds

$$Q_\theta h(x) - \theta(x - y) \leq h(y), \quad \forall x, y \in \mathbb{R}^d$$

and so $h \geq P_\theta(Q_\theta h)$. Since $Q_\theta h$ is convex (as an infimum convolution of convex functions), the function f defined by

$$f(y) = P_\theta(Q_\theta h)(y) = \sup_{u \in \mathbb{R}^d} \{Q_\theta h(y + u) - \theta(u)\}, \quad y \in \mathbb{R}^d$$

is also convex as a supremum of convex functions. Moreover, it is easily seen that $Q_\theta f = Q_\theta h$ and so $P_\theta(Q_\theta f) = f$. Therefore,

$$\int Q_\theta h d\mu - \int h d\nu \leq \int Q_\theta f d\mu - \int f d\nu,$$

which shows that the duality formula can be restricted to $f \in \mathcal{F}$ (the fact that one can always assume that $f(0) = 0$ is clear).

Step 2 - Dual attainment. Now, let us show that there is some convex function \bar{f} satisfying (4). First of all, if $f \in \mathcal{F}$, then f is $4R$ -Lipschitz. This comes from the fact that

$$f(y) = \sup_{x \in \mathbb{R}^d} \{Q_\theta f(x) - \theta(x - y)\}, \quad \forall y \in \mathbb{R}^d$$

which (the function $y \mapsto \theta(x - y)$ being $4R$ -Lipschitz on \mathbb{R}^d for every $x \in \mathbb{R}^d$) shows that f is a supremum of $4R$ -Lipschitz functions and is thus $4R$ -Lipschitz itself. Since $f(0) = 0$, this implies in particular that $|f| \leq 20R^2$ on $B' = 5B$ (the ball of radius $5R$ centered at 0). Also, since f is $4R$ -Lipschitz, it holds

$$f(y) + |x - y|^2 \geq f(x) - 4R|x - y| + |x - y|^2 \quad \forall x, y \in \mathbb{R}^d.$$

Therefore, $f(y) + |x - y|^2 \geq f(x)$ whenever $|x - y| \geq 4R$. Since $Q_2 f(x) \leq f(x)$, this implies that

$$Q_2 f(x) = \inf_{|y-x| \leq 4R} \{f(y) + |x - y|^2\}, \quad \forall x \in \mathbb{R}^d.$$

Now, let $f_n \in \mathcal{F}$ be some minimizing sequence. The functions f_n are $4R$ -Lipschitz and uniformly bounded on the ball B' . Therefore, it follows from Ascoli's theorem that f_n admits a sub-sequence (still denoted f_n in the sequel) converging uniformly to some \bar{f} on B' . This function \bar{f} is convex on B' as a pointwise limit of convex functions. It is easily seen that $Q_2 f_n \rightarrow Q_2 \bar{f}$ (uniformly) on B . Since $Q_2 f_n \geq Q_\theta f_n$ pointwise, it holds

$$\overline{\mathcal{T}}_2(\nu|\mu) \geq \int Q_2 f_n d\mu - \int f_n d\nu \geq \int Q_\theta f_n d\mu - \int f_n d\nu.$$

Since the right hand side goes to $\overline{\mathcal{T}}_2(\nu|\mu)$ and μ, ν are supported in B , letting $n \rightarrow \infty$ yields to (4). For the moment the convex function \bar{f} is defined only on B' but it can be easily extended outside B' as follows: the function $\tilde{f}(x) = \inf_{y \in B'} \{\bar{f}(y) + 4R|x - y|\}$, $x \in \mathbb{R}^d$, is convex as an infimum convolution of two convex functions, and since \bar{f} is $4R$ -Lipschitz it is easily seen that $\tilde{f} = \bar{f}$ on B' . In the sequence, we will thus assume that \bar{f} is a finite valued convex function defined on the whole \mathbb{R}^d . \square

In order to prove Item (b) of Theorem 1.2, we will need a technical lemma adapted from [23, Lemma 2.4].

Lemma 6.1. *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a convex function and let $h(x) = \frac{f(x) + |x|^2}{2}$, $x \in \mathbb{R}^d$.*

(a) *The function h^* is \mathcal{C}^1 smooth and it holds*

$$Q_2 f(x) = f(\nabla h^*(x)) + |x - \nabla h^*(x)|^2, \quad \forall x \in \mathbb{R}^d.$$

(b) *For all convex function $u : \mathbb{R}^d \rightarrow \mathbb{R}$,*

$$\lim_{t \rightarrow 0^+} \frac{Q_2(f + tu)(x) - Q_2(f)(x)}{t} = u(\nabla h^*(x)), \quad \forall x \in \mathbb{R}^d,$$

and for any compact set $K \subset \mathbb{R}^d$

$$\sup_{t \in [0,1], x \in K} \left| \frac{Q_2(f + tu)(x) - Q_2(f)(x)}{t} \right| < +\infty.$$

(c) *The conclusions of Item (b) also hold for the concave function $u = -f$.*

Let us admit the lemma for a moment and prove Item (b).

Proof of Item (b) of Theorem 1.2. According to Item (a) of Lemma 6.1, the function $\varphi = h^*$ is of class \mathcal{C}^1 . The fact that $\nabla \varphi$ is 1-Lipschitz follows from Lemma 2.1 (proved below). Let $\bar{\mu} = \nabla \varphi \# \mu$ and let us show that $\bar{\mu} = \mu^*$. First let us prove that $\bar{\mu} \preceq_c \nu$. Let $u : \mathbb{R}^d \rightarrow \mathbb{R}$ be some arbitrary convex function. By optimality of \bar{f} it holds, for all $t > 0$,

$$\int Q_2(\bar{f} + tu) d\mu - \int (\bar{f} + tu) d\nu \leq \int Q_2(\bar{f}) d\mu - \int \bar{f} d\nu$$

Therefore, for all $t > 0$,

$$\int \frac{Q_2(\bar{f} + tu) - Q_2(\bar{f})}{t} d\mu \leq \int u d\nu.$$

Using Item (b) of Lemma 6.1, one concludes that

$$\int u(\nabla \varphi) d\mu \leq \int u d\nu$$

for all convex function $u : \mathbb{R}^d \rightarrow \mathbb{R}$. This shows that $\bar{\mu} \preceq_c \nu$. In particular, taking $u = \bar{f}$, one gets $\int \bar{f} d\bar{\mu} \leq \int \bar{f} d\nu$. Actually, for this special function, equality holds. Indeed, if $u = -f$ then it is still true that $\bar{f} + tu$ is convex for $0 \leq t \leq 1$. So using Item (c) of

Lemma 6.1 and repeating the argument gives that $\int \bar{f} d\bar{\mu} \geq \int \bar{f} d\nu$, which shows equality. Therefore,

$$\begin{aligned} \bar{\mathcal{T}}_2(\nu|\mu) &= \int Q_2 \bar{f} d\mu - \int \bar{f} d\nu \\ &= \int \bar{f}(\nabla\varphi(x)) + |x - \nabla\varphi(x)|^2 d\mu(x) - \int \bar{f}(y) d\nu(y) \\ &= \int |x - \nabla\varphi(x)|^2 d\mu(x) \\ &\geq W_2^2(\bar{\mu}, \mu). \end{aligned}$$

Finally, if $\eta \preceq_c \nu$, then

$$\begin{aligned} \bar{\mathcal{T}}_2(\nu|\mu) &= \sup_{h \text{ convex}} \left\{ \int Q_2 h d\mu - \int h d\nu \right\} \\ &\leq \sup_{h \text{ convex}} \left\{ \int Q_2 h d\mu - \int h d\eta \right\} \\ &\leq W_2^2(\mu, \eta), \end{aligned}$$

where the last inequality follows easily from the inequality $Q_2 h(x) - h(y) \leq |x - y|^2$, $x, y \in \mathbb{R}^d$. In particular taking $\eta = \bar{\mu}$ shows that

$$\bar{\mathcal{T}}_2(\nu|\mu) = W_2^2(\bar{\mu}, \mu) = \inf_{\eta \preceq_c \nu} W_2^2(\mu, \eta)$$

and completes the proof. \square

Proof of Item (c) of Theorem 1.2. Since $\mu^* \preceq_c \nu$, Strassen Theorem implies that there exists a kernel q such that $\int y dq_x(y) = x$ for μ^* almost all x and $\nu(\cdot) = \int q_x(\cdot) d\mu^*(x)$. Let (X, X^*, Y) be a time inhomogeneous Markov chain with initial distribution μ and $\text{Law}(X^*|X) = \delta_{\nabla\varphi(X)}$ and $\text{Law}(Y|X^*) = q_{X^*}$ almost surely. Then it holds

$$\begin{aligned} \mathbb{E}[|X - \mathbb{E}[Y|X]|^2] &= \mathbb{E} \left[\left| X - \mathbb{E} \left[\int y dq_{X^*}(y) | X \right] \right|^2 \right] = \mathbb{E} \left[\left| X - \int y dq_{\nabla\varphi(X)}(y) \right|^2 \right] \\ &= \mathbb{E} \left[|X - \nabla\varphi(X)|^2 \right] = \bar{\mathcal{T}}_2(\nu|\mu), \end{aligned}$$

which shows the optimality of (X, Y) .

Now let $d\pi(x, y) = d\mu(x)dp_x(y)$ be the law of (X, Y) (with therefore $dp_x(y) = dq_{\nabla\varphi(x)}(y)$) and suppose that $d\pi'(x, y) = d\mu(x)dp'_x(y)$ is another weak optimal transport plan, then

$$\int \left| x - \int y d \left(\frac{p_x + p'_x}{2} \right) (y) \right|^2 d\mu(x) \leq \int \frac{|x - \int y dp_x(y)|^2 + |x - \int y dp'_x(y)|^2}{2} d\mu(x).$$

By optimality and strict convexity of $|\cdot|^2$ we deduce that $\int y dp_x = \int y dp'_x$, μ -almost surely. In other words, $\mathbb{E}[Y'|X' = u] = \mathbb{E}[Y|X = u] = \nabla\varphi(u)$ for μ almost all $u \in \mathbb{R}^d$. So if (X', Y') is a weak optimal coupling, one has $\mathbb{E}[Y'|X'] = \nabla\varphi(X')$ almost surely. In particular $\mathbb{E}[Y'|X'] \sim \mu^*$. The fact that $(\mathbb{E}[Y'|X'], Y')$ is a martingale is always true. \square

Finally let us prove Lemmas 6.1 and 2.1.

Proof of Lemma 6.1. (a) It is easily seen that h^* is everywhere finite on \mathbb{R}^d . Furthermore, the function h being strictly convex, its conjugate h^* is continuously differentiable on \mathbb{R}^d ([34, Theorem E.4.1.1]). Moreover,

$$Q_2 f(x) = \inf_{y \in \mathbb{R}^d} \{f(y) + |x - y|^2\} = |x|^2 - 2 \sup_{y \in \mathbb{R}^d} \{x \cdot y - h(y)\} = |x|^2 - 2h^*(x)$$

and one sees that y is optimal in the definition of $Q_2 f(x)$ if and only if $x \cdot y = h(y) + h^*(x)$, that is to say if and only if $y \in \partial h^*(x) = \{\nabla h^*(x)\}$. This proves the first formula.

(b) First observe that for any function $u : \mathbb{R}^d \rightarrow \mathbb{R}$ it holds

$$\begin{aligned} Q_2(f+tu)(x) &= \inf_{y \in \mathbb{R}^d} \{(f+tu)(y) + |x-y|^2\} \leq (f+tu)(\nabla h^*(x)) + |x - \nabla h^*(x)|^2 \\ &= Q_2f(x) + tu(\nabla h^*(x)) \end{aligned}$$

and so it holds

$$(9) \quad \frac{Q_2(f+tu)(x) - Q_2f(x)}{t} \leq u(\nabla h^*(x)), \quad \forall x \in \mathbb{R}^d, \forall t \in (0, 1].$$

Now let us bound this quantity from below. Since the functions f and u are both convex on \mathbb{R}^d , there exist $a, b \in \mathbb{R}^d$ and $c, d \in \mathbb{R}$ such that $f(y) \geq a \cdot y + c$ and $u(y) \geq b \cdot y + d$, for all $y \in \mathbb{R}^d$ and thus, for all $t \in [0, 1]$,

$$(10) \quad (f+tu)(y) \geq (a+tb) \cdot y + (c+td) \geq -(|a|+|b|)|y| - (|c|+|d|), \quad \forall y \in \mathbb{R}^d.$$

Therefore, for all $t \in [0, 1]$ and $x \in \mathbb{R}^d$, the continuous function $y \mapsto (f+tu)(y) + |x-y|^2$ tends to $+\infty$ when $|y| \rightarrow +\infty$ and so it achieves its minimum at some point $y_t(x)$ (which is actually unique by strict convexity). Moreover, it holds

$$Q_2(f+tu)(x) - Q_2f(x) \geq (f+tu)(y_t(x)) + |x-y_t(x)|^2 - \left(f(y_t(x)) + |x-y_t(x)|^2 \right) = tu(y_t(x))$$

and so

$$(11) \quad \frac{Q_2(f+tu)(x) - Q_2f(x)}{t} \geq u(y_t(x)), \quad \forall x \in \mathbb{R}^d, \forall t \in (0, 1].$$

The function u being convex it is continuous and so to complete the proof of Item (b) it is enough to show that $y_t \rightarrow \nabla h^*$ pointwise as $t \rightarrow 0^+$ and that for any compact set K , the family $\{y_t(x) : t \in [0, 1], x \in K\}$ is bounded. Fix some compact set $K \subset \mathbb{R}^d$ and let $C = \sup_{x \in K, t \in [0, 1]} \{f(x) + tu(x)\}$. For all $t \in [0, 1]$, it holds $Q_2(f+tu)(x) \leq (f+tu)(x) \leq C$, for all $x \in K$. So letting

$$L = \left\{ y \in \mathbb{R}^d : -(|a|+|b|)|y| - (|c|+|d|) + \inf_{x \in K} |x-y|^2 \leq C \right\}$$

one easily sees that $y_t(x) \in L$ for all $x \in K$ and $t \in [0, 1]$. This set L being compact, it follows that if $t_n \in (0, 1]$ is some sequence converging to 0, the sequence $y_{t_n}(x)$ admits a subsequence (still denoted $y_{t_n}(x)$ for simplicity) converging to some $\ell(x)$. According to (9) and (11) $Q_2(f+t_nu)(x) \rightarrow Q_2f(x)$. But on the other hand, $Q_2(f+t_nu)(x) = f(y_{t_n}(x)) + |x-y_{t_n}(x)|^2 + t_nu(y_{t_n}(x)) \rightarrow f(\ell(x)) + |x-\ell(x)|^2$. So we get $Q_2f(x) = f(\ell(x)) + |x-\ell(x)|^2$, which implies according to the proof of Item (a) that $\ell(x) = \nabla h^*(x)$. It follows that $\nabla h^*(x)$ is the unique limit point of the family $(y_t(x))_{t \in (0, 1]}$ when $t \rightarrow 0^+$.

(c) If $u = -f$, it is easy to see that

$$(f+tu)(y) = (1-t)f(y) \geq (1-t)(a \cdot y + c) \geq -|a||y| - |c|, \quad \forall y \in \mathbb{R}^d, \forall t \in [0, 1].$$

Using this inequality instead of (10) the preceding proof can be repeated step by step. \square

Proof of Lemma 2.1. Let us show that (a) implies (b). Let $f(x) = g(x) - \frac{|x|^2}{2}$, $x \in \mathbb{R}^d$ so that $g = f + \frac{|\cdot|^2}{2}$. According to e.g. [34, Theorem E.2.3.1] the convex conjugate of g is therefore given by

$$g^*(y) = \inf_{x \in \mathbb{R}^d} \left\{ g(x) + \frac{|x-y|^2}{2} \right\} = \frac{|y|^2}{2} - \sup_{x \in \mathbb{R}^d} \left\{ x \cdot y - \left(g(x) + \frac{|x|^2}{2} \right) \right\}$$

The function defined by the supremum being clearly convex, it follows that $y \mapsto \frac{|y|^2}{2} - g^*(y)$ is convex on \mathbb{R}^d , which shows (b). Conversely, let us show that (b) \Rightarrow (a). Let $k(y) =$

$\frac{|y|^2}{2} - g^*(y)$, $y \in \mathbb{R}^d$, which is convex by assumption. By Fenchel-Legendre duality, it holds

$$g(x) = \sup_{y \in \mathbb{R}^d} \{x \cdot y - g^*(y)\} = \sup_{y \in \mathbb{R}^d} \left\{ x \cdot y - \frac{|y|^2}{2} + k(y) \right\}, \quad \forall x \in \mathbb{R}^d$$

and so

$$g(x) - \frac{|x|^2}{2} = \sup_{y \in \mathbb{R}^d} \left\{ k(y) - \frac{|x-y|^2}{2} \right\} = \sup_{u \in \mathbb{R}^d} \left\{ k(x-u) - \frac{|u|^2}{2} \right\} \quad \forall x \in \mathbb{R}^d.$$

The function $x \mapsto g(x) - \frac{|x|^2}{2}$ is therefore convex as a supremum of convex functions.

Now let us show that (a) implies (c). We have already seen that g^* is of class \mathcal{C}^1 in the proof of Item (a) of Lemma 6.1. It remains to prove that ∇g^* is 1-Lipschitz. Since the function $x \mapsto g(x) - \frac{|x|^2}{2}$ is convex, its subgradient is a monotone operator, which means that

$$(b-a) \cdot (y-x) \geq |y-x|^2, \quad \forall x, y \in \mathbb{R}^d, \quad \forall b \in \partial g(y), \quad \forall a \in \partial g(x)$$

Since $u \in \partial g(v)$ is equivalent to $v \in \partial g^*(u) = \{\nabla g^*(u)\}$, the statement above is equivalent to

$$(\nabla g^*(b) - \nabla g^*(a)) \cdot (b-a) \geq |\nabla g^*(b) - \nabla g^*(a)|^2, \quad \forall a, b \in \mathbb{R}^d$$

which immediately implies that ∇g^* is 1-Lipschitz. Finally, let us show that (c) implies (b). Since ∇g^* is 1-Lipschitz, it holds

$$(\nabla g^*(y) - \nabla g^*(x)) \cdot (y-x) \leq |y-x|^2, \quad \forall x, y \in \mathbb{R}^d$$

which easily implies that $x \mapsto \frac{|x|^2}{2} - g^*(x)$ is convex. This completes the proof. \square

REFERENCES

1. Jean-Jacques Alibert, Guy Bouchitté, and Thierry Champion, *A new class of cost for optimal transport planning*, preprint, March 2018. [6](#), [14](#), [15](#)
2. L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, second ed., Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008. [1](#), [16](#)
3. L. Ambrosio and S. Rigot, *Optimal mass transportation in the Heisenberg group*, J. Funct. Anal. **208** (2004), no. 2, 261–301. MR 2035027 (2004m:49107) [1](#)
4. D. Bakry, I. Gentil, and M. Ledoux, *Analysis and geometry of Markov diffusion operators*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 348, Springer, Cham, 2014. [1](#)
5. M. Beiglböck and N. Juillet, *Shadow couplings*, ArXiv e-prints (2016). [2](#), [15](#)
6. Mathias Beiglböck and Nicolas Juillet, *On a problem of optimal transport under marginal martingale constraints*, Ann. Probab. **44** (2016), no. 1, 42–106. MR 3456332 [5](#), [9](#), [11](#)
7. Jérôme Bertrand, *Existence and uniqueness of optimal maps on Alexandrov spaces*, Adv. Math. **219** (2008), no. 3, 838–851. MR 2442054 (2009j:49094) [1](#)
8. V. I. Bogachev, *Measure theory. Vol. I*, Springer-Verlag, Berlin, 2007. MR 2267655 [15](#)
9. Charles Boubel and Nicolas Juillet, *The markov-quantile process attached to a family of marginals*. [2](#)
10. M. Bowles and N. Ghoussoub, *A Theory of Transfers: Duality and convolution*, ArXiv e-prints (2018). [6](#)
11. Yann Brenier, *Décomposition polaire et réarrangement monotone des champs de vecteurs*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), no. 19, 805–808. MR 923203 [1](#)
12. ———, *Polar factorization and monotone rearrangement of vector-valued functions*, Comm. Pure Appl. Math. **44** (1991), no. 4, 375–417. MR 1100809 [1](#)
13. Luis A. Caffarelli, *Monotonicity properties of optimal transportation and the FKG and related inequalities*, Comm. Math. Phys. **214** (2000), no. 3, 547–563. MR 1800860 [4](#), [6](#)
14. G. Conforti, *A second order equation for Schrödinger bridges with applications to the hot gas experiment and entropic transportation cost*, ArXiv e-prints (2017). [6](#)
15. G. Conforti and L. Ripani, Personal communication, 2018. [6](#)
16. Daskalakis Constantinos, Deckelbaum Alan, and Tzamos Christos, *Strong duality for a multiple-good monopolist*. [15](#)
17. Dario Cordero-Erausquin, *Some applications of mass transport to Gaussian-type inequalities*, Arch. Ration. Mech. Anal. **161** (2002), no. 3, 257–269. MR 1894593 [6](#)

18. Amir Dembo, *Information inequalities and concentration of measure*, Ann. Probab. **25** (1997), no. 2, 927–939. MR 1434131 [5](#)
19. Shizan Fang and Jinghai Shao, *Optimal transport maps for Monge-Kantorovich problem on loop groups*, J. Funct. Anal. **248** (2007), no. 1, 225–257. MR 2329689 (2008f:58011) [1](#)
20. Max Fathi and Yan Shu, *Curvature and transport inequalities for Markov chains in discrete spaces*, Bernoulli **24** (2018), no. 1, 672–698. MR 3706773 [5](#)
21. D. Feyel and A. S. Üstünel, *Monge-Kantorovich measure transportation and Monge-Ampère equation on Wiener space*, Probab. Theory Related Fields **128** (2004), no. 3, 347–385. MR 2036490 (2004m:60121) [1](#)
22. Alessio Figalli and Ludovic Rifford, *Mass transportation on sub-Riemannian manifolds*, Geom. Funct. Anal. **20** (2010), no. 1, 124–159. MR 2647137 (2012a:49081) [1](#)
23. Wilfrid Gangbo, *An elementary proof of the polar factorization of vector-valued functions*, Arch. Rational Mech. Anal. **128** (1994), no. 4, 381–399. MR 1308860 [17](#), [18](#)
24. Wilfrid Gangbo and Robert J. McCann, *The geometry of optimal transportation*, Acta Math. **177** (1996), no. 2, 113–161. MR 1440931 [1](#), [7](#)
25. Nassif Ghoussoub, Young-Heon Kim, and Tongseok Lim, *Structure of optimal martingale transport plans in general dimensions*, to appear in AoP. [6](#)
26. Nicola Gigli, *On the inverse implication of Brenier-McCann theorems and the structure of $(\mathcal{P}_2(M), W_2)$* , Methods Appl. Anal. **18** (2011), no. 2, 127–158. MR 2847481 [1](#), [4](#)
27. Nicola Gigli, Tapio Rajala, and Karl-Theodor Sturm, *Optimal maps and exponentiation on finite-dimensional spaces with Ricci curvature bounded from below*, J. Geom. Anal. **26** (2016), no. 4, 2914–2929. MR 3544946 [1](#)
28. N. Gozlan and C. Léonard, *Transport inequalities. A survey*, Markov Process. Related Fields **16** (2010), no. 4, 635–736. [5](#)
29. N. Gozlan, C. Roberto, P.-M. Samson, Y. Shu, and P. Tetali, *Characterization of a class of weak transport-entropy inequalities on the line*, preprint, to appear in Ann. Inst. Henri Poincaré Probab. Stat., 2015. [4](#), [5](#), [14](#)
30. N. Gozlan, C. Roberto, P.-M. Samson, and P. Tetali, *Displacement convexity of entropy and related inequalities on graphs*, Probab. Theory Related Fields **160** (2014), no. 1-2, 47–94. [5](#)
31. Nathael Gozlan, Cyril Roberto, Paul-Marie Samson, and Prasad Tetali, *Kantorovich duality for general transport costs and applications*, J. Funct. Anal. **273** (2017), no. 11, 3327–3405. MR 3706606 [1](#), [2](#), [3](#), [5](#), [16](#)
32. Gilles Hargé, *Inequalities for the Gaussian measure and an application to Wiener space*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), no. 8, 791–794. MR 1868955 [6](#)
33. Pierre Henry-Labordère and Nizar Touzi, *An explicit martingale version of the one-dimensional Brenier theorem*, Finance Stoch. **20** (2016), no. 3, 635–668. MR 3519164 [5](#)
34. J.-B. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of convex analysis*, Grundlehren Text Editions, Springer-Verlag, Berlin, 2001. [19](#), [20](#)
35. F. Hirsch and M. Yor, *Looking for martingales associated to a self-decomposable law*, Electron. J. Probab. **15** (2010), no. 29, 932–961. MR 2659753 (2011f:60147) [2](#)
36. Francis Hirsch, Christophe Profeta, Bernard Roynette, and Marc Yor, *Peacocks and associated martingales, with explicit constructions*, Bocconi & Springer Series, vol. 3, Springer, Milan; Bocconi University Press, Milan, 2011. MR 2808243 [2](#)
37. Francis Hirsch and Bernard Roynette, *On \mathbb{R}^d -valued peacocks*, ESAIM Probab. Stat. **17** (2013), 444–454. MR 3066388 [2](#)
38. Francis Hirsch, Bernard Roynette, and Marc Yor, *Kellerer’s theorem revisited*, Asymptotic Laws and Methods in Stochastics. Volume in Honour of Miklos Csorgo (Springer, ed.), Fields Institute Communications Series, 2014. [2](#)
39. David Hobson and Martin Klimmek, *Robust price bounds for the forward starting straddle*, Finance Stoch. **19** (2015), no. 1, 189–214. MR 3292129 [6](#)
40. David Hobson and Anthony Neuberger, *Robust bounds for forward start options*, Math. Finance **22** (2012), no. 1, 31–56. MR 2881879 [6](#)
41. David G. Hobson and Dominykas Norgilas, *The left-curtain martingale coupling in the presence of atoms*. [5](#)
42. Nicolas Juillet, *Peacocks parametrised by a partially ordered set*, Séminaire de Probabilités XLVIII, Lecture Notes in Math., vol. 2168, Springer, Cham, 2016, pp. 13–32. MR 3618125 [2](#)
43. ———, *Stability of the shadow projection and the left-curtain coupling*, Ann. Inst. Henri Poincaré Probab. Stat. **52** (2016), no. 4, 1823–1843. MR 3573297 [5](#)
44. Hans G. Kellerer, *Markov-Komposition und eine Anwendung auf Martingale*, Math. Ann. **198** (1972), 99–122. MR 0356250 [2](#)
45. ———, *Integraldarstellung von Dilationen*, (1973), 341–374. MR 0356249 [2](#)

46. M. Ledoux, *The concentration of measure phenomenon*, Mathematical Surveys and Monographs, vol. 89, American Mathematical Society, Providence, RI, 2001. [5](#)
47. Christian Léonard, *A survey of the Schrödinger problem and some of its connections with optimal transport*, Discrete Contin. Dyn. Syst. **34** (2014), no. 4, 1533–1574. MR 3121631 [6](#)
48. Tongseok Lim, *Optimal martingale transport between radially symmetric marginals in general dimensions*. [6](#)
49. George Lowther, *Limits of one-dimensional diffusions*, Ann. Probab. **37** (2009), no. 1, 78–106. MR 2489160 (2010g:60182) [2](#)
50. Hadrien De March, *Local structure of multi-dimensional martingale optimal transport*. [6](#)
51. K. Marton, *Bounding \bar{d} -distance by informational divergence: a method to prove measure concentration*, Ann. Probab. **24** (1996), no. 2, 857–866. [5](#)
52. ———, *A measure concentration inequality for contracting Markov chains*, Geom. Funct. Anal. **6** (1996), no. 3, 556–571. [5](#)
53. Robert J. McCann, *Polar factorization of maps on Riemannian manifolds*, Geom. Funct. Anal. **11** (2001), no. 3, 589–608. MR 1844080 [1](#)
54. Robert J. McCann and Nestor Guillen, *Five lectures on optimal transportation: geometry, regularity and applications*, Analysis and geometry of metric measure spaces, CRM Proc. Lecture Notes, vol. 56, Amer. Math. Soc., Providence, RI, 2013, pp. 145–180. MR 3060502 [1](#)
55. Emanuel Milman, *Spectral estimates, contractions and hypercontractivity*, J. Spectr. Theory **8** (2018), no. 2, 669–714. MR 3812812 [6](#)
56. Marcel Nutz and Florian Stebegg, *Canonical supermartingale couplings*, to appear in AoP. [6](#), [11](#)
57. Enrique Outerelo and Jesús M. Ruiz, *Mapping degree theory*, Graduate Studies in Mathematics, vol. 108, American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2009. MR 2566906 [13](#)
58. S. T. Rachev and L. Rüschendorf, *Mass transportation problems. Vol. I*, Probability and its Applications (New York), Springer-Verlag, New York, 1998, Theory. [7](#)
59. L. Rüschendorf and S. T. Rachev, *A characterization of random variables with minimum L^2 -distance*, J. Multivariate Anal. **32** (1990), no. 1, 48–54. MR 1035606 [7](#)
60. P.-M. Samson, *Concentration of measure inequalities for Markov chains and Φ -mixing processes*, Ann. Probab. **28** (2000), no. 1, 416–461. [5](#)
61. ———, *Concentration inequalities for convex functions on product spaces*, Stochastic inequalities and applications, Progr. Probab., vol. 56, Birkhäuser, Basel, 2003, pp. 33–52. [5](#)
62. ———, *Infimum-convolution description of concentration properties of product probability measures, with applications*, Ann. Inst. H. Poincaré Probab. Statist. **43** (2007), no. 3, 321–338. [5](#)
63. ———, *Concentration of measure principle and entropy inequalities*, Convexity and concentration, The IMA Volumes in Mathematics and its Applications, vol. 161, Springer, New York, 2017, pp. 55–105. [4](#), [5](#)
64. Paul-Marie Samson, *Transport-entropy inequalities on locally acting groups of permutations*, Electron. J. Probab. **22** (2017), Paper No. 62, 33. MR 3690287 [5](#)
65. Yan Shu, *Hamilton-Jacobi equations on graph and applications*, Potential Anal. **48** (2018), no. 2, 125–157. MR 3748388 [5](#)
66. V. Strassen, *The existence of probability measures with given marginals*, Ann. Math. Statist. **36** (1965), 423–439. [2](#), [9](#)
67. M. Talagrand, *Concentration of measure and isoperimetric inequalities in product spaces*, Inst. Hautes Études Sci. Publ. Math. (1995), no. 81, 73–205. [5](#)
68. ———, *New concentration inequalities in product spaces*, Invent. Math. **126** (1996), no. 3, 505–563. [5](#)
69. C. Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, RI, 2003. [1](#), [7](#), [15](#)
70. ———, *Optimal transport*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338, Springer-Verlag, Berlin, 2009, Old and new. [1](#), [7](#)

NG : UNIVERSITÉ PARIS DESCARTES, MAP5, UMR 8145, 45 RUE DES SAINTS PÈRES, 75270 PARIS CEDEX 06

E-mail address: natael.gozlan@parisdescartes.fr

NJ: UNIVERSITÉ DE STRASBOURG, IRMA, UMR 7501, 7 RUE RENÉ-DESCARTES 67084 STRASBOURG CEDEX

E-mail address: nicolas.juillet@math.unistra.fr