Spectrum of a linear differential equation over a field of formal power series

Tinhinane Amina Azzouz

SPECTRUM OF A LINEAR DIFFERENTIAL EQUATION OVER A FIELD OF FORMAL POWER SERIES

TINHINANE A. AZZOUZ

Abstract. In this paper we associate to a linear differential equation with coefficients in the field of Laurent formal power series a new geometric object, a spectrum in the sense of Berkovich. We compute this spectrum and show that it contains interesting informations about the equation.

1. Introduction

Differential equations constitute an important tool for investigation of algebraic and analytic varieties. Among ways to look for solutions of such equations, the study of formal Taylor solutions of the equation around singular and regular points and the computation of their radii of convergence plays an important role. This leads in particular to the study of differential equations with coefficients in the field of Laurent formal power series. In this special case, we have interesting classification results. Before discussing in more details these results, we shall fix the setting.

Let k be a field of characteristic 0. We set $k(\!(T)\!)$ to be the field of Laurent formal power series. Consider the differential field $(k(\!(T)\!), T\frac{\mathrm{d}}{\mathrm{dT}})$. We mean by a differential equation with coefficients in $k(\!(T)\!)$ a differential module (M, ∇) over $(k(\!(T)\!), T\frac{\mathrm{d}}{\mathrm{dT}})$. The main fondamental classification results are the following:

(1) *Decomposition theorem according to the slopes.* Considering the T-adic valuation of the coefficients of the operator ∇ in a cyclic basis, we can associate a Newton polygon, called formal. The formal slopes of (M, ∇) are the slopes of this polygon. The decomposition theorem is the following.

Theorem 1.1 ([DMR07, p. 97-107])**.** *Let $\gamma_1 < \cdots < \gamma_\mu$ be the slopes of the formal Newton polygon of (M, ∇), with multiplicity $n_1, \cdots, n_\mu$ respectively. Then*

$$(M, \nabla) = \bigoplus_{i=1}^{\mu} (M_{\gamma_i}, \nabla_{\gamma_i}),$$

where M_{γ_i} has dimension n_i and a unique slope γ_i with multiplicity n_i.

(2) *The Turrittin-Levelt-Hukuhara decomposition theorem* [Kat70], [Rob80], [Ked10]. It claims that for any differential module (M, ∇), there exists a suitable finite extension $k'(\!(T^{\frac{1}{n}})\!)$ of $k(\!(T)\!)$ for which the pull-back of (M, ∇) with respect to this extension is an extension of differential modules of rank one.

In this paper we propose a new geometric invariant for differential modules over $(k(\!(T)\!), T\frac{\mathrm{d}}{\mathrm{dT}})$, a spectrum in the sense of Berkovich. We will compute this spectrum and show that it contains interesting information about the equation. For this purpose we will use the classification results listed above. Before announcing the main result of the paper, we shall recall quickly the notion of the spectrum in the sense of Berkovich.

Recall that for an element f of a non-zero k-algebra E with unit, the classical spectrum of f is the set

$$\{a \in k;\ f - a.1_E \text{ is not invertible in } E\}.$$

This set may be empty, even if E is a k-Banach algebra. To deal with this issue Berkovich proposed to consider the spectrum not as a subset of k, but as a subset of the analytic affine line $\mathbb{A}_k^{1,\mathrm{an}}$ (which is a bigger space than k)[Ber90, Chapter 7]. Let E be a non-zero k-Banach algebra and $f \in E$. The spectrum $\Sigma_{f,k}(E)$ of f in the sense of Berkovich is the set of points of $\mathbb{A}_k^{1,\mathrm{an}}$ that correspond to a pair (Ω, c), where Ω is a complete extension of k and $c \in \Omega$, for which $f \otimes 1 - 1 \otimes c$ is not invertible in $E\hat{\otimes}_k\Omega$. This spectrum is non-empty, compact and satisfies other nice properties (cf.[Ber90, Theorem 7.1.2]).

Let (M, ∇) be a differential module over $(k(\!(T)\!), T\frac{\mathrm{d}}{\mathrm{dT}})$. Let $r > 0$ be a real positive number. From now on we endow $k(\!(T)\!)$ with the T-adic absolute value given by

$$|\sum_{i\geq N} a_i T^i| := r^N, \tag{1.1}$$

if $a_N \neq 0$. In this setting $k(\!(T)\!)$ is a complete valued field and the induced valuation on k is the trivial valuation. From now on we endow k with the trivial valuation. We can endow M with $k(\!(T)\!)$-Banach structure, moreover it induces a k-Banach structure for which $\nabla : M \to M$ is a bounded operator. As in our previous work [Azz18], the spectrum $\Sigma_{\nabla,k}(\mathcal{L}_k(M))$ of (M, ∇) will be the spectrum of ∇ as an element of the k-Banach algebra $\mathcal{L}_k(M)$ of bounded endomorphism of M with respect to operator norm.

Notice that we cannot use the classical index theorem of B. Malgrange [Mal74] to compute neither the spectrum in the sense of Berkovich nor the classical spectrum of ∇. Indeed, it is relatively easy to show that any non trivial rank one connection on $k(\!(T)\!)$ is set-theoretically bijective. However, the set-theoretical inverse of the connection may not be bounded. This is due to the fact that the base field k is trivially valued and Banach open mapping theorem does not hold in this setting.

For any positive real number l, we set $x_{0,l}$ to be the point of $\mathbb{A}_k^{1,\mathrm{an}}$ associated to l-Gauss norm on $k[T]$ (i.e. $\sum_i a_i T^i \mapsto \max_i |a_i| l^i$). The main result of the paper is the following:

Theorem 1.2. *Assume that k is algebraically closed. Let (M, ∇) be a differential module over $(k(\!(T)\!), T\frac{\mathrm{d}}{\mathrm{dT}})$. Let $\{\gamma_1, \cdots, \gamma_\nu\}$ be the set of the slopes of (M, ∇) and let $\{a_1, \cdots, a_\mu\}$ be the set of the exponents of the regular part of (M, ∇). Then the spectrum of ∇ as an element of $\mathcal{L}_k(M)$ is:*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \{x_{0,r^{-\gamma_1}}, \cdots, x_{0,r^{-\gamma_\nu}}\} \cup \bigcup_{i=1}^{\mu}(a_i + \mathbb{Z}).$$

This result clearly shows the importance of the points of the spectrum that are not in k. Indeed, form these points we can recover the slopes without multiplicity of the differential module. On other the hand, although differential modules over $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}})$ are algebraic objects, their spectra in the sense of Berkovich depend highly on the choice of the absolute value on $k(\!(S)\!)$.

The paper is organized as follows. Section 2 is devoted to providing setting and notation. It is divided into three parts, the first one is to recall the notion of differential module. In the second one, we recall the definition of the analytic affine line $\mathbb{A}_k^{1,\mathrm{an}}$, and give, in the setting of the paper, a precise topological description for disks and annuli of $\mathbb{A}_k^{1,\mathrm{an}}$. In the last part, we recall the definition of the spectrum in the sense of Berkovich and give some properties.

In Section 3, we introduce the spectrum of a differential module and recall some properties given in [Azz18]. We also show how the spectrum of a differential module behaves after ramified ground field extension. In the end of the section we recall the definition of the formal Newton polygon.

Section 4 is devoted to announcing and proving the main result of the paper. Using the decomposition theorem according to the slopes, we can reduce the probleme to the computation of the spectrum of regular singular differential modules and differential modules without regular part. The spectrum of a regular singular module is mainly obtained by the computation of the spectrum of $T\frac{\mathrm{d}}{\mathrm{dT}}$ as an element of $\mathcal{L}_k(k(\!(T)\!))$. For differential modules without regular part, we reduce the computation of the spectrum to the case of differential modules of rank one. This is possible by using Turrittin-Levelt-Hukura decomposition theorem and the behaviour of the spectrum after ramified ground field extension.

Acknowledgments. The author wishes to express her gratitude to her advisors Andrea Pulita and Jérôme Poineau for their precious advice and suggestions, and for careful reading. She also thank Francesco Baldassarri, Frits Beukers, Antoine Ducros and Françoise Truc for useful occasional discussions and suggestions.

2. Definitions and notations

All rings are with unit element. We will denote by $\mathbb{R}$ the field of real numbers, by $\mathbb{Z}$ the ring of integers and by $\mathbb{N}$ the set of natural numbers. We set

$$\mathbb{R}_+ := \{r \in \mathbb{R};\ r \geq 0\}.$$

In all the paper we fix $(k, |.|)$ to be an algebraically closed trivially valued field of characteristic 0. Let $E(k)$ be the category whose objects are $(\Omega, |.|_\Omega)$, where Ω is a field extension of k, complete with respect to the valuation $|.|_\Omega$, and whose morphisms are isometric rings morphisms. For $(\Omega, |.|_\Omega) \in E(k)$, we fix an algebraic closure Ω^{alg} of Ω. The absolute value $|.|_\Omega$ extends uniquely to Ω^{alg}. We denote by $\widehat{\Omega^{\mathrm{alg}}}$ the completion of Ω^{alg} with respect to this absolute value.

2.1. **Differential modules.** Let $F \in E(k)$ and d be a k-linear derivation on F. Consider the differential field (F, d). A differential module (M, ∇) over (F, d) of rank $n \in \mathbb{N}$ is an F-vector space M of dimension n together with a k-linear map $\nabla : M \to M$, called connection of M, satisfying $\nabla(fm) = df.m + f.\nabla(m)$ for all $f \in F$ and $m \in M$. If we fix a basis of $M$the operator ∇ is given in the induced basis $\{e_1, \cdots, e_n\}$ by the formula:

$$\nabla \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix} + G \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \tag{2.1}$$

where $G \in \mathcal{M}_n(F)$ is the matrix that whose i column is the vector $\nabla(e_i)$. Conversely the datum of such matrix defines a differential module structure on M by the formula (2.1).

A morphism between differential modules is a k-linear map $M \to N$ commuting with connections.

Notation 2.1. We denote by d-**Mod**(F) the category of differentiel modules over (F, d) whose arrows are morphisms of differential modules.

We set $\mathscr{D}_F := \bigoplus_{i \in \mathbb{N}} F.D^i$ to be the ring of differential polynomials on D with coefficients in F, where the multiplication is non-commutative and defined as follows: $D.f = d(f) + f.D$ for all $f \in A$. Let $P(D) = g_0 + \cdots + g_{n-1}D^{n-1} + D^n$ be a monic differential polynomial. The quotient $\mathscr{D}_F/\mathscr{D}_F.P(D)$ is an F-vector space of dimension n. Equipped with the multiplication by D, it is a differential module over (F, d). In the basis $\{1, D, \ldots, D^{n-1}\}$ the multiplication by D satisfies:

$$D \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & -g_0 \\ 1 & \ddots & 0 & \vdots \\ 0 & \ddots & 0 & \\ \vdots & \ddots & 0 & \\ 0 & \cdots & 0 & -g_{n-1} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \tag{2.2}$$

Proposition 2.2 (The cyclic vector theorem)**.** *Assume that $d \neq 0$. Let (M, ∇) be a differential module over (F, d). Then there exists $P(D) \in \mathscr{D}_F$ such that $(M, \nabla) \simeq (\mathscr{D}_F/\mathscr{D}_F.P(D), D)$.*

Proof. See [Ked10, Theorem 5.7.2.]. □

Let (M_1, ∇_1) and (M_2, ∇_2) be two differential modules over (F, d) of rank n_1 and n_2. The tensor product $M_1 \otimes M_2$ equipped with the connection:

$$\nabla = \nabla_1 \otimes 1 + 1 \otimes \nabla_2 \tag{2.3}$$

is a differential module over (F, d) of rank $n_1 n_2$.

2.2. **Analytic affine line.** In this part we give basic notions related to the analytic affine line $\mathbb{A}_k^{1,\mathrm{an}}$ in the sense of Berkovich, and describe some analytic domains of $\mathbb{A}_k^{1,\mathrm{an}}$ that are helpful to describe spectra of differential modules later.

We will consider k-analytic spaces in the sense of Berkovich (see [Ber90]). We denote by $\mathbb{A}_k^{1,\mathrm{an}}$ the affine analytic line over the ground field k. Recall that a point $x \in \mathbb{A}_k^{1,\mathrm{an}}$ corresponds to a multiplicative semi-norm $|.|_x$ on $k[T]$ (i.e. $|0|_x = 0$, $|1|_x = 1$, $|P - Q|_x \leq \max(|P|_x, |Q|_x)$ and $|P \cdot Q|_x = |P|_x \cdot |Q|_x$ for all P, $Q \in k[T]$) whose restriction coincides with the absolute value of k.

Notation 2.3. Let $c \in k$ and $r \in \mathbb{R}_+$. Denote by $x_{c,r}$ the point that corresponds to the semi-norm (norm if $r > 0$)

$$\begin{array}{rcl} k[T] & \longrightarrow & \mathbb{R}_+ \\ \sum_{i=0}^n a_i (T-c)^i & \mapsto & \max_i |a_i| r^i \end{array} . \tag{2.4}$$

Remark 2.4. Since $(k, |.|)$ is trivially valued, any point of $\mathbb{A}_k^{1,\mathrm{an}}$ is a point of the form $x_{c,r}$. The points of the form $x_{c,0}$ coincides with the element of k.

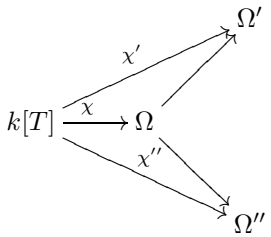
Notation 2.5. The set

$$\mathfrak{p}_x := \{P \in k[T];\ |f|_x = 0\}$$

is a prime ideal of $k[T]$ and the semi-norm extends to a multiplicative norm on the fraction field $\mathrm{Frac}(A/\mathfrak{p}_x))$. We denote by $\mathscr{H}(x)$ the completion of $\mathrm{Frac}(A/\mathfrak{p}_x)$ with respect to $|.|_x$, and by $|.|$ the absolute value on $\mathscr{H}(x)$ induced by $|.|_x$.

Let $\Omega \in E(k)$. A nonzero k-algebra morphism $\chi : k[T] \to \Omega$ is called a character of $k[T]$. Let $\sim$ be the equivalence relation defined as follows: let $\chi' : k[T] \to \Omega'$ and $\chi'' : k[T] \to \Omega''$ be two characters,

$$\chi' \sim \chi'' \Leftrightarrow \exists \chi : k[T] \to \Omega \text{ such that:}$$



For any point $x \in \mathbb{A}_k^{1,\mathrm{an}}$ we have a natural nonzero k-algebra morphism

$$\begin{array}{rcl} \chi_x : k[T] & \longrightarrow & \mathscr{H}(x) \\ P & \mapsto & P(x) \end{array} .$$

This induces a bijection between $\mathbb{A}_k^{1,\mathrm{an}}$ and the set of equivalence classes of characters of $k[T]$.

For a point $x_{c,r} \in \mathbb{A}_k^{1,\mathrm{an}}$ with $r > 0$ the field $\mathscr{H}(x_{c,r})$ can be described more concretely. The case where $r = 0$ is trivial, indeed $\mathscr{H}(x_{c,0}) \simeq k$.

In the case where $r < 1$, the field $\mathscr{H}(x_{c,r})$ coincides with the field of Laurent formal power series

$$k(\!(T-c)\!) := \left\{ \sum_{i\geq N} a_i (T-c)^i;\ a_i \in k,\ N \in \mathbb{Z} \right\}. \tag{2.5}$$

equipped with $(T-c)$-adic absolute value given by $|\sum_{i\geq N} a_i(T-c)^i| := r^N$, if $a_N \neq 0$.

If $r > 1$, then $\mathscr{H}(x_{c,r})$ coincides with $k(\!((T-c)^{-1})\!)$ equipped with the $(T-c)^{-1}$-adic absolute value given by $|\sum_{i\geq N} a_i(T-c)^{-i}| := r^{-N}$, if $a_N \neq 0$.

Otherwise, $\mathscr{H}(x_{c,1})$ coincides with $k(T)$ equipped with the trivial absolute value.

Let $\Omega \in E(k)$ and $c \in \Omega$. For $r \in \mathbb{R}_+ \setminus \{0\}$ we set

$$D^+_\Omega(c,r) := \{x \in \mathbb{A}^{1,\mathrm{an}}_\Omega;\ |T(x)-c| \leq r\}$$

and

$$D^-_\Omega(c,r) := \{x \in \mathbb{A}^{1,\mathrm{an}}_\Omega;\ |T(x)-c| < r\}.$$

The point $x_{c,r} \in \mathbb{A}^{1,\mathrm{an}}_\Omega$ corresponds to the disk $D^+_\Omega(c,r)$, more precisely it does not depend on the of the center c (cf. [Ber90, Section 1.4.4]).

Let $c \in k$. The map

$$\begin{array}{ccc} [0,+\infty) & \longrightarrow & \mathbb{A}^{1,\mathrm{an}}_k \\ r & \mapsto & x_{c,r} \end{array} \tag{2.6}$$

induces an homeomorphism between $[0,+\infty)$ and its image. Since the valuation is trivial on k, we can describe disks of $\mathbb{A}^{1,\mathrm{an}}_k$ as follows.

Notation 2.6. We denote by $[x_{c,r},\infty)$ (resp. $(x_{c,r},\infty)$) the image of $[r,\infty)$ (resp. (r,∞)), by $[x_{c,r},x_{c,r'}]$ (resp. $(x_{c,r},x_{c,r'}]$, $[x_{c,r},x_{c,r'})$, $(x_{c,r},x_{c,r'})$) the image of $[r,r']$ (resp. $(r,r']$, $[r,r')$, (r,r')).

Let $c \in k$ and $r \in \mathbb{R}_+$. In the case where $r < 1$, we have

$$D^+_k(c,r) = [c,x_{c,r}] \qquad D^-_k(c,r) = [c,x_{c,r}). \tag{2.7}$$

If $r > 1$, recall that for all $a \in k$, $x_{a,1} = x_{0,1}$ and we have

$$D^+_k(c,r) = \coprod_{a\in k} [a,x_{0,1}) \coprod [x_{0,1},x_{0,r}]. \tag{2.8}$$

$$D^-_k(c,r) = \begin{cases} [c,x_{0,1}) & \text{if } r = 1 \\ \coprod_{a\in k} [a,x_{0,1}) \coprod [x_{0,1},x_{0,r}) & \text{otherwise} \end{cases}. \tag{2.9}$$

For r_1, $r_2 \in \mathbb{R}_+$, such that $0 < r_1 \leq r_2$ we set

$$C^+_\Omega(c,r_1,r_2) := D^+_\Omega(c,r_2) \setminus D^-_\Omega(c,r_1)$$

and for $r_1 < r_2$ we set:

$$C^-_\Omega(c,r_1,r_2) := D^-_\Omega(c,r_2) \setminus D^+_\Omega(c,r_1).$$

We may suppress the index Ω when it is obvious from the context.

Let X be an affinoid domain of $\mathbb{A}^{1,\mathrm{an}}_{\Omega}$, we denote by $\mathcal{O}(X)$ the Ω-Banach algebra of global sections of X.

Since k is trivially valued, if $r<1$ we have

$$\mathcal{O}(D^+(c,r))=k[\![T-c]\!]:=\{\sum_{i\in\mathbb{N}}a_i(T-c)^i;\ a_i\in k\}, \tag{2.10}$$

otherwise,

$$\mathcal{O}(D^+(c,r))=k[T-c]. \tag{2.11}$$

In the both cases, the multiplicative norm on $\mathcal{O}(D^+(c,r))$ is :

$$\|\sum_{i\in\mathbb{N}}a_i(T-c)^i\|=\max_{i\in\mathbb{N}}|a_i|r^i. \tag{2.12}$$

Let $X=C^+_{\Omega}(c,r_1,r_2)$

$$\mathcal{O}(C^+_{\Omega}(c,r_1,r_2))=\left\{\sum_{i\in\mathbb{N}\setminus\{0\}}\frac{a_i}{(T-c)^i};\ a_i\in\Omega,\ |a_i|r_1^{-i}\to 0\right\}\oplus\mathcal{O}(D^+(c,r_2)). \tag{2.13}$$

where $\|\sum\limits_{i\in\mathbb{N}\setminus\{0\}}\frac{a_i}{(T-c)^i}\|=\max_i|a_i|r_1^{-i}$ and the sum above is equipped with the maximum norm.

In the case where $\Omega=k$, for $r_2<1$ we have

$$\mathcal{O}(C^+(c,r_1,r_2))=k(\!(T-c)\!). \tag{2.14}$$

If $r_1\leq 1\leq r_2$, we have

$$\mathcal{O}(C^+(c,r_1,r_2))=k[T,(T-c)^{-1}]. \tag{2.15}$$

If $r_1>1$, we have

$$\mathcal{O}(C^+(c,r_1,r_2))=k(\!(T-c)^{-1}\!)). \tag{2.16}$$

Notation 2.7. Let X be an analytic domain of $\mathbb{A}^{1,\mathrm{an}}_k$, and $f\in\mathcal{O}(X)$. We can see f as an analytic map $X\to\mathbb{A}^{1,\mathrm{an}}_k$ that we still denote by f.

2.3. **Berkovich spectrum.** This part is devoted to recalling the definition of the spectrum in the sense of Berkovich and the definition of the sheaf of analytic function with values in a k-Banach space over an analytic space, given by V. Berkovich in [Ber90, Chapter 7]. We recall here also the most important properties. We add in later sections some other properties that are necessary to compute the spectrum.

Definition 2.8. Let M and N be two k-Banach spaces. We defined a norm on the tensor product $M\otimes_k N$ as follows:

$$\begin{array}{rcl}\|.\|:M\otimes_k N & \longrightarrow & \mathbb{R}_+\\ f & \mapsto & \inf\{\max_i\{\|m_i\|_M\cdot\|n_i\|_N\}|\ f=\sum\limits_i m_i\otimes n_i\}\ .\end{array} \tag{2.17}$$

We denote by $M\hat{\otimes}_k N$ the completion of the tensor product with respect to this norm.

Definition 2.9. Let E be k-Banach algebra with unit and $f\in E$. The spectrum of f is the set $\Sigma_{f,k}(E)$ of points $x\in\mathbb{A}^{1,\mathrm{an}}_k$ such that the element $f\otimes 1-1\otimes T(x)$ is not invertible in the k-Banach algebra $E\hat{\otimes}_k\mathscr{H}(x)$. The resolvent of f is the function:

$$\begin{array}{rcl}R_f:\mathbb{A}^{1,\mathrm{an}}_k\setminus\Sigma_{f,k}(E) & \longrightarrow & \coprod\limits_{x\in\mathbb{A}^{1,\mathrm{an}}_k\setminus\Sigma_{f,k}(E)}E\hat{\otimes}_k\mathscr{H}(x)\\ x & \longmapsto & (f\otimes 1-1\otimes T(x))^{-1}\end{array}$$

Remark 2.10. If there is no confusion we denote the spectrum of f, as an element of E, just by Σ_f.

Remark 2.11. The set $\Sigma_f \cap k$ coincides with the classical spectrum, i.e.

$$\Sigma_f \cap k = \{a \in k;\ f - a \text{ is not invertible in } E\}.$$

Definition 2.12. Let X be a k-affinoid space and B be a k-Banach space. We define the sheaf of analytic functions with values in B over X to be the sheaf:

$$\mathcal{O}_X(B)(U) = \varprojlim_{V \subset U} B\hat{\otimes}_k A_V$$

where U is an open subset of X, V an affinoid domain and A_V the k-affinoid algebra associated to V.

As each k-analytic space is obtained by gluing k-affinoid spaces (see [Ber90], [Ber93]), we can extend the definition to k-analytic spaces.

Definition 2.13. Let E be a k-Banach algebra. The spectral semi-norm associated to the norm $\|.\|$ of E is the map:

$$\begin{array}{rccc} \|.\|_{\mathrm{Sp}} : & E & \longrightarrow & \mathbb{R}_+ \\ & f & \mapsto & \lim\limits_{n\to+\infty} \|f^n\|^{\frac{1}{n}} \end{array} . \tag{2.18}$$

If $\|.\|_{\mathrm{Sp}} = \|.\|$, we say that E is a uniform algebra.

This notion of spectrum satisfies the same properties of the classical spectrum in the complex case.

Theorem 2.14 ([Ber90, Theorem 7.1.2])**.** *Let E be a non-zero k-Banach algebra and $f \in E$. Then:*

1. *The spectrum Σ_f is a non-empty compact subset of $\mathbb{A}_k^{1,\mathrm{an}}$.*
2. *The radius of the smallest (inclusion) closed disk with center at zero which contains Σ_f is equal to $\|f\|_{\mathrm{Sp}}$.*
3. *The resolvent R_f is an analytic function on $\mathbb{P}_k^{1,an} \setminus \Sigma_f$ which is equal to zero at infinity.*

Lemma 2.15 ([Azz18, Lemma 2.20])**.** *Let E be a non-zero k-Banach algebra and $f \in E$. If $a \in (\mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_f) \cap k$, then the biggest open disk centred at a contained in $\mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_f$ has radius $R = \|(f-a)^{-1}\|_{\mathrm{Sp}}^{-1}$.*

Let M, N be two k-Banach spaces. Recall that a k-linear map $\varphi : M \to N$ is said to be bounded if

$$\exists C \in \mathbb{R}_+,\ \forall m \in M;\ \|\varphi(m)\| \leq C\|m\|.$$

If moreover φ is an isomorphism and if φ^{-1} is bounded we say that it is bi-bounded isomorphism.

Notation 2.16. Denote By $\mathcal{L}_k(M,N)$ the k-algebra of bounded k-linear maps $M \to N$. We set $\mathcal{L}_k(M,M) := \mathcal{L}_k(M)$.

The k-algebra $\mathcal{L}_k(M,N)$ equipped with the *operator norm*:

$$\varphi \mapsto \|\varphi\|_{\mathrm{op}} := \sup_{m \in M\setminus\{0\}} \frac{\|\varphi(m)\|}{\|m\|} \tag{2.19}$$

is a k-Banach algebra.

Lemma 2.17. *Let E and E' be two k-Banach algebras and $\varphi : E \to E'$ be a bounded morphism of k-algebras. If $f \in E$ then we have:*

$$\Sigma_{\varphi(f)}(E') \subset \Sigma_f(E).$$

If moreover φ is a bi-bounded morphism then we have the equality.

Proof. Consequence of the definition. □

The following Lemma allow us to compute the spectrum of a translate of a connection ∇ from the spectrum of ∇. We will see that in Sections 3.2 and 4.1.

Let $P(T) \in k[T]$, let E be a k-Banach algebra and let $f \in E$. We set $P(f)$ to be the image of $P(T)$ by the morphism $k[T] \to E$, $T \mapsto f$.

Lemma 2.18. *We have the equality of sets:*

$$\Sigma_{P(f)} = P(\Sigma_f)$$

Proof. The proof is analogous to the proof [Bou07, p. 2]. □

3. Differential module over $k(\!(S)\!)$ and spectrum

In this section we recall some properties of a differential module (M, ∇) over (F, d), where (F, d) is a finite differential extension of $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}})$. We will recall also some general properties of the spectrum associated to (M, ∇), and show how it behaves under ramification of the indeterminate S.

Convention 3.1. We fix $r \in (0, 1)$ and endow $k(\!(S)\!)$ with the S-adic absolute value given by $|\sum_{i\geq N} a_i S^i| := r^N$, if $a_N \neq 0$. In this setting the pair $(k(\!(S)\!), |.|)$ coincides with $\mathscr{H}(x_{0,r})$, where $x_{0,r} \in \mathbb{A}_k^{1,\mathrm{an}}$.

Remark 3.2. The derivation $S\frac{\mathrm{d}}{\mathrm{dS}}$ is bounded as an operator on $k(\!(S)\!)$.

3.1. **Spectrum of differential modules.** Recall that if F is a finite extension of $k(\!(S)\!)$ of degree m, then we have $F \simeq k(\!(S^{\frac{1}{m}})\!)$ [VS12, Proposition 3.3]. The absolute value $|.|$ on $k(\!(S)\!)$ extends uniquely to an absolute value on F. The pair $(F, |.|)$ is an element of $E(k)$ and can be identified with $\mathscr{H}(x_{0,r^{\frac{1}{m}}})$. The derivation $S\frac{\mathrm{d}}{\mathrm{dS}}$ extends uniquely to a derivation d on F, where $d(S^{\frac{1}{m}}) = \frac{1}{m}S^{\frac{1}{m}}$. Then (F, d) is a finite differential extension of $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}})$. Hence, the derivation d is also bounded as an operator on F. Conversely, any finite differential extension of $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}})$ is obtained this way.

Let (F, d) be a finite differential extension of $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}})$. Let (M, ∇) be a differential module over (F, d). In order to associate to this differential module a spectrum we need to endow M with a structure of k-Banach space. By fixing a basis on M we can pull-back the structure of F-Banach space of F^n equipped with the max norm to M. Any other choice of basis induces an equivalent F-Banach structure.

This induces a structure of k-Banach space on M. As ∇ satisfies the formula (2.1) and $d \in \mathcal{L}_k(M)$, we have $\nabla \in \mathcal{L}_k(M)$. The spectrum of (M, ∇) will be the spectrum of ∇ as an element of $\mathcal{L}_k(M)$, which we denote by $\Sigma_{\nabla,k}(\mathcal{L}_k(M))$ (or just by Σ_∇ if the dependence is obvious from the context). This spectrum does not depend on the choice of a basis on M. Indeed, since the norms are equivalent, by Lemma 2.17 we obtain the equality of spectra.

Let $\varphi : (M, \nabla) \to (N, \nabla')$ be a morphism of differential modules. If we endow M and N with structures of k-Banach spaces then it induces a bounded k-linear map. In the case where φ is an isomorphism, then it induces a bi-bounded k-linear isomorphism and according to Lemma 2.17 we have:

$$\Sigma_\nabla(\mathcal{L}_k(M)) = \Sigma_{\nabla'}(\mathcal{L}_k(M)) \tag{3.1}$$

This proves the following statement:

Proposition 3.3. *The spectrum of a connection is invariant by bi-bounded isomorphisms of differential modules.*

Remark 3.4. We observe here that spectrum depends on the choice of the derivation. In this paper we chose in particular the derivation $S\frac{\mathrm{d}}{\mathrm{dS}}$ to compute the spectrum.

This spectrum behaves nicely under exact sequences and direct sums.

Proposition 3.5. *Let (M,∇), (M_1,∇_1) and (M_2,∇_2) be three differential modules over (F,d). If we have two exact sequences of the form:*

$$0\to(M_1,\nabla_1)\to(M,\nabla)\to(M_2,\nabla_2)\to 0$$
$$0\to(M_2,\nabla_2)\to(M,\nabla)\to(M_1,\nabla_1)\to 0.$$

Then we have $\Sigma_\nabla(\mathcal{L}_k(M))=\Sigma_{\nabla_1}(\mathcal{L}_k(M_1))\cup\Sigma_{\nabla_2}(\mathcal{L}_k(M_2))$.

Proof. See [Azz18, Proposition 3.7] and [Azz18, Remark 3.8]. □

Corollary 3.6. *Let (M,∇), (M_1,∇_1) and (M_2,∇_2) be three differential modules over (F,d). If we suppose that $(M,\nabla)=(M_1,\nabla_1)\oplus(M_2,\nabla_2)$, then we have $\Sigma_\nabla(\mathcal{L}_k(M))=\Sigma_{\nabla_1}(\mathcal{L}_k(M_1))\cup\Sigma_{\nabla_2}(\mathcal{L}_k(M_2))$.*

Corollary 3.7. *Let $f\in F$. Consider the differential module $(M,\nabla):=(\mathscr{D}_F/\mathscr{D}_F.(D-f)^n,D)$. Then we have:*

$$\Sigma_\nabla(\mathcal{L}_k(M))=\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}}+f}(\mathcal{L}_k(F)).$$

3.2. **Spectrum of a differential module after ramified ground field extension.** In this section we show how the spectrum of a differential module behaves after ramified ground field extensions.

Let (F,d) be a finite differential extension of $(k(\!(S)\!),S\frac{\mathrm{d}}{\mathrm{dS}})$. Let $m\in\mathbb{N}$ such that $F\simeq k(\!(S^{\frac{1}{m}})\!)$. Now, if we set $Z=S^{\frac{1}{m}}$, then we have $F=k(\!(Z)\!)$ and $d=\frac{Z}{m}\frac{\mathrm{d}}{\mathrm{dZ}}$. Note that we can see $(F,\frac{Z}{m}\frac{\mathrm{d}}{\mathrm{dZ}})$ as a differential module over $(k(\!(S)\!),S\frac{\mathrm{d}}{\mathrm{dS}})$. In the basis $\{1,Z,\cdots,Z^{m-1}\}$ we have:

$$\frac{Z}{m}\frac{\mathrm{d}}{\mathrm{dZ}}\begin{pmatrix}f_1\\ \vdots\\ f_m\end{pmatrix}=\begin{pmatrix}S\frac{\mathrm{d}}{\mathrm{dS}}f_1\\ \\ \\ S\frac{\mathrm{d}}{\mathrm{dS}}f_m\end{pmatrix}+\begin{pmatrix}0 & 0 & \cdots & 0\\ 0 & \frac{1}{m} & & 0\\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \frac{m-1}{m}\end{pmatrix}\begin{pmatrix}f_1\\ \vdots\\ f_m\end{pmatrix}\tag{3.2}$$

We have a functor:

$$\begin{array}{rccc} I_F{}^*: S\frac{\mathrm{d}}{\mathrm{dS}}-\mathbf{Mod}(k(\!(S)\!)) & \longrightarrow & \frac{Z}{m}\frac{\mathrm{d}}{\mathrm{dZ}}-\mathbf{Mod}(F)\\ (M,\nabla) & \mapsto & (I_F{}^*M, I_F{}^*\nabla)\end{array}\tag{3.3}$$

where $I_F{}^*M=M\otimes_{k(\!(S)\!)}F$ and the connection $I_F{}^*\nabla$ is defined as follows:

$$I_F{}^*\nabla=\nabla\otimes 1+1\otimes\frac{Z}{m}\frac{\mathrm{d}}{\mathrm{dZ}}.$$

Let (M,∇) be an object of $S\frac{\mathrm{d}}{\mathrm{dS}}-\mathbf{Mod}(k(\!(S)\!))$ of rank n. If $\{e_1,\ldots,e_n\}$ is a basis of M such that we have:

$$\nabla\begin{pmatrix}f_1\\ \vdots\\ f_n\end{pmatrix}=\begin{pmatrix}S\frac{\mathrm{d}}{\mathrm{dS}}f_1\\ \vdots\\ S\frac{\mathrm{d}}{\mathrm{dS}}f_n\end{pmatrix}+G\begin{pmatrix}f_1\\ \vdots\\ f_n\end{pmatrix},$$

with $G\in\mathcal{M}(k(\!(S)\!))$, then $(I_F{}^*M,I_F{}^*\nabla)$ is of rank n and in the basis $\{e_1\otimes 1,\ldots,e_n\otimes 1\}$ we have:

$$I_F{}^*\nabla\begin{pmatrix}f_1\\ \vdots\\ f_n\end{pmatrix}=\begin{pmatrix}\frac{Z}{m}\frac{\mathrm{d}}{\mathrm{dZ}}f_1\\ \vdots\\ \frac{Z}{m}\frac{\mathrm{d}}{\mathrm{dZ}}f_n\end{pmatrix}+G\begin{pmatrix}f_1\\ \vdots\\ f_n\end{pmatrix}.\tag{3.4}$$

We have also the functor:

$$\begin{array}{rccc} I_{F*}: \frac{Z}{m}\frac{\mathrm{d}}{\mathrm{dZ}}-\mathbf{Mod}(F) & \longrightarrow & S\frac{\mathrm{d}}{\mathrm{dS}}-\mathbf{Mod}(k(\!(S)\!))\\ (M,\nabla) & \mapsto & (I_{F*}M, I_{F*}\nabla)\end{array}\tag{3.5}$$

where $I_{F*}M$ is the restriction of scalars of M via $k(\!(S)\!) \hookrightarrow F$, and $\nabla = I_{F*}\nabla$ are equal as k-linear maps. If (M,∇) has rank equal to n, the rank of $(I_{F*}M, I_{F*}\nabla)$ is equal to $n.m$.

Let (M,∇) be an object of $S\frac{\mathrm{d}}{\mathrm{dS}} - \mathbf{Mod}(k(\!(S)\!))$ of rank n. The differential module $(I_{F*}{I_F}^*M, I_{F*}{I_F}^*\nabla)$ has rank nm. Let $\{e_1,\cdots,e_n\}$ be a basis of (M,∇) and let G be the associated matrix in this basis. Then the matrix associated to $(I_{F*}{I_F}^*M, I_{F*}{I_F}^*\nabla)$ in the basis $\{e_1\otimes 1,\cdots,e_n\otimes 1, e_1\otimes Z,\cdots,e_n\otimes Z,\cdots,e_1\otimes Z^{m-1},\cdots,e_n\otimes Z^{m-1}\}$ is:

$$\begin{pmatrix} G & 0 & \cdots & 0 \\ 0 & G+\frac{1}{m}\cdot I_n & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & G+\frac{m-1}{m}\cdot I_n \end{pmatrix}.$$

Therefore we have the following isomorphism:

$$(I_{F*}{I_F}^*M, I_{F*}{I_F}^*\nabla) \simeq \bigoplus_{i=0}^{m-1}(M,\nabla+\frac{i}{m}). \tag{3.6}$$

As k-Banach spaces $I_{F*}{I_F}^*M$ and ${I_F}^*M$ are the same, and $I_{F*}{I_F}^*\nabla$ as a k-linear map coincides with ${I_F}^*\nabla$. Therefore, we have

$$\Sigma_{I_F^*\nabla,k}(\mathcal{L}_k(I_F^*M)) = \Sigma_{I_{F*}I_F^*\nabla,k}(\mathcal{L}_k(I_{F*}I_F^*M)). \tag{3.7}$$

By Lemma 2.18 and Corollary 3.6 we have:

$$\Sigma_{{I_F}^*\nabla,k}(\mathcal{L}_k({I_F}^*M)) = \bigcup_{i=0}^{m-1}\frac{i}{m} + \Sigma_{\nabla,k}(\mathcal{L}_k(M)). \tag{3.8}$$

3.3. **Newton polygon and the decomposition according to the slopes.** Let $v : k(\!(S^{\frac{1}{m}})\!) \to \mathbb{Z}\cup\{\infty\}$ be the valuation map associated to the $S^{\frac{1}{m}}$-adic valuation. Let $P = \sum_{i=0}^{n} g_i D^i$ be an element of $\mathscr{D}_{k(\!(S)\!)}$. Let L_P be the convex hull in $\mathbb{R}^2$ of the set of points

$$\{(i, v(g_i))|\ 0\le i\le n\}\cup\{(0, \min_{0\le i\le n} v(g_i))\}.$$

Definition 3.8 ([VS12, Definition 3.44])**.** The Newton polygon $\mathrm{NP}(P)$ of P is the boundary of L_P. The finite slopes γ_i of P are called the slopes of $\mathrm{NP}(P)$. The horizontal width of the segment of $NP(P)$ of slope γ_i is called the multiplicity of γ_i.

Definition 3.9. A differential module (M,∇) over $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}})$ is said to be regular singular if there exists a basis for which G (cf. (2.1)) has constant entries (i.e $G\in\mathcal{M}_n(k)$). We will call the eigenvalues of such G the exponents of (M,∇).

Proposition 3.10. *Let (M,∇) be a differential module over $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}})$. The following properties are equivalent:*

- *(M,∇) is regular singular;*
- *There exists a differential polynomial $P(D)$ with only one slope equal to 0 such that $(M,\nabla)\simeq \mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.P(D)$;*
- *There exists $P(D) = g_0 + g_1 D + \cdots + g_{n-1}D^{n-1} + D^n$ with $g_i\in k[\![S]\!]$, such that $(M,\nabla)\simeq (\mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.P(D), D)$;*
- *There exists $P(D) = g_0 + g_1 D + \cdots + g_{n-1}D^{n-1} + D^n$ with $g_i\in k$, such that $(M,\nabla)\simeq (\mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.P(D), D)$;*

Proof. See [Ked10, Corollary 7.1.3] and [Chr, Proposition 10.1]. □

Proposition 3.11 ([Ked10, Proposition 7.3.6])**.** *Let $P(D) = g_0 + g_1 D + \cdots + g_{n-1}D^{n-1} + D^n$ such that $g_i \in k[\![S]\!]$. Then we have the isomorphism in $S\frac{\mathrm{d}}{\mathrm{dS}} - \mathbf{Mod}(k(\!(S)\!))$:*

$$(\mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.P(D), D) \simeq (\mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.P_0(D)),$$

where $P_0(D) = g_0(0) + g_1(0)D + \cdots g_{n-1}(0)D^{n-1} + D^n$.

Remark 3.12. This proposition means in particular that for all $f \in k[\![S]\!]$ there exists $g \in k(\!(S)\!) \setminus \{0\}$ such that $f - \frac{S\frac{\mathrm{d}}{\mathrm{dS}}(g)}{g} = f(0)$. Indeed, by Proposition 3.11 we have $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}} + f) \simeq (k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}} + f(0))$. This is equivalente to saying that there exists $g \in k(\!(S)\!) \setminus \{0\}$ such that

$$g^{-1} \circ S\frac{\mathrm{d}}{\mathrm{dS}} \circ g + f = S\frac{\mathrm{d}}{\mathrm{dS}} + f(0).$$

Definition 3.13. Let (M, ∇) be an element of $S\frac{\mathrm{d}}{\mathrm{dS}} - \mathbf{Mod}(k(\!(S)\!))$, and let $P(D) \in \mathscr{D}_{k(\!(S)\!)}$ such that $(M, \nabla) \simeq (\mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.P(D), D)$. If all the slopes of $P(D)$ are different from 0, then we say that (M, ∇) is without regular part.

Proposition 3.14. *Let $P \in \mathscr{D}_{k(\!(S)\!)}$ and γ a slope of P. Let ν be the multiplicity of γ. Then there exist differential polynomials R, R', Q and Q' which satisfy the following properties:*

- *$P = RQ = Q'R'$.*
- *The degree of R and R' is equal to ν, and their only slopes are γ with multiplicity equal to ν.*
- *All the slopes of Q and Q' are different from γ.*
- *$\mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.P = \mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.R \oplus \mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.Q = \mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.Q' \oplus \mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.R'$.*

Proof. See [Chr, Proposition 12.1] and [VS12, Theorem 3.48]. □

Corollary 3.15. *Let (M, ∇) be a differential module over $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}})$. Then we have decomposition in $S\frac{\mathrm{d}}{\mathrm{dS}} - \mathbf{Mod}(k(\!(S)\!))$:*

$$(M, \nabla) = (M_{\mathrm{reg}}, \nabla_{\mathrm{reg}}) \oplus (M_{\mathrm{irr}}, \nabla_{\mathrm{irr}})$$

where $(M_{\mathrm{reg}}, \nabla_{\mathrm{reg}})$ is a regular singular differential module and $(M_{\mathrm{irr}}, \nabla_{\mathrm{irr}})$ is a differential module without regular part.

4. Spectrum of a differential module

In this section we compute the spectrum of a differential module (M, ∇) over $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}})$. The main statement of the paper is the following.

Theorem 4.1. *Let (M, ∇) be a differential module over $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}})$. Let $\{\gamma_1, \cdots, \gamma_\nu\}$ be the set of slopes of (M, ∇) and let $\{a_1, \cdots, a_\mu\}$ be the set of exponents of the regular part of (M, ∇). Then the spectrum of ∇ as an element of $\mathcal{L}_k(M)$ is:*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \{x_{0,r^{-\gamma_1}}, \cdots, x_{0,r^{-\gamma_\nu}}\} \cup \bigcup_{i=1}^{\mu}(a_i + \mathbb{Z})$$

Remark 4.2. We observe that although differential modules over $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}})$ are algebraic objects, their spectra in the sense of Berkovich depends highly on the choice of the absolute value on $k(\!(S)\!)$.

According to Corollary 3.15 we have the decomposition:

$$(M, \nabla) = (M_{\mathrm{reg}}, \nabla_{\mathrm{reg}}) \oplus (M_{\mathrm{irr}}, \nabla_{\mathrm{irr}})$$

We know that $\Sigma_\nabla = \Sigma_{\nabla_{\mathrm{reg}}} \cup \Sigma_{\nabla_{\mathrm{irr}}}$ (cf. Corollary 3.6). Therefore, in order to obtain the main statement, it is enough to know the spectrum of a regular singular differential module and the spectrum of differential module without regular part.

4.1. **Spectrum of regular singular differential module.** Let (M,∇) be a regular singular differential module. The behaviour of the spectrum of ∇ is recapitulated in the following theorem.

Theorem 4.3. *Let (M,∇) be a regular singular differential module over $(k(\!(S)\!),S\frac{\mathrm{d}}{\mathrm{dS}})$. Let G a matrix associated to ∇ with constant entries (i.e. $G\in\mathcal{M}_\nu(k)$), and let $\{a_1,\cdots,a_N\}$ be the set of eigenvalues of G. The spectrum of ∇ is*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M))=\bigcup_{i=1}^{N}(a_i+\mathbb{Z})\cup\{x_{0,1}\}.$$

Lemma 4.4 ([VS12, Proposition 3.12])**.** *There exists a basis for which the set of the eigenvalues $\{a_1,\cdots,a_N\}$ of G satisfies $a_i-a_j\notin\mathbb{Z}$ for each $i\neq j$.*

In order to prove this Theorem, we use the following Proposition to reduce the computation of the spectrum of $S\frac{\mathrm{d}}{\mathrm{dS}}$ as an element of $(k(\!(S)\!),\nabla)$.

Proposition 4.5 ([Azz18, Proposition 3.15])**.** *Let (M,∇) be a differential module over $(k(\!(S)\!),S\frac{\mathrm{d}}{\mathrm{dS}})$ such that:*

$$\nabla\begin{pmatrix}f_1\\ \vdots\\ f_n\end{pmatrix}=\begin{pmatrix}df_1\\ \vdots\\ df_n\end{pmatrix}+G\begin{pmatrix}f_1\\ \vdots\\ f_n\end{pmatrix},$$

with $G\in\mathcal{M}_n(k)$. The spectrum of ∇ is $\Sigma_{\nabla,k}(\mathcal{L}_k(M))=\bigcup_{i=1}^{N}(a_i+\Sigma_{d,k}(\mathcal{L}_k(k(\!(S)\!))))$, where $\{a_1,\ldots,a_N\}$ are the eigenvalues of G.

We now compute the spectrum of $S\frac{\mathrm{d}}{\mathrm{dS}}$.

Lemma 4.6. *The norm and spectral semi-norm of $S\frac{\mathrm{d}}{\mathrm{dS}}$ as an element of $\mathcal{L}_k(k(\!(S)\!))$ satisfy:*

$$\|S\frac{\mathrm{d}}{\mathrm{dS}}\|=1,\quad \|S\frac{\mathrm{d}}{\mathrm{dS}}\|_{\mathrm{Sp}}=1.$$

Proof. Since $\|S\|=|S|=r$ and $\|\frac{\mathrm{d}}{\mathrm{dS}}\|=\frac{1}{r}$ (cf. [Pul15, Lemma 4.4.1]), we have $\|S\frac{\mathrm{d}}{\mathrm{dS}}\|\leq 1$. Hence also, $\|S\frac{\mathrm{d}}{\mathrm{dS}}\|_{\mathrm{Sp}}\leq 1$. The map

$$\begin{array}{ccc}\mathcal{L}_k(\mathscr{H}(x)) & \longrightarrow & \mathcal{L}_k(\mathscr{H}(x))\\ \varphi & \mapsto & S^{-1}\circ\varphi\circ S\end{array}$$

is bi-bounded and induces change of basis. Therefore, as $S^{-1}\circ(S\frac{\mathrm{d}}{\mathrm{dS}})\circ S=S\frac{\mathrm{d}}{\mathrm{dS}}+1$, we have $\|S\frac{\mathrm{d}}{\mathrm{dS}}\|_{\mathrm{Sp}}=\|S\frac{\mathrm{d}}{\mathrm{dS}}+1\|_{\mathrm{Sp}}$. Since 1 commutes with $S\frac{\mathrm{d}}{\mathrm{dS}}$, we have:

$$1=\|1\|_{\mathrm{Sp}}=\|S\frac{\mathrm{d}}{\mathrm{dS}}+1-S\frac{\mathrm{d}}{\mathrm{dS}}\|_{\mathrm{Sp}}\leq\max(\|S\frac{\mathrm{d}}{\mathrm{dS}}+1\|_{\mathrm{Sp}},\|S\frac{\mathrm{d}}{\mathrm{dS}}\|_{\mathrm{Sp}}).$$

Consequently, we obtain

$$\|S\frac{\mathrm{d}}{\mathrm{dS}}\|=\|S\frac{\mathrm{d}}{\mathrm{dS}}\|_{\mathrm{Sp}}=1.$$

□

Proposition 4.7. *The spectrum of $S\frac{\mathrm{d}}{\mathrm{dS}}$ as an element of $\mathcal{L}_k(k(\!(S)\!))$ is equal to:*

$$\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}}}(\mathcal{L}_k(k(\!(S)\!)))=\mathbb{Z}\cup\{x_{0,1}\}$$

We will use the following lemma.

Lemma 4.8 ([Azz18, Lemma 2.29]). *Let M_1 and M_2 be k-Banach spaces and let $M = M_1 \oplus M_2$ endowed with the max norm. Let p_1, p_2 be the respective projections associated to M_1 and M_2 and i_1, i_2 be the respective inclusions. Let $\varphi \in \mathcal{L}_k(M)$and set $\varphi_1 = p_1\varphi i_1 \in \mathcal{L}_k(M_1)$ and $\varphi_2 = p_2\varphi i_2 \in \mathcal{L}_k(M_2)$. If $\varphi(M_1) \subset M_1$, then we have:*

i) $\Sigma_{\varphi_i}(\mathcal{L}_k(M_i)) \subset \Sigma_\varphi(\mathcal{L}_k(M)) \cup \Sigma_{\varphi_j}(\mathcal{L}_k(M_j))$, *where i, $j \in \{1,2\}$ and $i \neq j$.*
ii) $\Sigma_\varphi(\mathcal{L}_k(M)) \subset \Sigma_{\varphi_1}(\mathcal{L}_k(M_1)) \cup \Sigma_{\varphi_2}(\mathcal{L}_k(M_2))$. *Furthermore, if $\varphi(M_2) \subset M_2$, then we have the equality.*
iii) *If $\Sigma_{\varphi_1}(\mathcal{L}_k(M_1)) \cap \Sigma_{\varphi_2}(\mathcal{L}_k(M_2)) = \varnothing$, then $\Sigma_\varphi(\mathcal{L}_k(M)) = \Sigma_{\varphi_1}(\mathcal{L}_k(M_1)) \cup \Sigma_{\varphi_2}(\mathcal{L}_k(M_2))$.*

Proof of Proposition 4.7. We set $d := S\frac{\mathrm{d}}{\mathrm{d}S}$ and $\Sigma_{d-n} := \Sigma_{d-n,k}(\mathcal{L}_k(k(\!(S)\!)))$. As $\|d\|_{\mathrm{Sp}} = 1$ (cf. Lemma 4.6), we have $\Sigma_d \subset D^+(0,1)$.

Let $a \in k \cap D^+(0,1)$. If $a \in \mathbb{Z}$, then we have $(d-a)(S^a) = 0$. Hence, $d-a$ is not injective and $\mathbb{Z} \subset \Sigma_d$. As the spectrum is compact, we have $\mathbb{Z} \cup \{x_{0,1}\} \subset \Sigma_d$. If $a \notin \mathbb{Z}$, then $d-a$ is invertible in $\mathcal{L}_k(k(\!(S)\!))$. Indeed, let $g(S) = \sum_{i\in\mathbb{Z}} b_i S^i \in k(\!(S)\!)$. If there exists $f = \sum_{i\in\mathbb{Z}} a_i S^i \in k(\!(S)\!)$ such that $(d-a)f = g$, then for each $i \in \mathbb{Z}$ we have

$$a_i = \frac{b_i}{(i-a)}.$$

For each $i \in \mathbb{Z}$ we have $|a_i| = |b_i|$. This means that f it is unique and converges in $k(\!(S)\!)$. We obtain also $|f| = |g|$. Consequently, the set theoretical inverse $(d-a)^{-1}$ is bounded and $\|(d-a)^{-1}\| = 1$. Hence, we have $\|(d-a)^{-1}\|_{\mathrm{Sp}} = 1$. According to Lemma 2.15, we have $D^-(a,1) \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_d$.

Recall that $D^+(0,1) = \bigcup_{a\in k}[a, x_{0,1}]$ (cf. (2.8)). In order to end the proof, it is enough to show that $(n, x_{0,1}) \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_d$ for all $n \in \mathbb{Z}$. Let $n \in \mathbb{Z}$. Then we have

$$k(\!(S)\!) = k.S^n \oplus \widehat{\bigoplus}_{i\in\mathbb{Z}\setminus\{n\}} k.S^i.$$

The operator $(d-n)$ stabilises both $k.S^n$ and $\widehat{\bigoplus}_{i\in\mathbb{Z}\setminus\{n\}} k.S^i$. We set $(d-n)|_{k.S^n} = \nabla_1$ and $(d-n)|_{\widehat{\bigoplus}_{i\in\mathbb{Z}\setminus\{n\}} k.S^i} = \nabla_2$. We set $\Sigma_{\nabla_1} := \Sigma_{\nabla_1,k}(\mathcal{L}_k(k.S^n))$ and $\Sigma_{\nabla_2} := \Sigma_{\nabla_2,k}(\mathcal{L}_k(\widehat{\bigoplus}_{i\in\mathbb{Z}\setminus\{n\}} k.S^i))$. We have $\nabla_1 = 0$. By Lemma 4.8, we have:

$$\Sigma_{d-n} = \Sigma_{\nabla_1} \cup \Sigma_{\nabla_2} = \{0\} \cup \Sigma_{\nabla_2}.$$

We now prove that

$$D^-(0,1) \cap \Sigma_{\nabla_2} = \varnothing.$$

The operator ∇_2 is invertible in $\mathcal{L}_k(\widehat{\bigoplus}_{i\in\mathbb{Z}\setminus\{n\}} k.S^i)$. Indeed, let $g(S) = \sum_{i\in\mathbb{Z}\setminus\{n\}} b_i S^i \in \widehat{\bigoplus}_{i\in\mathbb{Z}\setminus\{n\}} k.S^i$. If there exists $f = \sum_{i\in\mathbb{Z}\setminus\{n\}} a_i S^i \in \widehat{\bigoplus}_{i\in\mathbb{Z}\setminus\{n\}} k.S^i$ such that $\nabla_2(f) = g$, then for each $i \in \mathbb{Z}\setminus\{n\}$ we have

$$a_i = \frac{b_i}{(i-n)}.$$

Since $|a_i| = |b_i|$, the element f exists and it is unique, moreover $|f| = |g|$. Hence, ∇_2 is invertible in $\mathcal{L}_k(\widehat{\bigoplus}_{i\in\mathbb{Z}\setminus\{n\}} k.S^i)$ and as a k-linear map it is isometric. Therefore, we have $\|\nabla_2^{-1}\|_{\mathrm{Sp}} = 1$. Hence, by Lemma 2.15 $D^-(0,1) \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_{\nabla_2}$. Consequently, $D^-(0,1) \cap \Sigma_{d-n} = \{0\}$. As $\Sigma_d = \Sigma_{d-n} + n$ (cf. Lemma 2.18), we have $D^-(n,1) \cap \Sigma_d = \{n\}$. Therefore, for all $n \in \mathbb{Z}$ we have $(n, x_{0,1}) \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_d$ and the claim follows. □

4.2. **Spectrum of a differential module without regular part.** Recall the following Theorem, which is the celebrated theorem of Turrittin. It ensures that any differential module becomes extension of rank one differential modules after pull-back by a suitable ramified extension.

Theorem 4.9 ([Tur55]). *Let (M,∇) be a differential module over $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{d}S})$. There exists a finite extension $F = k(\!(S^{\frac{1}{m}})\!)$ such that we have:*

$$(I_F{}^*M, I_F{}^*\nabla) = \bigoplus_{i=1}^{N} (\mathscr{D}_F/\mathscr{D}_F.(D-f_i)^{\alpha_i}, D) \tag{4.1}$$

where $f_i \in k[\![S^{-\frac{1}{m}}]\!]$ and $\alpha_i \in \mathbb{N}$.

Proof. See [VS12, Theorem 3.1]. □

Now, in order to compute the spectrum, we need the following result.

Proposition 4.10. *Let $f = \sum_{i\in\mathbb{Z}} a_i S^{\frac{i}{m}}$ an element of $F := k(\!(S^{\frac{1}{m}})\!)$ and let (F,∇) be the differential module of rank one such that $\nabla = S\frac{\mathrm{d}}{\mathrm{d}S} + f$. If $v(f) < 1$, then the spectrum of ∇ as an element of $\mathcal{L}_k(F)$ is:*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(F)) = \{x_{0,r^{v(f)}}\}.$$

The following results are necessary to prove this proposition.

Lemma 4.11. *Let $\Omega \in E(k)$. Consider the isometric embedding of k-algebras*

$$\begin{array}{ccc} \Omega & \longrightarrow & \mathcal{L}_k(\Omega) \\ a & \mapsto & b \mapsto a.b \end{array}.$$

With respect to this embedding, Ω is a maximal commutative subalgebra of $\mathcal{L}_k(\Omega)$.

Proof. Let A be a commutative subalgebra of $\mathcal{L}_k(\Omega)$ such that $\Omega \subset A$. Then each element of A is an endomorphism of Ω that commutes with the elements of Ω. Therefore, $A \subset \mathcal{L}_\Omega(\Omega) = \Omega$. Hence, we have $A = \Omega$. □

Lemma 4.12. *Let $\Omega \in E(k)$ and $\pi_{\Omega/k} : \mathbb{A}^{1,\mathrm{an}}_\Omega \to \mathbb{A}^{1,\mathrm{an}}_k$ be the canonical projection. Let $\alpha \in \Omega$. The spectrum of α as an element of $\mathcal{L}_k(\Omega)$ is $\Sigma_\alpha(\mathcal{L}_k(\Omega)) = \{\pi_{\Omega/k}(\alpha)\}$.*

Proof. By [Ber90, Proposition 7.1.4, i)], the spectrum of α as an element of Ω is the point which corresponds to the character $k[T] \to \Omega$, $T \mapsto \alpha$. Hence, $\Sigma_{\alpha,k}(\Omega) = \{\pi_{\Omega/k}(\alpha)\}$. By Lemma 4.11 and [Ber90, Proposition 7.2.4] we conclude. □

Lemma 4.13 ([Azz18, Lemma 2.3]). *Let $\Omega \in E(k)$ and let M be a k-Banach space. Then, the inclusion $M \hookrightarrow M \otimes_k \Omega$ is an isometry. In particular, for all $v \in M$ and $c \in K$ we have $\|v \otimes c\| = |c|\cdot\|v\|$.*

Proof of Proposition 4.10. We set $d := S\frac{\mathrm{d}}{\mathrm{d}S}$. We can assume that $f = \sum_{i\in\mathbb{N}} a_i S^{\frac{-i}{m}}$. Indeed, since $f = f_- + f_+$ with $f_- := \sum_{i<0} a_i S^{\frac{-i}{m}}$ and $f_+ := \sum_{i\geq 0} a_i S^{\frac{-i}{m}}$, according to Remark 3.12 there exists $g \in k(\!(S^{\frac{1}{m}})\!)$ such that $f_+ - a_0 = \frac{S\frac{\mathrm{d}}{\mathrm{d}S}(g)}{g}$. Therefore, we have $(k(\!(S^{\frac{1}{m}})\!), \nabla) \simeq (k(\!(S^{\frac{1}{m}})\!), S\frac{\mathrm{d}}{\mathrm{d}S} + f_- + a_0)$. Since the point $\pi_{F/k}(f)$ corresponds to the character $k[T] \to F$, $T \mapsto f$ and $F \simeq \mathscr{H}(x_{0,r^{\frac{1}{m}}})$, it coincides with $f(x_{0,r^{\frac{1}{m}}})$(cf. Notation 2.7). Moreover, we have $f(x_{0,r^{\frac{1}{m}}}) = x_{0,|f|} = x_{0,r^{v(f)}}$. By Lemma 4.12 $\Sigma_{f,k}(\mathcal{L}_k(F)) = \{x_{0,r^{v(f)}}\}$. Let us prove now that $\Sigma_{\nabla,k}(\mathcal{L}_k(F)) = \{x_{0,r^{v(f)}}\}$. Let $y \in \mathbb{A}^{1,\mathrm{an}}_k \setminus \{x_{0,r^{v(f)}}\}$. We know that $f \otimes 1 - 1 \otimes T(y)$ is invertible in $F\hat{\otimes}_k\mathscr{H}(y)$, hence invertible in $\mathcal{L}_k(F)\hat{\otimes}_k\mathscr{H}(y)$. Since $d \otimes 1 = (\nabla \otimes 1 - 1 \otimes T(y)) - (f \otimes 1 - 1 \otimes T(y))$, in order to prove that $\nabla \otimes 1 - 1 \otimes T(y)$ is invertible, it is enough to show that

$$\|d \otimes 1\| < \|(f \otimes 1 - 1 \otimes T(y))^{-1}\|^{-1}.$$

In order to do so, since $\|d\| = \|d \otimes 1\| = 1$ (cf. Lemmas 4.13 and 4.6), it is enough to show that $1 < \|(f \otimes 1 - 1 \otimes T(y))^{-1}\|^{-1}$. On the one hand, since $F \hookrightarrow \mathcal{L}_k(F)$ is an isometric embedding, then so

is $F\hat{\otimes}_k\mathscr{H}(y) \to \mathcal{L}_k(F)\hat{\otimes}_k\mathscr{H}(y)$ (cf. [Poi13, Lemme 3.1]). On the other hand, we have $F\hat{\otimes}_k\mathscr{H}(y) = \mathcal{O}(C^+_{\mathscr{H}(y)}(0, r^{\frac{1}{m}}, r^{\frac{1}{m}}))$. Therefore, we have

$$\|(f\otimes 1 - 1\otimes T(y))^{-1}\|^{-1} = \|f\otimes 1 - 1\otimes T(y)\| = \max(|f-a_0|, |T(y)-a_0|).$$

Consequently, we obtain $1 < \|(f\otimes 1 - 1\otimes T(y))^{-1}\|^{-1}$ in $F\hat{\otimes}_k\mathscr{H}(y)$, hence in $\mathcal{L}_k(F)\hat{\otimes}_k\mathscr{H}(y)$. Since the spectrum $\Sigma_{\nabla,k}(\mathcal{L}_k(F))$ is not empty, we conclude that $\Sigma_{\nabla,k}(\mathcal{L}_k(F)) = \{x_{0,r^{v(f)}}\}$. □

Proposition 4.14. *Let (M,∇) be a differential module over $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}})$ without regular part. The spectrum of ∇ as an element of $\mathcal{L}_k(M)$ is:*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \{x_{0,r^{v(f_1)}}, \cdots, x_{0,r^{v(f_N)}}\}$$

where the f_i are as in the formula (4.1).

Proof. We set $\Sigma_\nabla := \Sigma_{\nabla,k}(\mathcal{L}_k(M))$. By Theorem 4.9, there exists $F = k(\!(S^{\frac{1}{m}})\!)$ such

$$({I_F}^*M, {I_F}^*\nabla) = \bigoplus_{i=1}^{N} \mathscr{D}_F/\mathscr{D}_F.(D-f_i)^{\alpha_i}$$

where $f_i \in k[\![S^{-\frac{1}{m}}]\!]$. We set $\Sigma_{{I_F}^*\nabla} := \Sigma_{{I_F}^*\nabla,k}(\mathcal{L}_k({I_F}^*M))$. Since (M,∇) is without regular part, we have $f_i \in k[\![S^{-\frac{1}{m}}]\!] \setminus k$. By Corollaries 3.6 and 3.7, we have:

$$\Sigma_{{I_F}^*\nabla} = \bigcup_{i=1}^{N} \Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}}+f_i}(\mathcal{L}_k(F)).$$

By Proposition 4.10, we have $\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}}+f_i}(\mathcal{L}_k(F)) = \{x_{0,r^{v(f_i)}}\}$. Hence,

$$\Sigma_{{I_F}^*\nabla} = \{x_{0,r^{v(f_1)}}, \cdots, x_{0,r^{v(f_N)}}\}.$$

By the formula (3.8), we have:

$$\Sigma_{{I_F}^*\nabla} = \bigcup_{i=0}^{m-1} \frac{i}{m} + \Sigma_\nabla.$$

Since $r^{v(f_i)} > 1$ for all $1 \le i \le N$, then each element of $\Sigma_{{I_F}^*\nabla}$ is invariant by translation by $\frac{j}{m}$ where $1 \le j \le m$. This means that $\Sigma_\nabla = \Sigma_\nabla + \frac{j}{m}$. Therefore, we have $\Sigma_{{I_F}^*\nabla} = \Sigma_\nabla$. □

Remark 4.15. Note that, it is not easy to compute the f_i of the formula (4.1). However, the values $-v(f_i)$ coincide with the slopes of the differential module (cf. [Kat87] and [VS12, Remarks 3.55]).

We now prove the main statement of the paper that summarizes all the previous results.

Proof of Theorem 4.1. According to Theorem 4.3, Proposition 4.14 and Remark 4.15 we obtain the result. □

References

[Azz18] T. A. Azzouz. "Spectrum of a linear differential equation with constant coefficients". In: (Feb. 2018). eprint: 1802.07306. URL: https://arxiv.org/abs/1802.07306.

[Ber90] V. G. Berkovich. *Spectral Theory and Analytic Geometry Over non-Archimedean Fields*. AMS Mathematical Surveys and Monographs 33. AMS, 1990.

[Ber93] V. G. Berkovich. "Étale cohomology for non-Archimedean analytic spaces." English. In: *Publ. Math., Inst. Hautes Étud. Sci.* 78 (1993), pp. 5–161. ISSN: 0073-8301; 1618-1913/e. DOI: 10.1007/BF02712916.

[Bou07] N. Bourbaki. *Théories spectrales: Chapitres 1 et 2*. Bourbaki, Nicolas. Springer Berlin Heidelberg, 2007. ISBN: 9783540353317.

[Chr] G. Christol. "Le théorème de turritin p-adique (version du 11/06/2011)".

[DMR07] P. Deligne, B. Malgrange, and J. Ramis. *Singularités irrégulières. Correspondance et documents*. French. Vol. 5. Paris: Société Mathématique de France, 2007, pp. x + 188. ISBN: 978-2-85629-241-9/hbk.

[Kat70] N. M. Katz. "Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin." English. In: *Publ. Math., Inst. Hautes Étud. Sci.* 39 (1970), pp. 175–232. ISSN: 0073-8301; 1618-1913/e. DOI: 10.1007/BF02684688.

[Kat87] N. M. Katz. "On the calculation of some differential galois groups". In: *Inventiones mathematicae* 87.1 (Feb. 1987), pp. 13–61. ISSN: 1432-1297. DOI: 10.1007/BF01389152. URL: https://doi.org/10.1007/

[Ked10] K. S. Kedlaya. *p-adic differential equations*. English. Cambridge: Cambridge University Press, 2010, pp. xvii + 380. ISBN: 978-0-521-76879-5/hbk.

[Mal74] B. Malgrange. "Sur les points singuliers des équations différentielles". In: *Enseignement Math. (2)* 20 (1974), pp. 147–176. ISSN: 0013-8584.

[Poi13] J. Poineau. "Les espaces de Berkovich sont angéliques". In: *Bull. soc. Math. France* (2013).

[Pul15] A. Pulita. "The convergence Newton polygon of a p-adic differential equation. I: Affinoid domains of the Berkovich affine line." English. In: *Acta Math.* 214.2 (2015), pp. 307–355. ISSN: 0001-5962; 1871-2509/e. DOI: 10.1007/s11511-015-0126-9.

[Rob80] P. Robba. "Lemmes de Hensel pour les opérateurs différentiels. Application à la reduction formelle des équations différentielles." French. In: *Enseign. Math. (2)* 26 (1980), pp. 279–311. ISSN: 0013-8584; 2309-4672/e.

[Tur55] H. L. Turrittin. "Convergent solutions of ordinary linear homogeneous differential equations in the neighborhood of an irregular singular point". In: *Acta Math.* 93 (1955), pp. 27–66. DOI: 10.1007/BF02392519. URL: https://doi.org/10.1007/BF02392519.

[VS12] M. Van der Put and M. F. Singer. *Galois Theory of Linear Differential Equations*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2012. ISBN: 9783642557507.

Tinhinane Amina, AZZOUZ

Univ. Grenoble Alpes, CNRS, Institut Fourier, F-38000 Grenoble, France
E-mail address: tinhinane-amina.azzouz@univ-grenoble-alpes.fr