

Supplement to “On spike and slab empirical Bayes multiple testing”

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Abstract: This supplementary file contains additional materials for the proofs as well as the proof of Propositions 1–2 and Theorem 3; a study of ℓ -values and q -values; inequalities for the thresholds of the corresponding BMT procedures; properties of the moment functions \tilde{m} , m_1 and m_2 and additional numerical experiments.

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S-1. Intermediate lemmas used in the proof of main results

In the sequel we freely use that $s_n \leq n^\nu$ as assumed in the main results of the paper. We assume that the function g satisfies the assumptions from (38) up to and including (42) (recall that this is in particular the case if g arises from a convolution $g = \gamma \star \phi$ for γ satisfying (16)–(18), which is the case in the Bayesian setting with a slab density γ).

We start by two basic lemmas on $w_0 = w_0(n, M)$, $w_1 = w_1(n, M, \theta_0, \nu)$, $w_2 = w_2(n, M, \theta_0, \nu)$, quantities introduced in (60), (67), (68), respectively.

Lemma S-1. *Let w_0 as in (60) with $M > 1$ arbitrary. Let \tilde{m} be defined by (56). Then, for an integer $N_0(g) > 0$, and constants $c_1 = 1/\tilde{m}(1)$, $c_2 = c_2(g)$, we have for all $n \geq N_0(g)$,*

$$\frac{n}{M} \tilde{m}(Mc_1/n) \leq \frac{1}{w_0} \leq \frac{n}{M} \tilde{m}(\sqrt{Mc_2/n}).$$

In particular, for any $M \in [1, \log n]$, for C_1, C_2 depending only on g ,

$$C_1 \frac{\sqrt{\log n}}{n} \leq w_0 \leq \frac{\log n}{n} e^{C_2 \sqrt{\log n}}.$$

Proof. Lemma S-24 gives $\tilde{m}(w) \gtrsim w^c$ for any $c > 0$. Setting $c = 1$ and using the equation defining w_0 , that is $nw_0\tilde{m}(w_0) = M$, leads to $w_0 \leq (CM/n)^{1/2}$. Reinserting this estimate into \tilde{m} in the equation defining w_0 (by using that \tilde{m} is increasing by Lemma S-22) gives the first upper bound of the lemma. Next, one notes that $\tilde{m}(w) \leq \tilde{m}(1)$, which leads to $w_0 \geq M/(n\tilde{m}(1))$. Reinserting this estimate into \tilde{m} in the equation defining w_0 gives the first lower bound of the lemma.

To prove the second display of the lemma, one notes that the fact that $\log g$ is Lipschitz and $g(u) \lesssim (1+u^2)^{-1}$ by (41) imply for w small enough,

$$\zeta(w)^{\kappa-1} e^{-\Lambda\zeta(w)} \lesssim \tilde{m}(w) \lesssim \zeta(w)^{\kappa-3}.$$

Using the first display of the lemma together with Lemma S-15 on ζ and $1 \leq M \leq \log n$ leads to the result. \square

Lemma S-2. *For $M > 0$ and $\nu \in (0, 1)$, there exist an integer $N_0 = N_0(\nu, v, g) > 0$ and $r = r(\nu, v, g) \in (0, 1)$ such that for all $n \geq N_0$ and $\theta_0 \in \ell_0[s_n]$, if a solution $w_1 = w_1(n, M, \theta_0, \nu)$ of (67) exists, then*

$$w_0 \leq w_1 \leq n^{-r}.$$

Proof. The lower bound follows from the definition of w_0 and w_1 . For the upper bound, one uses the definition of w_1 and the global bound $|m_1(\mu, w)| \leq 1/(w \wedge c_1)$ (which follows from Lemma S-21) to get,

$$\frac{\sigma_0}{w_1 \wedge c_1} \geq (1-\nu)(n-\sigma_0)\tilde{m}(w_1).$$

As \tilde{m} is increasing and $\tilde{m}(w) \gtrsim w^c$ for arbitrary $c \in (0, 1)$ (see Lemma S-24), one gets $(w_1 \wedge c_1)^{1+c} \leq C\sigma_0/n \leq Cs_n/n$. Using $s_n \leq n^\nu$ gives the result. \square

Lemma S-3 (Bernstein w_0). *There exist an integer $N_0 = N_0(g, \nu) > 0$ and $C_0 = C_0(g) > 0$ such that the following holds for all $n \geq N_0$ and $\theta_0 \in \ell_0[s_n]$. Let $M \in [1, \log n]$ and w_0 as in (60). Let $\nu \in (0, 1)$ and assume (62) (which is implied by the fact that (61) has no solution). Then the MMLE estimate \hat{w} satisfies*

$$P_{\theta_0}(\hat{w} > w_0) \leq e^{-C_0 \nu^2 n w_0 \tilde{m}(w_0)} = e^{-C_0 \nu^2 M}. \quad (\text{S-1})$$

Proof of Lemma S-3. One first notes the almost sure equality of events $\{\hat{w} > w_0\} = \{S(w_0) > 0\}$. This follows since S is (strictly) decreasing and continuous on $[0, 1]$ (except in the case that $g(X_i) = \phi(X_i)$ for all i which happens with probability 0). Then, with $P = P_{\theta_0}$, $E = E_{\theta_0}$ as shorthand,

$$\begin{aligned} P(\hat{w} > w_0) &= P(S(w_0) > 0) = P(S(w_0) - ES(w_0) > -ES(w_0)) \\ &\leq P(S(w_0) - ES(w_0) > \nu(n - \sigma_0)\tilde{m}(w_0)), \end{aligned}$$

as $ES(w_0) = \sum_{i \in S_0} m_1(\theta_{0,i}, w_0) - (n - \sigma_0)\tilde{m}(w_0) < -\nu(n - \sigma_0)\tilde{m}(w_0)$ using (62). Now, the score function equals $S(w_0) = \sum_{i=1}^n \beta(X_i, w_0)$, a sum of independent variables. One applies Bernstein’s inequality (see Lemma S-35 and notation therein) to the variables $W_i = \beta(X_i, w_0) - E\beta(X_i, w_0)$. Note that $|W_i| \leq 2/w_0 =: \mathcal{M}$ as $|\beta| \leq (w_0 \wedge c_1)^{-1} = w_0^{-1}$ by Lemma S-21 for n large enough (indeed, w_0 goes to 0 with n by Lemma S-1). Also,

$$V := \sum_{i=1}^n \text{Var}(W_i) \leq \sum_{i=1}^n m_2(\theta_{0,i}, w_0).$$

One splits the last sum in two. Consider $\zeta_0 = \beta^{-1}(w_0^{-1})$ the pseudo-threshold associated to w_0 . Using Corollary S-29 (recall as noted above that w_0 goes to 0 with n), with M_0 the constant therein, combined with (62), one gets

$$\begin{aligned} \sum_{i: |\theta_{0,i}| > M_0} m_2(\theta_{0,i}, w_0) &\leq \frac{C_2}{w_0} \sum_{i: |\theta_{0,i}| > M_0} m_1(\theta_{0,i}, w_0) \\ &\leq \frac{C_2}{w_0} (1 - \nu)(n - \sigma_0)\tilde{m}(w_0) - \frac{C_2}{w_0} \sum_{i: |\theta_{0,i}| \leq M_0} m_1(\theta_{0,i}, w_0) \\ &\leq \frac{2C_2}{w_0} (1 - \nu)n\tilde{m}(w_0), \end{aligned}$$

because $\mu \in \mathbb{R}_+ \rightarrow m_1(\mu, w_0)$ is nondecreasing (see Lemma S-22) and bounded from below by $-\tilde{m}(w_0)$.

For small non-zero signals, one uses Lemma S-27 to get, with $\zeta_0 := \zeta(w_0)$,

$$\sum_{i: 0 < |\theta_{0,i}| \leq M_0} m_2(\theta_{0,i}, w_0) \leq C \sum_{i: 0 < |\theta_{0,i}| \leq M_0} \frac{\bar{\Phi}(\zeta_0 - |\theta_{0,i}|)}{w_0^2} \leq C\sigma_0 \frac{\bar{\Phi}(\zeta_0 - M_0)}{w_0^2},$$

and one uses $\bar{\Phi}(\zeta_0 - M_0) \leq C\phi(\zeta_0 - M_0)/\zeta_0 \leq C'e^{M_0\zeta_0}\phi(\zeta_0)/\zeta_0$. With Lemma S-24, one gets plutot $\phi(\zeta)/\zeta \asymp wg(\zeta)/\zeta \asymp w\tilde{m}(w)/\zeta^\kappa$ for small w , so that

$$\sum_{i: |\theta_{0,i}| \leq M_0} m_2(\theta_{0,i}, w_0) \lesssim \frac{s_n e^{M_0\zeta_0}}{n\zeta_0^\kappa} \frac{n\tilde{m}(w_0)}{w_0} \lesssim \frac{n\tilde{m}(w_0)}{\zeta_0^\kappa w_0},$$

where we use that $s_n e^{M_0 \zeta_0} / n \leq C$, as follows from $s_n = O(n^\nu)$ and $\zeta_0^2 \lesssim \log n$ (combining Lemmas S-1 on w_0 and Lemma S-15). With $A = (n - \sigma_0) \nu \tilde{m}(w_0)$, one gets, for $n \geq N_0$,

$$\frac{V + \frac{1}{3} \mathcal{M} A}{A^2} \lesssim \frac{\nu^{-2}}{n w_0 \tilde{m}(w_0)} + \frac{\nu^{-2}}{n w_0 \tilde{m}(w_0) \zeta_0^k} \lesssim \frac{\nu^{-2}}{n w_0 \tilde{m}(w_0)},$$

An application of Bernstein’s inequality (see Lemma S-35) now gives (S-1). \square

Lemma S-4 (Bernstein w_1, w_2). *There exist an integer $N_0 = N_0(g, \nu) > 0$ and $C_1 = C_1(g) > 0$ such that the following holds for all $n \geq N_0$ and $\theta_0 \in \ell_0[s_n]$: for $\nu \in (0, 1)$, suppose that a solution w_1 of (67) exists, and let w_2 be the solution of (68). Then the MMLE estimate \hat{w} satisfies*

$$P_{\theta_0}(\hat{w} \notin [w_2, w_1]) \leq e^{-C_1 \nu^2 n w_1 \tilde{m}(w_1)} + e^{-C_1 \nu^2 n w_2 \tilde{m}(w_2)}. \quad (\text{S-2})$$

Proof. One bounds successively each of the probabilities $P(\hat{w} > w_1)$ and $P(\hat{w} < w_2)$. The first bound is obtained in exactly the same way as in the proof of Lemma S-3, with w_0 replacing w_1 . We note the two minor differences: $ES(w_1) = \sum_{i \in S_0} m_1(\theta_{0,i}, w_1) - (n - \sigma_0) \tilde{m}(w_1)$ now equals $-\nu(n - \sigma_0) \tilde{m}(w_1)$ by the definition (67) of w_1 . Then bounds on m_2 can be carried out in the same way – now evaluated at $w = w_1$ – as in the proof of Lemma S-3. We note that w_1 goes to zero with n by Lemma S-2. This means that we can use the bounds of Lemma S-27 and Corollary S-29 as in the proof of Lemma S-3. Further, if $\zeta_1 := \zeta(w_1)$, we have $\zeta_1 \leq \zeta_0$, so one also has $s_n e^{M_0 \zeta_1} / n \leq C$ using the corresponding bound for ζ_0 . This shows the desired result for w_1 .

For w_2 , one proceeds similarly. If $w_2 = 0$ the result is immediate. Otherwise we have $\{\hat{w} < w_2\} = \{S(w_2) < 0\}$. Again, one applies Bernstein’s inequality to the score function $S(w) = \sum_{i=1}^n \beta(X_i, w)$ and set $W_i = \beta(X_i, w_2) - m_1(\theta_{0,i}, w_2)$. As W_i are centered independent variables with $|W_i| \leq \mathcal{M}$ and $\sum_{i=1}^n \text{Var}(W_i) \leq \sum_{i=1}^n E[\beta(X_i, w_2)^2] =: V_2$, for any $B > 0$,

$$P\left[\sum_{i=1}^n W_i < -B\right] \leq \exp\left\{-\frac{1}{2} B^2 / (V_2 + \frac{1}{3} \mathcal{M} B)\right\}.$$

One can take $\mathcal{M} = c_3/w$, using Lemma S-21. Set $B = \sum_{i=1}^n m_1(\theta_{0,i}, w_1)$. By definition of w_2 in (68), we have

$$B = -(n - \sigma_0) \tilde{m}(w_2) + \sum_{i \in S_0} m_1(\theta_{0,i}, w_2) = \nu(n - \sigma_0) \tilde{m}(w_2).$$

The term V_2 is bounded in a similar way as in the proof of Lemma S-3, using the bounds of Lemma S-27 and Corollary S-29. As for w_1 above, one notes that, if $\zeta_2 = \zeta(w_2)$, one has $s_n e^{M_0 \zeta_2} / n \leq C$ as, using Lemma S-5, we have $w_1 \lesssim w_2$, so that $w_2 \gtrsim 1/n$ and $\zeta_2 \lesssim \sqrt{\log n}$. One obtains $V_2 \lesssim (n w_2 \tilde{m}(w_2))^{-1}$ which leads to

$$\frac{V_2 + \frac{1}{3} \mathcal{M} B}{B^2} \lesssim \frac{\nu^{-2}}{n w_2 \tilde{m}(w_2)},$$

and the desired bound on w_2 is obtained. \square

Lemma S-5. *Let $\nu \in (0, 1)$. There exist some integer $N = N(\nu, v, g) > 0$ and $C = C(\nu, v, g) > 1$ such that, for all $n \geq N$ and $\theta_0 \in \ell_0[s_n]$, if (67) has a solution w_1 , the solution w_2 of (68) verifies*

$$w_1/C \leq w_2 \leq w_1. \quad (\text{S-3})$$

Proof. The behaviour of w_1, w_2 for a given specific true signal θ_0 is determined through properties of the function

$$H_{\theta_0}(w) = \sum_{i \in S_0} m_1(\theta_{0,i}, w) / \tilde{m}(w).$$

This function is decreasing, as $w \rightarrow m_1(\theta_{0,i}, w), 1 \leq i \leq n$, and $w \rightarrow \tilde{m}(w)^{-1}$ both are, by Lemma S-22. It suffices to show that for an appropriately large constant $z \geq 1$ (possibly depending on v, g, ν), for n large enough,

$$H_{\theta_0}\left(\frac{w_1}{z}\right) \geq \frac{1+\nu}{1-\nu} H_{\theta_0}(w_1), \quad (\text{S-4})$$

Indeed, by definition of w_1, w_2 , one has $H_{\theta_0}(w_2) = (1+\nu)(n-\sigma_0) = (1+\nu)(1-\nu)^{-1} H_{\theta_0}(w_1)$. So, if (S-4) holds, $H_{\theta_0}(w_2) \leq H_{\theta_0}(w_1/z)$ which in turn yields $w_2 \geq w_1/z$ by monotonicity.

Now, (S-4) is obtained in two steps. First, one shows that appropriately small signals do not contribute too much to the sum defining H_{θ_0} , so that one can replace the sum in (S-4) by a sum $H_{\theta_0}^\circ$, to be defined now, on large signals only. For $w \in (0, 1)$ and $K > 1$, set $\mathcal{C}_0(w, K) = \{1 \leq i \leq n : |\theta_{0,i}| \geq \zeta(w)/K\}$ and

$$H_{\theta_0}^\circ(w, K) = \sum_{i \in \mathcal{C}_0(w, K)} m_1(\theta_{0,i}, w) / \tilde{m}(w).$$

Set $K_2 = 4/(1-\nu)$. By Lemmas S-1 and S-2, both w_1 and w_1/z belong to the interval $[1/n, 1/\log n]$, provided $z \lesssim (\log n)^{1/4}$ (which will be the case below). Let us now use, with $K_1 = K_2/2$ and $D > 0$, both Lemmas S-31 and S-32, and $z = z(\nu, v, g)$ a constant to be chosen below,

$$\begin{aligned} H_{\theta_0}\left(\frac{w_1}{z}\right) &= H_{\theta_0}^\circ\left(\frac{w_1}{z}, K_2\right) + H_{\theta_0}\left(\frac{w_1}{z}\right) - H_{\theta_0}^\circ\left(\frac{w_1}{z}, K_2\right) \\ &\geq Cz^{1/(2K_2)} H_{\theta_0}^\circ(w_1, K_2/1.1) - C'n^{1-D} \\ &\geq Cz^{(1-\nu)/8} H_{\theta_0}^\circ(w_1, K_1) - C'n^{1-D}, \end{aligned}$$

where in the last inequality one uses that $K \rightarrow H_{\theta_0}^\circ(w, K)$ is nondecreasing by definition. Using Lemma S-31 again now shows that, for $D > 0$,

$$|H_{\theta_0}(w_1) - H_{\theta_0}^\circ(w_1, K_1)| \leq C'n^{1-D}.$$

One deduces that, for C the constant in the one but last display,

$$H_{\theta_0}\left(\frac{w_1}{z}\right) \geq Cz^{(1-\nu)/8} H_{\theta_0}(w_1) + o(n).$$

Since $H_{\theta_0}(w_1) \asymp n$ by definition of w_1 , the latter is bounded from below by $(C/2)z^{(1-\nu)/8} H_{\theta_0}(w_1)$ for n large enough. Taking $z = \{\max((2/C), 1)(1+\nu)/(1-\nu)\}^{8/(1-\nu)}$ shows (S-4) and the proof is complete. \square

S-2. Auxiliary proofs

S-2.1. Proof of Proposition 1

For any multiple testing procedure φ ,

$$\text{BFDR}(\varphi; w, \gamma) = \int_{\mathbb{R}^n} \text{FDR}(\theta, \varphi) d\Pi_{w, \gamma}(\theta) = E_{X, \theta} \left[\frac{\sum_{i=1}^n \mathbf{1}\{\theta_i = 0\} \varphi_i}{1 \vee \sum_{i=1}^n \varphi_i} \right].$$

For φ^ℓ , using the chain rule $E[\cdot] = E[E[\cdot | X]]$, one gets

$$\begin{aligned} \text{BFDR}(\varphi^\ell; w, \gamma) &= E_X \left[\frac{\sum_{i=1}^n \ell_i(X) \varphi_i^\ell}{1 \vee \sum_{i=1}^n \varphi_i^\ell} \right] = E_X \left[\frac{\sum_{i=1}^n \ell_i(X) \mathbf{1}\{\ell_i(X) \leq \alpha\}}{1 \vee \sum_{i=1}^n \mathbf{1}\{\ell_i(X) \leq \alpha\}} \right] \\ &\leq \alpha P(\exists i : \ell_i(X) \leq \alpha). \end{aligned}$$

For φ^q , conditioning this time on the variables $\varphi_1^q(X), \dots, \varphi_n^q(X)$ and using that for the prior $\Pi_{w, g}$ the conditional distribution of $\theta_i | X$ only depends on X_i for all i , so that $E[\mathbf{1}\{\theta_i = 0\} | \varphi_1^q, \dots, \varphi_n^q] \varphi_i^q = P(\theta_i = 0 | \varphi_i^q = 1) \varphi_i^q$ a.s., one obtains

$$\begin{aligned} \text{BFDR}(\varphi^q; w, \gamma) &= E_X \left[\frac{\sum_{i=1}^n P(\theta_i = 0 | \varphi_i^q = 1) \varphi_i^q}{1 \vee \sum_{i=1}^n \varphi_i^q} \right] \\ &= E_X \left[\frac{\sum_{i=1}^n P(\theta_i = 0 | q_i(X) \leq \alpha) \mathbf{1}\{q_i(X) \leq \alpha\}}{1 \vee \sum_{i=1}^n \mathbf{1}\{q_i(X) \leq \alpha\}} \right]. \end{aligned}$$

Now observe that from (43), $q_i(X) \leq \alpha$ if and only if $|X_i| \geq \Psi(\alpha)$, for some function Ψ such that $q(\Psi(\alpha); w, g) = \alpha$ (namely, Ψ is the inverse of $u \in (0, \infty) \rightarrow q(u; w, g)$). Now, the result follows from

$$P(\theta_i = 0 | q_i(X) \leq \alpha) = P(\theta_i = 0 | |X_i| \geq \Psi(\alpha)) = q(\Psi(\alpha); w, g) = \alpha.$$

Finally, the relation between (24) and (25) comes from Lemma S-10.

S-2.2. Proof of Proposition 2

For (i), we use Lemma S-33:

$$P_{\theta_0=0}(\ell_i(X) \leq t) = 2\bar{\Phi}(\xi(r(w, t))) \leq 2 \frac{\phi(\xi(r(w, t)))}{\xi(r(w, t))},$$

which provides (53) because $\phi(\xi(r(w, t))) = r(w, t)g(\xi(r(w, t)))$ by definition of $\xi(\cdot)$. Next, if $\xi(r(w, t)) \geq 1$, we also have the lower bound:

$$P_{\theta_0=0}(\ell_i(X) \leq t) \geq \frac{\phi(\xi(r(w, t)))}{\xi(r(w, t))},$$

which provides (54). Finally, (ii) follows from the definition of χ .

S-2.3. Proof of Theorem 3

We prove the result first for **EBayesq**. Recall that the exact number of nonzero coefficients σ_0 of θ_0 is s_n by definition of $\mathcal{L}_0[s_n]$. Set $b = (a + 1)/2 > 1$ and let \mathcal{A} be the event, for $K_n < s_n$ to be specified below,

$$\mathcal{A} = \left\{ \#\{i \in S_0, |X_i| > b\{2 \log(n/s_n)\}^{1/2}\} \geq s_n - K_n \right\}.$$

If \mathcal{A}^c denotes the complement of \mathcal{A} ,

$$\begin{aligned} \mathcal{A}^c &= \left\{ \#\{i \in S_0, |X_i| > b\{2 \log(n/s_n)\}^{1/2}\} < s_n - K_n \right\} \\ &= \left\{ \#\{i \in S_0, |X_i| \leq b\{2 \log(n/s_n)\}^{1/2}\} > K_n \right\} \\ &\subset \left\{ \#\{i \in S_0, |\varepsilon_i| > (a - b)\{2 \log(n/s_n)\}^{1/2}\} > K_n \right\} =: \mathcal{C}, \end{aligned}$$

where we have used $X_i = \theta_{0,i} + \varepsilon_i$ to get $|\varepsilon_i| \geq |\theta_{0,i}| - |X_i|$ by the triangle inequality. Let $c = \sqrt{2}(a - b) > 0$. By looking at the indicator variables $Z_i = 1_{|\varepsilon_i| \geq x_n}$ with $x_n = c\{2 \log(n/s_n)\}^{1/2}$, one can translate the event \mathcal{C} in the last display into an event for a binomial trial, leading to

$$\sup_{\theta_0 \in \mathcal{L}_0[s_n]} P_{\theta_0}[\mathcal{A}^c] \leq P[\text{Bin}(s_n, 2\bar{\Phi}(x_n)) > K_n].$$

Let $p_n = 2\bar{\Phi}(x_n)$, then using the expression of x_n above,

$$p_n \leq 2\phi(x_n)/x_n \leq C(s_n/n)^{c^2/2}/(c\sqrt{\log(n/s_n)}),$$

which goes to 0 with n as $s_n/n \rightarrow 0$.

Let $K_n = \max(2s_n p_n, s_n/\log s_n)$. By Bernstein’s inequality, see Lemma S-35, as $K_n \geq 2s_n p_n$ and $\sum_{i \in S_0} \text{Var}(Z_i) \leq s_n p_n$,

$$P\left[\sum_{i \in S_0} Z_i > K_n\right] \leq P\left[\sum_{i \in S_0} (Z_i - p_n) > K_n/2\right] \leq \exp\left\{-\frac{1}{8} \frac{K_n^2}{K_n/6 + s_n p_n}\right\},$$

which is less, using $s_n p_n \leq K_n/2$ again, than $\exp(-CK_n)$, which goes to 0 with n , since $K_n \geq s_n/\log s_n \rightarrow \infty$. So, we have obtained $P_{\theta_0}[\mathcal{A}^c] = o(1)$.

Now one can follow the proof of Theorems 1 and 2 and consider the fundamental equation (61), for some fixed $\theta_0 \in \mathcal{L}_0[s_n]$, and n large enough. The lower bound on w is given here by w_0 in (60), for some $M = M_n$ that we choose as $M_n = \min(c_0 s_n, \log n)$, so that $M_n \rightarrow \infty$ and c_0 a small enough constant to be chosen below.

Consider both sides of the equation (61) at the point $w = s_n/n$. On the one hand, by definition of $\mathcal{L}_0[s_n]$, we have $|\theta_{0,i}| \geq a\{2 \log(n/s_n)\}^{1/2}$ for $i \in S_0$. Lemma S-15 implies $\zeta(s_n/n) \sim \{2 \log(n/s_n)\}^{1/2}$, so one can apply Lemma S-30 (recall $\mu \rightarrow m_1(\mu, w)$ is even for all w) for a small $\varepsilon > 0$ to get, for large enough n ,

$$\sum_{i \in S_0} m_1(\theta_{0,i}, s_n/n) \geq (1 - \varepsilon) \frac{s_n}{(s_n/n)} = (1 - \varepsilon)n.$$

On the other hand, the right hand side of (61) equals $(1 - \nu)(n - s_n)\tilde{m}(s_n/n) = o(n)$, since $\tilde{m}(w)$ goes to 0 as $w \rightarrow 0$. Recall that $\sum_{i \in S_0} m_1(\theta_{0,i}, 1)$ is bounded from above by a constant times s_n (as $m_1(\theta_{0,i}, 1)$ is bounded, see Section 7.3.1) and that $(1 - \nu)n\tilde{m}(1)$ is of the order n . Combining the previous inequalities, the intermediate values theorem shows that (61) has a solution, at least on $[s_n/n, 1)$, for n large enough.

To show that w_1 exists, it is enough to check that the solution also belongs to $[w_0, 1)$. We distinguish two cases. If $w_0 \leq s_n/n$ then this is obvious by definition. In case $w_0 > s_n/n$, let us evaluate both sides of (61) this time at $w = w_0$. First, using the second display of Lemma S-1 (compatible with the present choice on $M_n = \min(c_0 s_n, \log n)$) combined with Lemma S-15 on ζ , one gets, for arbitrary $\varepsilon > 0$ and using $w_0 > s_n/n$, that

$$\zeta(w_0) \leq (1 + \varepsilon)\sqrt{2\log(1/w_0)} \leq (1 + \varepsilon)\sqrt{2\log(n/s_n)},$$

for large enough n . Deduce that one can apply Lemma S-30 as $(1 + \rho)\zeta(w_0) \leq |\theta_{0,i}|$ for small enough ρ . In particular

$$\sum_{i \in S_0} m_1(\theta_{0,i}, w_0) \geq (1 - \varepsilon)\frac{s_n}{w_0}.$$

On the other hand, the right hand side of (61) is $(1 - \nu)(n - s_n)\tilde{m}(w_0) = (1 - \nu)\{(n - s_n)/n\}M_n/w_0$ by definition of w_0 . As $M_n \leq c_0 s_n$, this quantity is thus smaller than the last display, provided c_0 is small enough. By the same reasoning as above, this shows that the solution to (61) indeed belongs to $[w_0, 1)$, so w_1 exists.

Now that we have the existence of w_1 , the fact that $w = s_n/n$ cannot be a solution of (61) (for n large enough) and the monotonicity of both sides of (61) show that $w_1 \geq s_n/n$, for n large enough. Using the same argument with equation (68) leads to $w_2 \geq s_n/n$, for n large enough.

As (61) has a solution, we can use the properties of the proof of Section 7 in this case (referred to as Case 2 in that proof). In particular, (72) provides for some constant $C > 0$,

$$\sup_{\theta_0 \in \mathcal{L}_0[s_n]} P_{\theta_0}(\hat{w} \notin [w_2, w_1]) \leq 2e^{-CM_n}.$$

Let us introduce the event $\Omega_0 = \mathcal{A} \cap \{\hat{w} \in [w_2, w_1]\}$. By the previous bounds, we have $P[\Omega_0^c] = o(1)$. Note that, on the event Ω_0 ,

$$\chi(r(\hat{w}, t)) \leq \zeta(\hat{w}) \leq \zeta(w_2)$$

using Lemma S-16 and the monotonicity of $\zeta(\cdot)$. We have seen that here $w_2 \geq s_n/n$, so $\zeta(w_2) \leq \zeta(s_n/n)$ and combining with the equivalent of $\zeta(w)$ as $w \rightarrow 0$ from Lemma S-15, one finally gets $\chi(r(\hat{w}, t)) \leq c(2\log(n/s_n))^{1/2}$ for any $c > 1$ for n large enough, so in particular for $c = b$ as defined above. One deduces that on Ω_0 , the q -value procedure $\varphi^{q\text{-val}}$ rejects the null hypotheses corresponding to the (at least $s_n - K_n$) indexes i in S_0 such that $|X_i| > b\{2\log(n/s_n)\}^{1/2}$,

because $b\{2\log(n/s_n)\}^{1/2} \geq \chi(r(\hat{w}, t))$ by using the previous bounds and the definition of the event \mathcal{A} .

Combining the above facts, we obtain

$$\begin{aligned} & \sup_{\theta_0 \in \mathcal{L}_0[s_n]} \text{FDR}(\theta_0, \varphi^{q\text{-val}}(t; \hat{w}, g)) \\ &= \sup_{\theta_0 \in \mathcal{L}_0[s_n]} E_{\theta_0} \left[\frac{\sum_{i=1}^n \mathbf{1}\{\theta_{0,i} = 0\} \varphi^{q\text{-val}}(t; \hat{w}, g)}{1 \vee \sum_{i=1}^n \varphi^{q\text{-val}}(t; \hat{w}, g)} \right] \\ &\leq \sup_{\theta_0 \in \mathcal{L}_0[s_n]} E_{\theta_0} \left[\frac{\sum_{i=1}^n \mathbf{1}\{\theta_{0,i} = 0\} \varphi^{q\text{-val}}(t; \hat{w}, g)}{1 \vee \sum_{i=1}^n \varphi^{q\text{-val}}(t; \hat{w}, g)} \mathbf{1}\{\Omega_0\} \right] + o(1). \end{aligned}$$

Therefore, since $\varphi^{q\text{-val}}(t; \hat{w}, g)$ makes at least $s_n - K_n$ correct rejections, that is, $\#\{i \in S_0 : \varphi_i^{q\text{-val}}(t; \hat{w}, g) = 1\} \geq s_n - K_n$, we derive

$$\begin{aligned} & \sup_{\theta_0 \in \mathcal{L}_0[s_n]} \text{FDR}(\theta_0, \varphi^{q\text{-val}}) \\ &\leq \sup_{\theta_0 \in \mathcal{L}_0[s_n]} E_{\theta_0} \left[\frac{\sum_{i=1}^n \mathbf{1}\{\theta_{0,i} = 0\} \varphi_i^{q\text{-val}}(t; w_1)}{\sum_{i=1}^n \mathbf{1}\{\theta_{0,i} = 0\} \varphi_i^{q\text{-val}}(t; w_1) + s_n - K_n} \right] + o(1) \\ &\leq \frac{\sup_{\theta_0 \in \mathcal{L}_0[s_n]} E_{\theta_0} [\sum_{i=1}^n \mathbf{1}\{\theta_{0,i} = 0\} \varphi_i^{q\text{-val}}(t; w_1)]}{\sup_{\theta_0 \in \mathcal{L}_0[s_n]} E_{\theta_0} [\sum_{i=1}^n \mathbf{1}\{\theta_{0,i} = 0\} \varphi_i^{q\text{-val}}(t; w_1)] + s_n - K_n} + o(1), \quad (\text{S-5}) \end{aligned}$$

by concavity and monotonicity of the function $x \in [0, +\infty) \rightarrow x/(x+1)$.

Now combine (55), Lemma S-17 and Lemma S-24 to get for any $\varepsilon \in (0, 1)$, for any $\theta_0 \in \mathcal{L}_0[s_n]$,

$$\begin{aligned} E_{\theta_0} \left[\sum_{i=1}^n \mathbf{1}\{\theta_{0,i} = 0\} \varphi_i^{q\text{-val}}(t; w_1) \right] &= (n - s_n) r(w_1, t) 2\bar{G}(\chi(r(w_1, t))) \\ &\leq (1 + \varepsilon) t (1 - t)^{-1} w_1 (n - s_n) 2\bar{G}(\zeta(w_1)) \\ &\leq (1 + \varepsilon)^2 t (1 - t)^{-1} (n - s_n) w_1 \tilde{m}(w_1). \end{aligned}$$

Next, since w_1 is a solution of (61), the latter is bounded above by

$$(1 + \varepsilon)^2 (1 - \nu)^{-1} t (1 - t)^{-1} \sum_{i \in S_0} w_1 m_1(\theta_{0,i}, w_1) \leq (1 + \varepsilon)^2 (1 - \nu)^{-1} t (1 - t)^{-1} s_n,$$

by using that $m_1(\cdot, w)$ is always upper-bounded by $1/w$ for small w , see Lemma S-22 (recall that w_1 goes to 0 with n by Lemma S-2). Putting this back into (S-5) gives for n large enough,

$$\sup_{\theta_0 \in \mathcal{L}_0[s_n]} \text{FDR}(\theta_0, \varphi^{q\text{-val}}) \leq \frac{(1 + \varepsilon)^2 (1 - \nu)^{-1} t (1 - t)^{-1} s_n}{(1 + \varepsilon)^2 (1 - \nu)^{-1} t (1 - t)^{-1} s_n + s_n - K_n} + o(1).$$

As $K_n = o(s_n)$ as shown above, taking the limsup as $n \rightarrow \infty$ and then letting ε, ν go to 0, we get, observing that $\frac{t(1-t)^{-1}}{t(1-t)^{-1}+1} = t$,

$$\overline{\lim}_n \sup_{\theta_0 \in \mathcal{L}_0[s_n]} \text{FDR}(\theta_0, \varphi^{q\text{-val}}) \leq t.$$

Let us now turn to prove

$$\liminf_n \inf_{\theta_0 \in \mathcal{L}_0[s_n]} \text{FDR}(\theta_0, \varphi^{q\text{-val}}) \geq t, \quad (\text{S-6})$$

which will lead to the conclusion. Fix some $\delta \in (0, 1)$ and for any $\theta_0 \in \mathcal{L}_0[s_n]$ consider w_1 and w_2 the associated solution of (61) and (68), respectively. The fact that both exist has been seen above. Let $\Omega_1 = \{\hat{w} \in [w_2, w_1]\}$, then

$$\begin{aligned} & \inf_{\theta_0 \in \mathcal{L}_0[s_n]} \text{FDR}(\theta_0, \varphi^{q\text{-val}}) \\ & \geq \inf_{\theta_0 \in \mathcal{L}_0[s_n]} E_{\theta_0} \left[\frac{V_q}{V_q + s_n} \mathbb{1}\{\Omega_1\} \right] \\ & \geq \inf_{\theta_0 \in \mathcal{L}_0[s_n]} E_{\theta_0} \left[\frac{E_{\theta_0} V_q (1 - \delta)}{E_{\theta_0} V_q (1 - \delta) + s_n} \mathbb{1}\{\Omega_1\} \mathbb{1}\{|V_q - E_{\theta_0} V_q| \leq \delta E_{\theta_0} V_q\} \right], \end{aligned}$$

where we have denoted $V_q = \sum_{i=1}^n \mathbb{1}\{\theta_{0,i} = 0\} \varphi_i^{q\text{-val}}(t; w_2)$, which is a Binomial variable. Similarly to the upper bound, combine (55), Lemma S-16 and Lemma S-24 to get for any $\varepsilon \in (0, 1)$ and $\theta_0 \in \mathcal{L}_0[s_n]$,

$$\begin{aligned} E_{\theta_0} V_q &= (n - s_n) r(w_2, t) 2\bar{G}(\chi(r(w_2, t))) \\ &\geq t(1 - t)^{-1} w_2 (1 - w_2)^{-1} (n - s_n) 2\bar{G}(\zeta(w_2)) \\ &\geq (1 - \varepsilon) t (1 - t)^{-1} w_2 (n - s_n) 2\bar{G}(\zeta(w_2)) \\ &\geq (1 - \varepsilon)^2 t (1 - t)^{-1} (n - s_n) w_2 \tilde{m}(w_2). \end{aligned}$$

Now using that w_2 is a solution of (68) and Lemma S-30, we obtain

$$\begin{aligned} E_{\theta_0} V_q &\geq (1 - \varepsilon)^2 (1 + \nu)^{-1} t (1 - t)^{-1} \sum_{i \in S_0} w_2 m_1(\theta_{0,i}, w_2) \\ &\geq (1 - \varepsilon)^3 (1 + \nu)^{-1} t (1 - t)^{-1} s_n. \end{aligned}$$

Next, observe that by Chebychev’s inequality, the supremum over $\theta_0 \in \mathcal{L}_0[s_n]$ of the following probability

$$P_{\theta_0}(|V_q - E_{\theta_0} V_q| > \delta E_{\theta_0} V_q) \leq \frac{\text{Var}_{\theta_0}(V_q)}{\delta^2 (E_{\theta_0} V_q)^2} \leq \frac{1}{\delta^2 E_{\theta_0} V_q}$$

goes to 0, because s_n tends to infinity. Combining the above facts leads to

$$\inf_{\theta \in \mathcal{L}_0[s_n]} \text{FDR}(\theta, \varphi^{q\text{-val}}) \geq \frac{(1 - \varepsilon)^3 (1 - \delta) (1 + \nu)^{-1} t (1 - t)^{-1}}{(1 - \varepsilon)^3 (1 - \delta) (1 + \nu)^{-1} t (1 - t)^{-1} + 1} + o(1),$$

and the result is proved by taking the liminf in n and then δ, ε, ν tending to zero.

Finally, to prove the result for **EBayesq.0**, one notes that by the previous arguments \hat{w} belongs to $[w_1, w_2]$ with probability tending to 1, and $w_2 \geq s_n/n$ which is larger than $2\omega_n$ by assumption. Deduce that the event $\{\hat{w} > \omega_n\}$ has probability going to 1 so the procedures **EBayesq** and **EBayesq.0** coincide with probability going to 1, which proves that **EBayesq.0** also satisfies the desired property.

S-3. Basic properties of ℓ -, q - and p -values

Let us assume that g satisfies (38) throughout this section. Recall that this assumption holds in particular whenever g is of the form $g = \phi \star \gamma$ as in the Bayesian setting.

Lemma S-6. *The q -value functional (14) has the explicit expression*

$$q(x; w, g) = \frac{(1-w)\bar{\Phi}(|x|)}{(1-w)\bar{\Phi}(|x|) + w\bar{G}(|x|)}, \quad x \in \mathbb{R}, \quad w \in [0, 1].$$

Proof. The latter comes from the fact that, for $s \geq 0$ and by symmetry of γ and ϕ ,

$$P(|X_i| \geq s \mid \theta_i = 0) = P(|\varepsilon_1| \geq s) = 2\bar{\Phi}(s),$$

$$\begin{aligned} P(|X_i| \geq s \mid \theta_i \neq 0) &= \int P(|\varepsilon_1 + u| \geq s) \gamma(u) du = \int (\bar{\Phi}(s-u) + \bar{\Phi}(s+u)) \gamma(u) du \\ &= 2 \int \bar{\Phi}(s-u) \gamma(u) du = 2 \int \int \mathbf{1}_{\{s-x \leq u\}} \gamma(u) du \phi(x) dx \\ &= 2 \int \int \mathbf{1}_{\{s \leq v\}} \gamma(v-x) dv \phi(x) dx = 2 \int \mathbf{1}_{\{s \leq v\}} g(v) dv. \end{aligned}$$

□

Lemma S-7. *For any fixed $x \in \mathbb{R}$, the ℓ -value functional $\ell(x; w, g)$ (12) and the q -value functional $q(x; w, g)$ (14) are both nonincreasing in w .*

Proof. This is immediate from their explicit expression. □

Lemma S-8. *Under (40), $\log \bar{G}$ is Lipschitz on \mathbb{R}^+*

Proof. We have $(\log \bar{G})' = -g/\bar{G}$. Now using (40), we have $(g/\bar{G})(x) \asymp x^{1-\kappa}$ ($x \rightarrow \infty$). This provides that $(\log \bar{G})'$ is a bounded function. □

Lemma S-9. *Assumption (42) implies (43).*

Proof. Let us consider the function

$$\Psi : u \in (0, 1/2) \rightarrow \bar{G}(\bar{\Phi}^{-1}(u)) = \int_{\bar{\Phi}^{-1}(u)}^{\infty} g(x) dx.$$

This defines a continuous function on $[0, 1/2)$ by setting $\Psi(0) = 0$. For all $u \in (0, 1/2)$, we have $\Psi'(u) = \frac{g}{\phi}(\bar{\Phi}^{-1}(u))$, which means by (42) that Ψ' is decreasing on $(0, 1/2)$ and therefore Ψ is strictly concave on $(0, 1/2)$. This implies that $u \in (0, 1/2) \rightarrow \Psi(u)/u$ is decreasing and thus that $x \in \mathbb{R}_+ \rightarrow \bar{G}(x)/\bar{\Phi}(x)$ is increasing by letting $u = \bar{\Phi}(x)$, $x > 0$. Moreover, since $\infty = \lim_{u \rightarrow 0^+} \Psi'(u) = \lim_{u \rightarrow 0^+} \Psi(u)/u = \lim_{x \rightarrow \infty} \bar{G}(x)/\bar{\Phi}(x)$ and $\bar{G}(0)/\bar{\Phi}(0) = 1$, (43) is proved. □

Lemma S-10. *Assume that g comes from (39)–(42). For $w \in [0, 1]$, the functions $x \rightarrow \ell(x; w, g)$ and $x \rightarrow q(x; w, g)$ are symmetric and decreasing on \mathbb{R}_+ . For all $x \in \mathbb{R}$, $w \in [0, 1]$, we have $q(x; w, g) \leq \ell(x; w, g)$. In particular, $q_i(X) \leq \ell_i(X)$ almost surely.*

Proof. The first claim comes from the explicit expressions of $\ell(x; w, g)$ and $q(x; w, g)$ together with (42) and (43), respectively. Now, denoting P the probability operator in the Bayesian setting, a simple relation is that for all $x \in \mathbb{R}$,

$$\begin{aligned} q(x; w, g) &= P(\theta_i = 0 \mid |X_i| \geq |x|) \\ &= E(\mathbf{1}\{\theta_i = 0\} \mid |X_i| \geq |x|) \\ &= E[P(\theta_i = 0 \mid X_i) \mid |X_i| \geq |x|] \\ &= E[\ell_i(X) \mid |X_i| \geq |x|] \\ &\leq \ell(x; w, g), \end{aligned}$$

by using the monotonicity of $x \rightarrow \ell(x; w, g)$. \square

Figure S-1 shows how the choice of the prior influences the quantities g and \bar{G} . The Laplace calculations are done thanks to Remark S-11. Strikingly, while the quantities g stays of the same order (which guided the choice $a = 1/2$), the difference for \bar{G} is more substantial.

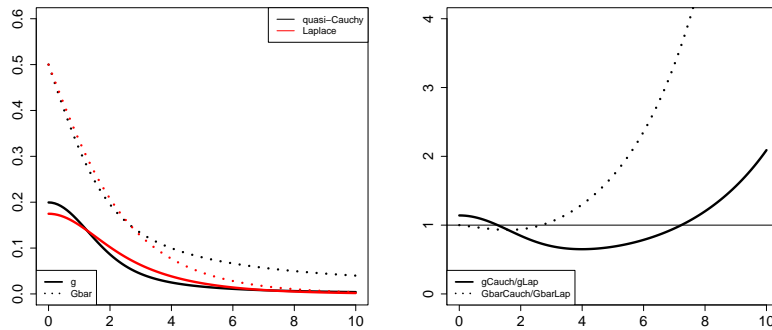


FIG S-1. Plots of the functions g and \bar{G} for the quasi-Cauchy and Laplace ($a = 1/2$) priors respectively (left) and ratio (right).

Figure S-2 below shows how the parameters w and g interplay in the quantities $q(x; w, g)$ and $\ell(x; w, g)$: for large values of $|x|$ (which play a central role in the multiple testing phase), the quantity $\ell(x; w, g)$ decreases as the prior puts its mass away from 0, that is, making the tail distribution heavier or increasing w .

Remark S-11 (Explicit expressions for Laplace prior). The Laplace prior of parameter $a > 0$ is given by

$$\gamma(x) = \gamma_a(x) = (a/2) e^{-a|x|}, \quad x \in \mathbb{R}. \quad (\text{S-7})$$

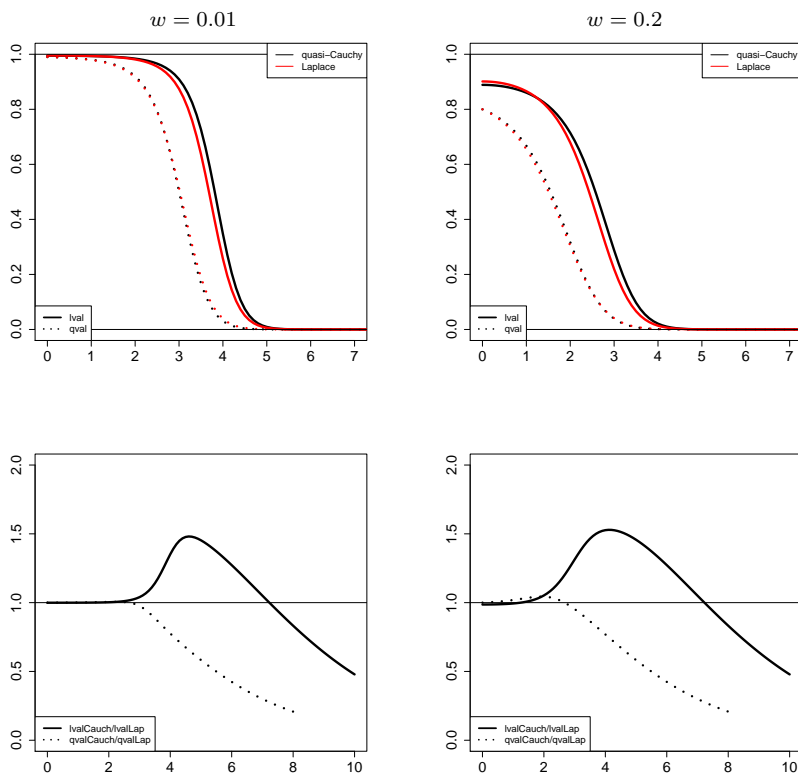


FIG S-2. Plot of the functions $x \rightarrow \ell(x, g, w)$ and $x \rightarrow q(x, g, w)$ for different values of w and g (see text, top) and ratio (bottom).

Straightforward calculations show, for γ as in (S-7),

$$\begin{aligned}
 g(x) &= (a/2)e^{a^2/2} (e^{-ax}\overline{\Phi}(a-x) + e^{ax}\overline{\Phi}(a+x)); \\
 g(x)/\phi(x) &= (a/2) \left(\frac{\overline{\Phi}(a-x)}{\phi(a-x)} + \frac{\overline{\Phi}(a+x)}{\phi(a+x)} \right); \\
 \overline{G}(x) &= (1/2) e^{a^2/2} (e^{-ax}\overline{\Phi}(a-x) - e^{ax}\overline{\Phi}(a+x)) + \overline{\Phi}(x).
 \end{aligned}$$

S-4. Threshold properties

We henceforth assume that g satisfies (38)–(42). In this section, all the non-universal constants appearing in the results depend on g .

S-4.1. Link between ξ , χ and ζ

Recall the definitions (47)-(49)-(50) of the thresholds ξ , ζ , χ . We start by a simple connection between ζ and ξ . Namely,

$$\frac{\phi(\zeta)}{g(\zeta)} = \frac{1}{\beta(\zeta) + 1} = 1/(1/w + 1) = w/(1 + w),$$

so

$$\zeta(w) = (\phi/g)^{-1}(w/(1 + w)) = \xi(w/(1 + w)), \quad (\text{S-8})$$

which implies in particular that $\zeta(w) \geq \xi(w)$. The next lemma relates these quantities to $\chi(w)$.

Lemma S-12. *For any $w \in (0, 1)$, we have $\chi(w) \leq \xi(w) \leq \zeta(w)$.*

Proof. From the proof of Lemma S-9, by concavity $\bar{G}(\bar{\Phi}^{-1}(u))/u \geq \frac{g}{\phi}(\bar{\Phi}^{-1}(u))$ holds for any $u \in (0, 1/2)$. Any $x > 0$ can be written $\bar{\Phi}^{-1}(u)$ for $u \in (0, 1/2)$, so for such x we have $(\bar{\Phi}/\bar{G})(x) \leq (\phi/g)(x)$. As $\bar{\Phi}/\bar{G}$ is decreasing by (43), so is its reciprocal, which implies $x \geq (\bar{\Phi}/\bar{G})^{-1}((\phi/g)(x))$. The inequality follows by setting $x = (\phi/g)^{-1}(w) = \xi(w)$. \square

S-4.2. Bounds for ξ , χ and ζ

Lemma S-13. *Consider ξ as in (47). Then for $C = (2\pi)^{1/2}\|g\|_\infty$ we have for $u \in (0, 1]$ small enough,*

$$\xi(u) \geq \left(-2 \log u - 2 \log g \left(\sqrt{-2 \log(Cu)}\right) - \log(2\pi)\right)^{1/2}; \quad (\text{S-9})$$

$$\xi(u) \leq \left(-2 \log u - 2 \log g \left(\sqrt{-4 \log u}\right) - \log(2\pi)\right)^{1/2}. \quad (\text{S-10})$$

We also have the following sharper bound: for $u \in (0, 1]$ small enough,

$$\xi(u) \leq \left(-2 \log u - 2 \log g \left(\left(-2 \log u + 5\Lambda(-\log u)^{1/2}\right)^{1/2}\right) - \log(2\pi)\right)^{1/2}. \quad (\text{S-11})$$

In particular, $\xi(u) \sim (-2 \log u)^{1/2}$ when u tends to zero.

Proof. Now fix $u \in (0, 1]$. Since $\phi(\xi(u)) = g(\xi(u))u$, we have $\phi(\xi(u)) \leq \|g\|_\infty u$ which implies $\xi(u) \geq \sqrt{-2 \log(Cu)}$, so $g(\xi(u)) \leq g\left(\sqrt{-2 \log(Cu)}\right)$ for u small enough. This in turn implies $\phi(\xi(u)) \leq ug\left(\sqrt{-2 \log(Cu)}\right)$ and thus (S-9). Conversely, using (39), $g(|x|) \geq g(0)e^{-\Lambda|x|}$ for all $x \in \mathbb{R}$ and thus $\phi(|x|)/g(|x|) \leq (g(0)\sqrt{2\pi})^{-1}e^{-x^2/2}e^{\Lambda|x|} \leq e^{-x^2/4}$ for $|x|$ larger than a constant, which in turn provides $|x| \leq \sqrt{-4 \log(\phi(|x|)/g(|x|))}$ and thus $\phi(|x|)/g(|x|) \leq (g(0)\sqrt{2\pi})^{-1}e^{-x^2/2}e^{\Lambda\sqrt{-4 \log(\phi(|x|)/g(|x|))}}$.

On the one hand, this gives that if u is small enough, $\phi(\xi(u)) \geq g(0)ue^{-\Lambda\sqrt{-4\log u}}$, so

$$\begin{aligned}\xi(u) &\leq \left(-2\log u + 4\Lambda(-\log u)^{1/2} - 2\log g(0) - \log(2\pi)\right)^{1/2} \\ &\leq \left(-2\log u + 5\Lambda(-\log u)^{1/2}\right)^{1/2}.\end{aligned}\tag{S-12}$$

As g decreases on a vicinity of ∞ , we have $g(\xi(u)) \geq g(\sqrt{-4\log u})$ for u small enough. Hence,

$$\phi(\xi(u)) \geq (\phi(\xi(u))/g(\xi(u))) g(\sqrt{-4\log u}) = u g(\sqrt{-4\log u}),$$

which leads to (S-10). To get (S-11) we use the same reasoning as above with the bound (S-12) instead of $\sqrt{-4\log u}$. \square

Lemma S-14. Consider χ as in (50). Then we have for all $u \in (0, 1]$,

$$\chi(u) \geq \bar{\Phi}^{-1}\left(u\bar{G}\left(\bar{\Phi}^{-1}(u)\right)\right); \tag{S-13}$$

$$\chi(u) \leq \bar{\Phi}^{-1}\left(u\bar{G}\left(\left(-2\log u + 4\Lambda(-\log u)^{1/2} + C\right)^{1/2}\right)\right) \text{ for } u \text{ small enough,} \tag{S-14}$$

and $C = -2\log g(0) - \log(2\pi)$. We also have the following sharper bound: for some constant $C' > 0$, for $u \in (0, 1]$ small enough,

$$\chi(u) \geq \left(-2\log\left(u\bar{G}\left(\bar{\Phi}^{-1}(u)\right)\right) - \log\log(1/u) - C'\right)^{1/2}. \tag{S-15}$$

Proof. Let $u \in (0, 1]$. Since $\bar{\Phi}(\chi(u)) = \bar{G}(\chi(u))u$, we have $\bar{\Phi}(\chi(u)) \leq u$ and thus $\chi(u) \geq \bar{\Phi}^{-1}(u)$, which in turn implies $\bar{\Phi}(\chi(u)) \leq \bar{G}(\bar{\Phi}^{-1}(u))u$ and (S-13). Conversely, as $\chi \leq \xi$ by Lemma S-12, using the bound on $\xi(u)$ just above (S-12) in the proof of Lemma S-13,

$$\chi(u) \leq \xi(u) \leq \left(-2\log u + 4\Lambda(-\log u)^{1/2} - 2\log g(0) - \log(2\pi)\right)^{1/2},$$

so the relation $\chi(u) = \bar{\Phi}^{-1}(\bar{G}(\chi(u))u)$ leads to (S-14). Let us now prove (S-15). First observe, by using (45), that $\bar{G}(\chi(u)) \gtrsim e^{-\Lambda\chi(u)}$. Next using the upper bound (S-12) on $\xi \geq \chi$ leads to $u\bar{G}(\chi(u)) \geq u^2$ for u small enough. Now, by the second part of Lemma S-33, for u small enough,

$$\begin{aligned}\chi(u) &= \bar{\Phi}^{-1}(\bar{G}(\chi(u))u) \\ &\geq \left\{2\log(1/\{u\bar{G}(\chi(u))\}) - \log\log(1/\{u\bar{G}(\chi(u))\}) - C\right\}^{1/2} \\ &\geq \left\{2\log(1/\{u\bar{G}(\chi(u))\}) - \log\log(1/u^2) - C\right\}^{1/2},\end{aligned}$$

for some constant $C > 0$, which gives the result. \square

Lemma S-15. Consider ζ as in (49). Then for a constant $C > 0$, we have for w small enough,

$$\zeta(w) \geq \left(-2 \log w - 2 \log g \left(\sqrt{-2 \log(Cw)}\right) - \log(2\pi)\right)^{1/2}; \quad (\text{S-16})$$

$$\zeta(w) \leq \left(-2 \log w - 2 \log g \left(\sqrt{-5 \log w}\right) + C\right)^{1/2}. \quad (\text{S-17})$$

We also have the following sharper bound: for $w \in (0, 1]$ small enough,

$$\zeta(w) \leq \left(-2 \log w - 2 \log g \left(\left(-2 \log w + 6\Lambda(-\log w)^{1/2}\right)^{1/2}\right) + C\right)^{1/2}. \quad (\text{S-18})$$

In particular, $\zeta(w) \sim (-2 \log w)^{1/2}$ as w tends to zero.

Proof. The result follows from Lemma S-13, combined with the relations $\zeta(w) \geq \xi(w)$ and $\zeta(w) = \xi(w/(1+w))$ established above. \square

S-4.3. Relations between $\xi(r(w, t))$, $\chi(r(w, t))$ and $\zeta(w)$

Let us recall the definition $r(w, t) = wt/\{(1-w)(1-t)\}$, see (46).

Lemma S-16. For any $t \in (0, 1)$, for $\omega_0 = \omega_0(t)$ small enough, for all $w \leq \omega_0$, we have $\chi(r(w, t)) \leq \zeta(w)$.

Proof. Denote by $T(u) = (-2 \log u + 4\Lambda(-\log u)^{1/2} + C)^{1/2}$ the term appearing in (S-14). By (S-14) and Lemma S-33, for u small enough,

$$\begin{aligned} \chi(u) &\leq \bar{\Phi}^{-1}(u \bar{G}(T(u))) \\ &\leq \left\{2 \log(1/u) - 2 \log \bar{G}(T(u)) - \log \log(1/u)\right\}^{1/2}. \end{aligned}$$

Now using that $\bar{G}(y) \geq Dg(y)$ for y large enough (see (40)), we have for u small enough,

$$\chi(u)^2 \leq 2 \log(1/u) - 2 \log D - 2 \log g(T(u)) - \log \log(1/u).$$

Hence, for w small enough, denoting $R = (1-t)(1-w)/t$ and recalling $r(w, t) = w/R$ via (46), and using (S-16) together with assumption (39),

$$\begin{aligned} &\chi(r(w, t))^2 - \zeta(w)^2 \\ &\leq 2 \log(1/r(w, t)) - 2 \log D - 2 \log g(T(r(w, t))) - \log \log(1/r(w, t)) \\ &\quad + 2 \log w + 2 \log g\left(\{-2 \log(Cw)\}^{1/2}\right) + \log(2\pi) \\ &\leq 2 \log R + 2\Lambda \left| \{-2 \log(Cw)\}^{1/2} - T(r(w, t)) \right| - \log \log(1/r(w, t)) + C', \end{aligned}$$

for some constant $C' > 0$. Now using $|\sqrt{a} - \sqrt{b}| = |a - b|/(\sqrt{a} + \sqrt{b})$ one gets, for w small enough,

$$\begin{aligned} & \left| \{-2 \log(Cw)\}^{1/2} - T(r(w, t)) \right| \\ & \leq \frac{\left| 2 \log(r(w, t)/(Cw)) - 4\Lambda(-\log r(w, t))^{1/2} - C \right|}{\{2 \log(1/(Cw))\}^{1/2}} \\ & \leq C'_1 \left(\frac{|\log((1-t)/t)|}{(\log 1/w)^{1/2}} + 1 \right) \end{aligned}$$

As a result, for w small enough and smaller than a threshold $\omega_0(t)$ (depending on t in a way such that $\log(1/w) \geq \log^2((1-t)/t)$ as well as $\log \log(1/w) \geq 2 \log R + C''$ for a large enough constant $C'' > 0$) we have $\chi(r(w, t))^2 - \zeta(w)^2 \leq 0$ and the result holds. \square

Lemma S-17. *There exists some constant $C = C(g) > 0$ such that for all $t \in (0, 1)$ there exists $\omega_0(t)$ such that for all $w \leq \omega_0(t)$,*

$$|\zeta(w) - \xi(r(w, t))| \leq \frac{2|\log(\frac{t}{1-t})| + C}{\zeta(w) + \xi(r(w, t))}. \quad (\text{S-19})$$

Furthermore, for all $\varepsilon > 0$ and $t \in (0, 1)$, there exists $\omega_0(t, \varepsilon)$ such that for $w \leq \omega_0(t, \varepsilon)$,

$$\frac{g(\xi(r(w, t)))}{g(\zeta(w))} \leq 1 + \varepsilon; \quad (\text{S-20})$$

$$\frac{\bar{G}(\chi(r(w, t)))}{\bar{G}(\zeta(w))} \leq 1 + \varepsilon. \quad (\text{S-21})$$

Proof. Let us set

$$S_1(w) = \left(-2 \log w + 6\Lambda(-\log w)^{1/2} \right)^{1/2}$$

and $S_2(w) = \sqrt{-2 \log(Cw)}$ the terms appearing in the bounds (S-18) and (S-9), respectively. Using these bounds, one obtains

$$\begin{aligned} & \zeta(w)^2 - \xi(r(w, t))^2 \\ & \leq 2 \log(r(w, t)/w) + 2 \log g(S_2(r(w, t))) - 2 \log g(S_1(w)) + D \\ & \leq 2|\log(t/(1-t))| + D', \end{aligned}$$

for w smaller than a threshold depending on t , by using that $\log g$ is Lipschitz and proceeding as in the proof of Lemma S-16 to bound the difference $|S_1(w) - S_2(r(w, t))|$ by a universal constant. Conversely, by using (S-11) and (S-16), we

have, with $S_3(w)$ as $S_1(w)$ except that 6Λ is replaced by 5Λ and $S_4(w)$ as $S_2(w)$ with C as in (S-16),

$$\begin{aligned} & \xi(r(w, t))^2 - \zeta(w)^2 \\ & \leq -2 \log(r(w, t)/w) - 2 \log g(S_3(w)) + 2 \log g(S_4(w)) + D'' \\ & \leq 2 |\log(t/(1-t))| + D''', \end{aligned}$$

as above, which leads to (S-19) by using $a^2 - b^2 = (a-b)(a+b)$. Next, (S-20) is a direct consequence of (S-19) by using that $\log g$ is Lipschitz. Finally, let us prove (S-21). By Lemma S-16 and the bounds (S-18) and (S-15), we have for $w \leq w_0(t)$ and $S_1(w)$ as above,

$$\begin{aligned} 0 & \leq \zeta(w)^2 - \chi(r(w, t))^2 \\ & \leq -2 \log w - 2 \log g(S_1(w)) + C \\ & \quad + 2 \log \left\{ r(w, t) \bar{G} \circ \bar{\Phi}^{-1}(r(w, t)) \right\} + \log \log \{1/r(w, t)\} + C' \\ & \leq |2 \log(t/(1-t))| + D + \log \log \{1/r(w, t)\} + 2 \log \left\{ \frac{\bar{G} \circ \bar{\Phi}^{-1}(r(w, t))}{g(S_1(w))} \right\}. \end{aligned}$$

Next, we have

$$\log \left\{ \frac{\bar{G} \circ \bar{\Phi}^{-1}(r(w, t))}{g(S_1(w))} \right\} = \log \left\{ \frac{\bar{G} \circ \bar{\Phi}^{-1}(r(w, t))}{\bar{G}(S_1(w))} \right\} + \log \left\{ \frac{\bar{G}(S_1(w))}{g(S_1(w))} \right\}.$$

The first term is bounded by a constant, by an argument similar to the proof of Lemma S-16, as $\log \bar{G}$ is Lipschitz. For the second term, by (40),

$$\log \left\{ \frac{\bar{G}(S_1(w))}{g(S_1(w))} \right\} \leq \log S_1(w).$$

This gives, upon dividing by $\zeta(w) + \chi(r(w, t))$ the obtained inequality on $\zeta(w)^2 - \chi(r(w, t))^2$, that $|\zeta(w) - \chi(r(w, t))|$ is arbitrary small when w is small, which leads to (S-21) by using again that $\log \bar{G}$ is Lipschitz. \square

Lemma S-18. *There exists a constant $C = C(g) > 0$ such that for all $t \in (0, 0.9)$ there exists $\omega_0(t)$ such that for $w \leq \omega_0(t)$ and $\mu \in \mathbb{R}$,*

$$\bar{\Phi}(\xi(r(w, t)) - \mu) \geq C t \bar{\Phi}(\zeta(w) - \mu). \quad (\text{S-22})$$

Proof. By Lemma S-17, for small w , $|\zeta(w) - \xi(r(w, t))| \leq 1/4$. Hence, we can apply Lemma S-34, which gives

$$\begin{aligned} \frac{\bar{\Phi}(\xi(r(w, t)) - \mu)}{\bar{\Phi}(\zeta(w) - \mu)} & \geq \frac{1}{4} e^{-|\xi(r(w, t))^2 - \zeta(w)^2|/2} \\ & \geq C e^{-|\log(\frac{t}{1-t})|}, \end{aligned}$$

by using again (S-19). This shows the desired result. \square

S-4.4. Variations of certain useful functions

For any $w \in (0, 1)$ and $\mu \neq 0$, let us denote

$$T_\mu(w) = 1 + \frac{|\zeta(w) - |\mu||}{|\mu|}. \quad (\text{S-23})$$

Lemma S-19. *First, for all $\varepsilon \in (0, 1)$, for any $z \geq 1$, there exists $\omega_0 = \omega_0(z, \varepsilon) \in (0, 1)$, such that for all $w \leq \omega_0$,*

$$\begin{cases} 1 - \varepsilon \leq g(\zeta(w/z))/g(\zeta(w)) \leq 1 \\ 1 - \varepsilon \leq \overline{G}(\zeta(w/z))/\overline{G}(\zeta(w)) \leq 1. \end{cases} \quad (\text{S-24})$$

Second, for any $K \geq 1$, one can find $d_1 = d_1(K)$ and $d_2 = d_2(K) > 0$ such that for all $z \geq 1$, for $w \leq \omega_0 = \omega_0(z, 1/2)$ as before and $|\mu| > \zeta(w)/K$,

$$d_1 \leq T_\mu(w/z)/T_\mu(w) \leq d_2. \quad (\text{S-25})$$

Proof. Since $\log g$ and $\log \overline{G}$ are Lipschitz and by monotonicity, it is sufficient to bound $\zeta(w/z) - \zeta(w)$ from above. For this, we combine (S-16) and (S-18) to obtain, with S_1, S_4 as in the proof of Lemma S-17,

$$\begin{aligned} & \zeta(w/z)^2 - \zeta(w)^2 \\ & \leq 2 \log w + 2 \log g(S_4(w)) + \log(2\pi) - 2 \log(w/z) - 2 \log g(S_1(w/z)) + C \\ & \leq 2 \log z + D + 2\Lambda |S_4(w) - S_1(w/z)|, \end{aligned}$$

by using that $\log g$ is Λ -Lipschitz by (39). Since the last bound is bounded by some constant for $w \leq \omega_0(z)$, we obtain (S-24).

To prove (S-25), one notes that since $|\mu| > \zeta(w)/K$, we have $1 \leq T_\mu(w/z) \leq 2 + K\zeta(w/z)/\zeta(w)$ which itself is less than $2 + K + K(\zeta(w/z) - \zeta(w))/\zeta(w)$. Using the previous bound on $\zeta(w/z) - \zeta(w)$ and the fact that $\zeta(w)$ goes to ∞ as w goes to 0 the last bound is no more than a constant $C = C(K)$ whenever $w \leq \omega(z, 1/2)$. On the other hand, $1 \leq T_\mu(w) \leq 2 + K$ for $|\mu| > \zeta(w)/K$. The desired inequality follows. \square

Let us denote, for $w \in (0, 1)$ and $\mu \in \mathbb{R}$,

$$G_\mu(w) = \frac{\overline{\Phi}(\zeta(w) - |\mu|)}{w}. \quad (\text{S-26})$$

Lemma S-20. *Consider G_μ defined by (S-26). For all $K_0 > 1$ and any $z \geq 1$, there exists $\omega_0 = \omega_0(K_0, z)$ such that for all $w \leq \omega_0$, any $\mu \in \mathbb{R}$ with $|\mu| \geq \zeta(w)/K_0$, we have*

$$G_\mu(w/z) \geq z^{1/(2K_0)} G_\mu(w). \quad (\text{S-27})$$

Proof. Let us focus on $\mu \geq 0$ without loss of generality. Let us rewrite the desired inequality as, with $\Gamma(u) = \log G_\mu(e^{-u})$,

$$\Gamma\left(\log \frac{z}{w}\right) - \Gamma\left(\log \frac{1}{w}\right) \geq \frac{1}{2K_0} \left(\log \frac{z}{w} - \log \frac{1}{w}\right).$$

To prove this, it is enough to check that $\Gamma'(u) \geq 1/(2K_0)$ for $u \in [\log \frac{1}{w}, \log \frac{z}{w}]$, for appropriately small w . To do so, one computes the derivative of Γ explicitly using the chain rule. First one notes that

$$\zeta'(w) = -\frac{1}{w^2\beta'(\zeta(w))},$$

and from this one deduces that

$$\Gamma'(u) = 1 - \frac{e^u}{\beta'(\zeta(e^{-u}))} \frac{\phi}{\Phi}(\zeta(e^{-u}) - \mu).$$

One further computes

$$\beta'(x) = (\beta(x) + 1)xQ(x), \quad \text{for } Q(x) = 1 + \frac{(\log g)'(x)}{x},$$

which gives $\beta'(\zeta(e^{-u})) = \zeta(e^{-u})Q(\zeta(e^{-u}))(\beta(\zeta(e^{-u})) + 1)$. Using the identity $\beta(\zeta(e^{-u})) = e^u$ leads to

$$\Gamma'(u) = 1 - \frac{e^u}{1 + e^u} \frac{1}{Q(\zeta(e^{-u}))} \frac{1}{\zeta(e^{-u})} \frac{\phi}{\Phi}(\zeta(e^{-u}) - \mu).$$

Now, by using (39) one sees that the map $u \rightarrow e^u(1 + e^u)^{-1}Q(\zeta(e^{-u}))^{-1}$ has limit 1 as u goes to infinity. So, for u large enough, $e^u(1 + e^u)^{-1}Q(\zeta(e^{-u}))^{-1} \leq 1 + \varepsilon$ for some $\varepsilon > 0$ to be chosen later on. Now using Lemma S-33, whenever $\mu \leq \zeta(e^{-u}) - 1$,

$$\begin{aligned} \frac{1}{\zeta(e^{-u})} \frac{\phi}{\Phi}(\zeta(e^{-u}) - \mu) &\leq \frac{1}{\zeta(e^{-u})} \frac{1 + (\zeta(e^{-u}) - \mu)^2}{\zeta(e^{-u}) - \mu} \\ &= \frac{\zeta(e^{-u}) - \mu}{\zeta(e^{-u})} + \frac{1}{\zeta(e^{-u})(\zeta(e^{-u}) - \mu)}. \end{aligned}$$

By definition of u , we have $e^{-u} \in [w/z, w]$, so $\zeta(e^{-u}) \leq \zeta(w/z)$. Deduce that, using that by assumption $\mu \geq \zeta(w)/K_0$,

$$\frac{\zeta(e^{-u}) - \mu}{\zeta(e^{-u})} \leq 1 - \frac{1}{K_0} \frac{\zeta(w)}{\zeta(w/z)}.$$

The behaviour of the difference $\zeta(w/z) - \zeta(w)$ was studied in the proof of Lemma S-19 where it is seen that this quantity is smaller a certain universal constant if w is small enough. By writing

$$\zeta(w/z)/\zeta(w) = \left(1 + \frac{\zeta(w/z) - \zeta(w)}{\zeta(w)}\right)^{-1},$$

one gets that this ratio is at least $1 - 1/8$ for w small enough, using $\zeta(w) \rightarrow \infty$ as $w \rightarrow 0$. This shows that for $w \leq \omega(z)$ small enough,

$$\frac{1}{\zeta(e^{-u})} \frac{\phi}{\Phi}(\zeta(e^{-u}) - \mu) \leq 1 - (1/K_0)(1 - 1/8) + \frac{1}{\zeta(e^{-u})},$$

where we have used $\zeta(e^{-u}) - \mu \geq 1$. On the other hand, if $\mu \geq \zeta(e^{-u}) - 1$,

$$\frac{1}{\zeta(e^{-u})} \frac{\phi}{\Phi}(\zeta(e^{-u}) - \mu) \leq \frac{\phi(0)}{\Phi(1)\zeta(e^{-u})},$$

which can be made arbitrarily small for w small enough. As a result, in both cases, for $w \leq \omega(K_0, z)$ small enough, for all $\mu \geq \zeta(w)/K_0$,

$$1 - \Gamma'(u) \leq (1 + \varepsilon)(1 - 7/8K_0 + 1/(4K_0)) \leq (1 + \varepsilon)(1 - 5/(8K_0)) = 1 - 1/(2K_0)$$

by choosing $\varepsilon^{-1} = 8K_0 - 5$. This proves the desired inequality. \square

S-5. Moment properties

The main results in this section concern the moments of the score function, $\tilde{m}(w) = -E_0\beta(X, w) = -\int_{-\infty}^{\infty} \beta(t, w)\phi(t)dt$ and $m_1(\tau, w) = E_\tau[\beta(X, w)]$, $m_2(\tau, w) = E_\tau[\beta(X, w)^2]$. Remember that g is assumed to enjoy (38)–(42). Also, since these functions only depends on g , all the constants appearing in the results of this section only depend on g (except in Section S-5.6 where the sparsity comes in). In this section, we freely use $\zeta = \zeta(w)$ as a shorthand notation.

S-5.1. Basic lemmas on moments

The following two lemmas are (mostly) small parts of Lemmas 7–9 in [1]. We include the proofs for completeness.

Lemma S-21. For $c_1 = (-\beta(0))^{-1} - 1 > 0$, for any $x \in \mathbb{R}$ and $w \in (0, 1]$,

$$|\beta(x, w)| \leq \frac{1}{w \wedge c_1}. \quad (\text{S-28})$$

Proof. It suffices to distinguish the cases $\beta(x) < 0$ and $\beta(x) \geq 0$ and to bound $|\beta(x, w)|$ by $|\beta(0)|/(1 + \beta(0))$ and $1/w$, respectively. \square

Lemma S-22. The function $w \in (0, 1] \rightarrow \tilde{m}(w)$ is continuous, nonnegative, increasing and $\tilde{m}(0) = 0$. The map $w \in (0, 1] \rightarrow m_1(\mu, w)$ is continuous and decreasing. In addition, $m_1(\mu, 0) > 0$ if $\mu \neq 0$ and $\mu \in \mathbb{R}_+ \rightarrow m_1(\mu, w)$ is nondecreasing for any $w \in [0, 1]$. Also, there exists a constant $\omega = \omega(g)$ such that, for any $w \leq \omega$ and any $\mu \in \mathbb{R}$,

$$m_1(\mu, w) \leq \frac{1}{w}, \quad m_2(\mu, w) \leq \frac{1}{w}.$$

Proof. Since $w \rightarrow \beta(u, w)$ is decreasing (for any u with $\beta(u) \neq 0$), so are $w \rightarrow -\tilde{m}(w)$ and $w \rightarrow m_1(\mu, w)$ for any real μ . The continuity of \tilde{m} follows by continuity of $\beta(u, w)$ and domination of $\beta(u, w)\phi(u)$ by $g(u) + \phi(u)$ (up to a constant). In addition, since, as g is a density, $\int \beta(u)\phi(u)du = 0$, and we have

$$\tilde{m}(w) = - \int \frac{\beta(u)}{1 + w\beta(u)} \phi(u)du = \int \frac{w\beta(u)^2}{1 + w\beta(u)} \phi(u)du. \quad (\text{S-29})$$

From this one deduces that \tilde{m} is nonnegative. For m_1 , the continuity follows by local domination using Lemma S-21. Next, if $\mu \neq 0$, say $\mu > 0$, we have

$$m_1(\mu, 0) = \int_{-\infty}^{\infty} \beta(u + \mu)\phi(u)du = \int_{-\infty}^{\infty} (\beta(u + \mu) - \beta(u))\phi(u)du.$$

Moreover, by (42), $u \rightarrow \beta(u + \mu) - \beta(u)$ is a positive function. Since it is also continuous, the integral is positive, which means that $m_1(\mu, 0) > 0$. To see that $\mu \in \mathbb{R}_+ \rightarrow m_1(\mu, w)$ is nondecreasing, we compute its derivative

$$\frac{\partial m_1(\mu, w)}{\partial \mu} = \int_0^{\infty} \frac{\partial \{\beta(x)/(1 + w\beta(x))\}}{\partial x} (\phi(x - \mu) - \phi(x + \mu))dx \geq 0.$$

Finally, the bounds on m_1, m_2 follow from Lemma S-21, with $\omega = c_1$. \square

The following is a reformulation of Corollary 1 in [1] (see (58) therein). We provide a proof below for completeness.

Lemma S-23. *Consider Λ as in (39). Then for all $z \geq 4\Lambda$ and all $\mu \geq 0$,*

$$\int_0^z \left(\frac{g(u)}{\phi(u)} \right)^2 \phi(u - \mu)du \leq \frac{8}{z} \left(\frac{g(z)}{\phi(z)} \right)^2 \phi(z - \mu). \quad (\text{S-30})$$

Proof. We have for all $u \in [0, z]$,

$$\left(\frac{g(u)}{\phi(u)} \right)^2 \phi(u - \mu) = \left(\frac{g(z)}{\phi(z)} \right)^2 \phi(z - \mu) \exp \left\{ - \int_u^z [\log \{g^2/\phi^2(\cdot)\phi(\cdot - \mu)\}]' (v)dv \right\}.$$

Now, by (39), for all $v \in [0, z]$ and $\mu \geq 0$,

$$(2 \log g - 2 \log \phi + \log \phi(\cdot - \mu))' (v) \geq -2\Lambda + 2v - (v - \mu) \geq v - 2\Lambda.$$

Therefore, inserting the latter in the above display, we obtain

$$\left(\frac{g(u)}{\phi(u)} \right)^2 \phi(u - \mu) \leq \left(\frac{g(z)}{\phi(z)} \right)^2 \phi(z - \mu) e^{-(z-2\Lambda)^2/2} e^{(u-2\Lambda)^2/2}.$$

One concludes because letting $s = z - 2\Lambda \geq z/2$ and noting that

$$\begin{aligned} e^{-s^2/2} \int_0^z e^{(u-2\Lambda)^2/2} du &\leq e^{-s^2/2} \int_{-s}^s e^{t^2/2} dt = 2 \int_0^s e^{-(s-t)(s+t)/2} dt \\ &\leq 2 \int_0^s e^{-(s-t)s/2} \leq \int_0^{\infty} e^{-xs/2} dx = 4/s \leq 8/z. \end{aligned}$$

\square

S-5.2. Behaviour of \tilde{m}

The next lemma refines Lemma 7 in [1].

Lemma S-24. For $\tilde{m}(w)$ defined by (56), we have, for $\zeta = \zeta(w)$ and asymptotically as $w \rightarrow 0$,

$$\frac{\tilde{m}(w)}{2\overline{G}(\zeta)} \sim 1. \quad (\text{S-31})$$

In particular, for κ as in (40), as $w \rightarrow 0$, $\tilde{m}(w) \asymp \zeta^{\kappa-1}g(\zeta)$ and $\tilde{m}(w) \gtrsim w^c$ for arbitrary $c \in (0, 1)$.

Proof. Using (S-29), symmetry of β and $\beta\phi = g - \phi$ on $[\zeta, \infty)$,

$$\tilde{m}(w) = 2 \int_0^\zeta \frac{w\beta(u)^2}{1+w\beta(u)}\phi(u)du - \int_\zeta^\infty \frac{2w\beta(u)}{1+w\beta(u)}\phi(u)du + \int_\zeta^\infty \frac{2w\beta(u)}{1+w\beta(u)}g(u)du. \quad (\text{S-32})$$

For the first term of (S-32), since for $u \in [0, \zeta]$, $1+w\beta(u) \geq 1+\beta(0)$,

$$\begin{aligned} 2 \int_0^\zeta \frac{w\beta(u)^2}{1+w\beta(u)}\phi(u)du &\leq 2w(1+\beta(0))^{-1} \int_0^\zeta \beta(u)^2\phi(u)du \\ &\leq \frac{C}{\zeta}w\beta(\zeta)(g/\phi)(\zeta) = \frac{Cg(\zeta)}{\zeta}, \end{aligned}$$

for $C = 20/(1+\beta(0))$, by Lemma S-23 ($\mu = 0$), where we use that $\beta(\zeta) \leq (g/\phi)(\zeta) \leq (5/4)\beta(\zeta)$ which holds for ζ large enough, or equivalently for $w \leq \omega_1$ with $\omega_1 = \omega_1(g)$ a universal constant. The second term of (S-32) is negative whenever $\zeta > \beta^{-1}(0)$ and of smaller order than the third term. For the third term we use that for $u \geq \zeta$, $w\beta(u) \geq 1$ and thus $1 \leq 2w\beta(u)/(1+w\beta(u)) \leq 2$, hence

$$\overline{G}(\zeta) \leq \int_\zeta^\infty \frac{2w\beta(u)}{1+w\beta(u)}g(u)du \leq 2\overline{G}(\zeta).$$

Now, by assumption $\overline{G}(\zeta) \asymp g(\zeta)\zeta^{\kappa-1}$, see (40). Hence, when w is small, the dominating term in (S-32) is the third one, which gives

$$\tilde{m}(w) \sim \int_\zeta^\infty \frac{2w\beta(u)}{1+w\beta(u)}g(u)du \quad (\text{S-33})$$

Now, let us prove

$$\int_\zeta^\infty \frac{w\beta(u)}{1+w\beta(u)}g(u)du \sim \overline{G}(\zeta) \quad (\text{S-34})$$

from which (S-31) follows. To prove (S-34), let us write

$$\int_\zeta^\infty \frac{w\beta(u)}{1+w\beta(u)}g(u)du = \overline{G}(\zeta) - \int_\zeta^\infty \frac{g(u)}{1+w\beta(u)}du.$$

Hence, we obtain

$$\begin{aligned} \left| \bar{G}(\zeta) - \int_{\zeta}^{\infty} \frac{w\beta(u)}{1+w\beta(u)} g(u) du \right| &\leq \int_{\zeta}^{\infty} \frac{g(u)}{1+w\beta(u)} du \\ &\leq w^{-1} \int_{\zeta}^{\infty} \phi(u) du = \frac{\bar{\Phi}(\zeta)}{w}, \end{aligned}$$

because $1+w\beta(u) = 1-w+wg(u)/\phi(u) \geq wg(u)/\phi(u)$. Now using that $\bar{\Phi}(\zeta) \sim \phi(\zeta)/\zeta \sim wg(\zeta)/\zeta$ and since $\bar{G}(\zeta) \asymp g(\zeta)\zeta^{\kappa-1}$ (see (40)), the difference in the last display is a $o(\bar{G}(\zeta))$ and (S-31) is proved. Then, $\tilde{m}(w) \asymp \zeta^{\kappa-1}g(\zeta)$ follows from (40) and this in turn implies by (44) and Lemma S-15, $\tilde{m}(w) \gtrsim e^{-\Lambda\zeta(w)} \gtrsim w^c$ for any $c > 0$. \square

S-5.3. Upper bound on m_1

The next lemma refines the bounds on m_1 of Lemma 9 in [1]. The refinement is important in that we obtain a precise upper-bound for any μ larger than a constant. Moreover, the bound is sharp in this regime of μ 's, as we shall see below.

Lemma S-25. *There exist constants $C > 0$ and $\omega_0 \in (0, 1)$ such that for any $w \leq \omega_0$, for any μ such that $\mu \geq \mu_0 := 2\Lambda$, with $T_{\mu}(w)$ as in (S-23),*

$$m_1(\mu, w) \leq C \frac{\bar{\Phi}(\zeta - |\mu|)}{w} T_{\mu}(w).$$

In particular, $m_1(\mu, w) \leq C\zeta^2\bar{\Phi}(\zeta - \mu)/w$ holds for any $\mu \geq \mu_0$ and $w \leq \omega_0$. For any $w \leq \omega_0$, one also has

$$\begin{aligned} m_1(\mu, w) &\leq \frac{C}{|\mu|} e^{-\mu^2/2+|\mu|\zeta}, & \text{for any } \zeta^{-1} \leq |\mu| \leq \mu_0, \\ |m_1(\mu, w)| &\leq C(1 + \zeta\mu^2), & \text{for any } |\mu| \leq \zeta^{-1}. \end{aligned}$$

Since $T_{\mu}(w) = 1+|\zeta-|\mu||/|\mu|$ can be written $1+(\zeta-|\mu|)_+/|\mu|+(|\mu|-\zeta)_+/|\mu| \leq 2+(\zeta/|\mu|-1)_+$, we deduce the following corollary.

Corollary S-26. *There exists $\omega_0 \in (0, 1)$ such that for any $K > 1$, there exist constants $C(K) > 0$ such that for any $w \leq \omega_0$, for any μ such that $\mu \geq \zeta/K$, we have*

$$m_1(\mu, w) \leq C(K) \frac{\bar{\Phi}(\zeta - |\mu|)}{w}.$$

We now prove Lemma S-25.

Proof. As $\mu \rightarrow m_1(\mu, w)$ is even by symmetry of β and ϕ , it suffices to consider the case $\mu \geq 0$. For $\mu > \zeta - 1$, the result directly follows from the global bound

$|m_1(\mu, w)| \leq Cw^{-1}$, a consequence of Lemma S-21. By definition

$$\begin{aligned} m_1(\mu, w) &= \int_{-\infty}^{\infty} \frac{\beta(x)}{1+w\beta(x)} \phi(x-\mu) dx \\ &= \int_{-\zeta}^{\zeta} \frac{\beta(x)}{1+w\beta(x)} \phi(x-\mu) dx + \int_{|x|>\zeta} \frac{\beta(x)}{1+w\beta(x)} \phi(x-\mu) dx \\ &= \quad (I) \quad \quad \quad + \quad \quad \quad (II). \end{aligned}$$

We first deal with the term (II), for which $\beta(x) \geq \beta(\zeta) \geq 0$ (for small enough universal ω_0), so (II) ≥ 0 , and using $1+w\beta(x) \geq w\beta(x)$ one obtains

$$(II) \leq \frac{1}{w} \int_{|x|>\zeta} \phi(x-\mu) dx \leq \frac{2}{w} \bar{\Phi}(\zeta - \mu).$$

Now one rewrites (I) as

$$\begin{aligned} (I) &= \int_{-\zeta}^{\zeta} \beta(x) \phi(x-\mu) dx - w \int_{-\zeta}^{\zeta} \frac{\beta(x)^2}{1+w\beta(x)} \phi(x-\mu) dx \\ &\leq \int_{-\zeta}^{\zeta} \beta(x) \phi(x-\mu) dx. \end{aligned}$$

Let us split

$$\begin{aligned} \int_{-\zeta}^{\zeta} \beta(x) \phi(x-\mu) dx &= \int_{|x| \leq 1/\mu} \beta(x) \phi(x-\mu) dx + \int_{1/\mu \leq |x| \leq \zeta} \beta(x) \phi(x-\mu) dx \\ &= \quad (a) \quad \quad \quad + \quad \quad \quad (b). \end{aligned}$$

First, the integral (a) can be written, by definition of β ,

$$\int_{|x| \leq 1/\mu} \beta(x) \phi(x-\mu) dx = \int_{-1/\mu}^{1/\mu} (g-\phi)(x) e^{\mu x - \frac{\mu^2}{2}} dx$$

Using $|g-\phi| \leq \|g-\phi\|_{\infty} \leq C$, one gets (a) $\lesssim e^{-\mu^2/2}/\mu$. For the integral (b), with $\beta(x) \leq (g/\phi)(x)$ (note that $\beta(x)$ is possibly negative here),

$$\begin{aligned} (b) &\leq \int_{-\zeta}^{-1/\mu} g(x) e^{\mu x - \frac{\mu^2}{2}} dx + \int_{1/\mu}^{\zeta} g(x) e^{\mu x - \frac{\mu^2}{2}} dx \\ &\leq \int_{1/\mu}^{\zeta} g(x) e^{-\mu x - \frac{\mu^2}{2}} dx + \int_{1/\mu}^{\zeta} g(x) e^{\mu x - \frac{\mu^2}{2}} dx \\ &\leq 2e^{-\frac{\mu^2}{2}} \int_1^{\mu\zeta} g(t/\mu) e^t dt / \mu. \end{aligned}$$

From this one deduces the global bound, for $\mu > 1/\zeta$,

$$\begin{aligned} m_1(\mu, w) &\leq \frac{2}{w} \bar{\Phi}(\zeta - \mu) + \frac{C}{\mu} \|g\|_{\infty} e^{-\mu^2/2 + \mu\zeta} \\ &\lesssim \frac{g(\zeta)}{\phi(\zeta)} \phi(\zeta - \mu) + \frac{1}{\mu} e^{-\mu^2/2 + \mu\zeta} \lesssim (\|g\|_{\infty} + \mu^{-1}) e^{-\mu^2/2 + \mu\zeta}, \end{aligned}$$

which leads to the second inequality of the lemma. Now turning to the first inequality, an integration by parts gives, with $0 \leq -g'/g \leq \Lambda$ from (39),

$$\begin{aligned} \int_1^{\mu\zeta} g(t/\mu)e^t dt &= [g(t/\mu)e^t]_1^{\mu\zeta} - \int_1^{\mu\zeta} \frac{1}{\mu} g'(t/\mu)e^t dt \\ &\leq g(\zeta)e^{\mu\zeta} + \frac{\Lambda}{\mu} \int_1^{\mu\zeta} g(t/\mu)e^t dt. \end{aligned}$$

One obtains

$$(b) \leq 2 \left(1 - \frac{\Lambda}{\mu}\right)^{-1} g(\zeta)e^{\mu\zeta} \frac{e^{-\frac{\mu^2}{2}}}{\mu}.$$

Noting that $g(\zeta)e^{\mu\zeta} \geq g(0)e^{(\mu-\Lambda)\zeta}$ using (39) again, and that this quantity is bounded away from 0 for $\mu \geq \mu_0 = 2\Lambda$, one concludes that for such μ 's the upper-bound for (b) dominates the one for (a), so that

$$(a) + (b) \leq Cg(\zeta) \frac{e^{\mu\zeta - \frac{\mu^2}{2}}}{\mu}.$$

Now one can note, using $\mu_0 \leq \mu \leq \zeta - 1$ and $(g/\phi)(\zeta) \asymp w^{-1}$,

$$\begin{aligned} g(\zeta) \frac{e^{\mu\zeta - \frac{\mu^2}{2}}}{\mu} &= g(\zeta) \frac{\phi(\zeta - \mu)}{\phi(\zeta)} \frac{1}{\mu} \\ &\leq C \frac{\bar{\Phi}(\zeta - \mu)}{w} \frac{|\zeta - \mu|}{\mu}. \end{aligned}$$

This gives the result in the case $\mu_0 \leq \mu \leq \zeta - 1$, which concludes the proof of the first inequality. The last part of the lemma follows by noting that $T_\mu(w) \leq C\zeta^2$.

For $|\mu| \leq 1/\zeta$, we can invoke Lemma 9, eq. (89) from [1], that is

$$m_1(\mu, w) \leq -\tilde{m}(w) + C\zeta\mu^2$$

which is at most $C + C\zeta\mu^2$. \square

S-5.4. Upper bound on m_2

Lemma S-27. *There exist constants $C > 0$ and $\omega_0 \in (0, 1)$ such that for any $w \leq \omega_0$, for any $\mu \in \mathbb{R}$,*

$$m_2(\mu, w) \leq C \frac{\bar{\Phi}(\zeta - |\mu|)}{w^2}.$$

Proof. Since $m_2(\mu, w) = E[\beta(Z + \mu, w)^2] = \int_{-\infty}^{\infty} \beta(u, w)^2 \phi(u - \mu) du$ by definition, we first bound

$$\beta(u, w)^2 = \left(\frac{\beta(u)}{1 + w\beta(u)} \right)^2 \leq C\beta(u)^2 1_{|u| \leq \zeta} + w^{-2} 1_{|u| > \zeta}.$$

Indeed, for $\beta(u) \geq 0$ this follows from bounding the denominator from below by 1 or $w\beta(u)$ respectively, and for $\beta(u) < 0$ (in which case $|u| < \zeta$, as soon as $w_0 < \beta^{-1}(0)$) one uses the fact that $1 + w\beta(u) \geq 1 + w\beta_{\min} \geq c_0 > 0$. Deduce that

$$\begin{aligned} m_2(\mu, w) &\leq C \int_{-\zeta}^{\zeta} \beta(z)^2 \phi(z - \mu) dz + \int_{|z| > \zeta} w^{-2} \phi(z - \mu) dz \\ &\leq \quad (A) \quad + \quad (B). \end{aligned}$$

By definition of (B),

$$(B) = w^{-2}(\bar{\Phi}(\zeta - \mu) + \bar{\Phi}(\zeta + \mu)) \leq 2w^{-2}\bar{\Phi}(\zeta - |\mu|).$$

To bound (A), we note

$$(A) = C \left(\int_0^{\zeta} \beta(z)^2 \phi(z + \mu) dz + \int_0^{\zeta} \beta(z)^2 \phi(z - \mu) dz \right) \leq 2C \int_0^{\zeta} \beta(z)^2 \phi(z - |\mu|) dz.$$

As the last bound is symmetric in μ , it is enough to obtain the desired bound for $\mu \geq 0$, which we thus assume for the remaining of the proof. For large enough C , it holds $(\frac{g}{\phi} - 1)^2 \leq C(\frac{g}{\phi})^2$ (e.g. expanding the square and using that g/ϕ is bounded away from 0) which with Lemma S-23 leads to

$$\int_0^{\zeta} \beta(z)^2 \phi(z - \mu) dz \leq C \int_0^{\zeta} (g/\phi)(z)^2 \phi(z - \mu) dz \leq C \frac{8}{\zeta} \left(\frac{g}{\phi}\right)^2(\zeta) \phi(\zeta - \mu).$$

Also, $(g/\phi)(\zeta) = \beta(\zeta) + 1 = w^{-1} + 1 \leq 2w^{-1}$. To conclude one writes

$$\frac{\phi(\zeta - \mu)}{\zeta} = \frac{\phi(\zeta - \mu)}{\zeta - \mu + \mu}.$$

If $\zeta - \mu \geq 1$, one can use Lemma S-33 to obtain that the previous quantity is less than $2\bar{\Phi}(\zeta - \mu)$ (bound the denominator from below by $\zeta - \mu$). If $\zeta - \mu \leq 1$, there exist $C_1, C_2 > 0$ with

$$\sup_{\mu: \mu \geq \zeta - 1} \frac{\phi(\zeta - \mu)}{\zeta} \leq C_1 \leq C_2 \bar{\Phi}(1) \leq C_2 \bar{\Phi}(\zeta - \mu).$$

The lemma follows by combining the previous bounds. \square

S-5.5. Lower bound on m_1

Lemma S-28. *There exist constants $M_0, C_1 > 0$ and $\omega_0 \in (0, 1)$ such that for any $w \leq \omega_0$, and any $\mu \geq M_0$, with $T_\mu(w)$ defined by (S-23),*

$$m_1(\mu, w) \geq C_1 \frac{\bar{\Phi}(\zeta - \mu)}{w} T_\mu(w).$$

Proof. By definition, using $\zeta = \zeta(w)$ as shorthand,

$$\begin{aligned} m_1(\mu, w) &= \int_{-\zeta}^{\zeta} \frac{\beta(x)}{1+w\beta(x)} \phi(x-\mu) dx + \int_{|x|>\zeta} \frac{\beta(x)}{1+w\beta(x)} \phi(x-\mu) dx \\ &= \quad (I) \quad \quad \quad + \quad \quad \quad (II). \end{aligned}$$

To bound (II) from below, one notes that $1+w\beta(x) \leq 2w\beta(x)$ for $|x| \geq \zeta$, so

$$(II) \geq \frac{1}{2w} \int_{|x|>\zeta} \phi(x-\mu) dx = \frac{1}{2w} (\bar{\Phi}(\zeta-\mu) + \bar{\Phi}(\zeta+\mu)) \geq \frac{1}{2w} \bar{\Phi}(\zeta-\mu).$$

To bound (I) from below, let us introduce $d = \max(d_1, d_2)$, where d_1 verifies $\beta(d_1) = 1$ and d_2 is such that for $x \geq d_2$, the map $x \rightarrow g(x)$ is decreasing (such d_2 exists by (38)). We isolate first the possibly negative part of the integral defining (I) and write

$$\begin{aligned} \int_{|x| \leq d} \frac{\beta(x)}{1+w\beta(x)} \phi(x-\mu) dx &\geq - \int_{|x| \leq d} \frac{|\beta(x)|}{1+w\beta(0)} \phi(x-\mu) dx \\ &\geq - \int_{|x| \leq d} \frac{|\beta(x)|}{1+w\beta(0)} \frac{dx}{\sqrt{2\pi}} =: -D_1. \end{aligned}$$

Let I_1 be the part of the integral (I) corresponding to x in $\Gamma := \{x : d \leq |x| \leq \zeta\}$. If $\zeta > d$,

$$\begin{aligned} I_1 &\geq \int_{\Gamma} \beta(x) \phi(x-\mu) dx - w \int_{\Gamma} \frac{\beta(x)^2}{1+w\beta(x)} \phi(x-\mu) dx \\ &\geq \frac{1}{2} \int_{\Gamma} \beta(x) \phi(x-\mu) dx \\ &\geq \frac{1}{4} \int_{\Gamma} g(x) \frac{\phi(x-\mu)}{\phi(x)} dx, \end{aligned}$$

where we have used that $w\beta(\cdot)/(1+w\beta(\cdot)) \leq 1/2$ on Γ and that $g/\phi - 1 \geq g/(2\phi)$ on Γ by definition of this set. An integration by parts now shows that

$$\begin{aligned} \int_d^{\zeta} g(x) e^{\mu x} dx &= \frac{1}{\mu} \int_{\mu d}^{\mu \zeta} g(t/\mu) e^t dt \\ &= \mu^{-1} [g(t/\mu) e^t]_{\mu d}^{\mu \zeta} - \mu^{-2} \int_{\mu d}^{\mu \zeta} g'(t/\mu) e^t dt \\ &\geq \mu^{-1} [g(\zeta) e^{\mu \zeta} - g(d) e^{\mu d}], \end{aligned}$$

as $g'(u) < 0$ for $u > d \geq d_2$. We now claim that $g(\zeta) e^{\mu \zeta} \geq 2g(d) e^{\mu d}$ for any $\mu \geq 2\Lambda$ and $\zeta \geq d + \log(2)/\Lambda$. Indeed, for such μ, ζ ,

$$e^{\mu(\zeta-d)} \geq e^{2\Lambda(\zeta-d)} \geq 2e^{\Lambda(\zeta-d)},$$

while, using that $-\Lambda \leq (\log g)' < 0$ on (d, ∞) by (39) and the definition of d , one obtains

$$2 \frac{g(d)}{g(\zeta)} = 2e^{-\{\log g(\zeta) - \log g(d)\}} \leq 2e^{\Lambda(\zeta-d)} \leq e^{\mu(\zeta-d)}.$$

Putting the two previous bounds together leads to, for such μ, ζ ,

$$I_1 \geq \frac{1}{8\mu} g(\zeta) e^{\mu\zeta - \mu^2/2}.$$

Let us now distinguish two cases. Suppose first that $M_0 \leq \mu \leq \zeta - 1$ for $M_0 := 2\Lambda$. The map $\mu \rightarrow \mu\zeta - \mu^2/2$ is increasing on this interval, so its minimum is attained for $\mu = M_0$. Combining this with $g(\zeta) \geq Ce^{-\Lambda\zeta}$ and using the rough bound $\mu^{-1} \geq \zeta^{-1}$ leads to, uniformly for $\mu \in [M_0, \zeta - 1]$,

$$I_1 \geq \frac{e^{-\Lambda\zeta + M_0\zeta - M_0^2/2}}{8\zeta} \gtrsim \frac{e^{\Lambda\zeta}}{\zeta}.$$

Since $e^{\Lambda u}/u \rightarrow \infty$ as $u \rightarrow \infty$ and $\zeta = \zeta(w) \rightarrow \infty$ as $w \rightarrow 0$, we have $I_1 \geq 2D_1$ for any $\mu \geq [M_0, \zeta - 1]$ and any $w \geq \omega_0$ for ω_0 small enough. One deduces that for such w and μ ,

$$I_1 - D_1 \geq \frac{g(\zeta)}{16} \frac{e^{\zeta\mu - \mu^2/2}}{\mu} \gtrsim \frac{1}{\mu} \frac{\phi(\zeta - \mu)}{\phi(\zeta)} g(\zeta).$$

Noting that $\phi(\zeta)/g(\zeta) \sim w$ and combining with the bound on (II) above, one deduces, for $w \leq \omega_0$ and $\mu \in [M_0, \zeta - 1]$,

$$m_1(\mu, w) \geq \frac{\bar{\Phi}(\zeta - \mu)}{2w} + C \frac{\phi(\zeta - \mu)}{\mu w}.$$

Using that $\mu \leq \zeta - 1$, one deduces that

$$\frac{\phi(\zeta - \mu)}{\mu w} \geq \frac{\zeta - \mu}{\mu} \frac{\bar{\Phi}(\zeta - \mu)}{w}.$$

This gives the desired inequality if $\mu \in [M_0, \zeta - 1]$. The second case is now $\mu > \zeta - 1$. In this case, we simply use $I_1 \geq 0$ to get

$$m_1(\mu, w) \geq -D_1 + (II) \geq -D_1 + \frac{1}{2w} \bar{\Phi}(\zeta - \mu).$$

As $\bar{\Phi}(\zeta - \mu)/(2w) \geq \bar{\Phi}(1)/(2w)$ for small enough w , the last display is bounded from below by $\bar{\Phi}(\zeta - \mu)/(4w)$. Noting that the bound

$$m_1(\mu, w) \geq C \frac{\bar{\Phi}(\zeta - \mu)}{w} \left[1 + \frac{|\zeta - \mu|}{\mu} \right]$$

holds in the two cases, for C a small enough constant, leads to the result, recalling the definition of $T_w(\mu)$ in (S-23). \square

Combining Lemmas S-27 and S-28 (and using $T_\mu(w) \geq 1$) one obtains the following bound.

Corollary S-29. *There exist constants $M_0, C_2 > 0$ and $\omega_0 \in (0, 1)$ such that for any $w \leq \omega_0$, and any $\mu \geq M_0$,*

$$m_2(\mu, w) \leq C_2 \frac{m_1(\mu, w)}{w}.$$

Here is another lower bound for m_1 when the signal is large

Lemma S-30. *For any $\varepsilon \in (0, 1)$ and $\rho > 0$, there exist $\omega_0 = \omega_0(\varepsilon, \rho) \in (0, 1)$ such that for any $w \leq \omega_0$, and any $\mu \geq (1 + \rho)\zeta(w)$,*

$$m_1(\mu, w) \geq (1 - \varepsilon)/w.$$

Proof. Let $a = 1 + (\rho/2)$ and let us write, for w small enough,

$$\begin{aligned} wm_1(\mu, w) &= \int_{-a\zeta}^{a\zeta} \frac{w\beta(x)}{1+w\beta(x)} \phi(x-\mu) dx + \int_{|x|>a\zeta} \frac{w\beta(x)}{1+w\beta(x)} \phi(x-\mu) dx \\ &\geq \int_{x>a\zeta} \frac{w\beta(x)}{1+w\beta(x)} \phi(x-\mu) dx - \int_{-a\zeta}^{a\zeta} \phi(x-\mu) dx \\ &\geq \frac{w\beta(a\zeta)}{1+w\beta(a\zeta)} \bar{\Phi}(a\zeta-\mu) - (1 - \bar{\Phi}(a\zeta-\mu)). \end{aligned}$$

Since for $\mu \geq (1 + \rho)\zeta$, we have that $\bar{\Phi}(a\zeta - \mu) \geq \bar{\Phi}(-(\rho/2)\zeta)$ tends to 1 when w tends to zero, we only have to prove that $w\beta(a\zeta) = \beta(a\zeta)/\beta(\zeta)$ tends to infinity. The latter comes from

$$\beta(a\zeta)/\beta(\zeta) \gtrsim e^{-a\Lambda\zeta} \frac{\phi(\zeta)}{\phi(a\zeta)} = e^{(a^2-1)\zeta^2 - a\Lambda\zeta},$$

by using the definition of β and (44). \square

S-5.6. Results for m_1 and \tilde{m} ratio

In the next lemmas, we study the behaviour of the functionals, for given $\theta_0 \in \mathbb{R}^n$,

$$H_{\theta_0}(w) = \frac{\sum_{i \in S_0} m_1(\theta_{0,i}, w)}{\tilde{m}(w)}, \quad w \in (0, 1), \quad (\text{S-35})$$

$$H_{\theta_0}^\circ(w, K) = \frac{\sum_{i \in \mathcal{C}_0(\theta_0, w, K)} m_1(\theta_{0,i}, w)}{\tilde{m}(w)}, \quad w \in (0, 1), \quad K \geq 1, \quad (\text{S-36})$$

where we denoted $S_0 = \{1 \leq i \leq n : \theta_{0,i} \neq 0\}$ and

$$\mathcal{C}_0(\theta_0, w, K) = \{1 \leq i \leq n : |\theta_{0,i}| \geq \zeta(w)/K\} \subset S_0.$$

The set $\mathcal{C}_0(\theta_0, w, K)$ is sometimes denoted by $\mathcal{C}_0(w, K)$ or \mathcal{C}_0 for short.

Lemma S-31. Consider a sparsity $s_n \leq n^v$ for $v \in (0, 1)$. Consider H_{θ_0} and $H_{\theta_0}^\circ$ as in (S-35) and (S-36), respectively. There exist constants $C = C(v, g) > 0$ and $D = D(v, g) \in (0, 1)$ such that

$$\sup_{\theta_0 \in \ell_0[s_n]} \sup_{w \in [\frac{1}{n}, \frac{1}{\log n}], K \in [\frac{2}{1-v}, \frac{4}{1-v}]} |H_{\theta_0}(w) - H_{\theta_0}^\circ(w, K)| \leq Cn^{1-D}, \quad (\text{S-37})$$

for any n larger than an integer $N = N(v, g)$.

Proof. For $\theta_0 \in \ell_0[s_n]$ and $w \in [n^{-1}, 1/\log n]$, denote

$$\mathcal{C}_1 = S_0 \setminus \mathcal{C}_0 = \{1 \leq i \leq n : 0 < |\theta_{0,i}| < \zeta(w)/K\}.$$

By using the upper bounds on m_1 obtained in Lemma S-25 (and μ_0 defined therein), with $\zeta = \zeta(w)$, and for now taking $K \geq 2$ arbitrary,

$$\begin{aligned} \sum_{i \in \mathcal{C}_1} m_1(\theta_{0,i}, w) &= \left\{ \sum_{0 < |\theta_{0,i}| \leq \zeta^{-1}} + \sum_{\zeta^{-1} < |\theta_{0,i}| \leq \mu_0} + \sum_{\mu_0 < |\theta_{0,i}| < \zeta/K} \right\} m_1(\theta_{0,i}, w) \\ &\lesssim s_n \left\{ (1 + \zeta^{-1}) + \zeta e^{\mu_0 \zeta} + \zeta w^{-1} \bar{\Phi}(\zeta - \zeta/K) \right\}, \end{aligned}$$

where to bound the third sum we use $\bar{\Phi}(\zeta - |\theta_{0,i}|) \leq \bar{\Phi}(\zeta - \zeta/K)$ and $T_\mu(w) \lesssim \zeta(w)$. Now, by Lemma S-33,

$$\bar{\Phi}\left(\zeta - \frac{\zeta}{K}\right) \leq \frac{K}{K-1} \zeta^{-1} \exp\left(-\frac{\zeta^2 (K-1)^2}{2K^2}\right) \lesssim \frac{1}{\zeta} w^{(1-1/K)^2}$$

for n large enough, where we used $\zeta(w)^2 \geq -2 \log w$ via (S-16) in the last step. Now using that for $w \geq n^{-1}$, we have $\zeta \leq 2\sqrt{\log n}$ for large n by Lemma S-15, so that $e^{\mu_0 \zeta}$ is negligible compared to any positive power of n . One deduces that, for n large enough, using $w \geq n^{-1}$ and $s_n \lesssim n^v$ by assumption, and any $K \geq 2$,

$$\begin{aligned} \sum_{i \in \mathcal{C}_1} m_1(\theta_{0,i}, w) &\leq C s_n \left\{ 1 + e^{C\zeta} + w^{-2/K+1/K^2} \right\} \\ &\leq C n^v e^{C\zeta} + C n n^{v-1+2/K-1/K^2}. \end{aligned}$$

Now if $v - 1 + 2/K \leq 0$, which holds for K as in the statement, one gets

$$\sup_{\theta_0 \in \ell_0[s_n]} \sup_{w \in [n^{-1}, 1/\log n]} \frac{\sum_{i \in \mathcal{C}_1} m_1(\theta_{0,i}, w)}{\tilde{m}(w)} \leq \frac{C}{\tilde{m}(n^{-1})} \{n^v e^{2C\sqrt{\log n}} + n^{1-1/K^2}\}.$$

For K as in the statement, we further have $1 - K^{-2} \leq 1 - (1-v)^2/16$. Since $\tilde{m}(n^{-1})$ decreases to 0 slower than any power of n (see Lemma S-24, combined with (44) and the bound (S-17) on ζ), the last display can be bounded by Cn^{1-D} , for D small enough, which shows (S-37). \square

Lemma S-32. Consider $H_{\theta_0}^\circ$ as in (S-36) for some choice of $K > 1$. Then there exists a constant $C = C(K, g) > 0$ such that, for all $z \geq 1$, there exists $\omega_0 = \omega_0(z, K, g) \in (0, 1)$ such that for all $w \in (0, \omega_0)$ and for all $\theta_0 \in \mathbb{R}^n$, we have

$$H_{\theta_0}^\circ(w/z, K) \geq C z^{1/(2K)} H_{\theta_0}^\circ(w, K/1.1). \quad (\text{S-38})$$

Proof. According to Lemma S-25 and Lemma S-28, there exists constants $C_1, C_2 > 0$ and $\omega_0 \in (0, 1)$ such that for $w \in (0, \omega_0)$ and any θ_0 ,

$$C_1 \sum_{i \in \mathcal{C}_0(w, K)} G_{\theta_0, i}(w) \frac{T_{\theta_0, i}(w)}{\tilde{m}(w)} \leq H_{\theta_0}^\circ(w, K) \leq C_2 \sum_{i \in \mathcal{C}_0(w, K)} G_{\theta_0, i}(w) \frac{T_{\theta_0, i}(w)}{\tilde{m}(w)},$$

where T_μ, G_μ are defined by (S-23), (S-26) respectively. Now, by Lemmas S-19 and S-20, for all $z \geq 1$, there exists $\omega_0(z, K) \in (0, 1)$ such that for $w \leq \omega_0(z, K)$ and any $\mu \geq \zeta(w)/K$,

$$\begin{aligned} G_\mu(w/z) &\geq z^{1/(2K)} G_\mu(w) \\ d_1 T_\mu(w) &\leq T_\mu(w/z) \leq d_2 T_\mu(w), \end{aligned}$$

for some constants $d_1 = d_1(K), d_2 = d_2(K)$. Combining Lemma S-24 on \tilde{m} with Lemma S-19 on \bar{G} , one can find $D_1, D_2 > 0$ with, for $w \leq \omega(z)$,

$$D_1 \tilde{m}(w) \leq \tilde{m}(w/z) \leq D_2 \tilde{m}(w).$$

Hence, by combining these results one gets, for $w \leq \omega_0(z, K)$ (and then $w/z \leq \omega_0(z, K)$ also holds),

$$\begin{aligned} H_{\theta_0}^\circ(w/z, K) &\geq C_1 \sum_{i \in \mathcal{C}_0(w/z, K)} G_{\theta_0, i}(w/z) \frac{T_{\theta_0, i}(w/z)}{\tilde{m}(w/z)} \\ &\geq (C_1 d_1 / D_2) z^{1/(2K)} \sum_{i \in \mathcal{C}_0(w/z, K)} G_{\theta_0, i}(w) \frac{T_{\theta_0, i}(w)}{\tilde{m}(w)}. \end{aligned}$$

Now we claim that $\mathcal{C}_0(w, K/1.1) \subset \mathcal{C}_0(w/z, K)$ for w small enough depending on z . Indeed, $\zeta(w/z)/\zeta(w) \leq 1 + (\zeta(w/z) - \zeta(w))/\zeta(w) \leq 1.1$ for w small enough depending on z , as in the proof of Lemma S-19. So,

$$\begin{aligned} \mathcal{C}_0(w/z, K) &= \{1 \leq i \leq n : |\theta_{0, i}| \geq \zeta(w/z)/K\} \\ &\supset \{1 \leq i \leq n : |\theta_{0, i}| \geq 1.1\zeta(w)/K\} = \mathcal{C}_0(w, K/1.1). \end{aligned}$$

One deduces that $H_{\theta_0}^\circ(w/z, K) \geq C z^{1/(2K)} H_{\theta_0}^\circ(w, K/1.1)$ for $w \leq \omega_0(z, K)$ as announced. \square

S-6. Auxiliary lemmas

Lemma S-33. For any $x > 0$,

$$\frac{x^2}{1+x^2} \frac{\phi(x)}{x} \leq \bar{\Phi}(x) \leq \frac{\phi(x)}{x}.$$

In particular, for any $x \geq 1$, $\bar{\Phi}(x) \geq \frac{1}{2} \frac{\phi(x)}{x}$ and $\bar{\Phi}(x) \sim \frac{\phi(x)}{x}$ when $x \rightarrow \infty$. Furthermore, for any $y \in (0, 1/2)$,

$$\{(2 \log(1/y) - \log \log(1/y) - \log(16\pi))_+\}^{1/2} \leq \bar{\Phi}^{-1}(y) \leq \{2 \log(1/y)\}^{1/2}.$$

and also for y small enough,

$$\bar{\Phi}^{-1}(y) \leq \{2 \log(1/y) - \log \log(1/y)\}^{1/2}.$$

In particular, $\bar{\Phi}^{-1}(y) \sim \{2 \log(1/y)\}^{1/2}$ when $y \rightarrow 0$.

Proof. The first display of the lemma are classical bounds on $\bar{\Phi}$. The second display follows using the first one and similar inequalities as those used to derive bounds on ξ, ζ, χ . Let us prove the last relation: for all $y \in (0, 1/2)$,

$$y \{(2 \log(1/y) - \log \log(1/y) - \log(16\pi))_+\}^{1/2} \leq y \bar{\Phi}^{-1}(y) \leq \phi(\bar{\Phi}^{-1}(y))$$

Hence,

$$\begin{aligned} \bar{\Phi}^{-1}(y) &\leq \left\{ -2 \log \left(y \{(2 \log(1/y) - \log \log(1/y) - \log(16\pi))_+\}^{1/2} \right) \right\}^{1/2} \\ &\leq \left\{ -2 \log y - \log \left((2 \log(1/y) - \log \log(1/y) - \log(16\pi))_+ \right) \right\}^{1/2} \end{aligned}$$

which provides the result. \square

Lemma S-34. For any $x, y \in \mathbb{R}$, with $|x - y| \leq 1/4$, we have

$$\bar{\Phi}(x) \geq \bar{\Phi}(y) \frac{1}{4} e^{-(x^2 - y^2)_+/2}. \quad (\text{S-39})$$

Proof. Let us assume $x > y$ (otherwise the result is trivial). If $y \leq 0$, we have $\bar{\Phi}(x) \geq \bar{\Phi}(1/4) \geq 1/4 \geq 1/4 \bar{\Phi}(y)$ so the inequality is true. Assume now $y > 0$. By Lemma S-33,

$$\begin{aligned} \frac{\bar{\Phi}(x)}{\bar{\Phi}(y)} &\geq \frac{\bar{\Phi}(y + 1/4)}{\bar{\Phi}(y)} \mathbf{1}\{y \leq 1\} + \frac{xy}{1 + x^2} e^{-(x^2 - y^2)/2} \mathbf{1}\{y \geq 1\} \\ &\geq \frac{\bar{\Phi}(5/4)}{\bar{\Phi}(1)} \mathbf{1}\{y \leq 1\} + \frac{x^2}{2(1 + x^2)} e^{-(x^2 - y^2)/2} \mathbf{1}\{y \geq 1\} \end{aligned}$$

because $y \in (0, \infty) \rightarrow \frac{\bar{\Phi}(y+1/4)}{\bar{\Phi}(y)}$ is decreasing and $y \geq x/2$ when $y \geq 1$. This concludes the proof. \square

Lemma S-35. [Bernstein’s inequality] Let W_i , $1 \leq i \leq n$ centered independent variables with $|W_i| \leq \mathcal{M}$ and $\sum_{i=1}^n \text{Var}(W_i) \leq V$, then for any $A > 0$,

$$P \left[\sum_{i=1}^n W_i > A \right] \leq \exp \left\{ -\frac{1}{2} A^2 / (V + \mathcal{M}A/3) \right\}.$$

Lemma S-36. There exists a constant $C > 1$ such that, for any $M \geq 1$,

$$\frac{e^M}{M^2} - 1 \leq \int_1^M \frac{e^v}{v^2} dv \leq C \frac{e^M}{M^2}.$$

Proof. For $M \leq 3$ the result is immediate for C chosen large enough beforehand. For $M > 3$, one writes

$$\int_3^M \frac{e^v}{v^2} dv = \left[\frac{e^v}{v^2} \right]_3^M + 2 \int_3^M \frac{e^v}{v^3} dv \leq \frac{e^M}{M^2} + \frac{2}{3} \int_3^M \frac{e^v}{v^2} dv,$$

so that $\int_3^M \frac{e^v}{v^2} dv \leq 3e^M/M^2$, from which the upper bound follows. The lower bound follows from integrating by parts between 1 and M and noting that the second term is nonnegative. \square

Lemma S-37. For $m \geq 1$, $p_1, \dots, p_m \in (0, 1)$, consider $U = \sum_{i=1}^m B_i$, where $B_i \sim \mathcal{B}(p_i)$, $1 \leq i \leq m$, are independent. For any nonnegative variable T independent of U , we have

$$E \left(\frac{T}{T+U} \mathbf{1}\{T > 0\} \right) \leq e^{-EU} + \frac{12ET}{EU}. \quad (\text{S-40})$$

Proof. Let us prove the two following inequalities: for all $u > 0$,

$$P(U = 0) \leq e^{-\sum_{i=1}^m p_i}.$$

$$E \left(\frac{u \sum_{i=1}^m p_i}{u \sum_{i=1}^m p_i + U \vee 1} \right) \leq 12u.$$

For the first inequality, using $\log(1-x) \leq -x$ for all $x \in (0, 1)$,

$$P(U = 0) = \prod_{i=1}^m (1-p_i) = e^{\sum_{i=1}^m \log(1-p_i)} \leq e^{-\sum_{i=1}^m p_i} = e^{-EU}.$$

For the second assertion, we have

$$E \left(\frac{u \sum_{i=1}^m p_i}{u \sum_{i=1}^m p_i + U \vee 1} \right) \leq E \left(\frac{\sum_{i=1}^m p_i}{U \vee 1} \right) u.$$

Now applying Bernstein's inequality, we have

$$\begin{aligned} P \left(U \leq \sum_{i=1}^m p_i/2 \right) &= P \left(U - \sum_{i=1}^m p_i \leq -\sum_{i=1}^m p_i/2 \right) \\ &\leq \exp \left\{ -\frac{1}{2} \sum_{i=1}^m p_i (1/2)^2 / (1 + 1/6) \right\} \leq e^{-0.1 \sum_{i=1}^m p_i}. \end{aligned}$$

As a result, one obtains, using $xe^{-x} \leq 1$ for $x \geq 0$,

$$\begin{aligned} &E \left(\frac{\sum_{i=1}^m p_i}{U \vee 1} \right) \\ &\leq E \left(\frac{\sum_{i=1}^m p_i}{U \vee 1} \mathbf{1}\{U > \sum_{i=1}^m p_i/2\} \right) + E \left(\frac{\sum_{i=1}^m p_i}{U \vee 1} \mathbf{1}\{U \leq \sum_{i=1}^m p_i/2\} \right) \\ &\leq 2 + 10 \left(0.1 \sum_{i=1}^m p_i \right) e^{-0.1 \sum_{i=1}^m p_i} \leq 12, \end{aligned}$$

as announced. To show (S-40), we now use the independence assumption and the concavity of $x \rightarrow \frac{x}{x+u}$ (for $u > 0$), to obtain

$$\begin{aligned} E \left[\frac{T}{T+U} \mathbf{1}\{T > 0\} \right] &= P(U = 0, T > 0) + E \left[\frac{T}{T+U} \mathbf{1}\{U > 0\} \right] \\ &\leq P(U = 0) + E \left[\frac{ET}{ET+U} \mathbf{1}\{U > 0\} \right] \\ &\leq P(U = 0) + E \left[\frac{ET}{ET+U \vee 1} \right]. \end{aligned}$$

The two previous inequalities for $u = ET/EU$ thus give the result. \square

S-7. Additional numerical experiments

The following pages present further numerical experiments along the lines of the comments of Section 4.

References

- [1] I. M. Johnstone and B. W. Silverman. Needles and straw in haystacks: empirical Bayes estimates of possibly sparse sequences. *Ann. Statist.*, 32(4):1594–1649, 2004.

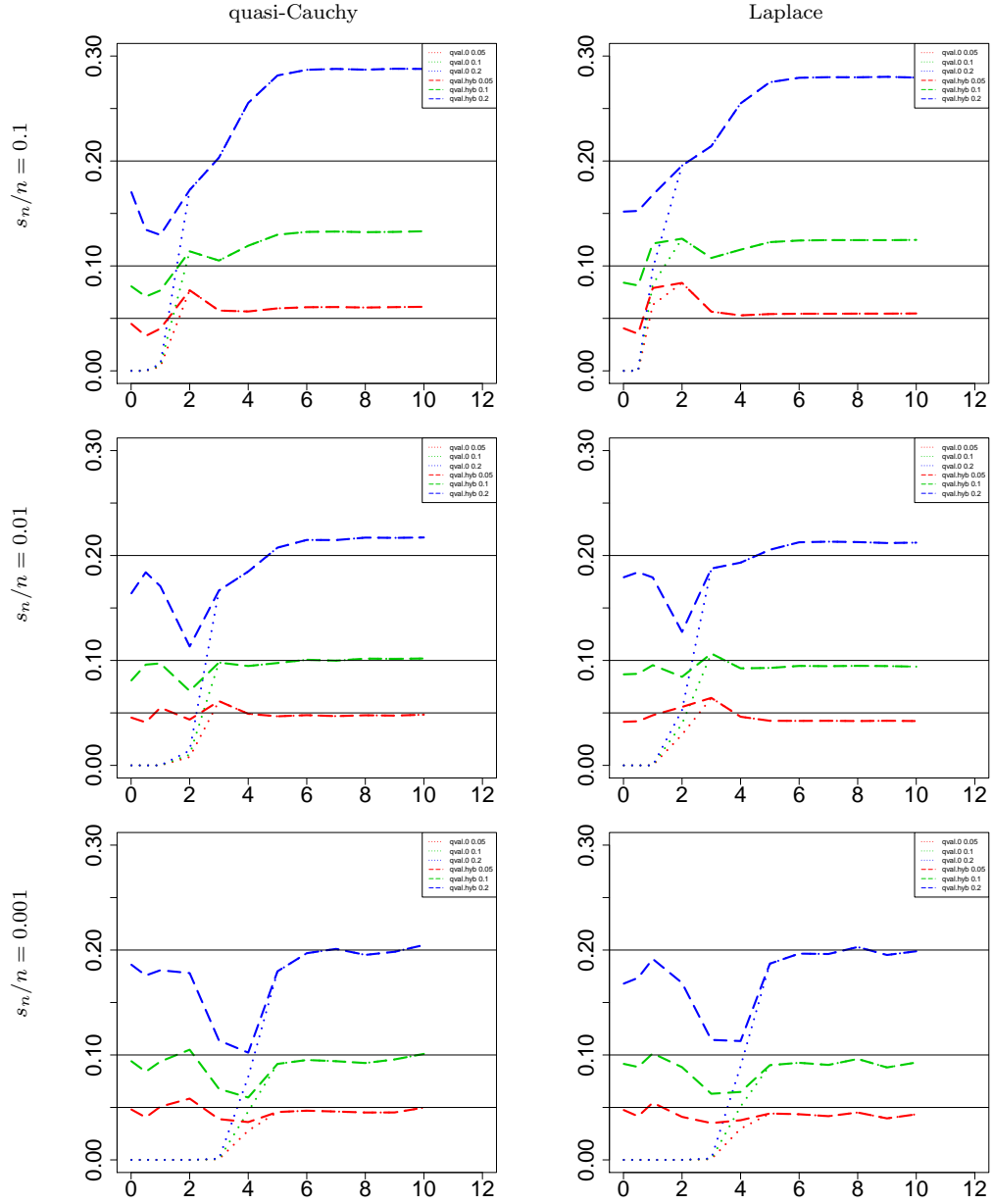


FIG S-3. *FDR of EBayesq.0 and EBayesq.hybrid procedures with threshold $t \in \{0.05, 0.1, 0.2\}$. $\alpha = 0.2$; $n = 10,000$; 500 replications; alternative all equal to μ (on the X-axis).*

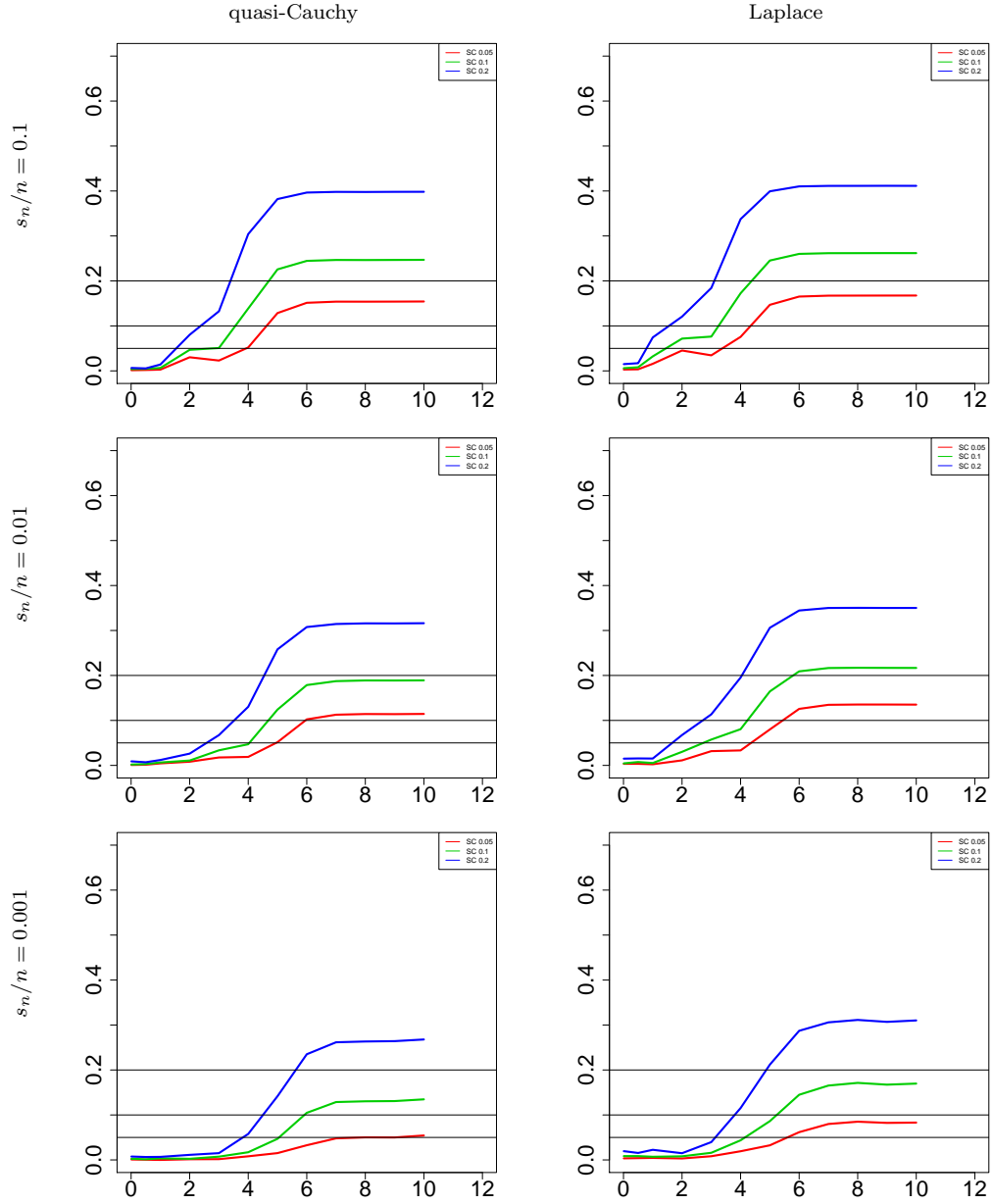


FIG S-4. FDR for SC procedure with threshold $t \in \{0.05, 0.1, 0.2\}$. $\alpha = 0.2$; $n = 10,000$; 2000 replications; alternative all equal to μ (on the X-axis).

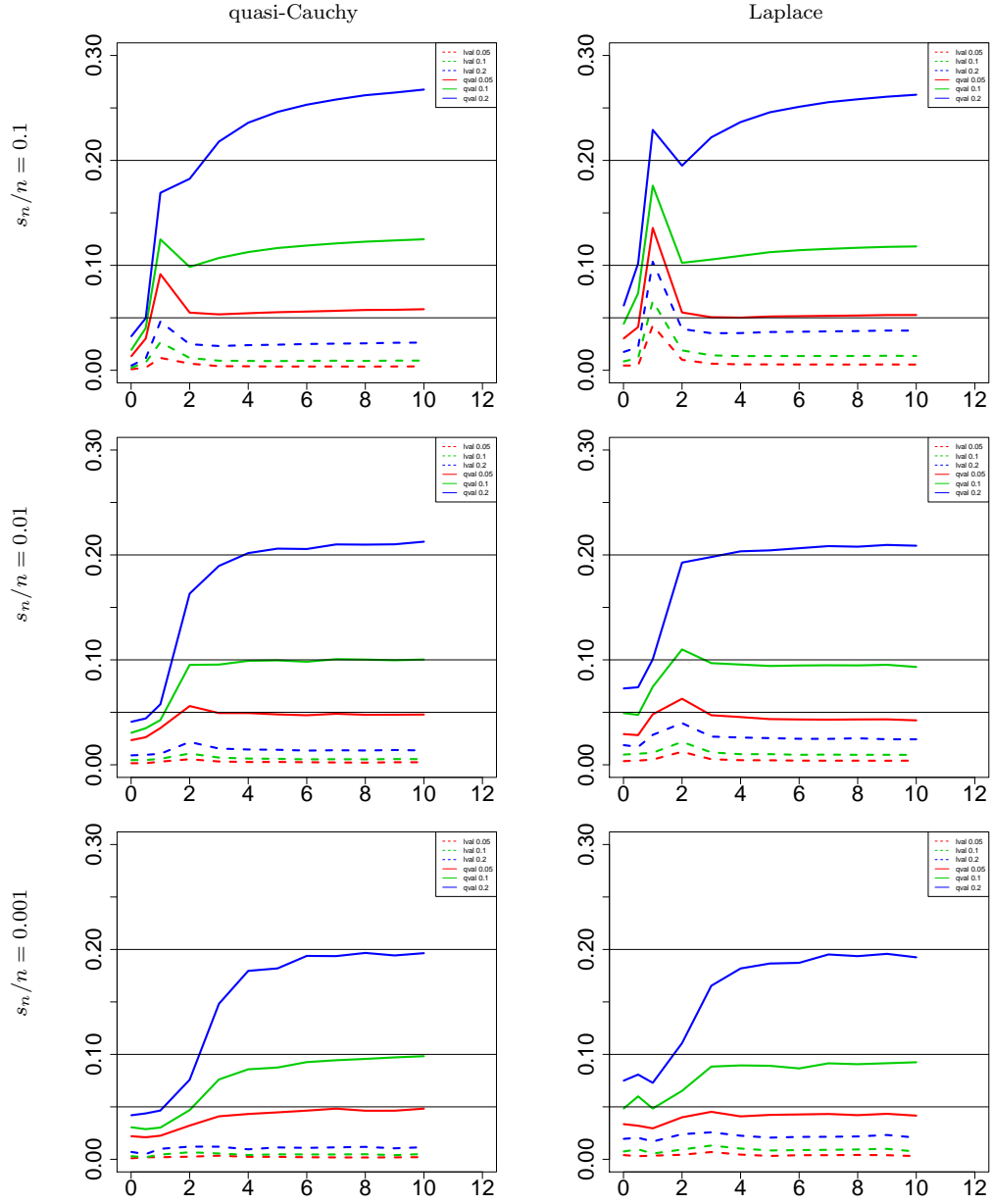


FIG S-5. FDR of EBayesL and EBayesq procedures with threshold $t \in \{0.05, 0.1, 0.2\}$. $\alpha = 0.2$; $n = 10,000$; 2000 replications; alternative values i.i.d. uniformly drawn into $[0, 2\mu]$ (μ on the X-axis).

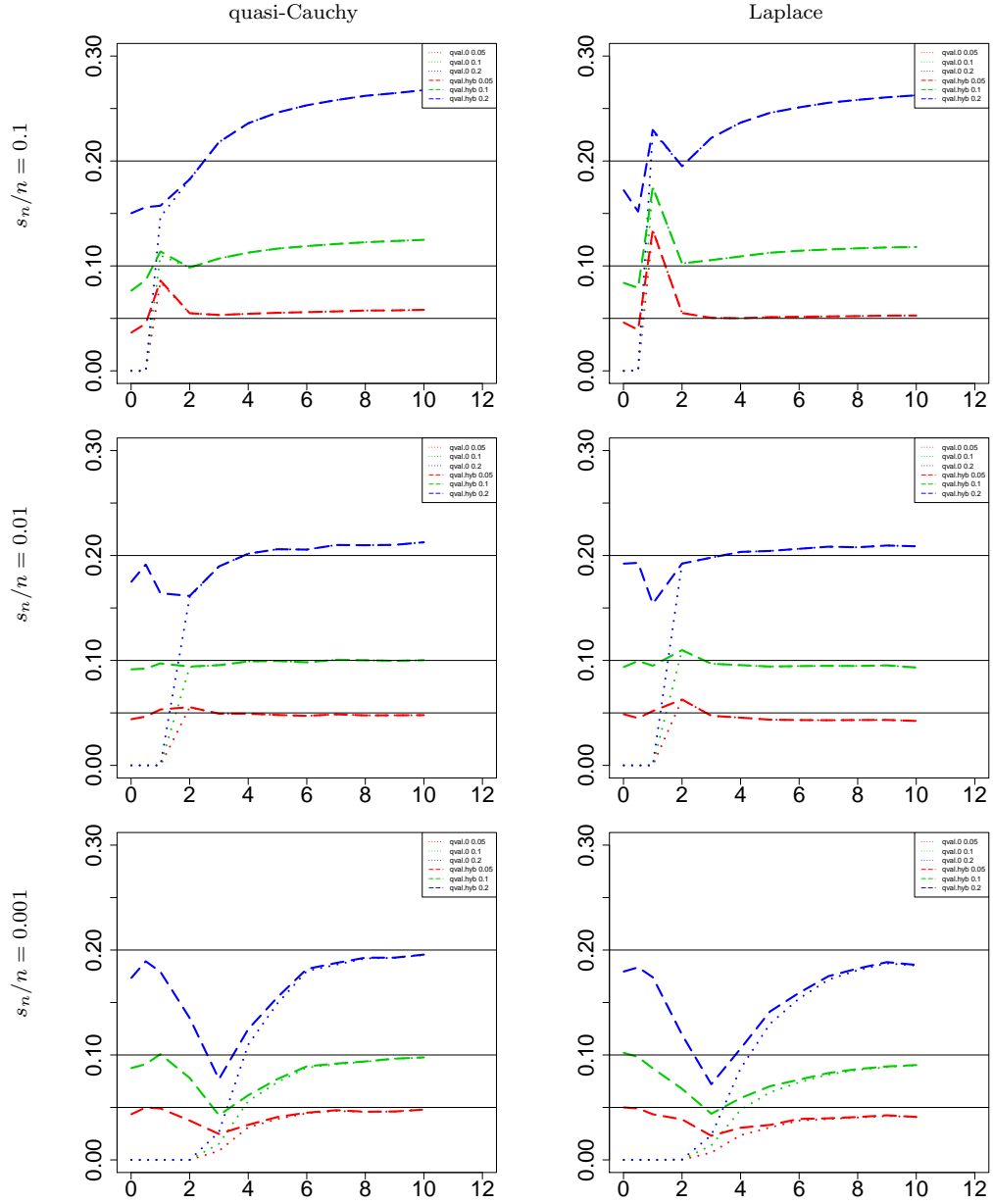


FIG S-6. FDR of EBayesq.0 and EBayesq.hybrid procedures with threshold $t \in \{0.05, 0.1, 0.2\}$. $\alpha = 0.2$; $n = 10,000$; 2000 replications; alternative values *i.i.d.* uniformly drawn into $[0, 2\mu]$ (μ on the X-axis).

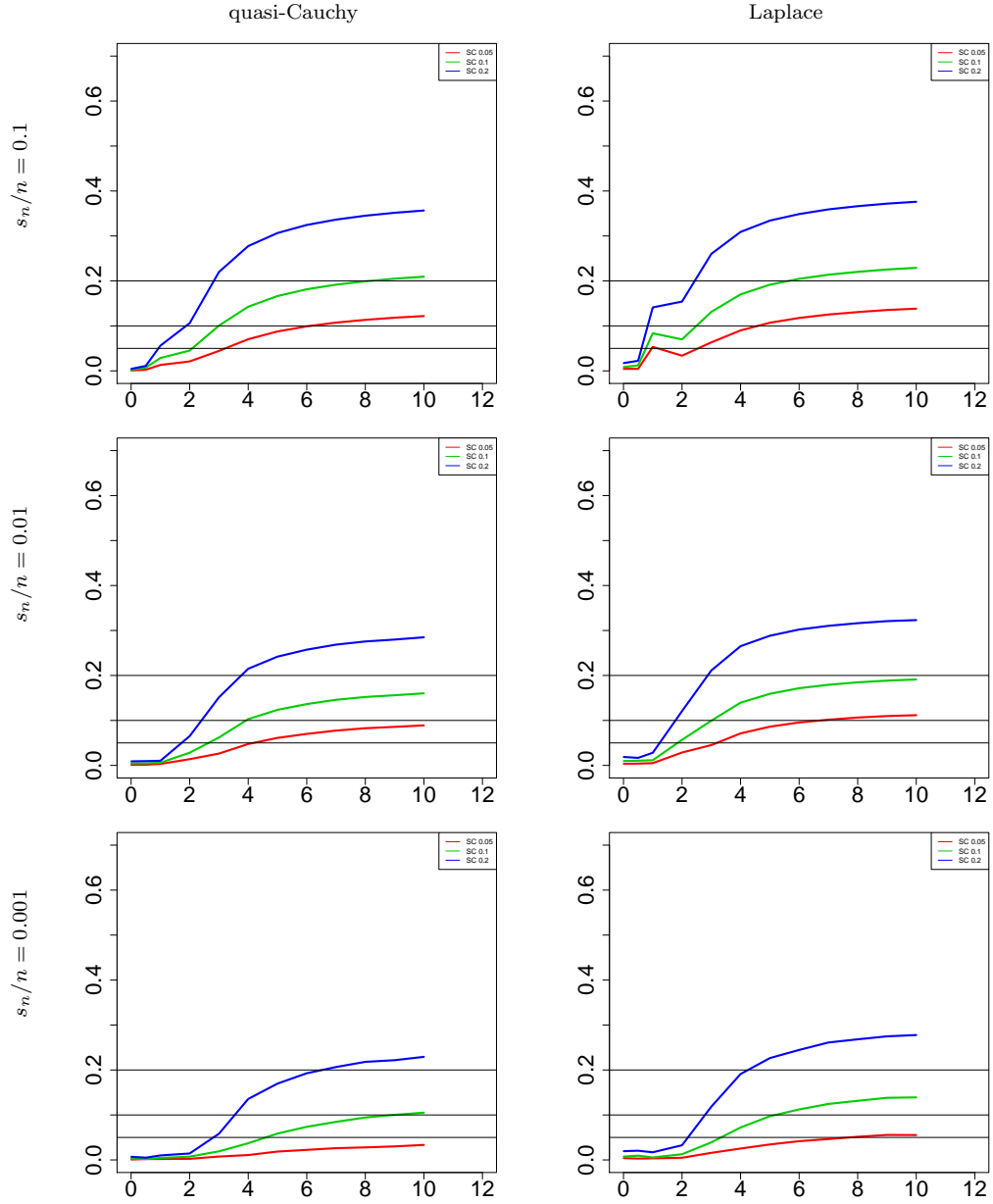


FIG S-7. FDR for SC procedure with threshold $t \in \{0.05, 0.1, 0.2\}$. $\alpha = 0.2$; $n = 10,000$; 2000 replications; alternative values *i.i.d.* uniformly drawn into $[0, 2\mu]$ (μ on the X -axis).