# Transfer of quadratic forms and of quaternion algebras over quadratic field extensions

Karim Johannes Becher, Nicolas Grenier-Boley, Jean-Pierre Tignol

# TRANSFER OF QUADRATIC FORMS AND OF QUATERNION ALGEBRAS OVER QUADRATIC FIELD EXTENSIONS

KARIM JOHANNES BECHER, NICOLAS GRENIER-BOLEY, AND JEAN-PIERRE TIGNOL

ABSTRACT. Two different proofs are given showing that a quaternion algebra $Q$ defined over a quadratic étale extension $K$ of a given field has a corestriction that is not a division algebra if and only if $Q$ contains a quadratic algebra that is linearly disjoint from $K$. This is known in the case of a quadratic field extension in characteristic different from two. In the case where $K$ is split the statement recovers a well-known result on biquaternion algebras due to Albert and Draxl.

KEYWORDS: isotropy, Witt index, corestriction, Albert form, characteristic two

CLASSIFICATION (MSC 2010): 11E04, 11E81, 12G05, 16H05

## 1. INTRODUCTION

A well-known theorem of Albert states that if a tensor product of two quaternion division algebras $Q_1$, $Q_2$ over a field $F$ of characteristic different from 2 is not a division algebra, then there exists a quadratic extension $L$ of $F$ that embeds as a subfield in $Q_1$ and in $Q_2$; see [6, (16.29)]. The same property holds in characteristic 2, with the additional condition that $L/F$ is separable: this was proved by Draxl [2], and several proofs have been proposed: see [3, Th. 98.19], and [7] for a list of earlier references.

Our purpose in this note is to extend the Albert–Draxl Theorem by substituting for the tensor product of two quaternion algebras the corestriction of a single quaternion algebra over a quadratic extension.

Let $F$ always denote a field and let $\mathsf{char}\, F$ denote its characteristic. Let $K$ be a quadratic étale $F$-algebra. In other words, either $K/F$ is a separable quadratic field extension or $K \simeq F \times F$.

---

*Date*: 18 April 2018.

The first author was supported by the FWO Odysseus Programme (project *Explicit Methods in Quadratic Form Theory*), funded by the Fonds Wetenschappelijk Onderzoek – Vlaanderen. The third author acknowledges support from the Fonds de la Recherche Scientifique–FNRS under grants n° J.0014.15 and J.0149.17.

A *quaternion algebra over* $F$ is an $F$-algebra obtained from an étale quadratic $F$-algebra $L$ and an element $a \in F^\times$ by endowing the 4-dimensional $F$-vector space $L \oplus Lz$ with the multiplication determined by the equations

$$z^2 = a \qquad \text{and} \qquad z\ell = \iota(\ell)z \quad \text{for } \ell \in L,$$

where $\iota$ is the nontrivial $F$-automorphism of $L$; this $F$-algebra is denoted by $(L/F, a)$.

Our main result is the following.

1.1. **Theorem.** *Let $F$ be an arbitrary field and let $K$ be a quadratic étale $F$-algebra. For every quaternion $K$-algebra $Q$, the following conditions are equivalent:*

(*i*) *$Q$ contains a quadratic $F$-algebra linearly disjoint from $K$;*
(*ii*) *$Q$ contains a quadratic étale $F$-algebra linearly disjoint from $K$;*
(*iii*) $\operatorname{Cor}_{K/F} Q$ *is not a division algebra.*

Note that when $K = F \times F$ the quaternion $K$-algebra $Q$ has the form $Q_1 \times Q_2$ for some quaternion $F$-algebras $Q_1$, $Q_2$, and $\operatorname{Cor}_{K/F} Q = Q_1 \otimes_F Q_2$. Thus, in this particular case Theorem 1.1 is equivalent to the Albert–Draxl Theorem. The more general case is needed for the proof of the main result in [1]: see [1, Lemma 7.5].

In the case where $\mathsf{char}\, F \neq 2$ Theorem 1.1 is proved in [6, (16.28)]. The proof below is close to that in [6], but it does not require any restriction on the characteristic. The idea is to use a transfer of the norm form $n_Q$ of $Q$ to obtain an Albert form of $\operatorname{Cor}_{K/F} Q$, which allows us to substitute for (*iii*) the condition that the transfer of $n_Q$ has Witt index at least 2. To complete the argument, we need to relate totally isotropic subspaces of the transfer to subforms of $n_Q$ defined over $F$. This is slightly more delicate in characteristic 2. Therefore, we first discuss the transfer of quadratic forms in Section 2, and give a first proof of Theorem 1.1 in Section 3. In the last section, we sketch an alternative proof of Theorem 1.1 based on a proof of the Albert–Draxl Theorem due to Knus [5]. This alternative proof relies on an explicit construction of an Albert form for the corestriction of a quaternion algebra.

## 2. Isotropic transfers

For quadratic and bilinear forms, we generally follow the conventions of [3]. Let $V$ be a finite-dimensional $F$-vector space and let $\varphi\colon V \to F$ be a quadratic form on $V$. We denote by $\mathfrak{b}_\varphi\colon V \times V \to F$ the polar form of $\varphi$, which is defined by

$$\mathfrak{b}_\varphi(x, y) = \varphi(x + y) - \varphi(x) - \varphi(y) \qquad \text{for } x,\, y \in V.$$

We set

$$\begin{aligned} \operatorname{rad} \mathfrak{b}_\varphi &= \{x \in V \mid \mathfrak{b}_\varphi(x,y) = 0 \text{ for all } y \in V\} \\ \operatorname{rad} \varphi &= \{x \in \operatorname{rad} \mathfrak{b}_\varphi \mid \varphi(x) = 0\} \end{aligned}$$

and observe that these sets are $F$-subspaces of $V$ with $\operatorname{rad}\varphi \subseteq \operatorname{rad}\mathfrak{b}_\varphi$. If $\mathsf{char}\, F \neq 2$ then $\varphi(x) = \frac{1}{2}\mathfrak{b}_\varphi(x,x)$ for all $x \in V$ and thus $\operatorname{rad}\varphi = \operatorname{rad}\mathfrak{b}_\varphi$. We call the quadratic form $\varphi$ *nonsingular* if $\operatorname{rad}\mathfrak{b}_\varphi = \{0\}$, *regular* if $\operatorname{rad}\varphi = \{0\}$ and *nondegenerate* if $\varphi_K$ is regular for every field extension $K/F$ or equivalently (by [3, Lemma 7.16]) if $\varphi$ is regular and $\dim_F \operatorname{rad}\mathfrak{b}_\varphi \leqslant 1$. (The last two terms are defined in [3], but nonsingular quadratic forms are not defined there.) Note that every nonsingular form is nondegenerate and every nondegenerate form is regular; moreover, all three conditions are equivalent when $\mathsf{char}\, F \neq 2$.

An $F$-subspace $U \subseteq V$ such that $\varphi(u) = 0$ for all $u \in U$ is called *totally isotropic* (for $\varphi$). The *Witt index* of $\varphi$ is the maximal dimension of a totally isotropic subspace of $V$; see [3, Prop. 8.11]. We write $\mathfrak{i}_0(\varphi)$ for the Witt index of $\varphi$.

2.1. **Lemma.** *Suppose that the quadratic form $\varphi$ on $V$ is regular and isotropic. Then the $F$-vector space $V$ is spanned by the isotropic vectors of $\varphi$.*

*Proof.* Let $V_0$ be the $F$-subspace of $V$ spanned by the isotropic vectors of $V$. Let $v \in V \setminus \{0\}$ be an isotropic vector. Since $\operatorname{rad}(\varphi) = \{0\}$ there exists $w \in V$ such that $\mathfrak{b}_\varphi(v,w) = 1$. If $x \in V$ is such that $\mathfrak{b}_\varphi(v,x) \neq 0$, then the vector $x - \varphi(x)\mathfrak{b}_\varphi(v,x)^{-1}v$ is isotropic, hence it belongs to $V_0$, whereby $x \in V_0$. Hence $V_0$ contains all vectors that are not orthogonal to $v$. In particular $w \in V_0$. If $x \in V$ is orthogonal to $v$, then $\mathfrak{b}_\varphi(v, x+w) = 1$, hence $x + w \in V_0$, and therefore $x \in V_0$. This shows that $V_0 = V$. □

Let $K$ be a quadratic field extension of $F$. We fix a nonzero $F$-linear functional $s\colon K \to F$ with $s(1) = 0$. Let $V$ be a finite-dimensional $K$-vector space and let $\varphi\colon V \to K$ be a quadratic form over $K$. The transfer $s_*\varphi$ is the quadratic form over $F$ defined on $V$, viewed as an $F$-vector space, by

$$s_*\varphi(x) = s\big(\varphi(x)\big) \qquad \text{for } x \in V.$$

If $\varphi$ is nonsingular, then $s_*\varphi$ is nonsingular, by [3, Lemma 20.4]. For every quadratic form $\psi$ over $F$, we denote by $\psi_K$ the quadratic form over $K$ obtained from $\psi$ by extending scalars to $K$.

The following result is well-known (and easy to prove) in characteristic different from 2: see [3, Proposition 34.1]. It appears to be new in characteristic 2.

2.2. **Theorem.** *Assume that the quadratic form $\varphi : V \to K$ is nonsingular. There exists a nondegenerate quadratic form $\psi$ over $F$ with $\dim \psi = \mathfrak{i}_0(s_*\varphi)$ such that $\psi_K$ is a subform of $\varphi$.*

*Proof.* Suppose that $\mathfrak{i}_0(s_*\varphi) \geqslant 1$, for otherwise there is nothing to show. If $\varphi$ is isotropic, then we fix $u \in V \setminus \{0\}$ such that $\varphi(u) = 0$, otherwise we fix $u \in V \setminus \{0\}$ such that $s_*\varphi(u) = 0$. Note that if $\varphi$ is isotropic over $K$, then $Ku$ is totally isotropic for $s_*\varphi$, whereby $\mathfrak{i}_0(s_*\varphi) \geqslant 2$. Hence, if $\mathfrak{i}_0(s_*\varphi) = 1$, then we may choose $\psi = \varphi|_{Fu}$.

Suppose now that $\mathfrak{i}_0(s_*\varphi) \geqslant 2$. This implies that $\dim_F V \geqslant 4$. Since $\varphi$ is nonsingular, there exists $v \in V \setminus Ku$ such that $\mathfrak{b}_\varphi(u, v) = 1$. We fix $\lambda \in K$ with $s(\lambda) = 1$. We have $s_*\varphi(u) = 0$ and

$$\mathfrak{b}_{s_*\varphi}(u, \lambda v) = s\big(\mathfrak{b}_\varphi(u, \lambda v)\big) = s(\lambda) = 1.$$

Therefore the restriction of $s_*\varphi$ to the $F$-subspace $U = Fu \oplus F\lambda v$ is nonsingular and isotropic. Hence $s_*\varphi|_U$ is hyperbolic.

Let $U'$ be the orthogonal complement of $U$ in $V$ with respect to $\mathfrak{b}_{s_*\varphi}$ and let $U''$ be the orthogonal complement of $Ku$ in $V$ with respect to $\mathfrak{b}_\varphi$.

Since $u$ and $v$ are $K$-linearly independent, we may find $x \in U''$ such that $\mathfrak{b}_\varphi(\lambda v, x) = 1$, whereby $\lambda x \in U'' \setminus U'$. Hence $U'' \not\subseteq U'$. Since we have $\dim_F U' = \dim_F V - 2 = \dim_F U''$, we conclude that $U' \not\subseteq U''$.

As $\mathfrak{i}_0(s_*\varphi) \geqslant 2$, the form $s_*\varphi|_{U'}$ is isotropic. It follows by Lemma 2.1 that $U'$ is spanned by isotropic vectors for $s_*\varphi$. As $U' \not\subseteq U''$, it follows that we can find a vector $w \in U' \setminus U''$ with $s_*\varphi(w) = 0$. We obtain that $\varphi(w) \in F$ and, furthermore, $\mathfrak{b}_{s_*\varphi}(u, w) = 0$ and $\mathfrak{b}_\varphi(u, w) \neq 0$, whereby $\mathfrak{b}_\varphi(u, w) \in F^\times$. As $\mathfrak{b}_{s_*\varphi}(u, \lambda v) = 1$ and $\mathfrak{b}_{s_*\varphi}(w, \lambda v) = 0$, the vectors $u$ and $w$ are $F$-linearly independent. Hence, by restricting $\varphi$ to $Fu \oplus Fw$ we obtain a 2-dimensional quadratic form $\beta$ over $F$.

For $\alpha \in K$ we have $\mathfrak{b}_\varphi(u, \alpha u) = \alpha \mathfrak{b}_\varphi(u, u) = 2\alpha\varphi(u)$, and as $\varphi(u) \in F$ it follows that if $\mathfrak{b}_\varphi(u, \alpha u) \in F^\times$ then $\alpha \in F^\times$. In particular, $u$ and $w$ are even $K$-linearly independent. Thus $\beta_K$ is a 2-dimensional subform of $\varphi$. Since $\mathfrak{b}_\varphi(u, w) \in F^\times$ and since by our choice of $u$, either $\varphi$ is anisotropic or $\varphi(u) = 0$, we conclude that $\beta$ is nonsingular.

Since $s_*(\beta_K)$ is hyperbolic, we obtain for the orthogonal complement $\varphi'$ of $\beta_K$ in $\varphi$ that $\mathfrak{i}_0(s_*\varphi') = \mathfrak{i}_0(s_*\varphi) - 2$. We may now repeat the same argument for $\varphi'$ in place of $\varphi$. The statement thus follows by induction. □

2.3. **Remarks.**

(1) In the proof above, the fact that $\mathfrak{b}_\varphi(u, w) \neq 0$ readily implies that $u$ and $w$ are $K$-linearly independent if $\mathsf{char}\, F = 2$.
(2) If $s_*\varphi$ is hyperbolic, then $\mathfrak{i}_0(s_*\varphi) = \dim \varphi$, hence Theorem 2.2 shows that $\varphi = \psi_K$ for some quadratic form $\psi$ over $F$. This particular case of Theorem 2.2 is established in [3, Theorem 34.9].

(3) If $\mathsf{char}\, F = 2$ and if $\mathfrak{i}_0(s_*\varphi)$ is odd, then the quadratic form $\psi$ in Theorem 2.2 cannot be nonsingular, since a regular quadratic form in characteristic 2 is nonsingular if and only if its dimension is even. In particular, $\psi_K$ is not an orthogonal direct summand of $\varphi$. By contrast, if $\mathfrak{i}_0(s_*\varphi)$ is even, then $\psi$ is nonsingular hence $\varphi \simeq \psi_K \perp \varphi'$ for some nonsingular quadratic form $\varphi'$. Since $s_*(\psi_K)$ is hyperbolic it follows that $\mathfrak{i}_0(s_*\varphi) = \mathfrak{i}_0\big(s_*(\psi_K)\big)$, hence $s_*(\varphi')$ is anisotropic. In this case we thus have an analogue of the result for symmetric bilinear forms [3, Proposition 34.1].
(4) If the extension $K/F$ is purely inseparable, then $\mathfrak{i}_0(s_*\varphi)$ is necessarily even. This follows because the $K$-subspace spanned by each isotropic vector for $s_*\varphi$ is a 2-dimensional $F$-subspace that is totally isotropic for $s_*\varphi$.

## 3. Proof of the main theorem

As in [3], we write $I_q(F)$ for the Witt group of nonsingular quadratic forms of even dimension over $F$, and $I(F)$ for the ideal of even-dimensional forms in the Witt ring $W(F)$ of nondegenerate symmetric bilinear forms over $F$, and we let $I_q^n(F) = I^{n-1}(F)I_q(F)$ for $n \geqslant 2$. Let also $\mathrm{Br}_2(F)$ denote the 2-torsion subgroup of the Brauer group of $F$. Recall from [3, Th. 14.3] the group homomorphism

$$e_2\colon I_q^2(F) \to \mathrm{Br}_2(F)$$

defined by mapping the Witt class of a quadratic form $\varphi$ to the Brauer class of its Clifford algebra.

3.1. **Lemma.** *Let $K$ be a quadratic field extension of an arbitrary field $F$, and let $s\colon K \to F$ be a nonzero $F$-linear functional such that $s(1) = 0$. The following diagram is commutative:*

$$\begin{array}{ccc} I_q^2(K) & \xrightarrow{s_*} & I_q^2(F) \\ {\scriptstyle e_2}\big\downarrow & & \big\downarrow{\scriptstyle e_2} \\ \mathrm{Br}_2(K) & \xrightarrow{\mathrm{Cor}_{K/F}} & \mathrm{Br}_2(F). \end{array}$$

Here, $\mathrm{Cor}_{K/F}\colon \mathrm{Br}_2(K) \to \mathrm{Br}_2(F)$ is the zero map if $K$ is a purely inseparable extension of $F$.

*Proof.* We have $I_q^2(K) = I(F)I_q(K) + I(K)I_q(F)$ by [3, Lemma 34.16], hence $I_q^2(K)$ is generated by Witt classes of 2-fold Pfister forms that have a slot in $F$. Commutativity of the diagram follows by Frobenius reciprocity [3, Prop. 20.2], the computation of transfers of 1-fold Pfister forms in [3, Lemma 34.14] and [3, Cor. 34.19], and the projection formula in cohomology, see [6, Ex. 9 p. 63-64] or more generally [4, Prop. 3.4.10]. □

*Proof of Theorem 1.1.* Since $(ii) \Rightarrow (i)$ is clear, it suffices to prove $(i) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$.

If $(i)$ holds, then we may represent $Q$ in the form $(LK/K, b)$ where $L$ is a quadratic étale $F$-algebra linearly disjoint from $K$ and $b \in K^\times$, or in the form $(M/K, b)$ where $M$ is a quadratic étale $K$-algebra and $b \in F^\times$. In each case the projection formula in cohomology shows that $\operatorname{Cor}_{K/F} Q$ is Brauer-equivalent to a quaternion algebra, hence $(iii)$ holds.

Now, assume $(iii)$ holds. If $Q$ is split, then it contains an $F$-algebra isomorphic to $F \times F$, so $(ii)$ holds. For the rest of the proof, we assume $Q$ is a division algebra. Let $n_Q$ be the norm form of $Q$, which is a 2-fold Pfister quadratic form in $I_q^2(K)$ such that $e_2(n_Q) = Q$ in $\operatorname{Br}(K)$. Since $n_Q$ represents 1, the transfer $s_*(n_Q)$ is isotropic, hence Witt-equivalent to a 6-dimensional nonsingular quadratic form $\varphi$ in $I_q^2(F)$. This form satisfies $e_2(\varphi) = \operatorname{Cor}_{K/F}(Q)$ in $\operatorname{Br}(F)$ by Lemma 3.1, hence $\varphi$ is an Albert form of $\operatorname{Cor}_{K/F}(Q)$ as per the definition in [6, (16.3)]. In particular, since $\operatorname{Cor}_{K/F}(Q)$ is not a division algebra, $\varphi$ is isotropic by [6, (16.5)], and therefore $\mathfrak{i}_0\big(s_*(n_Q)\big) \geqslant 2$. By Theorem 2.2 there exists a nonsingular quadratic form $\psi$ over $F$ with $\dim \psi = 2$ such that $\psi_K$ is a subform of $n_Q$. Since $Q$ is a division algebra, we have that $\psi_K$ is anisotropic, hence $\psi$ is similar to the norm form of a unique separable quadratic field extension $L/F$. The field $L$ is linearly disjoint from $K$ over $F$ because $\psi_K$ is anisotropic. On the other hand, $\psi_{KL}$ is hyperbolic, hence $KL$ splits the form $n_Q$, and it follows that there exists a $K$-algebra embedding of $KL$ in $Q$. Therefore, $(ii)$ holds. □

3.2. **Remark.** It follows from the proof above that $\operatorname{Cor}_{K/F} Q$ is split if and only if $Q$ is extended from a quaternion $F$-algebra. If $Q$ is extended from a quaternion $F$-algebra, the fact that $\operatorname{Cor}_{K/F} Q$ is split comes from the projection formula in cohomology or [6, (3.13)]. Conversely, if $\operatorname{Cor}_{K/F} Q$ is split, then the quadratic form $\varphi$ is hyperbolic by [6, (16.5)]. As in the proof above, we deduce that there exists a nonsingular quadratic form $\psi$ over $F$ with $\dim \psi = 4$ and $\psi_K = n_Q$, whence $Q$ is extended from a quaternion $F$-algebra.

If $K$ is a purely inseparable quadratic extension of $F$, all the statements of Theorem 1.1 hold for every quaternion algebra over $K$. To see this, recall from our definition of $\operatorname{Cor}_{K/F}$ that the corestriction of every quaternion $K$-algebra is split. Moreover, if $Q = (M/K, b)$ with $M$ a separable quadratic extension of $K$, then the separable closure of $F$ in $M$ is a separable quadratic extension of $F$ contained in $Q$ and linearly disjoint from $K$.

## 4. The Albert form of a corestriction

Let $Q$ be a quaternion algebra over a separable quadratic field extension $K$ of an arbitrary field $F$. By definition (see [6, (16.3)]), the Albert forms of

$\operatorname{Cor}_{K/F} Q$ are the 6-dimensional nonsingular quadratic forms in $I_q^2(F)$ such that $e_2(\varphi) = \operatorname{Cor}_{K/F} Q$ in $\operatorname{Br}_2(F)$; they are all similar. As observed in the proof of Theorem 1.1, an Albert form of $\operatorname{Cor}_{K/F} Q$ may be obtained from the Witt class of the (8-dimensional) transfer $s_*(n_Q)$ of the norm form of $Q$ for an arbitrary nonzero $F$-linear functional $s\colon K \to F$ such that $s(1) = 0$. In this section, we sketch a more explicit construction of an Albert form of $\operatorname{Cor}_{K/F} Q$, inspired by Knus's proof of the Albert–Draxl Theorem in [5], and we use it to give an alternative proof of Theorem 1.1. The arguments below also hold when $K \simeq F \times F$.

We first recall the construction of the corestriction $\operatorname{Cor}_{K/F} Q$. Let $\gamma$ be the nontrivial $F$-automorphism of $K$ and let ${}^\gamma Q$ denote the conjugate quaternion algebra ${}^\gamma Q = \{{}^\gamma x \mid x \in Q\}$ with the operations

$$ {}^\gamma x + {}^\gamma y = {}^\gamma(x+y), \quad {}^\gamma x \cdot {}^\gamma y = {}^\gamma(xy), \quad \lambda \cdot {}^\gamma x = {}^\gamma(\gamma(\lambda)x) $$

for $x$, $y \in Q$ and $\lambda \in K$. The algebra ${}^\gamma Q \otimes_K Q$ carries a $\gamma$-semilinear automorphism $s$ defined by

$$ s({}^\gamma x \otimes y) = {}^\gamma y \otimes x \qquad \text{for } x,\, y \in Q. $$

By definition, the corestriction (or norm) $\operatorname{Cor}_{K/F}(Q)$ is the $F$-algebra of fixed points (see [6, (3.12)]):

$$ \operatorname{Cor}_{K/F}(Q) = \big({}^\gamma Q \otimes_K Q\big)^s. $$

Let Trd and Nrd denote the reduced trace and the reduced norm on $Q$. Let also $\sigma$ be the canonical (conjugation) involution on $Q$. Consider the following $K$-submodule of ${}^\gamma Q \otimes_K Q$:

$$ V = \{{}^\gamma x_1 \otimes 1 - 1 \otimes x_2 \mid x_1, x_2 \in Q \text{ and } \gamma\big(\operatorname{Trd}(x_1)\big) = \operatorname{Trd}(x_2)\}. $$

This $K$-module is free of rank 6 and is preserved by $s$, and one can show that the $F$-space of $s$-invariant elements has the following description, where $T_{K/F}\colon K \to F$ is the trace form:

$$ V^s = \{{}^\gamma y \otimes 1 + 1 \otimes y \mid y \in Q \text{ and } T_{K/F}(\operatorname{Trd}(y)) = 0\}. $$

Now, pick an element $\kappa \in K^\times$ such that $\gamma(\kappa) = -\kappa$. (If $\mathsf{char}\, F = 2$ we may pick $\kappa = 1$.) The following formula defines a quadratic form $\varphi\colon V^s \to F$: for $y \in Q$ such that $T_{K/F}(\operatorname{Trd}(y)) = 0$, let

$$ \varphi({}^\gamma y \otimes 1 + 1 \otimes y) = \kappa \cdot \big(\gamma(\operatorname{Nrd}(y)) - \operatorname{Nrd}(y)\big). $$

Nonsingularity of the form $\varphi$ is easily checked after scalar extension to an algebraic closure of $F$, and computation shows that the linear map

$$ f\colon V^s \to M_2\big(\operatorname{Cor}_{K/F}(Q)\big) \quad \text{given by} \quad \xi \mapsto \begin{pmatrix} 0 & \kappa \cdot (\sigma \otimes \mathsf{id})(\xi) \\ \xi & 0 \end{pmatrix} $$

satisfies $f(\xi)^2 = \varphi(\xi)$ for all $\xi \in V^s$. Therefore, $f$ induces an $F$-algebra homomorphism $f_*$ defined on the Clifford algebra $C(V^s,\varphi)$. Dimension count shows that $f_*$ is an isomorphism

$$f_*\colon C(V^s,\varphi) \xrightarrow{\sim} M_2(\operatorname{Cor}_{K/F} Q). \tag{4.1}$$

The restriction to the even Clifford algebra is an isomorphism

$$C_0(V^s,\varphi) \simeq (\operatorname{Cor}_{K/F} Q) \times (\operatorname{Cor}_{K/F} Q),$$

hence the discriminant (or Arf invariant) of $\varphi$ is trivial since the center of its even Clifford algebra is split. This means $\varphi \in I_q^2(F)$, and (4.1) shows that $e_2(\varphi) = \operatorname{Cor}_{K/F} Q$ in $\operatorname{Br}_2(F)$, so $\varphi$ is an Albert form of $\operatorname{Cor}_{K/F} Q$.

We use the Albert form $\varphi$ to sketch an alternative proof of Theorem 1.1. If the base field $F$ is finite, then $Q$ is split and all the conditions in Theorem 1.1 trivially hold. Therefore, we may assume $F$ is infinite.

Suppose condition $(i)$ of Theorem 1.1 holds. If $x \in Q$ generates a quadratic $F$-algebra disjoint from $K$, then $\operatorname{Trd}(x) \in F$ and $\operatorname{Nrd}(x) \in F$ (and $x \notin K$), hence ${}^\gamma(\kappa x) \otimes 1 + 1 \otimes (\kappa x) \in V^s$ is an isotropic vector of $\varphi$. Since $\varphi$ is an Albert form of $\operatorname{Cor}_{K/F} Q$, it follows that $\operatorname{Cor}_{K/F} Q$ is not a division algebra. Therefore, $(i)$ implies $(iii)$.

For the converse, suppose $(iii)$ holds. Then there exists $y \in Q$ such that $\xi = {}^\gamma y \otimes 1 + 1 \otimes y \in V^s$ is an isotropic vector for $\varphi$. A density argument shows that we may find such an element $y$ with $\operatorname{Trd}(y) \neq 0$. We obtain that $\gamma(\operatorname{Nrd}(y)) - \operatorname{Nrd}(y) = 0$, hence $\operatorname{Nrd}(y) \in F$, and $T_{K/F}(\operatorname{Trd}(y)) = 0$.

We claim that $y \notin K$. Suppose on the contrary that $y \in K$. Then we have $\xi = 1 \otimes (y + \gamma(y)) \neq 0$ and therefore $T_{K/F}(y) \neq 0$. Since $2T_{K/F}(y) = T_{K/F}(\operatorname{Trd}(y)) = 0 \neq T_{K/F}(y)$ we conclude that $\mathsf{char}(F) = 2$ and $y \notin F$. But $y^2 = \operatorname{Nrd}(y)$ since $y \in K$ and $\operatorname{Nrd}(y) \in F$ because $\varphi(\xi) = 0$, hence $y^2 \in F$. This is a contradiction since $K$ is an étale $F$-algebra.

Thus $y \notin K$. Note that $\kappa y \in Q$ satisfies

$$\operatorname{Trd}(\kappa y) \in F^\times \quad \text{and} \quad \operatorname{Nrd}(\kappa y) = \kappa^2 \operatorname{Nrd}(y) \in F.$$

Therefore, $\kappa y$ generates a quadratic étale $F$-subalgebra of $Q$ linearly disjoint from $K$, proving that $(ii)$ (hence also $(i)$) holds.

## References


[1] K.-J. Becher, N. Grenier-Boley, J.-P. Tignol, Involutions and stable subalgebras, in preparation.

[2] P. Draxl, Über gemeinsame separabel-quadratische Zerfällungskörper von Quaternionenalgebren, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1975**, no. 16, 251–259.

[3] R. Elman, N. Karpenko, A. Merkurjev. *The algebraic and geometric theory of quadratic forms.* American Mathematical Society Colloquium Publications, **56**, Amer. Math. Soc., Providence, RI, 2008.

[4] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology.* Cambridge Studies in Advanced Mathematics, 101, Cambridge Univ. Press, Cambridge, 2006.

[5] M.-A. Knus, Sur la forme d'Albert et le produit tensoriel de deux algèbres de quaternions, Bull. Soc. Math. Belg. Sér. B **45** (1993), no. 3, 333–337.
[6] M.-A. Knus, A. S. Merkurjev, M. Rost, and J.-P. Tignol. *The book of involutions. American Mathematical Society Colloquium Publications* **44**. American Mathematical Society, Providence, RI, 1998.
[7] J. Tits, Sur les produits tensoriels de deux algèbres de quaternions, Bull. Soc. Math. Belg. Sér. B **45** (1993), no. 3, 329–331.

Universiteit Antwerpen, Departement Wiskunde en Informatica, Middelheimlaan 1, B-2020 Antwerpen, Belgium
*E-mail address*: KarimJohannes.Becher@uantwerpen.be

LDAR (EA4434), UA, UCP, UPD, UPEC, URN, Université de Rouen Normandie, Mont-Saint-Aignan, France.
*E-mail address*: nicolas.grenier-boley@univ-rouen.fr

Université catholique de Louvain, ICTEAM Institute, Avenue G. Lemaître 4, Box L4.05.01, B-1348 Louvain-La-Neuve, Belgium.
*E-mail address*: jean-pierre.tignol@uclouvain.be