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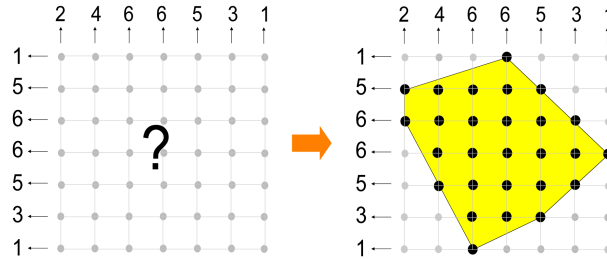
# Polynomial Time Reconstruction of Regular Convex Lattice Sets from their Horizontal and Vertical X-Rays

Yan Gérard

LIMOS, University Clermont Auvergne, France  
yan.gerard@uca.fr

**Abstract.** We consider a problem of Discrete Tomography that has been open for 20 years: the reconstruction of convex lattice sets from their horizontal and vertical *X-rays* (X-rays is the mathematical term for the number of points of a set in a sequence of consecutive lines). We prove that it can be solved in polynomial time for the subclass of the *regular* lattice sets. Regularity is a property related to the relative position of the points of the set with extreme abscissa and ordinate. This algorithm that we call **ConvexTomo** follows the classical strategy initiated by E. Barucci et al. for the reconstruction of horizontally and vertically convex 4-connected lattice sets. The approach introduced for the reconstruction of this class of lattice sets can be adapted until the creation of combinatorial structures called *switching components*. They are used to express horizontal and vertical convexity as a conjunction of 2-clauses. Then polynomial time algorithms solving 2-SAT provide polynomial time algorithms of reconstruction. The difficulty to overcome is that convexity (and no more directional convexities) requires 3-clauses which makes this approach no more polynomial.

In this paper, we present a new approach encoding the research of a convex configuration of the switching components in the research of a path between two sets of vertices in a Directed Acyclic Graph. This reduction passes through the introduction of a new class of problems of computational and discrete geometry that we call *Convex Aggregation*: given a convex lattice set  $A \subset \mathbb{Z}^2$  and an ordered finite family of lattice sets  $B^i \subset \mathbb{Z}^2$  called *blocks* (blocks are around  $A$ ), does there exist a non empty subset of blocks such that their union with  $A$  remains convex? We reduce the question to the research of a path connecting two sets of vertices in a Directed Acyclic Graph. Then we investigate its variant related to the research of a convex configuration of the switching components. This problem is made of four related problems of Convex Aggregation. We reduce it again in a more complex manner to the research of a path in Discrete Acyclic Graph. It provides the final step of the algorithm **ConvexTomo** with a polynomial time complexity whereas the clauses approaches might be exponential.



**Fig. 1. Considered problem of Discrete Tomography:** Find a convex polygon with given numbers of interior lattice points on the horizontal and vertical lines. The solution is a convex lattice set.

1 **Keywords:** Digital Geometry, Discrete Tomography, Convex lat-  
 2 **tice sets, Filling operations, Switching Component, 3-SAT, Di-**  
 3 **rected Acyclic Graph**

#### 4 **1 Introduction**

##### 5 **1.1 About Discrete Tomography**

6 In the mid 1990s, researchers in Material Science and especially in three  
 7 dimensional Electron Microscopy previewed the development of an up-  
 8 coming technology able to count the number of atoms of a material  
 9 crossed by a beam of straight lines [8]. Under the same principle than  
 10 Computerized Tomography, they intended to use this process in order to  
 11 reconstruct the 3D structure of different materials (proteins, crystals...)  
 12 with a very high level of precision. They started to use the algorithms  
 13 of Computerized Tomography well-known in Medical Imaging. They dis-  
 14 covered that these algorithms designed for the investigation of materials  
 15 at a scale where it can be assumed to be continuous were absolutely not  
 16 well-suited at a level where the set of atoms is closer to a discrete set  
 17 of points. The discrete nature of the objects to be reconstructed is the  
 18 first difficulty which makes CT algorithms ineffective at the atomic scale.  
 19 A second difficulty comes from the very low number of X-rays -from 2  
 20 to 10- which can be used in Material Science since the X-rays damage  
 21 the atomic structure. As comparison, CT-scans provide usually hundreds  
 22 of X-rays. The third difference with Computerized Tomography is that  
 23 for the reconstruction of the atomic structure of crystals (see [3, 26] for  
 24 crystalline structures of nano-particules computed with Discrete Tomog-  
 25 raphy in the 2010s years), the atoms are centered on a lattice so that the

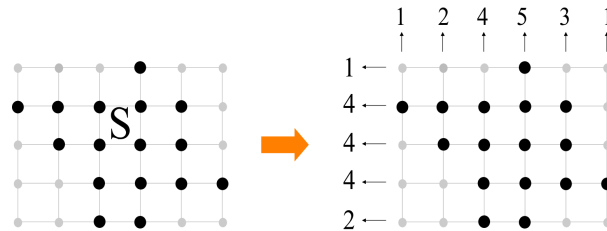
26 problem becomes the reconstruction of a lattice set, namely in dimension  
 27 2 a binary matrix.

28 The development of the technology for counting the number of atoms  
 29 on straight lines took finally more time than expected but the impulse  
 30 was given to explore this new range of questions dealing with the re-  
 31 construction of discrete sets of points. The sequence of cardinalities of  
 32 the intersections of a discrete set with consecutive parallel lines has been  
 33 called by keeping the physical term of *X-ray* while the reconstruction of  
 34 a discrete set from X-rays took the name of *Discrete Tomography* [15, 20,  
 35 21]. Due to the technical principle providing the measurements and the  
 36 complexity of the considered problems, a special attention has been given  
 37 on the problem in dimension 2.

## 38 1.2 Problem Statement

39 An *X-ray* is the sequence of the cardinalities of the intersection between  
 40 a given lattice set and the consecutive diophantine lines in a chosen direc-  
 41 tion. In the two-dimensional case of the vertical and horizontal directions,  
 42 it leads to the following definition:

43 **Definition 1.** *Given a finite lattice set  $S \subset [1..m] \times [1..n]$ , its vertical X-*  
 44 *ray  $V(S) \subset \mathbb{Z}^m$  is the vector of coordinates  $v_i(S) = |\{(x, y) \in S | x = i\}|$*   
 45 *for  $1 \leq i \leq m$  and its horizontal X-ray  $H(S) \subset \mathbb{Z}^n$  is the vector of*  
 46 *coordinates  $h_j(S) = |\{(x, y) \in S | y = j\}|$  for  $1 \leq j \leq n$  (Fig.2).*



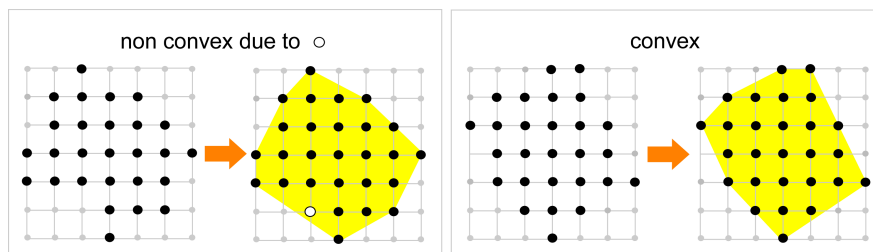
**Fig. 2.** The horizontal and vertical X-rays of the lattice set  $S$  are the vectors  $V(S) = (1, 2, 4, 5, 3, 1)$  and  $H(S) = (2, 4, 4, 5, 1)$ .

47 It leads to introduce a generic problem of Discrete Tomography. The  
 48 question is the existence of a lattice set with given X-rays and belonging  
 49 to a given class  $A$  of lattice sets:

50 *Problem 1* ( $DT_{\mathcal{A}}(h, v)$ ).  
 51 Given a class  $\mathcal{A}$  of finite lattice sets,  
 52 **Input:** two vectors  $v \in \mathbb{Z}^m$  and  $h \in \mathbb{Z}^n$ .  
 53 **Output:** does there exist a lattice set  $S \in \mathcal{A}$  included in the rectangle  
 54  $[1..m] \times [1..n]$  with  $V(S) = v$  and  $H(S) = h$  ?

55 The class  $\mathcal{A}$  is a parameter of Problem  $DT_{\mathcal{A}}(h, v)$ . We introduce the  
 56 class  $\mathcal{C}$  of convex lattice sets.

57 **Definition 2.** A lattice set  $S \subset \mathbb{Z}^d$  is convex if it is equal to its intersec-  
 58 tion with its real convex hull  $S = \text{conv}_{\mathbb{R}^d}(S) \cap \mathbb{Z}^d$  (Fig.3). The class of  
 59 the convex lattice sets is denoted  $\mathcal{C}$ .



**Fig. 3.** A convex lattice set is equal to the intersection of its convex hull (in yellow) with the lattice  $\mathbb{Z}^2$ .

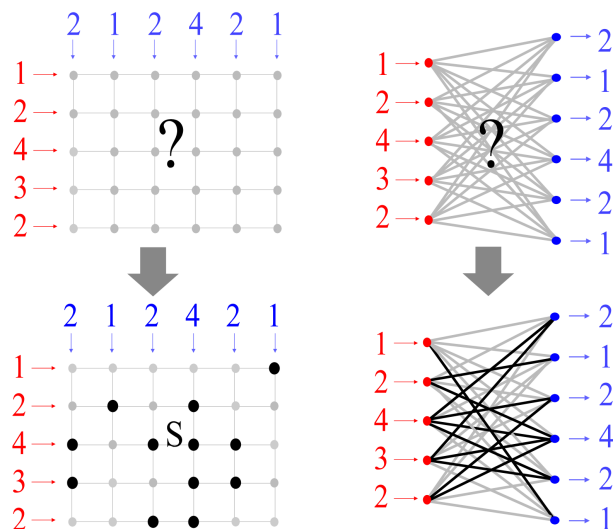
60 In other words, the convex lattice sets are the intersections of con-  
 61 vex polygons with the lattice  $\mathbb{Z}^2$ . The complexity of their reconstruction  
 62  $DT_{\mathcal{C}}(h, v)$  is a twenty years old open question (Fig.1). The purpose of the  
 63 paper is to break the status quo and provide a partial answer opening  
 64 new perspectives.

## 65 2 State of the Art

66 While Computerized Tomography has been stated on the prior works of J.  
 67 Radon (1917) [24] or Fourier Analysis [4], Discrete Tomography found its  
 68 basis in results of D. Gale and H.J. Ryser (1957) [13, 25] or in the more  
 69 general theory of flows in networks by L.R. Ford and D.R. Fulkerson  
 70 (1956) [12].

71 According to these fundamental results, if we consider the whole class  
 72 denote  $\mathcal{W}$  of all lattice sets, the problem  $DT_{\mathcal{W}}(h, v)$  can be solved in

73 polynomial time [13, 25]. Another way to consider the problem is on a set  
 74 of edges of a complete bipartite graph. It can be solved by any max-flow  
 75 algorithm (Fig.4).



**Fig. 4. Classical problem of reconstruction.** The problem of reconstruction of a lattice set with prescribed horizontal and vertical X-rays can be reformulated in terms of flows in a bipartite graph. It can also be solved with the polynomial time algorithm of H.J. Ryser [25].

76 Many variants of this problem have been investigated, not only with  
 77 horizontal and vertical X-rays but in different dimensions, with different  
 78 directions of X-rays and different kinds of atoms. In dimension 3, the  
 79 problem is related with timetables or data security. Both variants are  
 80 NP-complete [11, 22] and this extension is related with multi-commodity  
 81 flow problems [18]. The reconstruction of sets with different kinds of atoms  
 82 can also be considered with one X-ray per type of material. The problem  
 83 becomes again NP-hard from two different kinds of atoms [17, 10]. Still  
 84 in dimension 2, the number of X-rays can be increased with a result of  
 85 NP-completeness from three X-rays [16].

86 In the two-dimensional case with horizontal and vertical X-rays, the  
 87 complexities of the problem  $DT_{\mathcal{A}}(h, v)$  have already been determined  
 88 for many classes  $\mathcal{A}$ . The problem is NP-complete for the class of the 4-  
 89 connected lattice set (4-connected finite subsets of  $\mathbb{Z}^2$  are called *polyomi-*

90 *noes* and their class is denoted  $\mathcal{P}$ ). We have also results with directional  
 91 convexities. By denoting  $\mathcal{H}$  the class of horizontally convex (H-convex)  
 92 lattice sets i.e with consecutive points in any row (Fig.??), the problem  
 93 is NP-complete [2]. We have of course the same result for the class de-  
 94 noted  $\mathcal{V}$  of the vertically convex (V-convex) lattice sets. If we consider  
 95 the class of the horizontally and vertically convex (HV-convex) lattice  
 96 sets  $\mathcal{H} \cap \mathcal{V}$ , the problem remains again NP-complete [27] (Fig.??). No-  
 97 tice that the reconstruction of HV-convex lattice sets is a particular case  
 98 of puzzle games called nonograms. In summary, while the initial prob-  
 99 lem  $DT_{\mathcal{W}}(h, v)$  without complementary constraints on the solutions can  
 100 be solved in polynomial time, all the variants  $DT_{\mathcal{A}}(h, v)$  with  $\mathcal{P}$ ,  $\mathcal{H}$ ,  $\mathcal{V}$ ,  
 101  $\mathcal{H} \cap \mathcal{V}$  as class  $\mathcal{A}$  are NP-complete. These complexities in  $NP$  are how-  
 102 ever counter-balanced by two major results of the field published in two  
 103 seminal papers [2, 14].

- 104 – Horizontally and vertically 4-connected subsets of  $\mathbb{Z}^2$  can be recon-  
 105 structed in polynomial time:  $DT_{\mathcal{A}}(h, v)$  is polynomial for the class  
 106  $\mathcal{A} = \mathcal{H} \cap \mathcal{V} \cap \mathcal{P}$  [2].
- 107 – On the other side, results of uniqueness have been obtained for the  
 108 class  $\mathcal{C}$  of convex lattice sets with different number of directions of  
 109 X-rays. R. Gardner and P. Grizmann characterized the sets of  $d$  di-  
 110 rections for which any convex lattice set is uniquely determined by  
 111 its X-rays [14]. For  $n = 2$  or  $n = 3$  directions, for any directions,  
 112 there exist ambiguous pairs or triplet of X-rays. For  $n \geq 7$  directions,  
 113 all convex lattice sets are uniquely determined by their X-rays. For  
 114  $3 < n < 7$ , the so-called cross-ratios of the directions provide a charac-  
 115 terization of the sets of direction providing uniqueness or ambiguous  
 116 X-rays [14]. With the directions of X-rays providing uniqueness, these  
 117 results have been completed by a polynomial time algorithm of recon-  
 118 struction [6]. This algorithm follows the same principle than the one  
 119 used for the reconstruction of HV-convex polyominoes [2].

Class $\mathcal{A}$	H and V X-rays	4 directions or more
$\mathcal{H} \cap \mathcal{V} \cap \mathcal{P}$ (HV-convex polyominoes)	$DT_{\mathcal{H} \cap \mathcal{V} \cap \mathcal{P}}(h, v)$ polynomial time [1]	another open question
$\mathcal{C}$ (convex lattice sets)	$DT_{\mathcal{C}}(h, v)$ open question	polynomial time (if uniqueness) [14, 6]

Table 1. Milestones results

120 The problem  $DT_{\mathcal{C}}(h, v)$  that we consider in the paper is very close  
 121 to the two milestones results (Tab.1). It deals with very simple objects,  
 122 convex lattice sets, and the most simple directions of X-rays: horizontal  
 123 and vertical. After twenty years of silence, the question of its complexity  
 124 became recently subject of a new attention [9].

125 The complexity of  $DT_{\mathcal{C}}(h, v)$  has not been yet determined because the  
 126 principles used for providing the polynomial time algorithms of Tab.1 [2,  
 127 6] do not hold. First, we don't have the uniqueness property used for the  
 128 polynomial time algorithm from 4 directions of X-rays [6]. There are many  
 129 ambiguities expressed by boolean variables. Secondly, the combinatorial  
 130 expression of the convexity constraint requires 3-clauses whereas HV-  
 131 convexity is expressed by a conjunction of 2-clauses which can be solved  
 132 in a polynomial time.

## 133 2.1 Main Result

134 We prove in the paper that the problem  $DT_{\mathcal{C}}(h, v)$  of the reconstruction  
 135 of convex lattice sets with given X-rays can be solved in polynomial for  
 136 the subclass of the *regular* convex lattice sets (Fig.6). Regularity is re-  
 137 lated with the positions of the points of the lattice set with minimal and  
 138 maximal abscissa and ordinate. These extreme points are the *feet* of the  
 139 lattice set (Fig.5).

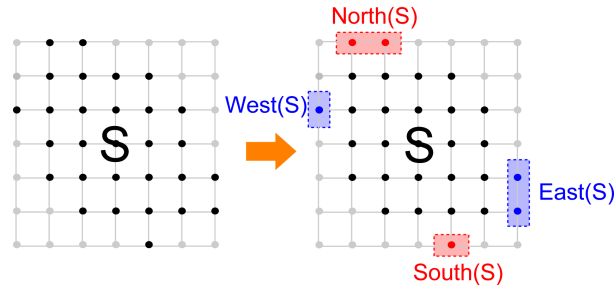


Fig. 5. The four feet of a lattice set  $S$  are denoted South, West, North and East.

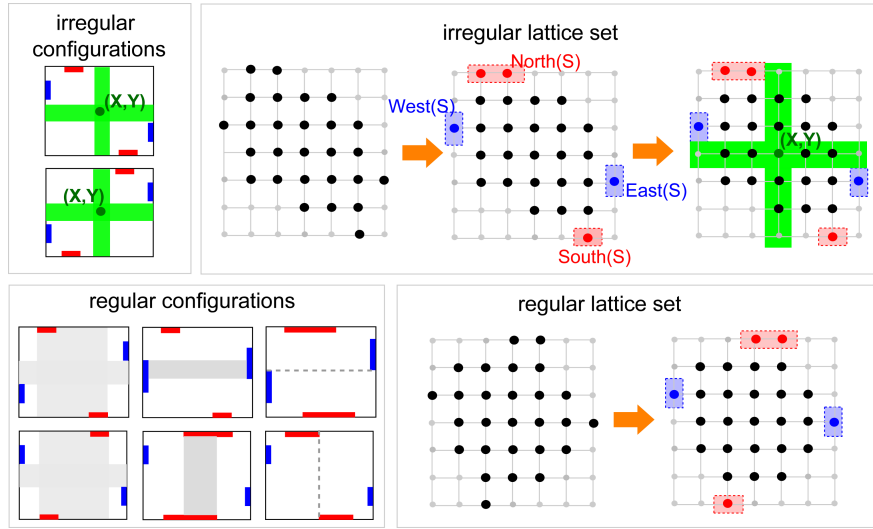
140 **Definition 3.** Given a lattice set  $S \subset \mathbb{Z}^2$ , the South, West, North and  
 141 East feet are its four subsets

142  $\text{South}(S) = \{(x, y) \in S \mid \forall (x', y') \in S, y' \geq y\},$

143  $\text{West}(S) = \{(x, y) \in S \mid \forall (x', y') \in S, x' \geq x\},$



144  $\text{North}(S) = \{(x, y) \in S \mid \forall (x', y') \in S, y' \leq y\}$ ,  
 145  $\text{East}(S) = \{(x, y) \in S \mid \forall (x', y') \in S, x' \leq x\}$ .  
 146 The lattice set  $S$  is said irregular if there exists  $(X, Y) \in \mathbb{Z}^2$  verifying  
 147 either  $x(\text{South}(S)) < X < x(\text{North}(S))$  and  $y(\text{West}(S)) < Y < y(\text{East}(S))$ ,  
 148 or  $x(\text{South}(S)) > X > x(\text{North}(S))$  and  $y(\text{West}(S)) > Y > y(\text{East}(S))$ .  
 149 Otherwise,  $S$  is said regular (Fig.6).



**Fig. 6. Regular lattice sets** are the lattice sets such that there exists no integer point  $(X, Y)$  (represented by the green cross) separating the pairs of feet in opposite corners.

150 The class of the regular lattice sets (Fig.6) is denoted  $\mathcal{R}$  (then the  
 151 class of the regular convex lattice sets is  $\mathcal{C} \cap \mathcal{R}$ ). The main result of the  
 152 paper is the following theorem:

153 **Theorem 1.** *The algorithm `ConvexTomo` solves  $DT_{\mathcal{C} \cap \mathcal{R}}(h, v)$  with a worst  
 154 case time complexity in  $O(m^4 n^4 + m^{11} n^2)$  where we can assume  $m \leq n$ .*

155 The time complexity of `ConvexTomo` is high but polynomial.

## 156 2.2 A Strategy Passing through Intermediary Results on a 157 New Convex Aggregation Problem

158 The algorithm `ConvexTomo` follows the guidelines of the three first steps  
 159 of the classical polynomial time algorithm designed for the reconstruction

160 of HV-convex polyominoes [2]. The first step fixes the feet in a regular  
 161 position (there are a polynomial number of possible positions). Under this  
 162 assumption, the second step determines some points belonging to all the  
 163 solutions (the set In) and some other points which can be excluded from  
 164 all solutions (the set Out). It ends while the so-called *filling operations* do  
 165 not allow to determine any new point or if a contradiction follows from a  
 166 non empty intersection  $\text{In} \cap \text{Out} \neq \emptyset$  (in this case, the considered position  
 167 of the feet admits no solution).

	<b>Data:</b> $m, n, h \in \mathbb{Z}^n, v \in \mathbb{Z}^m$	
	<b>Result:</b> Regular convex lattice set $\text{In} \subset [1, m] \times [1, n]$ verifying $H(\text{In}) = h$ and $V(\text{In}) = v$	
1	<b>for</b> <i>regular configuration</i> South, East, North, West <b>do</b>	
2	/* Step 1 - Initialization of the feet	*/
3	Out $\leftarrow \text{RectangleBorder} \setminus (\text{South} \cup \text{East} \cup \text{North} \cup \text{West})$	
4	In $\leftarrow \text{conv}_{\mathbb{R}^2}(\text{South} \cup \text{East} \cup \text{North} \cup \text{West}) \cap \mathbb{Z}^2$	
5	NW $\cup$ NE $\cup$ SE $\cup$ SW $\leftarrow \text{Decomposition}_{\text{In}}(\text{Shell})$	
6	/* Step 2 - Filling operations	*/
7	FillTomo( $m, n, h, v, \text{NW}, \text{NE}, \text{SE}, \text{SW}, \text{In}, \text{Out}$ )	
8	/* Step 3 - Compute Extended Switching Components	*/
9	Extended Switching Components $\leftarrow \text{SwitchingComponents}(m, n, h, v, \text{In}, \text{Out})$	
10	/* Step 4 - Convex Aggregation	*/
11	In $\leftarrow \text{ConvexAggregation}(m, n, v, h, \text{NW}, \text{NE}, \text{SE}, \text{SW}, \text{In}, \text{Extended Switching Components})$	
12	<b>end</b>	

**Algorithm 1:** Main algorithm ConvexTomo( $m, n, h, v$ )

168 The set of the remaining undetermined points is a shell between the  
 169 points of In and the ones of Out. It is denoted Shell. It is the set of the  
 170 points which might belong to some solutions and be outside from oth-  
 171 ers. These ambiguities are formalized with a decomposition of the shell  
 172 in combinatorial structures called *switching components*. After the fill-  
 173 ing operations, the prescribed X-rays are guaranteed but it remains to  
 174 determine a configuration of the switching components providing a con-  
 175 vex lattice set. Unfortunately, the classical approach encoding convexities  
 176 with clauses is unable to provide a polynomial algorithm.

177 This combinatorial difficulty is the main challenge to overcome. Our  
 178 main contribution is a new approach which allows to solve it in polyno-  
 179 mial time in the case of the regular lattice sets. This result requires first  
 180 recent results on regular switching components stated in Property 2 [19].  
 181 It requires also a better understanding of the combinatorial problem. It

182 can be seen as a specific problem of *Convex Aggregation* of either the odd  
183 parts, or the even parts of the switching components to the set In. This  
184 problem being quite hard to handle, we provide first results on a more  
185 generic problem of same nature:  
186 Given a finite convex lattice set  $A \subset \mathbb{Z}^2$  and  $q$  finite lattice sets  $B^i$  called  
187 blocks, is it possible to add blocks to In so that their union remains  
188 convex?

189 It is the problem denoted *ConvexAggregation*( $A, B^i$ ). It is stated  
190 more precisely in Sect.4 in Problem 2. Under the assumption that the  
191 blocks  $B^i$  can be ordered (their rows and columns are ordered), we pro-  
192 vide a polynomial time algorithm for solving it although its expression  
193 with boolean variables requires again 3-clauses. The result is stated in  
194 Theorem 2. The main idea is that convexity can be controlled through  
195 the local property that all the turning angles of the border have the same  
196 orientation. The only information that we need to keep in memory for  
197 building a convex contour is the last edge. This deep property allows to  
198 solve *ConvexAggregation*( $A, B^i$ ) in polynomial time by reducing it to  
199 the research of a path connecting two subsets of vertices in a Directed  
200 Acyclic Graph (DAG) (Property 3).

201 If we come back to the problem of Discrete Tomography  $DT_{\mathcal{C} \cap \mathcal{R}}(h, v)$ ,  
202 it can be seen as four related problems of Convex Aggregation. We build  
203 one DAG per problem and call them the *slave* DAGs. Then the relations  
204 between the four solutions that we search for are controlled by building  
205 a fifth DAG that we call *master* DAG. It allows to encode the existence  
206 of a solution of  $DT_{\mathcal{C} \cap \mathcal{R}}(h, v)$  with the considered feet in the existence of  
207 a path joining two regions of the master DAG (Theorem 3). It provides  
208 the fourth and final step of the algorithm **ConvexTomo** reconstructing a  
209 convex solution of  $DT_{\mathcal{C} \cap \mathcal{R}}(h, v)$  in polynomial time if there exists one.

210 The benefit of the paper is to solve this highly non trivial combinato-  
211 rial problem  $DT_{\mathcal{C} \cap \mathcal{R}}(h, v)$  with a completely new approach. The reduction  
212 of the problem of Convex Aggregation to the research of path in a DAG  
213 (Property 3) is an intermediary result which has its own interest.

### 214 **2.3 Plan**

215 The polynomial time algorithm **ConvexTomo** is presented in the following  
216 order. We start with the presentation of its three first steps in Sec.3. Then,  
217 the problem can be reformulated into a particular question of Convex  
218 Aggregation. Before solving it, we need first to investigate a more simple  
219 problem of the same kind. Sec.4 is devoted the presentation of generic  
220 Convex Aggregation and its reduction to the research of a path in a DAG

221 (Property 3). We adapt the approach for the fourth and last step of the  
 222 algorithm `ConvexTomo` with a master DAG and four slave DAGs in Sec.5.  
 223 Its final complexity analysis provides a proof of Theorem 1.

### 224 **3 ConvexTomo - Steps 1/2/3 - From X-rays to Extended** 225 **Switching Components**

226 Following the guidelines of the original algorithm presented in [2] for  
 227  $DT_{\mathcal{H} \cap \mathcal{V} \cap \mathcal{P}}(h, v)$ , we start by fixing the position of the feet.

#### 228 **3.1 Step 1 - Fixing the Position of the Feet and Initialization**

229 The algorithm `ConvexTomo` investigates all the regular configuration of  
 230 the feet. One of them being chosen, we can determine that some points  
 231 are necessary in a solution, if it exists one, while others can be excluded  
 232 from any solution. This principle is formalized by working with a partition  
 233 of  $[1..m] \times [1..n]$  in three sets of points:

- 234 – The set `In` contains the points which are known to belong to all solu-  
 235 tions.
- 236 – The set `Out` contains the points which are known to be excluded from  
 237 all solutions.
- 238 – The set `Shell` is the set of the undetermined points.

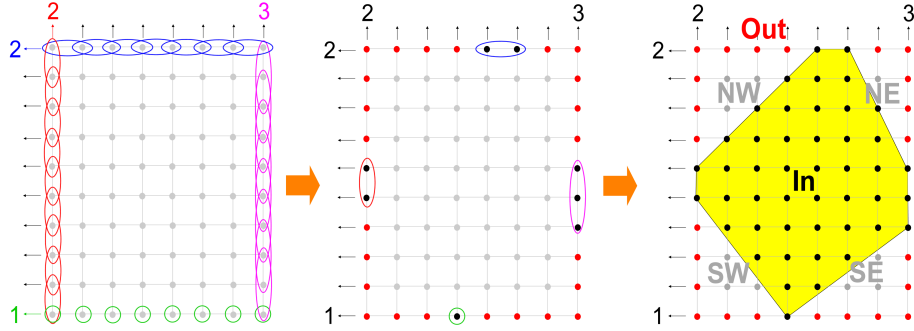
239 Once that the feet have been fixed, we add in `Out` the points with  $x = 1$   
 240 or  $x = m$  or  $y = 1$  or  $y = n$  which are not in the feet since under the  
 241 assumption of the considered feet, we are sure that they don't belong to  
 242 any solution. As we search for a convex solution, we initialize the set `In`  
 243 with the convex hull of the feet (Fig.7).

244 The convex hull of `In` provides a partition of the undetermined points  
 245 (the shell) in four subsets: the North West, North East, South East and  
 246 South West borders. They are respectively denoted NW, NE, SE and SW  
 247 (Fig.7). The assignation of the points of the shell to its four borders NW,  
 248 NE, SE and SW is done by the function denoted `DecompositionIn`(Shell)  
 249 in Alg.1.

250

#### 251 *Complexity Analysis*

252 There are at most  $m - 1$  possible positions for each one of the South  
 253 and the North feet and  $n - 1$  cases for the West and East feet. It makes  
 254 less than  $m^2 n^2$  configurations of the feet to explore.



**Fig. 7. Fix the feet and Initialize In, Out, NW, NE, SE and SW.** On the left, the different possible positions of the feet. We do not consider the irregular configurations. In the middle, a regular position of the feet is chosen. The other points with  $x = 1$  or  $x = m$  or  $y = 1$  or  $y = n$  are added in Out (in red). The shell (in grey) is the set of the points which are not yet determined. On the right, due to convexity, we can add to In not only the four feet but directly the lattice points of their convex hull. The shell is thus decomposed in four subsets NW, NE, SE and SW according to their position relatively to the convex hull of In.

255 With the configuration of the feet, the computation of the convex hull  
 256 of In takes a constant time. The assignation of the points of the lattice  
 257 to In, Out, NW, NE, SE and SW requires less than  $O(mn)$  operations.

258 **Proposition 1.** *The initialization takes  $O(mn)$  time.*

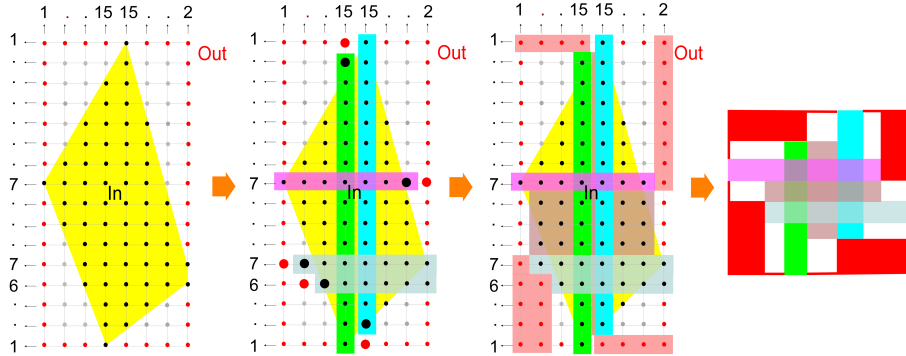
### 259 3.2 Step 2 - Filling Operations - FillTomo

260 Filling operations are widely used in Discrete Tomography and we refer  
 261 to [7, 6] for a more complete presentation of the operations with suitable  
 262 data structures.

263 Starting from the feet, according to the X-rays, the first run of the  
 264 filling operations fills directly the rows of the West and East feet and  
 265 the columns of the South and North feet so that it does not remain any  
 266 undetermined points on these lines. They are all either in In, or in Out.  
 267 Notice that just with HV-convexity (we recall that convexity implies HV-  
 268 convexity), a whole part of the lattice can be quickly determined (Fig.8).

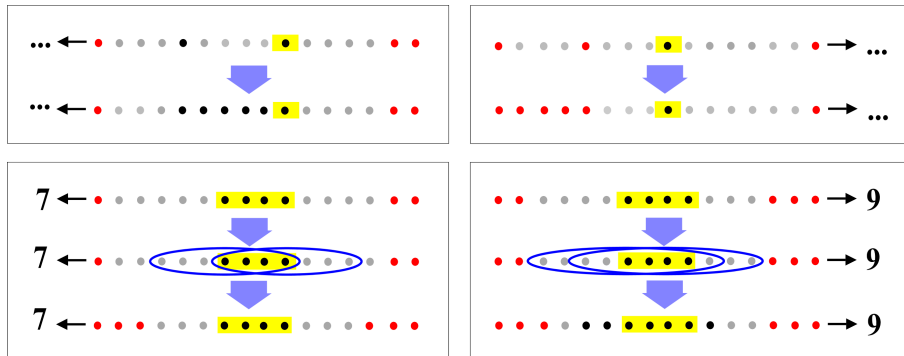
269 A run of the filling operations is organized as follows.

- 270 1. Fill the rows with the procedure `FillRows(In, Out, h)`. It includes the
- 271 four filling operations illustrated Fig.10.
- 272 2. Fill the columns with the procedure `FillColumns(In, Out, v)` under
- 273 the same principle than for rows.

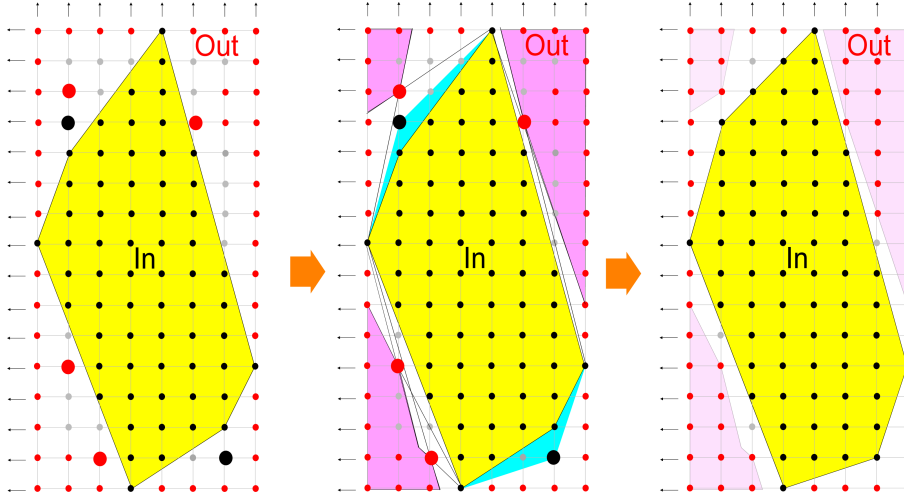


**Fig. 8. Filling from the feet.** Starting from the feet, the filling operations allow to fill directly an important part of the lattice. In one run, we obtain a figure that can be summarized in the right drawing, where the undetermined points are only in the white zones.

- 274 3. As we search for convex solutions, if In has been updated in the previous steps, we complete the update by replacing it by its discrete convex hull ( $\text{In} \leftarrow \text{convv}(\text{In}) \cap \mathbb{Z}^2$ ).
- 275
- 276
- 277 4. If In or Out have been updated in the three previous steps, we complete Out by all the hidden points (a point  $x$  is hidden if  $\text{conv}_{\mathbb{R}^2}(\{x\} \cup \text{In}) \cap \text{Out} \neq \emptyset$  - Fig.10).
- 278
- 279
- 280 5. Update the four borders SE, NE, NW and SW of the shell.



**Fig. 9. The filling operations** used in  $\text{FillRows}(\text{In}, \text{Out}, h)$  and  $\text{FillColumns}(\text{In}, \text{Out}, v)$ . Notice that even if there no point of In in a row, there is always some points of its convex hull (in yellow) that can be used in a similar way.



**Fig. 10. The filling operations** including the points of the convex hull of In and excluding the undetermined points hidden by a point of Out. On the left, we assume that new points of In and Out have been determined (the six large disks). Then we compute the new convex hull with the new points of In and use the tangent lines of the points of Out to compute the hidden zones (in pink).

281 We run the filling operations until falling in one of the two following  
 282 cases:

- 283 – We stop if a run does not allow to determine any new point. It is  
 284 for instance the case if we have no more undetermined point. In this  
 285 case, we just have to check the solution. We might also have a set  
 286 of undetermined points expressing the ambiguity of the input. They  
 287 express the possibility that different lattice sets can be solutions.
- 288 – We stop if a point of In is added in Out or conversely. This contradic-  
 289 tion means that the considered position of the feet does not provide  
 290 solutions.

291 *Complexity Analysis*

292 With suitable data structures, each run of the classical filling oper-  
 293 ations can be performed in  $O(mn)$  time [7, 6]. The complexity do not  
 294 differ for Algorithm FillTomo. We refer to [5] for the dynamic update of  
 295 the convex hull as well as the computation of the hidden points (through  
 296 the tangents). The time of computation of these two parts can be easily  
 297 bounded by  $O(mn)$  per new determined point.

298 As FillTomo stops if no new point is determined, the number of runs  
 299 is at most the number of undetermined points which is bounded by  $mn$ .

```

1 FillTomo Data:  $m, n, v \in \mathbb{Z}^m, h \in \mathbb{Z}^n$  and four borders In, Out, NW, NE, SE,
   SW
   Result: Sets In, Out, SE, NE, NW and SW under the assumption of the given
   feet
2 while SE, NE, NW or SW have been decreased do
3   /* Filling operations */
4   FillRows(In, Out, h);
5   FillColumns(In, Out, h);
6   In  $\leftarrow \text{conv}_{\mathbb{R}^2}(\text{In}) \cap \mathbb{Z}^2$ ;
7   Out  $\leftarrow \text{Shadow}_{I_n}(\text{Out}) \cap \mathbb{Z}^2$ ;
8   /* Remove new determined points from SE, NE, NW or SW */
9   Decrease(SE, NE, NW, SW, In, Out);
10  if In  $\cap$  Out  $\neq \emptyset$  then
11    | return("no solution");
12  end
13 end

```

**Algorithm 2:**  $\text{FillTomo}(m, n, v, h, \text{In}, \text{Out}, \text{NW}, \text{NE}, \text{SE}, \text{SW})$

300 Therefore, the overall time complexity of the function `FillTomo` is in  
301  $O(m^2n^2)$ .

302 **Proposition 2.** *Algorithm `FillTomo` requires  $O(m^2n^2)$  operations.*

### 303 **3.3 Step 3 - Computing the Switching Components -** 304 **SwitchingComponents**

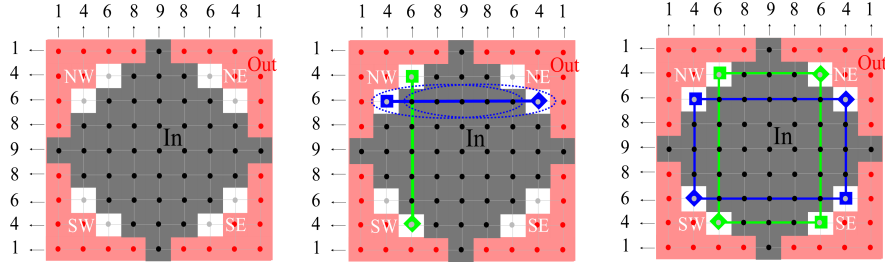
305 The third step of `ConvexTomo` occurs only if it remains undetermined  
306 points. Their set, the shell has a lot of properties. If an undetermined  
307 point  $\underline{p} = (i, j)$  is in a South border  $\text{SW} \cup \text{SE}$ , then the point denoted  
308  $\bar{p} = (i, j + v_i)$  is a North undetermined point (otherwise the filling op-  
309 erations would have determined  $(i, j)$ ). We define them as *vertical corre-*  
310 *spondents*. In the same way, any West undetermined point  $|p = (i, j)$  has  
311 an horizontal East correspondent  $p| = (i + h_j, j)$  (Fig.11). Horizontal and  
312 vertical correspondences are symmetric relations.

313 Their main property is the following:

314 *Property 1.* We consider an instance  $DT_{\mathcal{C}}(h, v)$  with a position of the feet  
315 leading to undetermined points at the end of the filling operations. For  
316 any solution  $S$  of  $DT_{\mathcal{C}}(h, v)$ , an undetermined point  $p$  is in  $S$  if and only  
317 if its correspondents are not in  $S$ .

318 *Proof.* Corresponding points cannot be both in  $S$  because their distance  
319 is  $h_j$  (horizontally) or  $v_i$  (vertically) would lead to have too many points





**Fig. 11. Corresponding points.** On the left, the undetermined points are drawn in grey in white cells. In the middle, a pair of vertical correspondents (green) and a pair of horizontal correspondents (blue). We represent the corresponding points alternatively with squares or diamonds. Notice that the segment represented by the dotted ellipse has only two possible positions. Due to its length, if it contains the square, it does not contain the diamond and conversely. On the right, the correspondences define closed paths called switching components which provide a partition of the undetermined points. For each switching components, either the squares, or the diamonds belong to a solution.

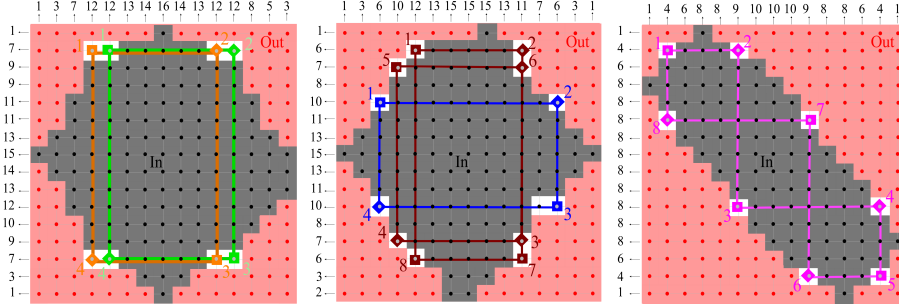
320 in the row or column of  $p$ . Conversely, they cannot be both outside from  
 321  $S$  because otherwise, it does not remain enough points between them to  
 322 have the prescribed number of points on its row or column.  $\square$

323 Starting from an undetermined point  $p_1$ , a sequence of correspondents  
 324 can be defined by induction: the point  $p_{2k}$  is the horizontal correspondent  
 325 of  $p_{2k-1}$  while  $p_{2k+1}$  is the vertical correspondent of  $p_{2k}$ . As the set of the  
 326 undetermined points is finite, the sequence is cyclic.

327 **Definition 4.** A switching component  $P$  is a closed path of alternatively  
 328 horizontal and vertical corresponding undetermined points (Fig.12).

329 The switching components provide a partition of the shell. Due to  
 330 Property 1, either the points with even indices, or the points with odd  
 331 indices belong to a solution  $S$ . This binary state of the switching compo-  
 332 nent with regard to a solution  $S$  can be encoded by a boolean variable  
 333 denoted  $P(S)$ . We choose  $P(S) = 1$  if the points with odd indices are in  
 334  $S$  and  $P(S) = 0$  otherwise.

335 The classical approach developed in [2] for reconstructing HV-convex  
 336 polyominoes is to search for an assignment of the boolean variables leading  
 337 to an HV-convex solution  $S$ . The approach passes through the encoding  
 338 of the HV-convexity constraint in a conjunction of 2-clauses. It leads to  
 339 a 2-SAT instance (Fig.13).



**Fig. 12. Switching components.** In each switching component, the squares represent the points with odd indices and the diamonds the ones with even indices. Either the squares, or the diamonds of a switching component are in a solution. In the two left cases, the feet are in a regular position, while we have an irregular position of the feet on the right. In this case, the switching components might have a different structure with turning sometimes clockwise and sometimes anticlockwise. Such switching components are said irregular but they don't occur in the regular case that we investigate.

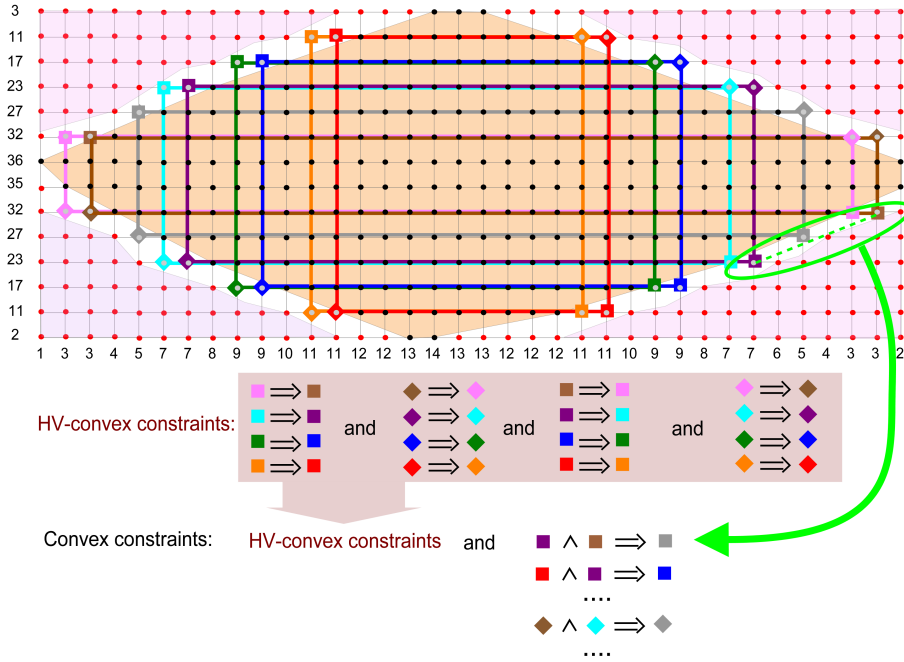
340 Convexity can be also be encoded with clauses (Fig.13) but 3-clauses  
 341 might be necessary with the difficulty that 3-SAT is no more polyno-  
 342 mial but NP-complete. This obstruction is the main difficulty of the re-  
 343 construction of convex lattice sets. The new approach presented in next  
 344 Sec.5 requires some properties proved very recently on regular switching  
 345 components [19]. We summarize these properties as follows.

346 *Property 2.* (i) With a regular position of the feet, all the switching com-  
 347 ponents have a constant turning angle (they are called regular).

348 (ii) By choosing  $p_1$  in NW (it is always possible), for any  $k$ , we have  
 349  $p_{1+4k} \in \text{NW}$ ,  $p_{2+4k} \in \text{NE}$ ,  $p_{3+4k} \in \text{SE}$ ,  $p_{4k} \in \text{SW}$ .

350 (iii) The switching components which have two points at Euclidean  
 351 distance 1 are considered as connected. This symmetric relation leads to  
 352 define the connected components of switching components that we call  
 353 extended switching components. The extended switching component of  
 354 a switching component  $P$  is denoted  $\bar{P}$ . HV-convexity enforces all the  
 355 switching component of  $\bar{P}$  to be equal to  $P$ . Either the points of the NW  
 356 and SE borders of  $\bar{P}$  (represented by squares in Fig.12) belong to  $S$ , or the  
 357 points of  $\text{NE} \cup \text{SW}$  (represented by diamonds) are in  $S$ . For the following,  
 358 we assume that the extended switching components are unstructured sets  
 359 obtained by union of the points of the connected sequences.

360 (iv) The points of different extended switching components can share  
 361 neither a row, nor a column.



**Fig.13. 2 and 3 clauses encoding convexity.** HV-convexity is expressed by a conjunction of 2-clauses. Thus, the research of an HV-convex solution is reduced to a 2-SAT instance that can be solved in polynomial time. Expressing convexity in the same manner, might require 3-clauses with the difficulty that 3-SAT is NP-complete.

362 (v) The switching components can be ordered according to their rows  
 363 (or equivalently according to their columns). They cannot interlace.

364 The algorithm `SwitchingComponents` decomposes the shell in switch-  
 365 ing components and then merges them in extended switching components.

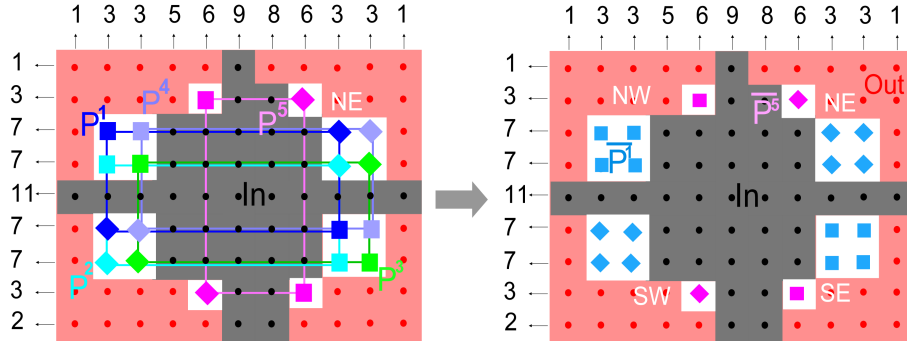
366 *Complexity analysis*

367 The computation is linear in the number of undetermined points. It  
 368 requires no more than  $O(mn)$  operations.

369 **Proposition 3.** *Algorithm `SwitchingComponents` takes  $O(mn)$  time.*

### 370 4 Problem of Convex Aggregation

371 The three first steps of Algorithm `ConvexTomo` led to a stage where any  
 372 assignment of the switching components provides the requested X-rays.



**Fig. 14. Extended switching components.** On the left, due to convexity, the switching components  $P^1, P^2, P^3, P^4$  are necessarily equal while  $P^5$  is independent. On the right, we merge  $P^1, P^2, P^3, P^4$  in the extended switching component  $\bar{P}^1$  and  $P^5$  becomes  $\bar{P}^5$ . The extended switching components are on different rows and columns. They can also be ordered according to their rows (or columns).

373 The convexity of the solution is however not yet guaranteed. The ap-  
 374 proaches of the state of the art do not allow to overcome this difficulty.  
 375 Therefore, we explore a new class of problems dealing with Convex Ag-  
 376 gregation. We reduce it to a question of reachability in a Directed Acyclic  
 377 Graph. In the following section, we use this tool to provide the final step  
 378 of Algorithm ConvexTomo.

#### 379 4.1 Problem Statement

380 We consider a new problem of Convex Aggregation of blocks along a  
 381 convex set. The problem can be stated as follows:

382 *Problem 2 (ConvexAggregation( $A, B^i$ )).*

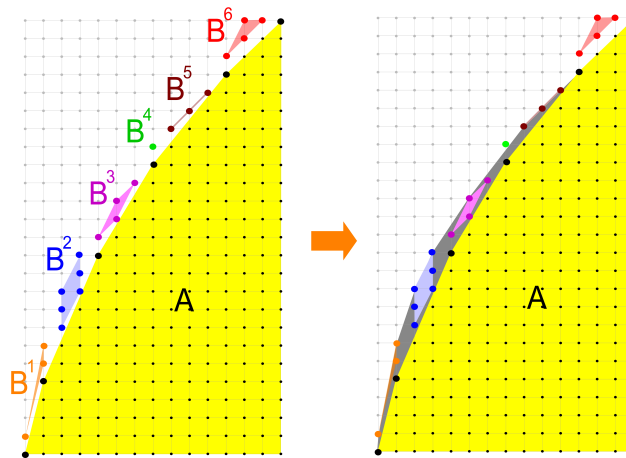
383 **Input:** - A convex lattice set  $A \subset [0..s] \times [0..t]$  with  $\{(0, 0), (s, t)\} \subset A$   
 384 - a finite sequence of  $q$  lattice sets that we call *blocks*  $B^i \subset [0..s] \times [0..t]$   
 385 disjoint from  $A$ , above the diagonal from  $(0, 0)$  to  $(s, t)$ , with increasing  
 386 abscissa and ordinates so that for any  $i < j$ ,  $(x, y) \in B^i, (x', y') \in B^j$ , we  
 387 have  $x < x'$  and  $y < y'$ .

388 **Output:** Does there exist a convex union  $A \cup (\cup_{i \in I} B^i)$  where  $I$  is a non  
 389 empty subset of the indices from 1 to  $q$  (Fig.15)?

390 We could also generalize the problem with a given non convex lattice  
 391 set  $A$ . In this case, the first step of an approach would be to fill the  
 392 non-convex parts of  $A$  with blocks and repeat this process until finding

393 a Convex Aggregation. There is no combinatorial difficulty with such an  
 394 instance. The problem becomes of interest if we start from a set which is  
 395 already convex and search for a non trivial Convex Aggregation of blocks.

396 We can assume without loss of generality that the set  $A$  contains the  
 397 point  $(s, 0)$  and fills the triangle below the diagonal from  $(0, 0)$  to  $(s, t)$ .  
 398 The reason is that the blocks  $B^i$  being above the diagonal, the edges of the  
 399 convex hull of  $A$  below the diagonal remain unchanged. The combinatorial  
 400 problem is above.



**Fig. 15. An instance of  $ConvexAggregation(A, B^i)$  and its solution.** On the left, the input is a convex lattice set  $A$  and a sequence of lattice sets  $B^i$ . On the right the output is a non empty union of  $B^i$  so that their union with  $A$  is still convex. It means that some of the sets  $B^i$  have to be aggregated with  $A$  and some others discarded (for instance the green and the red sets in the suggested solution).

401 We could encode the choice to add or reject a block  $B^i$  by a boolean  
 402 variable but as previously (Fig.13), the convexity is expressed by a con-  
 403 junction of 3-clauses that 3-SAT algorithms cannot necessarily solve in  
 404 polynomial time. It is the same difficulty than the one presented in the  
 405 framework of  $DT_C(h, v)$ .

#### 406 4.2 Rewriting $ConvexAggregation(A, B^i)$ with a Directed 407 Acyclic Graph

408 We provide a new approach by reducing the problem to the research of a  
 409 path joining two subsets in a Directed Acyclic Graph. Let us consider a

410 solution  $A \cup_{i \in I} B^i$  of the instance  $ConvexAggregation(A, B^i)$  as drawn in  
 411 Fig.15. The possible vertices of the upper hull of a solution are necessarily  
 412 upper vertices of  $A$  or upper vertices of a block  $B^i$ . The set of all these  
 413 upper vertices is denoted  $U$ . Then the set of the upper edges of a solution  
 414 is included  $U \times U$ . We orient them and call them *bipoints* in order to avoid  
 415 the possible ambiguities with other edges considered in the following.

416 Let us consider a bipoint  $\overrightarrow{vv'}$  between two vertices  $v =$   
 417  $(x_0, y_0)$  and  $v' = (x_1, y_1)$  in  $U$  with  $y_0 \leq y_1$ . We define its left  $\text{Left}(\overrightarrow{vv'})$   
 418 as the set of points  $p \in [0..m] \times [y_0..y_1]$ , verifying  $\det(\overrightarrow{vv'}, \overrightarrow{vp}) > 0$  (strict  
 419 inequality) while its right is the set of points  $p \in [0..m] \times [y_0..y_1]$  verifying  
 420  $\det(\overrightarrow{vv'}, \overrightarrow{vp}) \leq 0$  (large inequality) (Fig.16).

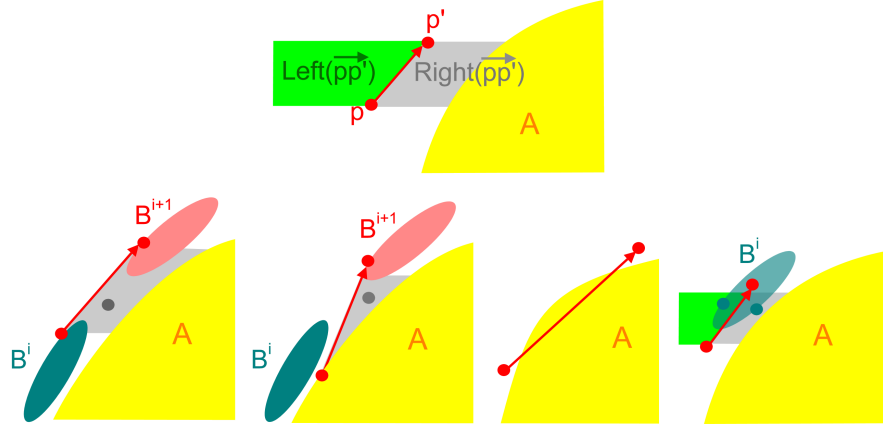
421 To complete the notations, given a block  $B^i$ , we denote  $H^i$  the minimal  
 422 horizontal strip containing all the points of the block ( $H^i = \{(x, y) |$   
 423  $\exists(x', y') \in B^i, \exists(x'', y'') \in B^i, y' \leq y \leq y''\}$ ). Notice that due to the  
 424 increasing assumption on the blocks in  $ConvexAggregation(A, B^i)$ , the  
 425 horizontal strips  $H^j$  are ordered and disjointed.

426 There are bipoints with vertices in  $U$  which can clearly not appear  
 427 in the contour of a solution of  $ConvexAggregation(A, B^i)$ . We want to  
 428 exclude them. We define the set  $V$  as the set of bipoints of  $U \times U$  obtained  
 429 by removing from  $U \times U$  any bipoint  $\overrightarrow{pp'} \in V \times V$  (Fig.16)

- 430 1. with  $x > x'$  or with  $x = x'$  and  $y > y'$  (we keep only the bipoints going  
 431 to the right or upward),
- 432 2. or having on its right (in  $\text{Right}(\overrightarrow{pp'})$ ) a lattice point which is not in  
 433  $A \cup_{1 \leq i \leq n} B^i$  (such point are called *outliers*),
- 434 3. or with a point of  $A$  on its left (in  $\text{Left}(\overrightarrow{pp'})$ ),
- 435 4. or with points of the same block  $B^i$  on its right ( $B^i \cap \text{Right}(\overrightarrow{pp'})$  non  
 436 empty) and on its left ( $B^i \cap \text{Left}(\overrightarrow{pp'})$  non empty).

437 Let us come back to a solution  $A \cup_{i \in I} B^i$  of the instance  $Convex$   
 438  $Aggregation(A, B^i)$ . The upper path connecting the origin to the point  
 439  $(s, t)$  is the concatenation of bipoints of  $V$ . Convexity is expressed by the  
 440 property that two consecutive bipoints  $\overrightarrow{pp'}$  and  $\overrightarrow{p'p''}$  turn always clockwise.  
 441 This condition is the key to build the DAG that we use to solve Problem  
 442  $ConvexAggregation(A, B^i)$  (Fig.17).

443 We have however to take care to avoid inconsistent concatenation of  
 444 bipoints as for instance drawn in Fig.17. A path which is passing on the  
 445 left of some points of a block  $B^i$  should not be able to pass further on the  
 446 right of  $B^i$ . A solution to forbid such inconsistent concatenation is first  
 447 to duplicate the bipoints included in the horizontal strip  $H^i$  of a block



**Fig. 16. The excluded bipoints.** Let us precise that, although the shapes drawn in this figure appear to be continuous, they represent discrete lattice sets. Above, the sets  $\text{Left}(\overrightarrow{pp'})$  and  $\text{Right}(\overrightarrow{pp'})$ . Below, we exclude the bipoints with an outlier on their right, with a point of  $A$  on their left, and with a pair of points of the same block on both sides. They are excluded because they are surely not contour bipoints of a solution of  $\text{ConvexAggregation}(A, B^i)$ .

448  $B^i$  but also, without any point of  $B^i$  neither on its left, nor on its right.  
 449 This case is illustrated in the lower part of Fig.17. Such duplication leads  
 450 to define a new set of bipoints that we denote  $V'$ . We complete now the  
 451 definitions:

452 **Definition 5.** Let  $U$  be the set of the upper vertices of the convex hulls  
 453 of the sets  $A$  and of the blocks  $B^i$ . Let  $p(x, y)$  and  $p'(x', y')$  be a pair of  
 454 vertices in  $U$ .

455 The set  $V$  is the set of bipoints  $\overrightarrow{pp'} \in V \times V$  satisfying the four condi-  
 456 tions:

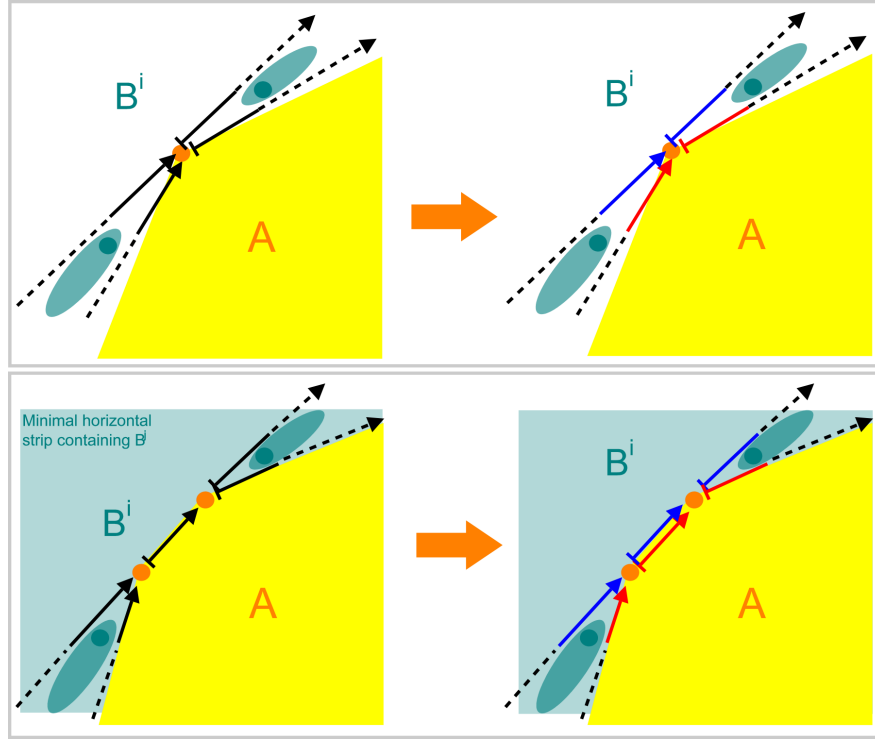
457 (i)  $y' > y$  or if  $y' = y$ ,  $x' > x$  where  $(x, y)$  and  $(x', y')$  are respectively the  
 458 coordinates of  $p$  and  $p'$ ,

459 (ii) there is no outlier on the right of  $\overrightarrow{pp'}$  ( $([0..s] \times [0..t] \setminus (A \cup_{1 \leq i \leq n} B^i)) \cap$   
 460  $\text{Right}(\overrightarrow{pp'}) = \emptyset$ ),

461 (iii) there is no point of  $A$  on the left of  $\overrightarrow{pp'}$  ( $A \cap \text{Left}(\overrightarrow{pp'}) = \emptyset$ ),

462 (iv) there are no pair of points of the same block  $B^i$  on the right and  
 463 on the left of  $\overrightarrow{pp'}$  (for any index  $i$ ,  $B^i \cap \text{Left}(\overrightarrow{pp'})$  or  $B^i \cap \text{Right}(\overrightarrow{pp'})$  is  
 464 empty).

465 The final set  $V'$  is obtained from  $V$  by duplicating the bipoints  $\overrightarrow{pp'}$  in-



**Fig. 17. Critical cases.** On the left, we have two configurations which might lead to inconsistent concatenation of bipoints with points of  $B^i$  on both sides of the path. We avoid this problem by adding labels (or colors). Labeling is sufficient to solve the case above, but not the one below. The reason is that the intermediary bipoint might be used for a path with the block  $B^i$  on its left or on its right (we did no assumption on the connectivity or convexity of the blocks). To avoid inconsistency, we duplicate this bipoints and provide a copy with the two possible labels (or colors).

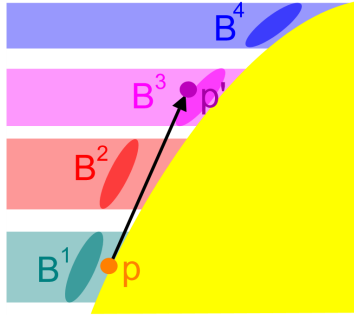
466 *cluded in the horizontal strip  $H^j$  of a block  $B^i$  but with no point of  $B^i$  in*  
 467  *$\text{Left}(\vec{pp}') \cup \text{Right}(\vec{pp}')$ . The copy is denoted  $\vec{pp}^*$ .*

468 We label now the bipoints of  $V'$  in order to avoid inconsistent paths.  
 469 The labels are registered in a vector of dimension  $q$  where  $q$  is the number  
 470 of blocks  $B^i$ .

471 **Definition 6.** *For any index  $i$  from 1 to  $q$ , the label of the bipoint  $\vec{pp}' \in V$*   
 472 *of index  $i$  is denoted  $\vec{pp}'[i]$  and we have:  $\vec{pp}'[i] = 1$  if  $B^i \cap \text{Right}(\vec{pp}')$  is*  
 473 *not empty,*  
 474  *$\vec{pp}'[i] = -1$  if  $B^i \cap \text{Left}(\vec{pp}')$  is not empty,*



475  $\overrightarrow{pp'}[i] = 0$  otherwise, except if the bipoint is included in the strip  $H^i$ . In  
 476 this specific case,  $\overrightarrow{pp'}[i] = -1$  for the original bipoint  $\overrightarrow{pp'}$  and  $\overrightarrow{pp'^*}[i] = 1$   
 477 for its copy.



**Fig. 18. Labels.** The labels of the bipoint  $\overrightarrow{pp'}$  are  $[-1, -1, 1, 0]$  since it passes to the right of the two previous blocks  $B^1$  and  $B^2$ , to the left of  $B^3$  and does not cross the horizontal strip  $H^4$  of  $B^4$ .

478 The labels allow to avoid the concatenation of inconsistent bipoints.  
 479 Concatenation of the bipoint  $\overrightarrow{pp'}$  with  $\overrightarrow{p'p''}$  is only accepted if they turn  
 480 clockwise and if  $\overrightarrow{pp'}[i]$  and  $\overrightarrow{p'p''}[i]$  are equal in the case where the point  $p'$   
 481 in the horizontal strip  $H^i$  of block  $B^i$ . As example, the concatenation of a  
 482 bipoint  $\overrightarrow{pp'}[i]$  of labels  $[0, 1, -1, 0, 0, 0]$  with  $\overrightarrow{p'p''}[i]$  of label  $[0, 0, 1, -1, 1, 0]$  is  
 483 excluded since the third label is different. The clockwise constraint guar-  
 484 antees the convexity of the path while the label consistency guarantees  
 485 that the blocks are either on the left, or on the right of a path obtained  
 486 by concatenation. Formally, we build the following DAG.

487 **Definition 7.** We consider the DAG  $G_{A, B^i} = (V', E)$ . Its vertices are  
 488 the bipoints of  $V'$ . We have an edge from the bipoint  $\overrightarrow{pp'} \in V'$  to the  
 489 bipoint  $\overrightarrow{p'p''} \in V'$  if  
 490 - the angle  $(\overrightarrow{pp'}, \overrightarrow{p'p''})$  turns clockwise ( $\det(\overrightarrow{pp'}, \overrightarrow{p'p''}) < 0$ ),  
 491 - and, if  $p'$  is in the horizontal strip  $H^i$  of the block  $B^i$ , then  $\overrightarrow{pp'}[i] =$   
 492  $\overrightarrow{p'p''}[i]$ .

493 We reduce the problem  $ConvexAggregation(A, B^i)$  to the research of  
 494 a path from a bipoint starting at the origin to a bipoint ending at point  
 495  $(s, t)$ .

496 *Property 3.* The set of bipoins  $\overrightarrow{pp'}$  where  $p = (0, 0)$  is denoted  $V^0$  while  
 497 the set of bipoins  $\overrightarrow{pp'}$  where  $p' = (m, n)$  is denoted  $V^1$ .

498 The instance  $ConvexAggregation(A, B^i)$  admits a solution if and only  
 499 if the DAG  $G_{A, B^i}$  admits a non trivial path from the set  $V^0$  to the set  $V^1$   
 500 (the trivial path is the convex border of  $A$  excluding all the blocks  $B^i$ .  
 501 All labels are  $-1$ ).

502 *Proof.* First, a solution of  $ConvexAggregation(A, B^i)$  has a convex path  
 503 from the origin to  $(s, t)$ . By taking its consecutive pairs of vertices as  
 504 bipoins, we have a sequence of bipoins turning clockwise, starting from  
 505 a bipoint with the origin in  $(0, 0)$  and going to a bipoint with its end in  
 506  $(s, t)$ . If they are all in  $V^1$  and verify the conditions of concatenation, they  
 507 provide a path going from  $V^0$  to  $V^1$  in the DAG. Both conditions follow  
 508 directly from the definitions.

509 Secondly, we consider a path going from  $V^0$  to  $V^1$  in the DAG  $G_{A, B^i}$ .  
 510 The concatenation condition on the labels guarantees that the path passes  
 511 either on the left, or on the right of any block. The orientation condition  
 512 (clockwise angle) guarantees the convexity of the path, and the condition  
 513 that each bipoint has no outlier on its right guarantees that the set of the  
 514 points on the right of the path is the union of  $A$  with the blocks providing  
 515 labels equal to  $+1$ .

516 It remains to notice that the trivial solution of  $ConvexAggregation(A,$   
 517  $B^i)$  corresponds to the excluded path with all labels equal to  $-1$ .  $\square$

518 According to Property 3, we can solve  $ConvexAggregation(A, B^i)$  by  
 519 searching for a non trivial path from  $V^0$  to  $V^1$  in the DAG  $G_{A, B^i}$ . Starting  
 520 from the bipoins of  $V^0$ , we use a depth-first search to determine if  $V^1$   
 521 is reached. At each vertex (bipoins) of the graph, we give the priority of  
 522 exploration to the following bipoins having a first non null label equal to  
 523  $+1$  or before  $-1$ . With this strategy, the trivial path is the last one to be  
 524 explored, so that the requested result is obtained before considering it.

525 *Complexity Analysis* The number of upper vertices of the sets  $A \subset$   
 526  $[0..s] \times [0..t]$  and of the disjointed blocks  $B^i \subset [0..s] \times [0..t]$  is at most  $s + s$ .  
 527 Therefore, the number of bipoins in  $V$  is in  $O(s^2)$ . With duplications, it  
 528 provides the number of vertices of the DAG in  $O(s^2)$ . It remains to count  
 529 the number of edges. Edges are no more than triangles. Their number is  
 530 in  $O(s^3)$ .

531 About the time to create the DAG  $G_{A, B^i}$ , we need first to compute all  
 532 the upper vertices of the convex hulls of  $A$  and the blocks. We can assume  
 533 that the lattice points are ordered so that the total time is bounded by  
 534  $O(st)$  (ordered lattice sets provide simple polygons whose convex hulls

535 can be computed in linear time [23]). It provides the set  $U$ .  
 536 Then we have to check the conditions required for a bipoint of  $U \times U$  to  
 537 belong to  $V$ . The condition (i) is trivial while each one of the conditions  
 538 (ii) (iii) and (iv) can be resolved in  $O(st)$  with a naive algorithm. The  
 539 labels are also computed in  $O(st)$ . With  $O(s^2)$  possible bipoints in  $V$ , we  
 540 obtain  $O(s^3t)$  for the computation of  $V'$ .

541 The determination of the valid edges of the DAG (clockwise turning angle  
 542 and label's consistency) can be done in constant time for each pair of  
 543 bipoints (or triangle) and therefore with a total time in  $O(s^3)$ .

544 The depth-first search algorithm used to determine whether  $V^1$  can  
 545 be reached from  $V^0$  takes a linear time in the number of vertices and  
 546 edges i.e  $O(s^2 + s^3) = O(s^3)$ . By counting the time necessary to build  
 547 the graph and to search for a solution, we have (without optimization) a  
 548 worst-case time complexity in  $O(s^3t)$ . It proves the following theorem.

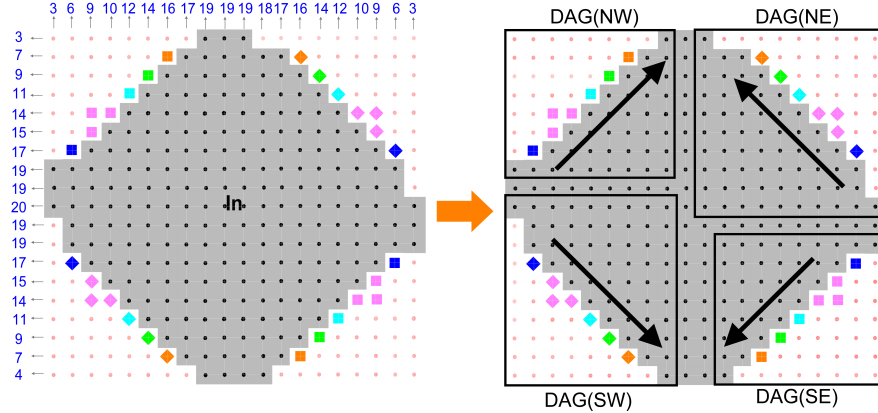
549 **Theorem 2.** *The problem  $ConvexAggregation(A, B^i)$  in  $[0..s] \times [0..t]$   
 550 can be solved in  $O(s^3t)$ .*

## 551 5 ConvexTomo - Steps 4 - Convex Aggregation of the 552 Extended Switching Components

### 553 5.1 Reformulation of the Problem in Terms of Convex 554 Aggregation

555 At the end of the third step of the algorithm `ConvexTomo`, the set `In`  
 556 and the extended switching components have been computed. It remains  
 557 to find a assignment of the extended switching components providing a  
 558 convex solution. According to Property 2, for each extended switching  
 559 component, we have the choice between aggregating its North West and  
 560 South East parts (represented with squares if the figures) or its South  
 561 West and North East parts (the diamonds in the figures) to the set `In`.  
 562 Thus, the blocks are the subsets of the extended switching components in  
 563 each border NW, NE, SE and SW. The block  $B^i$  of the North West border  
 564 is for instance  $B_{NW}^i = P^i \cap NW$ . Property 2 (different extended switching  
 565 components have neither rows, nor columns in common) guarantees the  
 566 growing property of the blocks assumed in  $ConvexAggregation(A, B^i)$ .

567 Without the relations induced by the switching components on each  
 568 border, we would have four independent problems of Convex Aggregation  
 569  $ConvexAggregation(A, B^i)$  as solved in previous section (Fig.19). The  
 570 difficulty comes from the fact the four problems are not independent but  
 571 deeply related: If the block  $B_{NW}^i$  is added in the North West border NW,



**Fig. 19. Convex Aggregation of switching components and corresponding DAGs.** After the third step, the algorithm `ConvexTomo` provides a decomposition of the undetermined points in extended switching components (with colored squares and diamonds). For each color, we aggregate either the squares, or the diamonds to the set `In` with the goal to provide a convex lattice set. It's the problem of Convex Aggregation that we address to conclude the paper. We start by building four DAGs  $G_{NW}$ ,  $G_{NE}$ ,  $G_{SE}$  and  $G_{SW}$  corresponding to the four rectangles drawn on the right. The four problems are considered upward for the North borders and downward for the South borders so that the switching components provide blocks in the same order.

572 the block  $B_{SE}^i$  of the same extended switching component in SE has also  
 573 to be aggregated while the corresponding blocks  $B_{NE}^i$  and  $B_{SW}^i$  in NE and  
 574 SW have to be discarded (Fig.19). We could think about exploring the sets  
 575 of solutions on the four border in order determine whether a combination  
 576 of solutions might be consistent but the potential exponential number  
 577 of solutions makes this approach potentially non polynomial. We have  
 578 to solve simultaneously the four related problems of Convex Aggregation.  
 579 The principle that is we use is to build a new DAG that we call the *master*  
 580 DAG coordinating the aggregation on each border.

## 581 5.2 Building the Master DAG

582 We place us after the third step of the algorithm `ConvexTomo` for an  
 583 instance an instance  $DT_{C \cap \mathcal{R}}(h, v)$  with a regular position of the feet. At  
 584 this stage, the extended switching components  $P^i$  and the convex set `In`  
 585 have been computed. We start by defining the four DAGs  $G_{NW}$ ,  $G_{NE}$ ,  
 586  $G_{SE}$  and  $G_{SW}$  with a subset of `In` as convex set  $A$  and with the switching  
 587 components  $B_{NW}^i = P^i \cap NW$  as block  $B^i$  and the same for the three  
 588 other borders. According to the ordering of the blocks, the fours DAGs

589 are oriented upward for the North borders and downward for the South  
590 borders (Fig.19) so that the blocks  $B^i$  of the switching components appear  
591 in the same order. The vertices of the DAGs are  $V'_{NW}$ ,  $V'_{NE}$ ,  $V'_{SE}$  and  $V'_{SW}$   
592 and their edges  $E_{NW}$ ,  $E_{NE}$ ,  $E_{SE}$  and  $E_{SW}$  as defined in Definition 7. We  
593 call them the *slave* DAGs. Then we define the master DAG as follows:

594 **Definition 8.** *The master DAG  $G = (V, E)$  has a set of vertices  $V$  con-*  
595 *taining the 4-tuples of bipoints  $(\overrightarrow{w_{NW}}, \overrightarrow{w_{NE}}, \overrightarrow{w_{SE}}, \overrightarrow{w_{SW}}) \in V'_{NW} \times V'_{NE} \times$*   
596  *$V'_{SE} \times V'_{SW}$  verifying two conditions (letter  $w$  represents bipoints):*

- 597 – *Label’s consistency-* for any index  $i$  from 1 to  $q$ , the labels  $\overrightarrow{w_{NW}}[i]$ ,  
598  $\overrightarrow{w_{SE}}[i]$  are of same sign (with the convention that 0 is both positive  
599 and negative). The labels  $\overrightarrow{w_{NE}}[i]$  and  $\overrightarrow{w_{SW}}[i]$  are also of same sign and  
600 it is the opposite from the previous pair.
- 601 – *Label’s continuity-* If there is null label for the four bipoints ( $\overrightarrow{w_{NW}}[i] =$   
602  $\overrightarrow{w_{NE}}[i] = \overrightarrow{w_{SE}}[i] = \overrightarrow{w_{SW}}[i] = 0$ ), either all the labels with indices  $i \leq i_0$   
603 are null ( $\forall i \leq i_0, \overrightarrow{w_{NW}}[i] = \overrightarrow{w_{NE}}[i] = \overrightarrow{w_{SE}}[i] = \overrightarrow{w_{SW}}[i] = 0$ ), either  
604 all labels with indices  $i \geq i_0$  are null ( $\forall i \geq i_0, \overrightarrow{w_{NW}}[i] = \overrightarrow{w_{NE}}[i] =$   
605  $\overrightarrow{w_{SE}}[i] = \overrightarrow{w_{SW}}[i] = 0$ ).

606 We have an edge from  $(\overrightarrow{w_{NW}}, \overrightarrow{w_{NE}}, \overrightarrow{w_{SE}}, \overrightarrow{w_{SW}})$  to  $(\overrightarrow{w'_{NW}}, \overrightarrow{w'_{NE}}, \overrightarrow{w'_{SE}}, \overrightarrow{w'_{SW}})$   
607 if three of the bipoints are unchanged (for instance  $\overrightarrow{w_{NW}} = \overrightarrow{w'_{NW}}$ ,  $\overrightarrow{w_{NE}} =$   
608  $\overrightarrow{w'_{NE}}$ ,  $\overrightarrow{w_{SE}} = \overrightarrow{w'_{SE}}$ ) and the pair of different bipoints is an edge of the  
609 corresponding slave DAG: for instance,  $(\overrightarrow{w_{SW}}, \overrightarrow{w'_{SW}}) \in E_{SW}$ .

610 The vertices of the master DAG are made of four bipoints, one on  
611 each border. The edges between these vertices do not allow to progress  
612 simultaneously on different border. The contour advances from the East  
613 and West feet border per border until reaching or not the South and  
614 North feet.

615 The condition of label’s consistency is not sufficient to guarantee the  
616 coherence of the contour obtained this way. Without the condition of  
617 continuity, we could imagine that the contour could progress first in the  
618 North West border until reaching the North foot, before advancing on the  
619 other borders. In such a case, there would be no guarantee that the choices  
620 (or labels) done in the North West border would be consistent with the  
621 ones done afterwards. The condition of continuity avoids such a lost of  
622 information. The contour does not leave a switching component before  
623 all the four paths arrived to it. It’s a keypoint to provide the following  
624 equivalence.

625 **5.3 Equivalence between a Path in the Master DAG and a**  
626 **Convex Aggregation of the Related Blocks**

627 We prove the following theorem:

628 **Theorem 3.** *We consider an instance  $DT_{\mathcal{C} \cap \mathcal{R}}(h, v)$  with a regular posi-*  
629 *tion of the feet providing after the third step of algorithm `ConvexTomo`  $q$*   
630 *switching components  $P^i$ .*

631 *The slave DAGs  $G_{\text{NW}}, G_{\text{NE}}, G_{\text{SE}}$  and  $G_{\text{SW}}$  and the master DAG  $G =$*   
632  *$(V, E)$  being defined according to Definition 8, we denote  $V^0$  the 4-tuples*  
633 *of vertices  $(w_{\text{NW}}, w_{\text{NE}}, w_{\text{SE}}, w_{\text{SW}})$  with all bipoints  $w$  starting from the*  
634 *West and East feet  $V^0 \subset V_{\text{NW}}^0 \times V_{\text{NE}}^0 \times V_{\text{SE}}^0 \times V_{\text{SW}}^0$ . In the same way,  $V^1$*   
635 *is the set the valid 4-tuples of bipoints ending at the North and South feet*  
636  *$V^1 \subset V_{\text{NW}}^1 \times V_{\text{NE}}^1 \times V_{\text{SE}}^1 \times V_{\text{SW}}^1$ .*

637 *The instance  $DT_{\mathcal{C} \cap \mathcal{R}}(h, v)$  admits a solution if and only if there is path*  
638 *in the master DAG  $G = (V, E)$  from  $V^0$  to  $V^1$ .*

639 *Proof.* We have to prove that a solution of  $DT_{\mathcal{C} \cap \mathcal{R}}(h, v)$  provides a path  
640 from  $V^0$  to  $V^1$  in the master DAG  $G$  and conversely that a path in the  
641 DAG provides a convex lattice set with the prescribed X-rays.

642 For the first assertion, we have a convex solution with a contour be-  
643 tween the four feet. The main point is to show that we can advance on the  
644 different borders from the East and West feet to the South and North feet  
645 by using only valid 4-tuples of bipoints and valid edges (valid according  
646 to the definition of the master DAG). The consistency of the 4-tuples of  
647 bipoints that can be extracted from the contour is straightforward but  
648 not the continuity. There are however strategies which allow to guarantee  
649 this property at each step: we move the bipoint with the less advanced  
650 starting point with respect to the horizontal strips  $H^i$  of the switching  
651 components (as illustrated in Fig.20). By induction, the continuity of 4-  
652 tuples of bipoints (which is true in  $V^0$ ) is guaranteed. It provides the first  
653 assertion.

654 For the converse, each path in the master DAG from  $V^0$  to  $V^1$  provides  
655 a path in the slave DAGs. Theorem 2 insures that each one of these paths  
656 defines a convex contour with blocks on the left and blocks on the right.  
657 It remains to prove that the contours on each border are consistent.

658 We have a first property: following the path in the DAG until a 4-tuple  
659 of bipoints  $(\overrightarrow{w_{\text{NW}}}, \overrightarrow{w_{\text{NE}}}, \overrightarrow{w_{\text{SE}}}, \overrightarrow{w_{\text{SW}}})$ , if a label of one of the bipoints (for  
660 instance  $\overrightarrow{w_{\text{NW}}}[i]$ ) is not null, then all the labels which have been visited  
661 and determined by the path after the index  $i$  have non null values for at  
662 least one of the bipoints of the 4-tuple. This property denoted (i) can be

663 established by induction. It is obvious for the initial 4-tuple of bipoints  
664 (due to the consistency condition) and it remains true by induction due  
665 to the continuity condition of the 4-tuple of bipoints.  
666 Let us show now that the contour gradually built from the path in the  
667 DAG is consistent. We prove it again by induction. With the consistency  
668 condition, it is true for the initial 4-tuple of bipoints. Then we assume  
669 that the contour generated by the path is consistent until the vertex  
670  $(\overrightarrow{w_{NW}}, \overrightarrow{w_{NE}}, \overrightarrow{w_{SE}}, \overrightarrow{w_{SW}})$ . According to the set of edges  $E$ , we can assume  
671 w.l.g that the next vertex is  $(\overrightarrow{w'_{NW}}, \overrightarrow{w'_{NE}}, \overrightarrow{w'_{SE}}, \overrightarrow{w'_{SW}})$ . If the new bipoint  
672  $\overrightarrow{w'_{SW}}$  is inconsistent for the switching component of index  $j$ , it means that  
673 the index already had in the path a non null label. Due to property (i),  
674 this label is also non null value in the current 4-tuple. As there was no  
675 inconsistency before, it follows that  $\overrightarrow{w'_{SW}}$  has an inconsistent label with  
676 one of the three bipoints  $\overrightarrow{w_{NW}}, \overrightarrow{w_{NE}}, \overrightarrow{w_{SE}}$ . It's in contradiction with  
677 the consistency of the 4-tuple  $(\overrightarrow{w_{NW}}, \overrightarrow{w_{NE}}, \overrightarrow{w_{SE}}, \overrightarrow{w_{SW}})$ . It proves that the  
678 contour built from the path in the master DAG is a valid solution and  
679 therefore provides a convex lattice set with the prescribed X-rays.

#### 680 5.4 Algorithm ConvexAggregation

681 The fourth step of Algorithm ConvexTomo solves the problem of Convex  
682 Aggregation issued from the switching components through the research  
683 of a path in the master DAG  $G$ . The algorithm ConvexAggregation starts  
684 by computing the four slave DAGs. Then it considers all the 4-tuples of  
685 bipoints in  $V_{NW} \times V_{NE} \times V_{SE} \times V_{SW}$  and checks their validity. Afterward,  
686 it computes the edges of the master DAG by checking the validity and  
687 the turning angle of the pairs of vertices.

688 The final part of the algorithm is the research of a path starting  
689 from  $V^0$  and reaching  $V^1$ . It can be done with depth-first or breadth-first  
690 search.

##### 691 *Complexity Analysis*

692 We start with the computation of the slave DAGs. According to the  
693 previous analysis, by replacing  $(s, t)$  by the upper bound  $(m, n)$ , it can  
694 be done in  $O(m^3n)$ .

695 For the master DAG, as we have at most  $m^2$  bipoints in each slave DAG,  
696 the number of 4-tuples of bipoints in  $V$  is bounded by  $O(m^8)$ . For each  
697 one of them, testing their validity requires  $O(q)$ . The computation of the  
698 vertices of the master DAG takes  $O(qm^8)$ .

699 For going from the vertex  $(\overrightarrow{p_{NW}p'_{NW}}, \overrightarrow{p_{NW}p'_{NW}}, \overrightarrow{p_{NW}p'_{NW}}, \overrightarrow{p_{NW}p'_{NW}})$  to one  
700 of its successors, we have to introduce a new upper point, for instance

<p><b>Data:</b> <math>m, n, \text{In}</math> and the extended switching components <math>(P^i)_{1 \leq i \leq q}</math></p> <p><b>Result:</b> An assignment of the extended switching component providing a convex aggregated set</p> <pre> 1 /* Compute the slave DAGs <math>G_{NW}, G_{NE}, G_{SE}</math> and <math>G_{SW}</math> with labels */ 2 <math>(V_{NW}, E_{NW}) \leftarrow \text{DAG}(\text{In}, (P^i)_{1 \leq i \leq q} \cap \text{NW});</math> 3 <math>(V_{NE}, E_{NE}) \leftarrow \text{DAG}(\text{In}, (P^i)_{1 \leq i \leq q} \cap \text{NE});</math> 4 <math>(V_{SE}, E_{SE}) \leftarrow \text{DAG}(\text{In}, (P^i)_{1 \leq i \leq q} \cap \text{SE});</math> 5 <math>(V_{SW}, E_{SW}) \leftarrow \text{DAG}(\text{In}, (P^i)_{1 \leq i \leq q} \cap \text{SW});</math> 6 /* Compute the master DAG */ 7 <math>(V, E) \leftarrow \text{masterDAG}((V_{NW}, V_{NE}, V_{SE}, V_{SW}, E_{NW}, E_{NE}, E_{SE}, E_{SW}, \text{labels});</math> 8 <math>V^0 \leftarrow V \cap V_{NW}^0 \times V_{NE}^0 \times V_{SE}^0 \times V_{SW}^0;</math> 9 <math>V^1 \leftarrow V \cap V_{NW}^1 \times V_{NE}^1 \times V_{SE}^1 \times V_{SW}^1;</math> 10 /* Search for a path from <math>V^0</math> to <math>V^1</math> */ 11 <math>\text{path} \leftarrow \text{searchPath}(V, E, V^0, V^1);</math> 12 <math>\text{readLabels}(\text{path})</math> </pre>
---

**Algorithm 3:** ConvexAggregation( $m, n, \text{In}, (P^i)_{1 \leq i \leq q}$ )

701  $p''_{NW} \in \text{NW}$ , and then checking if  $\overrightarrow{(p'_{NW} p''_{NW}, p_{NW} p'_{NW}, p_{NW} p'_{NW}, p_{NW} p'_{NW})}$   
702 is a valid vertex. It checks also whether the pair  $(p_{NW} p'_{NW}, p'_{NW} p''_{NW})$  is  
703 in  $E_{NW}$ .

704 The number of edges is bounded by the number of vertices  $O(m^8)$  times  
705 the number of possible new points  $2m$ . It makes a number of edges in  
706  $O(m^9)$ . The time necessary to check their validity is constant with a suit-  
707 able data structure containing the edges of the slave DAGs.

708 With the number of vertices and edges, the breadth-first search can be  
709 done in  $O(m^9)$ . As  $q$  is lower than  $m$ , it provides a total time of compu-  
710 tation in  $O(m^9)$ . It proves the following theorem.

711 **Theorem 4.** *Algorithm ConvexAggregation( $m, n, \text{In}, (P^i)_{1 \leq i \leq q}$ ) solves*  
712 *the problem of Convex Aggregation of the blocks issued from the switching*  
713 *components in  $[0..m] \times [0..n]$  with a worst case time complexity in  $O(m^9)$ .*

714 Notice that if  $n < m$ , with a rotation of  $\frac{\pi}{2}$ , it could be as well  $O(n^9)$ .

## 715 5.5 Proof of Theorem 1

716 Theorem 4 provides the time complexity in  $O(m^9)$  of the algorithm **Convex**  
717 **Aggregation**. According to the propositions 1, 2, 3, the three first steps  
718 (**Initialization**, **FillTomo**, **SwitchingComponents**) require respectively  
719  $O(mn)$ ,  $O(m^2 n^2)$  and  $O(mn)$  operations. Since there are at most  $m^2 n^2$   
720 regular positions of the feet to explore, the overall time complexity of  
721 Algorithm **FillTomo** is  $O(m^4 n^4 + m^{11} n^2)$ , as stated in Theorem 1.



## 722 6 Conclusion

723 The problem of the reconstruction of convex lattice sets with prescribed  
724 horizontal and vertical X-rays is the most challenging problem of com-  
725 plexity in the framework of Discrete Tomography. The question is to  
726 determine whether we can handle with ambiguous X-rays and convexity  
727 constraints in polynomial time.

728 The paper is a first step in the direction of a better understanding  
729 of the relations between ambiguities and convexity. The ambiguities are  
730 expressed by the switching components. After the three first steps of  
731 the classical algorithm, they lead to reduce the problem to a question  
732 of Convex Aggregation. The new idea that we present is that discrete  
733 convexity is a local constraint. It can be reduced to the property that  
734 consecutive edges have a clockwise turning angle. Such a constraint can  
735 be handled with a DAG or more generally with Dynamic Programming.  
736 The crucial information is the last reconstructed edge so that we can try  
737 to go further. These two elements (switching components and encoding  
738 of the last edge) allowed us to develop the polynomial time algorithm  
739 `ConvexTomo` in the case of a regular position of the feet. It is a good  
740 news but why not a more general result ? What happens in the case of  
741 an irregular position of the feet which is different than for the regular  
742 positions ? The answers can be reduced to only one word: "structure".

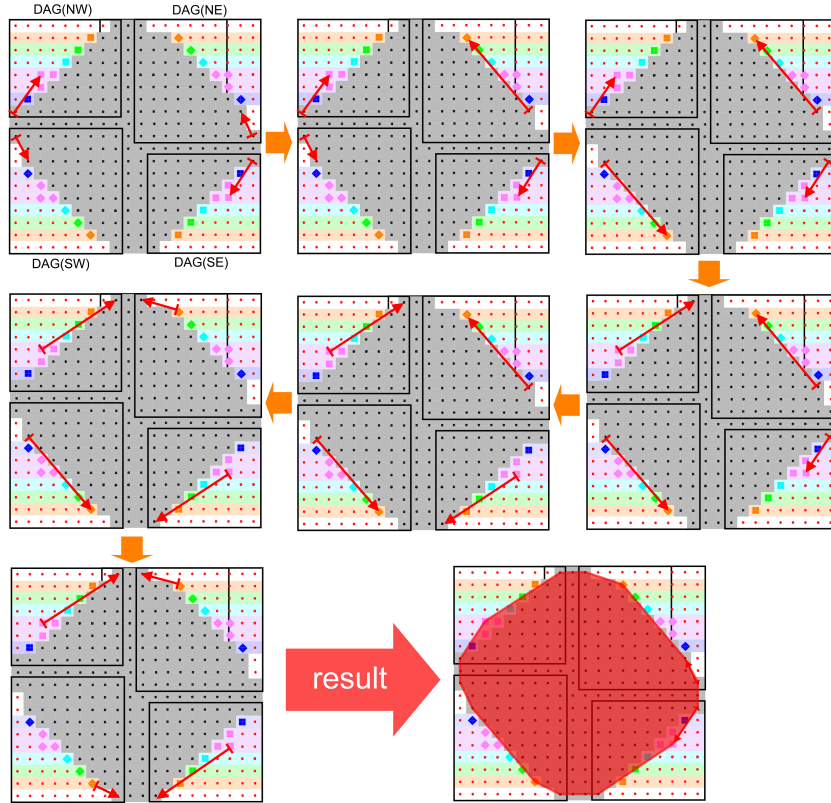
743 The structure of the regular switching components can be simplified  
744 by merging them into extended switching components. According to re-  
745 cent combinatorial results (Property 2 [19]), extended switching compo-  
746 nents can be ordered (their rows and columns are ordered) so that the  
747 consecutive blocks to consider in a Convex Aggregation framework are  
748 also well ordered. This order (or increasing property) is the only assump-  
749 tion that we did on the blocks in problem *ConvexAggregation*( $A, B^i$ ).  
750 Without it, there is no doubt that Convex Aggregation would become  
751 NP-hard. Structural properties are necessary to provide polynomial time  
752 algorithms of Convex Aggregation. This remark leads to the following  
753 question: with irregular positions of the feet, do switching components  
754 (Fig.12) have enough structures to be handled with polynomial time al-  
755 gorithms ? It's an open question whose answer could as well lead to a  
756 polynomial time algorithm as to a result of NP-completeness.

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**Fig.20. A path in the master DAG and the corresponding solution of  $DT_{C \cap \mathcal{R}}(h, v)$ .** We show why any solution of  $DT_{C \cap \mathcal{R}}(h, v)$  can be obtained through a path from  $V^0$  to  $V^1$  in the master DAG. The path is obtained with the strategy to advance the bipoint  $\overrightarrow{vv'}$  with the less advanced point  $v$  regarding the horizontal strips  $H^i$  of the switching components. With this strategy, the continuity of the labels is guaranteed by induction.