Determinantal expressions and recurrence relations for Fubini and Eulerian polynomials

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DETERMINANTAL EXPRESSIONS AND RECURRENCE RELATIONS FOR FUBINI AND EULERIAN POLYNOMIALS

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Abstract. In the paper, by virtue of an explicit formula for higher order derivatives of the ratio between two differentiable functions in terms of the Hessenberg determinant, the author presents determinantal expressions and recurrence relations for generalized Fubini polynomials. Hereafter, by virtue of relations between Eulerian polynomials and the (generalized) Fubini numbers and polynomials, the author derives a determinantal expression for Eulerian polynomials and several recurrence relations for the Fubini and Eulerian polynomials.

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1. Introduction

For $n \ge 0$, let

$$F_n = \sum_{k=0}^{n} k! S(n,k), \tag{1.1}$$

where $S(n,k)$, which can be generated by the exponential function

$$\frac{(e^x-1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!}$$

and can be computed by the explicit formula

$$S(n,k) = \frac{1}{k!}\sum_{\ell=1}^{k}(-1)^{k-\ell}\binom{k}{\ell}\ell^n,$$

denotes the Stirling numbers of the second kind [16]. One calls these numbers F_n the Fubini numbers [7], ordered Bell numbers [10], or geometric numbers [36]. A Fubini number F_n has been interpreted in [10, 36] combinatorially: it counts all the possible set partitions of an n element set such that the order of the blocks matters. In [10], the Fubini numbers F_n were connected

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with preference arrangements and the recursion for F_n was derived. In [10, 11], the exponential generating function

$$G(t)=\frac{1}{2-e^t}=\sum_{n=0}^{\infty}F_n\frac{t^n}{n!} \tag{1.2}$$

and an asymptotic estimate for F_n were established.

In [36], the Fubini polynomials $F_n(y)$ were defined by

$$F_n(y)=\sum_{k=0}^{n}S(n,k)k!y^k. \tag{1.3}$$

These polynomials can be generated [36, pp. 735–736] by

$$G(t,y)=\frac{1}{1-y(e^t-1)}=\sum_{n=0}^{\infty}F_n(y)\frac{t^n}{n!}. \tag{1.4}$$

It is clear that $F_n(1)=F_n$. Due to the relation

$$\left(y\frac{\mathrm{d}}{\mathrm{d}\,y}\right)^m\frac{1}{1-y}=\sum_{k=0}^{\infty}k^my^k=\frac{1}{1-y}F_m\left(\frac{y}{1-y}\right),\quad |y|<1$$

in [2], one also calls $F_n(y)$ the geometric polynomials.

In [12, Section 2], the Fubini polynomials $F_n(x,y)$ of two variables x,y are defined by means of the generating function

$$G(t,y,x)=\frac{e^{xt}}{1-y(e^t-1)}=\sum_{n=0}^{\infty}F_n(x,y)\frac{t^n}{n!}.$$

It is apparent that $F_n(0,y)=F_n(y)$.

In [13], the bivariate Fubini polynomials $F_n^{(r)}(x,y)$ of order r for $r\in\mathbb{N}$, generated by

$$G(t,y,x,r)=\frac{e^{xt}}{[1-y(e^t-1)]^r}=\sum_{n=0}^{\infty}F_n^{(r)}(x,y)\frac{t^n}{n!},$$

were studied. It is obvious that $F_n^{(1)}(x,y)=F_n(x,y)$.

The generating functions of F_n, $F_n(y)$, $F_n(x,y)$, and $F_n^{(r)}(x,y)$ remind us to consider the generating function

$$G(t,y,x,q,z)=\frac{e^{xt}}{[z-y(e^t-1)]^q}=\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!},\quad x,q\in\mathbb{R} \tag{1.5}$$

and generalized Fubini polynomials $F_n(x,y,z,q)$ of four variables x,y,z,q. It is clear that, since

$$G(t,y,x,q,z)=\frac{1}{z^q}G\left(t,\frac{y}{z},x,r\right),$$

we have

$$F_n(x,y,z,r)=\frac{F_n^{(r)}(x,y/z)}{z^r}.$$

In this paper, we will establish some properties of generalized Fubini polynomials $F_n(x,y,z,q)$ and, consequently, derive some known results.

2. A new determiantal expression of the Fubini polynomials

At first, we simply establish a new determinantal expressions of the Fubini polynomials and numbers $F_n(y)$ and F_n respectively.

Theorem 2.1. *The Fubini polynomials $F_n(y)$ for $n \ge 0$ can be represented determinantally by*

$$F_n(y)=\begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & y & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & y & \binom{2}{1}y & -1 & \cdots & 0 & 0 & 0 \\ 0 & y & \binom{3}{1}y & \binom{3}{2}y & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ 0 & y & \binom{n-2}{1}y & \binom{n-2}{2}y & \cdots & \binom{n-2}{n-3}y & -1 & 0 \\ 0 & y & \binom{n-1}{1}y & \binom{n-1}{2}y & \cdots & \binom{n-1}{n-3}y & \binom{n-1}{n-2}y & -1 \\ 0 & y & \binom{n}{1}y & \binom{n}{2}y & \cdots & \binom{n}{n-3}y & \binom{n}{n-2}y & \binom{n}{n-1}y \end{vmatrix}. \tag{2.1}$$

Consequently, the Fubini numbers F_n can be represented determinantally by

$$F_n=\begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \binom{2}{1} & -1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ 0 & 1 & \binom{n-2}{1} & \binom{n-2}{2} & \cdots & \binom{n-2}{n-3} & -1 & 0 \\ 0 & 1 & \binom{n-1}{1} & \binom{n-1}{2} & \cdots & \binom{n-1}{n-3} & \binom{n-1}{n-2} & -1 \\ 0 & 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-3} & \binom{n}{n-2} & \binom{n}{n-1} \end{vmatrix}.$$

Proof. Let $u(t)$ and $v(t) \ne 0$ be differentiable functions, let $U_{(n+1)\times 1}(t)$ be an $(n+1)\times 1$ matrix whose elements $u_{k,1}(t) = u^{(k-1)}(t)$ for $1 \le k \le n+1$, let $V_{(n+1)\times n}(t)$ be an $(n+1)\times n$ matrix whose elements

$$v_{i,j}(t)=\begin{cases}\binom{i-1}{j-1}v^{(i-j)}(t), & i-j\ge 0;\\ 0, & i-j<0\end{cases}$$

for $1 \le i \le n+1$ and $1 \le j \le n$, and let $|W_{(n+1)\times(n+1)}(t)|$ denote the lower Hessenberg determinant of the $(n+1)\times(n+1)$ lower Hessenberg matrix

$$W_{(n+1)\times(n+1)}(t)=\begin{bmatrix}U_{(n+1)\times 1}(t) & V_{(n+1)\times n}(t)\end{bmatrix}.$$

Then the nth derivative of the ratio $\frac{u(t)}{v(t)}$ can be computed by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left[\frac{u(t)}{v(t)}\right]=(-1)^n\frac{\left|W_{(n+1)\times(n+1)}(t)\right|}{v^{n+1}(t)}. \tag{2.2}$$

See [1, p. 40, Exercise 5)], the papers [14, 21] and closely related references. Taking $u(t)=1$ and $v(t)=1-y\big(e^t-1\big)$ in (2.2) yields

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}G(t,y)=(-1)^nG^{n+1}(t,y)$$

$$\times\begin{vmatrix} 1 & \binom{0}{0}\left[1-y\left(e^t-1\right)\right] & 0 & \cdots & 0 \\ 0 & \binom{1}{0}\left(-ye^t\right) & \binom{1}{1}\left[1-y\left(e^t-1\right)\right] & \cdots & 0 \\ 0 & \binom{2}{0}\left(-ye^t\right) & \binom{2}{1}\left(-ye^t\right) & \cdots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & \binom{n-2}{0}\left(-ye^t\right) & \binom{n-2}{1}\left(-ye^t\right) & \cdots & 0 \\ 0 & \binom{n-1}{0}\left(-ye^t\right) & \binom{n-1}{1}\left(-ye^t\right) & \cdots & \binom{n-1}{n-1}\left[1-y\left(e^t-1\right)\right] \\ 0 & \binom{n}{0}\left(-ye^t\right) & \binom{n}{1}\left(-ye^t\right) & \cdots & \binom{n}{n-1}\left(-ye^t\right) \end{vmatrix}$$

$$\to (-1)^n \begin{vmatrix} 1 & \binom{0}{0} & 0 & \cdots & 0 \\ 0 & \binom{1}{0}(-y) & \binom{1}{1} & \cdots & 0 \\ 0 & \binom{2}{0}(-y) & \binom{2}{1}(-y) & \cdots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & \binom{n-2}{0}(-y) & \binom{n-2}{1}(-y) & \cdots & 0 \\ 0 & \binom{n-1}{0}(-y) & \binom{n-1}{1}(-y) & \cdots & \binom{n-1}{n-1} \\ 0 & \binom{n}{0}(-y) & \binom{n}{1}(-y) & \cdots & \binom{n}{n-1}(-y) \end{vmatrix}, \quad t\to 0$$

$$= \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & \binom{1}{0}y & -1 & \cdots & 0 \\ 0 & \binom{2}{0}y & \binom{2}{1}y & \cdots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & \binom{n-2}{0}y & \binom{n-2}{1}y & \cdots & 0 \\ 0 & \binom{n-1}{0}y & \binom{n-1}{1}y & \cdots & -1 \\ 0 & \binom{n}{0}y & \binom{n}{1}y & \cdots & \binom{n}{n-1}y \end{vmatrix}.$$

The determinantal expression (2.1) is thus proved. The proof of Theorem 2.1 is complete. □

3. Two recurrence relations of the Fubini polynomials

Basing on Theorem 2.1 and the binomial inversion theorem, we now derive two recurrence relations of the Fubini polynomials $F_n(y)$.

Theorem 3.1. *The Fubini polynomials $F_n(y)$ satisfy*

$$F_n(y) = y\sum_{r=0}^{n-1}\binom{n}{r}F_r(y), \quad n\in\mathbb{N} \tag{3.1}$$

and

$$F_n(y) = (-1)^{n+1}(1+y)\left[\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}F_k(y)-\frac{1}{y}\right], \quad n\ge 0. \tag{3.2}$$

Proof. Let $M_0 = 1$ and

$$M_n = \begin{vmatrix} m_{1,1} & m_{1,2} & 0 & \dots & 0 & 0 \\ m_{2,1} & m_{2,2} & m_{2,3} & \dots & 0 & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{n-2,1} & m_{n-2,2} & m_{n-2,3} & \dots & m_{n-2,n-1} & 0 \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \dots & m_{n-1,n-1} & m_{n-1,n} \\ m_{n,1} & m_{n,2} & m_{n,3} & \dots & m_{n,n-1} & m_{n,n} \end{vmatrix}$$

for $n \in \mathbb{N}$. Then the main theorem on page 222 in [5] reads that the sequence M_n for $n \geq 0$ satisfies $M_1 = m_{1,1}$ and

$$M_n = m_{n,n} M_{n-1} + \sum_{r=1}^{n-1} (-1)^{n-r} m_{n,r} \left(\prod_{j=r}^{n-1} m_{j,j+1} \right) M_{r-1}, \quad n \geq 2. \tag{3.3}$$

Applying (3.3) to the determinantal expression (2.1) and simplifying yield the recurrence relation (3.1).

It is well known [6, pp. 143–144] that the binomial inversion theorem reads that

$$s_n = \sum_{k=0}^{n} \binom{n}{k} S_k \quad \text{if and only if} \quad S_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} s_k$$

for $n \geq 0$, where s_n and S_n are sequences of complex numbers. The relation (3.1) can be rearranged as

$$\left(1 + \frac{1}{y}\right) F_{n+1}(y) = \sum_{r=0}^{n+1} \binom{n+1}{r} F_r(y), \quad n \geq 0.$$

Therefore, we obtain

$$F_n(y) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left(1 + \frac{1}{y}\right) F_k(y) - \frac{(-1)^n}{y}, \quad n \geq 0$$

which can be rewritten as (3.2). The proof of Theorem 3.1 is complete. □

4. A determinantal expression for generalized Fubini polynomials

In this section, We establish a determinantal expression for generalized Fubini polynomials $F_n(x, y, z, q)$ of four variables x, y, z, q.

Theorem 4.1. *Generalized Fubini polynomials $F_n(x, y, z, q)$ can be determinantally expressed as*

$$F_n(x,y,z,q) = \frac{(-1)^n}{z^{q(n+1)}} \times \begin{vmatrix} 1 & \binom{0}{0} z^q & 0 & \cdots & 0 \\ x & \binom{1}{0} m_1(z,y,q) & \binom{1}{1} z^q & \cdots & 0 \\ x^2 & \binom{2}{0} m_2(z,y,q) & \binom{2}{1} m_1(z,y,q) & \cdots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ x^{n-2} & \binom{n-2}{0} m_{n-2}(z,y,q) & \binom{n-2}{1} m_{n-3}(z,y,q) & \cdots & 0 \\ x^{n-1} & \binom{n-1}{0} m_{n-1}(z,y,q) & \binom{n-1}{1} m_{n-2}(z,y,q) & \cdots & \binom{n-1}{n-1} z^q \\ x^n & \binom{n}{0} m_n(z,y,q) & \binom{n}{1} m_{n-1}(z,y,q) & \cdots & \binom{n}{n-1} m_1(z,y,q) \end{vmatrix} \tag{4.1}$$

and satisfy the recurrence relation

$$F_n(x,y,z,q)=\frac{2}{z^q}\left[x^n-\sum_{r=0}^{n-1}\binom{n}{r}m_{n-r}(z,y,q)F_r(x,y,z,q)\right], \tag{4.2}$$

where

$$m_k(z,y,q)=\sum_{\ell=0}^{k}(-1)^\ell S(k,\ell)\langle q\rangle_\ell z^{q-\ell}y^\ell,\quad k\ge0.$$

Proof. In [6, p. 134, Theorem A] and [6, p. 139, Theorem C], the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ for $n\ge k\ge0$ are defined by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1}i\ell_i=n\\ \sum_{i=1}^{n-k+1}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}f\circ h(t)=\sum_{k=0}^{n}f^{(k)}(h(t))\,\mathrm{B}_{n,k}\bigl(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\bigr). \tag{4.3}$$

In [6, p. 135], it is given that

$$\mathrm{B}_{n,k}\bigl(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\bigr)=a^kb^n\,\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}) \tag{4.4}$$

and

$$\mathrm{B}_{n,k}(1,1,\dots,1)=S(n,k), \tag{4.5}$$

where a and b are complex numbers.

By the Faà di Bruno formula (4.3) and the identities (4.4) and (4.5), we obtain

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}\,t^k}\bigl[z-y\bigl(e^t-1\bigr)\bigr]^q&=\sum_{\ell=0}^{k}\frac{\mathrm{d}^\ell\,u^q}{\mathrm{d}\,u^\ell}\,\mathrm{B}_{k,\ell}\bigl(-ye^t,-ye^t,\dots,-ye^t\bigr)\\&=\sum_{\ell=0}^{k}\langle q\rangle_\ell u^{q-\ell}\bigl(-ye^t\bigr)^\ell\,\mathrm{B}_{k,\ell}(1,1,\dots,1)\\&=\sum_{\ell=0}^{k}\langle q\rangle_\ell\bigl[z-y\bigl(e^t-1\bigr)\bigr]^{q-\ell}\bigl(-ye^t\bigr)^\ell S(k,\ell)\\&\to\sum_{\ell=0}^{k}(-1)^\ell S(k,\ell)\langle q\rangle_\ell z^{q-\ell}y^\ell\end{aligned}$$

as $t\to0$, where $u=u(t)=z-y\bigl(e^t-1\bigr)$ and

$$\langle q\rangle_n=\prod_{k=0}^{n-1}(q-k)=\begin{cases}q(q-1)\cdots(q-n+1), & n\ge1\\ 1, & n=0\end{cases}$$

is the falling factorial of the number q.

In the formula (2.2), taking $u(t)=e^{xt}$ and $v(t)=\bigl[z-y\bigl(e^t-1\bigr)\bigr]^q$ yields

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}G(t,y,x,q,z)=\frac{(-1)^n}{[z-y(e^t-1)]^{q(n+1)}}$$

$$\times\begin{vmatrix} e^{xt} & \binom{0}{0}\bigl(z+y-ye^t\bigr)^q & 0 & \cdots & 0\\ xe^{xt} & \binom{1}{0}\bigl[\bigl(z+y-ye^t\bigr)^q\bigr]' & \binom{1}{1}\bigl(z+y-ye^t\bigr)^q & \cdots & 0\\ x^2e^{xt} & \binom{2}{0}\bigl[\bigl(z+y-ye^t\bigr)^q\bigr]'' & \binom{2}{1}\bigl[\bigl(z+y-ye^t\bigr)^q\bigr]' & \cdots & 0\\ \cdots & \cdots & \cdots & \ddots & \cdots\\ x^{n-2}e^{xt} & \binom{n-2}{0}\bigl[\bigl(z+y-ye^t\bigr)^q\bigr]^{(n-2)} & \binom{n-2}{1}\bigl[\bigl(z+y-ye^t\bigr)^q\bigr]^{(n-3)} & \cdots & 0\\ x^{n-1}e^{xt} & \binom{n-1}{0}\bigl[\bigl(z+y-ye^t\bigr)^q\bigr]^{(n-1)} & \binom{n-1}{1}\bigl[\bigl(z+y-ye^t\bigr)^q\bigr]^{(n-2)} & \cdots & \binom{n-1}{n-1}\bigl(z+y-ye^t\bigr)^q\\ x^ne^{xt} & \binom{n}{0}\bigl[\bigl(z+y-ye^t\bigr)^q\bigr]^{(n)} & \binom{n}{1}\bigl[\bigl(z+y-ye^t\bigr)^q\bigr]^{(n-1)} & \cdots & \binom{n}{n-1}\bigl[\bigl(z+y-ye^t\bigr)^q\bigr]' \end{vmatrix}$$

$$\to\frac{(-1)^n}{z^{q(n+1)}}\begin{vmatrix} 1 & \binom{0}{0}z^q & 0 & \cdots & 0\\ x & \binom{1}{0}m_1(z,y,q) & \binom{1}{1}z^q & \cdots & 0\\ x^2 & \binom{2}{0}m_2(z,y,q) & \binom{2}{1}m_1(z,y,q) & \cdots & 0\\ \cdots & \cdots & \cdots & \ddots & \cdots\\ x^{n-2} & \binom{n-2}{0}m_{n-2}(z,y,q) & \binom{n-2}{1}m_{n-3}(z,y,q) & \cdots & 0\\ x^{n-1} & \binom{n-1}{0}m_{n-1}(z,y,q) & \binom{n-1}{1}m_{n-2}(z,y,q) & \cdots & \binom{n-1}{n-1}z^q\\ x^n & \binom{n}{0}m_n(z,y,q) & \binom{n}{1}m_{n-1}(z,y,q) & \cdots & \binom{n}{n-1}m_1(z,y,q) \end{vmatrix}$$

as t tends to 0. The determinantal expression (4.1) is thus proved.

Applying (3.3) to the determinant in (4.1) gives

$$\begin{aligned}(-1)^nz^{q(n+1)}F_n(x,y,z,q)&=\binom{n}{n-1}m_1(z,y,q)(-1)^{n-1}z^{qn}F_{n-1}(x,y,z,q)\\&\quad+(-1)^{n-1+1}x^nz^{qn}(-1)^{1-2}z^{q(1-1)}F_{1-2}(x,y,z,q)\\&\quad+\sum_{r=2}^{n}(-1)^{n-r+1}\binom{n}{r-2}m_{n-r+2}(z,y,q)z^{q(n-r+1)}(-1)^{r-2}z^{q(r-1)}F_{r-2}(x,y,z,q),\end{aligned}$$

where $F_{-1}(x,y,z,q)=-1$. This can be simplified as (4.2). The proof of Theorem 4.1 is complete. □

5. Relations between Fubini and Eulerian polynomials

It is known [23, 29] that Eulerian polynomials $A_n(t)$ for $n\ge0$ can be generated by

$$\frac{1-t}{e^{x(t-1)}-t}=\sum_{n=0}^{\infty}A_n(t)\frac{x^n}{n!},\quad t\ne1 \tag{5.1}$$

and that higher order Eulerian polynomials $A_n^{(\alpha)}(t)$ for $n\ge0$ and $\alpha\in\mathbb{R}$ can be generated by

$$\left[\frac{1-t}{e^{x(t-1)}-t}\right]^{\alpha}=\sum_{n=0}^{\infty}A_n^{(\alpha)}(t)\frac{x^n}{n!},\quad t\ne1. \tag{5.2}$$

Comparing (1.2) with (5.1) and (5.2) reveals that

$$A_n(2)=A_n^{(1)}(2)=F_n=F_n(1)=F_n(0,1)=F_n^{(1)}(0,1)=yF_n(0,y,y,1),\quad n\ge0. \tag{5.3}$$

The equations (1.4) and (5.1) can be rewritten as

$$\frac{1}{e^w-\left(1+\frac{1}{y}\right)}=-y\sum_{n=0}^{\infty}F_n(y)\frac{w^n}{n!}$$

and

$$\frac{1}{e^w-t}=-\sum_{n=0}^{\infty}\frac{A_n(t)}{(t-1)^{n+1}}\frac{w^n}{n!}.$$

This means that

$$F_n(y)=y^nA_n\left(1+\frac{1}{y}\right) \tag{5.4}$$

and

$$A_n(t)=(t-1)^nF_n\left(\frac{1}{t-1}\right). \tag{5.5}$$

Similarly, the equations(1.5) and (5.2) can be rewritten as

$$\frac{1}{\left[e^w-\left(1+\frac{z}{y}\right)\right]^q}=y^qe^{-xw}\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{w^n}{n!}$$

and

$$\frac{1}{(e^w-t)^\alpha}=\sum_{n=0}^{\infty}\frac{A_n^{(\alpha)}(t)}{(1-t)^{n+\alpha}}\frac{w^n}{n!}.$$

Then we can conclude that

$$F_n(0,y,z,q)=(-1)^{n+q}\frac{y^n}{z^{n+q}}A_n^{(q)}\left(1+\frac{z}{y}\right)$$

and

$$A_n^{(\alpha)}(t)=(1-t)^{n+\alpha}F_n(0,1,t-1,\alpha). \tag{5.6}$$

In [23, Theorem 1], it was obtained that Eulerian polynomials $A_n(t)$ and higher order Eulerian polynomials $A_n^{(\alpha)}(t)$ for $n\ge0$ and $\alpha>0$ can be computed by

$$A_n(t)=\sum_{k=0}^{n}k!S(n,k)(t-1)^{n-k} \tag{5.7}$$

and

$$A_n^{(\alpha)}(t)=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n}\Gamma(k+\alpha)S(n,k)(t-1)^{n-k}, \tag{5.8}$$

where $\Gamma(z)$ denotes the classical Euler gamma function [17, 25]. Taking $t=2$ in (5.7) and using (5.3) recover (1.1). Substituting (5.5) into (5.7) yields

$$(t-1)^nF_n\left(\frac{1}{t-1}\right)=\sum_{k=0}^{n}k!S(n,k)(t-1)^{n-k}$$

which is equivalent to (1.3). Combining (5.6) and (5.8) results in

$$F_n(0,1,t-1,\alpha)=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n}(-1)^{n-k}\frac{\Gamma(k+\alpha)S(n,k)}{(1-t)^{k+\alpha}}.$$

Substituting (5.4) into (2.1), (3.1), and (3.2) arrives at a nice determinantal expression

$$A_n(t)=\begin{vmatrix} 1 & \frac{1}{1-t} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{1-t} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \binom{2}{1} & \frac{1}{1-t} & \cdots & 0 & 0 & 0 \\ 0 & 1 & \binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ 0 & 1 & \binom{n-2}{1} & \binom{n-2}{2} & \cdots & \binom{n-2}{n-3} & \frac{1}{1-t} & 0 \\ 0 & 1 & \binom{n-1}{1} & \binom{n-1}{2} & \cdots & \binom{n-1}{n-3} & \binom{n-1}{n-2} & \frac{1}{1-t} \\ 0 & 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-3} & \binom{n}{n-2} & \binom{n}{n-1} \end{vmatrix}, \quad n\geq 0$$

and two recurrence relations

$$A_n(t)=(t-1)^{n-1}\sum_{r=0}^{n-1}\binom{n}{r}\frac{A_r(t)}{(t-1)^r}, \quad n\in\mathbb{N},$$

$$A_n(t)=(-1)^{n+1}(t-1)^{n-1}t\left[1-t+\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\frac{A_k(t)}{(t-1)^k}\right], \quad n\geq 0.$$

Theorem 2 in [23] reads that the function

$$F(t,x)=\frac{1}{e^{x(t-1)}-t}, \quad t\neq 1$$

satisfies the nonlinear ordinary differential equation

$$\frac{\partial^n F(t,x)}{\partial x^n}=(t-1)^n\sum_{i=0}^{n}\left[\sum_{k=i}^{n}(-1)^k k!S(n,k)\binom{k}{i}\right]t^i F^{i+1}(t,x), \quad n\geq 0. \tag{5.9}$$

The equation (5.9) was simplified in [29, Theorem 5] as

$$\frac{\partial^n F(t,x)}{\partial x^n}=(1-t)^n\sum_{k=0}^{n}S(n+1,k+1)k!t^k F^{k+1}(t,x), \quad n\geq 0. \tag{5.10}$$

As an inversion formula of (5.10), Theorem 4 in [29] states that

$$F^{n+1}(t,x)=\frac{1}{n!t^n}\sum_{i=0}^{n}\frac{s(n+1,i+1)}{(1-t)^i}\frac{\partial^i F(t,x)}{\partial x^i}, \quad n\geq 0, \tag{5.11}$$

where $s(n,k)$, which can be generated by

$$\frac{[\ln(1+x)]^k}{k!}=\sum_{n=k}^{\infty}s(n,k)\frac{x^n}{n!}, \quad |x|<1,$$

stands for the Stirling numbers of the first kind. Taking $t=2$ in (5.10) and (5.11) leads to

$$G^{(n)}(x)=(-1)^n\sum_{k=0}^{n}(-1)^k 2^k k!S(n+1,k+1)G^{k+1}(x), \quad n\geq 0 \tag{5.12}$$

and

$$G^{n+1}(x)=\frac{(-1)^n}{n!2^n}\sum_{i=0}^{n}(-1)^i s(n+1,i+1)G^{(i)}(x), \quad n\geq 0. \tag{5.13}$$

The equations (5.12) and (5.13) were also established in [19, Theorem 1] by three alternative approaches. By similar methods to those in [23, 19, 28, 29, 31] and closely related references therein, we can easily deduce

$$\frac{\partial^n G(t,y)}{\partial t^n}=(-1)^n\sum_{m=0}^{n}(-1)^m(1+y)^m m!S(n+1,m+1)G^{m+1}(t,y),\quad n\ge0$$

and

$$G^{n+1}(t,y)=\frac{(-1)^n}{n!(1+y)^n}\sum_{m=0}^{n}(-1)^m s(n+1,m+1)\frac{\partial^m G(t,y)}{\partial t^m},\quad n\ge0.$$

In [23, Theorem 3] and [29, Theorem 6], it was obtained that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[\frac{S(n-k,0)}{(1-t)^{k-1}}+\frac{S(n-k,1)}{(1-t)^k}\right]A_k(t)=0,\quad n\in\mathbb{N}\tag{5.14}$$

and

$$\sum_{k=0}^{n}\frac{s(n,k)}{(t-1)^k}A_k(t)=\frac{n!}{(t-1)^n},\quad n\ge0.\tag{5.15}$$

Substituting (5.5) into (5.14) and (5.15) arrives at

$$\sum_{k=0}^{n}\binom{n}{k}\left[S(n-k,1)-\frac{S(n-k,0)}{t}\right]F_k(t)=0,\quad n\in\mathbb{N}$$

and

$$\sum_{k=0}^{n}s(n,k)F_k(t)=n!t^n,\quad n\ge0.$$

6. Remarks

Finally, we list several remarks.

Remark 6.1. When $y=1$, the recurrence relations (3.1) and (3.2) becomes

$$F_n=\sum_{r=0}^{n-1}\binom{n}{r}F_r,\quad n\in\mathbb{N}$$

and

$$F_n=(-1)^n2\left[1-\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}F_k\right],\quad n\ge0.$$

These two recurrence relations appeared in [8] in different forms.

Remark 6.2. Let $u=u(t)=y(e^t-1)$. Then, by the Faà di Bruno formula (4.3) and the identities (4.4) and (4.5), we have

$$\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\frac{1}{1-y(e^t-1)}=\lim_{t\to0}\sum_{k=0}^{n}\frac{\mathrm{d}^k}{\mathrm{d}\,u^k}\frac{1}{1-u}\,\mathrm{B}_{n,k}\left(ye^t,ye^t,\dots,ye^t\right)$$

$$=\lim_{t\to0}\sum_{k=0}^{n}\frac{k!}{(1-u)^{k+1}}\left(ye^t\right)^k\mathrm{B}_{n,k}(1,1,\dots,1)=\sum_{k=0}^{n}S(n,k)k!y^k=F_n(y).$$

This recovers once again that the Fubini numbers and polynomials F_n and $F_n(y)$ have generating functions $\frac{1}{2-e^t}$ and $\frac{1}{1-y(e^t-1)}$.

Remark 6.3. On 24 July 2018, when commenting on the paper [29] on ResearchGate, Professor Dr. Boyadzhiev (Ohio Northern University) recommended the website `http://www.luschny.de/math/euler/EulerianPolynomials.html` and two papers [3, 9]. He said that, when writing about Eulerian polynomials, it is good to mention that they originate from Euler and their theory is very classical—generating function and other formulas have been known for very long time. He pointed out that the paper [3] is a review where Eulerian polynomials are linked to geometric polynomials and Apostol–Bernoulli polynomials.

On 15 May 2019, Professor Dr. Boyadzhiev (Ohio Northern University) commented on ResearchGate again that the author should take in consideration the paper [4] and that the name "geometric polynomials" is established in the mathematics community while "Fubini polynomials" is not.

Remark 6.4. The ideas in this paper and the formulas (2.2) and (3.3) have been utilized in the papers [20, 22, 24, 26, 27, 30, 32, 33, 34, 35] and closely related references therein.

Remark 6.5. This paper is an extension of the preprint [18] and a slight revision of the preprint [15].

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