



**HAL**  
open science

# Transfer of quadratic forms and of quaternion algebras over quadratic field extensions

Karim Johannes Becher, Nicolas Grenier-Boley, Jean-Pierre Tignol

► **To cite this version:**

Karim Johannes Becher, Nicolas Grenier-Boley, Jean-Pierre Tignol. Transfer of quadratic forms and of quaternion algebras over quadratic field extensions. *Archiv der Mathematik*, 2018, 111 (2), pp.135-143. <10.1007/s00013-018-1198-5>. <hal-01853557>

**HAL Id: hal-01853557**

**<https://hal.science/hal-01853557v1>**

Submitted on 3 Aug 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# TRANSFER OF QUADRATIC FORMS AND OF QUATERNION ALGEBRAS OVER QUADRATIC FIELD EXTENSIONS

KARIM JOHANNES BECHER, NICOLAS GRENIER-BOLEY,  
AND JEAN-PIERRE TIGNOL

ABSTRACT. Two different proofs are given showing that a quaternion algebra  $Q$  defined over a quadratic étale extension  $K$  of a given field has a corestriction that is not a division algebra if and only if  $Q$  contains a quadratic algebra that is linearly disjoint from  $K$ . This is known in the case of a quadratic field extension in characteristic different from two. In the case where  $K$  is split the statement recovers a well-known result on biquaternion algebras due to Albert and Draxl.

KEYWORDS: isotropy, Witt index, corestriction, Albert form, characteristic two

CLASSIFICATION (MSC 2010): 11E04, 11E81, 12G05, 16H05

## 1. INTRODUCTION

A well-known theorem of Albert states that if a tensor product of two quaternion division algebras  $Q_1, Q_2$  over a field  $F$  of characteristic different from 2 is not a division algebra, then there exists a quadratic extension  $L$  of  $F$  that embeds as a subfield in  $Q_1$  and in  $Q_2$ ; see [6, (16.29)]. The same property holds in characteristic 2, with the additional condition that  $L/F$  is separable: this was proved by Draxl [2], and several proofs have been proposed: see [3, Th. 98.19], and [7] for a list of earlier references.

Our purpose in this note is to extend the Albert–Draxl Theorem by substituting for the tensor product of two quaternion algebras the corestriction of a single quaternion algebra over a quadratic extension.

Let  $F$  always denote a field and let  $\text{char } F$  denote its characteristic. Let  $K$  be a quadratic étale  $F$ -algebra. In other words, either  $K/F$  is a separable quadratic field extension or  $K \simeq F \times F$ .

---

*Date:* 18 April 2018.

The first author was supported by the FWO Odysseus Programme (project *Explicit Methods in Quadratic Form Theory*), funded by the Fonds Wetenschappelijk Onderzoek – Vlaanderen. The third author acknowledges support from the Fonds de la Recherche Scientifique–FNRS under grants n° J.0014.15 and J.0149.17.

A *quaternion algebra over  $F$*  is an  $F$ -algebra obtained from an étale quadratic  $F$ -algebra  $L$  and an element  $a \in F^\times$  by endowing the 4-dimensional  $F$ -vector space  $L \oplus Lz$  with the multiplication determined by the equations

$$z^2 = a \quad \text{and} \quad z\ell = \iota(\ell)z \quad \text{for } \ell \in L,$$

where  $\iota$  is the nontrivial  $F$ -automorphism of  $L$ ; this  $F$ -algebra is denoted by  $(L/F, a)$ .

Our main result is the following.

**1.1. Theorem.** *Let  $F$  be an arbitrary field and let  $K$  be a quadratic étale  $F$ -algebra. For every quaternion  $K$ -algebra  $Q$ , the following conditions are equivalent:*

- (i)  $Q$  contains a quadratic  $F$ -algebra linearly disjoint from  $K$ ;
- (ii)  $Q$  contains a quadratic étale  $F$ -algebra linearly disjoint from  $K$ ;
- (iii)  $\text{Cor}_{K/F} Q$  is not a division algebra.

Note that when  $K = F \times F$  the quaternion  $K$ -algebra  $Q$  has the form  $Q_1 \times Q_2$  for some quaternion  $F$ -algebras  $Q_1, Q_2$ , and  $\text{Cor}_{K/F} Q = Q_1 \otimes_F Q_2$ . Thus, in this particular case Theorem 1.1 is equivalent to the Albert–Draxl Theorem. The more general case is needed for the proof of the main result in [1]: see [1, Lemma 7.5].

In the case where  $\text{char } F \neq 2$  Theorem 1.1 is proved in [6, (16.28)]. The proof below is close to that in [6], but it does not require any restriction on the characteristic. The idea is to use a transfer of the norm form  $n_Q$  of  $Q$  to obtain an Albert form of  $\text{Cor}_{K/F} Q$ , which allows us to substitute for (iii) the condition that the transfer of  $n_Q$  has Witt index at least 2. To complete the argument, we need to relate totally isotropic subspaces of the transfer to subforms of  $n_Q$  defined over  $F$ . This is slightly more delicate in characteristic 2. Therefore, we first discuss the transfer of quadratic forms in Section 2, and give a first proof of Theorem 1.1 in Section 3. In the last section, we sketch an alternative proof of Theorem 1.1 based on a proof of the Albert–Draxl Theorem due to Knus [5]. This alternative proof relies on an explicit construction of an Albert form for the corestriction of a quaternion algebra.

## 2. ISOTROPIC TRANSFERS

For quadratic and bilinear forms, we generally follow the conventions of [3]. Let  $V$  be a finite-dimensional  $F$ -vector space and let  $\varphi: V \rightarrow F$  be a quadratic form on  $V$ . We denote by  $\mathfrak{b}_\varphi: V \times V \rightarrow F$  the polar form of  $\varphi$ , which is defined by

$$\mathfrak{b}_\varphi(x, y) = \varphi(x + y) - \varphi(x) - \varphi(y) \quad \text{for } x, y \in V.$$

We set

$$\begin{aligned}\mathrm{rad} \mathfrak{b}_\varphi &= \{x \in V \mid \mathfrak{b}_\varphi(x, y) = 0 \text{ for all } y \in V\} \\ \mathrm{rad} \varphi &= \{x \in \mathrm{rad} \mathfrak{b}_\varphi \mid \varphi(x) = 0\}\end{aligned}$$

and observe that these sets are  $F$ -subspaces of  $V$  with  $\mathrm{rad} \varphi \subseteq \mathrm{rad} \mathfrak{b}_\varphi$ . If  $\mathrm{char} F \neq 2$  then  $\varphi(x) = \frac{1}{2}\mathfrak{b}_\varphi(x, x)$  for all  $x \in V$  and thus  $\mathrm{rad} \varphi = \mathrm{rad} \mathfrak{b}_\varphi$ . We call the quadratic form  $\varphi$  *nonsingular* if  $\mathrm{rad} \mathfrak{b}_\varphi = \{0\}$ , *regular* if  $\mathrm{rad} \varphi = \{0\}$  and *nondegenerate* if  $\varphi_K$  is regular for every field extension  $K/F$  or equivalently (by [3, Lemma 7.16]) if  $\varphi$  is regular and  $\dim_F \mathrm{rad} \mathfrak{b}_\varphi \leq 1$ . (The last two terms are defined in [3], but nonsingular quadratic forms are not defined there.) Note that every nonsingular form is nondegenerate and every nondegenerate form is regular; moreover, all three conditions are equivalent when  $\mathrm{char} F \neq 2$ .

An  $F$ -subspace  $U \subseteq V$  such that  $\varphi(u) = 0$  for all  $u \in U$  is called *totally isotropic* (for  $\varphi$ ). The *Witt index* of  $\varphi$  is the maximal dimension of a totally isotropic subspace of  $V$ ; see [3, Prop. 8.11]. We write  $i_0(\varphi)$  for the Witt index of  $\varphi$ .

**2.1. Lemma.** *Suppose that the quadratic form  $\varphi$  on  $V$  is regular and isotropic. Then the  $F$ -vector space  $V$  is spanned by the isotropic vectors of  $\varphi$ .*

*Proof.* Let  $V_0$  be the  $F$ -subspace of  $V$  spanned by the isotropic vectors of  $V$ . Let  $v \in V \setminus \{0\}$  be an isotropic vector. Since  $\mathrm{rad}(\varphi) = \{0\}$  there exists  $w \in V$  such that  $\mathfrak{b}_\varphi(v, w) = 1$ . If  $x \in V$  is such that  $\mathfrak{b}_\varphi(v, x) \neq 0$ , then the vector  $x - \varphi(x)\mathfrak{b}_\varphi(v, x)^{-1}v$  is isotropic, hence it belongs to  $V_0$ , whereby  $x \in V_0$ . Hence  $V_0$  contains all vectors that are not orthogonal to  $v$ . In particular  $w \in V_0$ . If  $x \in V$  is orthogonal to  $v$ , then  $\mathfrak{b}_\varphi(v, x+w) = 1$ , hence  $x+w \in V_0$ , and therefore  $x \in V_0$ . This shows that  $V_0 = V$ .  $\square$

Let  $K$  be a quadratic field extension of  $F$ . We fix a nonzero  $F$ -linear functional  $s: K \rightarrow F$  with  $s(1) = 0$ . Let  $V$  be a finite-dimensional  $K$ -vector space and let  $\varphi: V \rightarrow K$  be a quadratic form over  $K$ . The transfer  $s_*\varphi$  is the quadratic form over  $F$  defined on  $V$ , viewed as an  $F$ -vector space, by

$$s_*\varphi(x) = s(\varphi(x)) \quad \text{for } x \in V.$$

If  $\varphi$  is nonsingular, then  $s_*\varphi$  is nonsingular, by [3, Lemma 20.4]. For every quadratic form  $\psi$  over  $F$ , we denote by  $\psi_K$  the quadratic form over  $K$  obtained from  $\psi$  by extending scalars to  $K$ .

The following result is well-known (and easy to prove) in characteristic different from 2: see [3, Proposition 34.1]. It appears to be new in characteristic 2.

**2.2. Theorem.** *Assume that the quadratic form  $\varphi : V \rightarrow K$  is nonsingular. There exists a nondegenerate quadratic form  $\psi$  over  $F$  with  $\dim \psi = i_0(s_*\varphi)$  such that  $\psi_K$  is a subform of  $\varphi$ .*

*Proof.* Suppose that  $i_0(s_*\varphi) \geq 1$ , for otherwise there is nothing to show. If  $\varphi$  is isotropic, then we fix  $u \in V \setminus \{0\}$  such that  $\varphi(u) = 0$ , otherwise we fix  $u \in V \setminus \{0\}$  such that  $s_*\varphi(u) = 0$ . Note that if  $\varphi$  is isotropic over  $K$ , then  $Ku$  is totally isotropic for  $s_*\varphi$ , whereby  $i_0(s_*\varphi) \geq 2$ . Hence, if  $i_0(s_*\varphi) = 1$ , then we may choose  $\psi = \varphi|_{Fu}$ .

Suppose now that  $i_0(s_*\varphi) \geq 2$ . This implies that  $\dim_F V \geq 4$ . Since  $\varphi$  is nonsingular, there exists  $v \in V \setminus Ku$  such that  $\mathfrak{b}_\varphi(u, v) = 1$ . We fix  $\lambda \in K$  with  $s(\lambda) = 1$ . We have  $s_*\varphi(u) = 0$  and

$$\mathfrak{b}_{s_*\varphi}(u, \lambda v) = s(\mathfrak{b}_\varphi(u, \lambda v)) = s(\lambda) = 1.$$

Therefore the restriction of  $s_*\varphi$  to the  $F$ -subspace  $U = Fu \oplus F\lambda v$  is nonsingular and isotropic. Hence  $s_*\varphi|_U$  is hyperbolic.

Let  $U'$  be the orthogonal complement of  $U$  in  $V$  with respect to  $\mathfrak{b}_{s_*\varphi}$  and let  $U''$  be the orthogonal complement of  $Ku$  in  $V$  with respect to  $\mathfrak{b}_\varphi$ .

Since  $u$  and  $v$  are  $K$ -linearly independent, we may find  $x \in U''$  such that  $\mathfrak{b}_\varphi(\lambda v, x) = 1$ , whereby  $\lambda x \in U'' \setminus U'$ . Hence  $U'' \not\subseteq U'$ . Since we have  $\dim_F U' = \dim_F V - 2 = \dim_F U''$ , we conclude that  $U' \not\subseteq U''$ .

As  $i_0(s_*\varphi) \geq 2$ , the form  $s_*\varphi|_{U'}$  is isotropic. It follows by Lemma 2.1 that  $U'$  is spanned by isotropic vectors for  $s_*\varphi$ . As  $U' \not\subseteq U''$ , it follows that we can find a vector  $w \in U' \setminus U''$  with  $s_*\varphi(w) = 0$ . We obtain that  $\varphi(w) \in F$  and, furthermore,  $\mathfrak{b}_{s_*\varphi}(u, w) = 0$  and  $\mathfrak{b}_\varphi(u, w) \neq 0$ , whereby  $\mathfrak{b}_\varphi(u, w) \in F^\times$ . As  $\mathfrak{b}_{s_*\varphi}(u, \lambda v) = 1$  and  $\mathfrak{b}_{s_*\varphi}(w, \lambda v) = 0$ , the vectors  $u$  and  $w$  are  $F$ -linearly independent. Hence, by restricting  $\varphi$  to  $Fu \oplus Fw$  we obtain a 2-dimensional quadratic form  $\beta$  over  $F$ .

For  $\alpha \in K$  we have  $\mathfrak{b}_\varphi(u, \alpha u) = \alpha \mathfrak{b}_\varphi(u, u) = 2\alpha\varphi(u)$ , and as  $\varphi(u) \in F$  it follows that if  $\mathfrak{b}_\varphi(u, \alpha u) \in F^\times$  then  $\alpha \in F^\times$ . In particular,  $u$  and  $w$  are even  $K$ -linearly independent. Thus  $\beta_K$  is a 2-dimensional subform of  $\varphi$ . Since  $\mathfrak{b}_\varphi(u, w) \in F^\times$  and since by our choice of  $u$ , either  $\varphi$  is anisotropic or  $\varphi(u) = 0$ , we conclude that  $\beta$  is nonsingular.

Since  $s_*(\beta_K)$  is hyperbolic, we obtain for the orthogonal complement  $\varphi'$  of  $\beta_K$  in  $\varphi$  that  $i_0(s_*\varphi') = i_0(s_*\varphi) - 2$ . We may now repeat the same argument for  $\varphi'$  in place of  $\varphi$ . The statement thus follows by induction.  $\square$

### 2.3. Remarks.

- (1) In the proof above, the fact that  $\mathfrak{b}_\varphi(u, w) \neq 0$  readily implies that  $u$  and  $w$  are  $K$ -linearly independent if  $\text{char } F = 2$ .
- (2) If  $s_*\varphi$  is hyperbolic, then  $i_0(s_*\varphi) = \dim \varphi$ , hence Theorem 2.2 shows that  $\varphi = \psi_K$  for some quadratic form  $\psi$  over  $F$ . This particular case of Theorem 2.2 is established in [3, Theorem 34.9].

- (3) If  $\text{char } F = 2$  and if  $\mathfrak{i}_0(s_*\varphi)$  is odd, then the quadratic form  $\psi$  in Theorem 2.2 cannot be nonsingular, since a regular quadratic form in characteristic 2 is nonsingular if and only if its dimension is even. In particular,  $\psi_K$  is not an orthogonal direct summand of  $\varphi$ . By contrast, if  $\mathfrak{i}_0(s_*\varphi)$  is even, then  $\psi$  is nonsingular hence  $\varphi \simeq \psi_K \perp \varphi'$  for some nonsingular quadratic form  $\varphi'$ . Since  $s_*(\psi_K)$  is hyperbolic it follows that  $\mathfrak{i}_0(s_*\varphi) = \mathfrak{i}_0(s_*(\psi_K))$ , hence  $s_*(\varphi')$  is anisotropic. In this case we thus have an analogue of the result for symmetric bilinear forms [3, Proposition 34.1].
- (4) If the extension  $K/F$  is purely inseparable, then  $\mathfrak{i}_0(s_*\varphi)$  is necessarily even. This follows because the  $K$ -subspace spanned by each isotropic vector for  $s_*\varphi$  is a 2-dimensional  $F$ -subspace that is totally isotropic for  $s_*\varphi$ .

### 3. PROOF OF THE MAIN THEOREM

As in [3], we write  $I_q(F)$  for the Witt group of nonsingular quadratic forms of even dimension over  $F$ , and  $I(F)$  for the ideal of even-dimensional forms in the Witt ring  $W(F)$  of nondegenerate symmetric bilinear forms over  $F$ , and we let  $I_q^n(F) = I^{n-1}(F)I_q(F)$  for  $n \geq 2$ . Let also  $\text{Br}_2(F)$  denote the 2-torsion subgroup of the Brauer group of  $F$ . Recall from [3, Th. 14.3] the group homomorphism

$$e_2: I_q^2(F) \rightarrow \text{Br}_2(F)$$

defined by mapping the Witt class of a quadratic form  $\varphi$  to the Brauer class of its Clifford algebra.

**3.1. Lemma.** *Let  $K$  be a quadratic field extension of an arbitrary field  $F$ , and let  $s: K \rightarrow F$  be a nonzero  $F$ -linear functional such that  $s(1) = 0$ . The following diagram is commutative:*

$$\begin{array}{ccc} I_q^2(K) & \xrightarrow{s_*} & I_q^2(F) \\ e_2 \downarrow & & \downarrow e_2 \\ \text{Br}_2(K) & \xrightarrow{\text{Cor}_{K/F}} & \text{Br}_2(F). \end{array}$$

Here,  $\text{Cor}_{K/F}: \text{Br}_2(K) \rightarrow \text{Br}_2(F)$  is the zero map if  $K$  is a purely inseparable extension of  $F$ .

*Proof.* We have  $I_q^2(K) = I(F)I_q(K) + I(K)I_q(F)$  by [3, Lemma 34.16], hence  $I_q^2(K)$  is generated by Witt classes of 2-fold Pfister forms that have a slot in  $F$ . Commutativity of the diagram follows by Frobenius reciprocity [3, Prop. 20.2], the computation of transfers of 1-fold Pfister forms in [3, Lemma 34.14] and [3, Cor. 34.19], and the projection formula in cohomology, see [6, Ex. 9 p. 63-64] or more generally [4, Prop. 3.4.10].  $\square$

*Proof of Theorem 1.1.* Since  $(ii) \Rightarrow (i)$  is clear, it suffices to prove  $(i) \Rightarrow (iii)$  and  $(iii) \Rightarrow (ii)$ .

If  $(i)$  holds, then we may represent  $Q$  in the form  $(LK/K, b)$  where  $L$  is a quadratic étale  $F$ -algebra linearly disjoint from  $K$  and  $b \in K^\times$ , or in the form  $(M/K, b)$  where  $M$  is a quadratic étale  $K$ -algebra and  $b \in F^\times$ . In each case the projection formula in cohomology shows that  $\text{Cor}_{K/F} Q$  is Brauer-equivalent to a quaternion algebra, hence  $(iii)$  holds.

Now, assume  $(iii)$  holds. If  $Q$  is split, then it contains an  $F$ -algebra isomorphic to  $F \times F$ , so  $(ii)$  holds. For the rest of the proof, we assume  $Q$  is a division algebra. Let  $n_Q$  be the norm form of  $Q$ , which is a 2-fold Pfister quadratic form in  $I_q^2(K)$  such that  $e_2(n_Q) = Q$  in  $\text{Br}(K)$ . Since  $n_Q$  represents 1, the transfer  $s_*(n_Q)$  is isotropic, hence Witt-equivalent to a 6-dimensional nonsingular quadratic form  $\varphi$  in  $I_q^2(F)$ . This form satisfies  $e_2(\varphi) = \text{Cor}_{K/F}(Q)$  in  $\text{Br}(F)$  by Lemma 3.1, hence  $\varphi$  is an Albert form of  $\text{Cor}_{K/F}(Q)$  as per the definition in [6, (16.3)]. In particular, since  $\text{Cor}_{K/F}(Q)$  is not a division algebra,  $\varphi$  is isotropic by [6, (16.5)], and therefore  $i_0(s_*(n_Q)) \geq 2$ . By Theorem 2.2 there exists a nonsingular quadratic form  $\psi$  over  $F$  with  $\dim \psi = 2$  such that  $\psi_K$  is a subform of  $n_Q$ . Since  $Q$  is a division algebra, we have that  $\psi_K$  is anisotropic, hence  $\psi$  is similar to the norm form of a unique separable quadratic field extension  $L/F$ . The field  $L$  is linearly disjoint from  $K$  over  $F$  because  $\psi_K$  is anisotropic. On the other hand,  $\psi_{KL}$  is hyperbolic, hence  $KL$  splits the form  $n_Q$ , and it follows that there exists a  $K$ -algebra embedding of  $KL$  in  $Q$ . Therefore,  $(ii)$  holds.  $\square$

**3.2. Remark.** It follows from the proof above that  $\text{Cor}_{K/F} Q$  is split if and only if  $Q$  is extended from a quaternion  $F$ -algebra. If  $Q$  is extended from a quaternion  $F$ -algebra, the fact that  $\text{Cor}_{K/F} Q$  is split comes from the projection formula in cohomology or [6, (3.13)]. Conversely, if  $\text{Cor}_{K/F} Q$  is split, then the quadratic form  $\varphi$  is hyperbolic by [6, (16.5)]. As in the proof above, we deduce that there exists a nonsingular quadratic form  $\psi$  over  $F$  with  $\dim \psi = 4$  and  $\psi_K = n_Q$ , whence  $Q$  is extended from a quaternion  $F$ -algebra.

If  $K$  is a purely inseparable quadratic extension of  $F$ , all the statements of Theorem 1.1 hold for every quaternion algebra over  $K$ . To see this, recall from our definition of  $\text{Cor}_{K/F}$  that the corestriction of every quaternion  $K$ -algebra is split. Moreover, if  $Q = (M/K, b)$  with  $M$  a separable quadratic extension of  $K$ , then the separable closure of  $F$  in  $M$  is a separable quadratic extension of  $F$  contained in  $Q$  and linearly disjoint from  $K$ .

#### 4. THE ALBERT FORM OF A CORESTRICTION

Let  $Q$  be a quaternion algebra over a separable quadratic field extension  $K$  of an arbitrary field  $F$ . By definition (see [6, (16.3)]), the Albert forms of

$\text{Cor}_{K/F} Q$  are the 6-dimensional nonsingular quadratic forms in  $I_q^2(F)$  such that  $e_2(\varphi) = \text{Cor}_{K/F} Q$  in  $\text{Br}_2(F)$ ; they are all similar. As observed in the proof of Theorem 1.1, an Albert form of  $\text{Cor}_{K/F} Q$  may be obtained from the Witt class of the (8-dimensional) transfer  $s_*(n_Q)$  of the norm form of  $Q$  for an arbitrary nonzero  $F$ -linear functional  $s: K \rightarrow F$  such that  $s(1) = 0$ . In this section, we sketch a more explicit construction of an Albert form of  $\text{Cor}_{K/F} Q$ , inspired by Knus's proof of the Albert–Draxl Theorem in [5], and we use it to give an alternative proof of Theorem 1.1. The arguments below also hold when  $K \simeq F \times F$ .

We first recall the construction of the corestriction  $\text{Cor}_{K/F} Q$ . Let  $\gamma$  be the nontrivial  $F$ -automorphism of  $K$  and let  ${}^\gamma Q$  denote the conjugate quaternion algebra  ${}^\gamma Q = \{{}^\gamma x \mid x \in Q\}$  with the operations

$${}^\gamma x + {}^\gamma y = \gamma(x + y), \quad {}^\gamma x \cdot {}^\gamma y = \gamma(xy), \quad \lambda \cdot {}^\gamma x = \gamma(\gamma(\lambda)x)$$

for  $x, y \in Q$  and  $\lambda \in K$ . The algebra  ${}^\gamma Q \otimes_K Q$  carries a  $\gamma$ -semilinear automorphism  $s$  defined by

$$s({}^\gamma x \otimes y) = {}^\gamma y \otimes x \quad \text{for } x, y \in Q.$$

By definition, the corestriction (or norm)  $\text{Cor}_{K/F}(Q)$  is the  $F$ -algebra of fixed points (see [6, (3.12)]):

$$\text{Cor}_{K/F}(Q) = ({}^\gamma Q \otimes_K Q)^s.$$

Let  $\text{Trd}$  and  $\text{Nrd}$  denote the reduced trace and the reduced norm on  $Q$ . Let also  $\sigma$  be the canonical (conjugation) involution on  $Q$ . Consider the following  $K$ -submodule of  ${}^\gamma Q \otimes_K Q$ :

$$V = \{{}^\gamma x_1 \otimes 1 - 1 \otimes x_2 \mid x_1, x_2 \in Q \text{ and } \gamma(\text{Trd}(x_1)) = \text{Trd}(x_2)\}.$$

This  $K$ -module is free of rank 6 and is preserved by  $s$ , and one can show that the  $F$ -space of  $s$ -invariant elements has the following description, where  $T_{K/F}: K \rightarrow F$  is the trace form:

$$V^s = \{{}^\gamma y \otimes 1 + 1 \otimes y \mid y \in Q \text{ and } T_{K/F}(\text{Trd}(y)) = 0\}.$$

Now, pick an element  $\kappa \in K^\times$  such that  $\gamma(\kappa) = -\kappa$ . (If  $\text{char } F = 2$  we may pick  $\kappa = 1$ .) The following formula defines a quadratic form  $\varphi: V^s \rightarrow F$ : for  $y \in Q$  such that  $T_{K/F}(\text{Trd}(y)) = 0$ , let

$$\varphi({}^\gamma y \otimes 1 + 1 \otimes y) = \kappa \cdot (\gamma(\text{Nrd}(y)) - \text{Nrd}(y)).$$

Nonsingularity of the form  $\varphi$  is easily checked after scalar extension to an algebraic closure of  $F$ , and computation shows that the linear map

$$f: V^s \rightarrow M_2(\text{Cor}_{K/F}(Q)) \quad \text{given by} \quad \xi \mapsto \begin{pmatrix} 0 & \kappa \cdot (\sigma \otimes \text{id})(\xi) \\ \xi & 0 \end{pmatrix}$$

satisfies  $f(\xi)^2 = \varphi(\xi)$  for all  $\xi \in V^s$ . Therefore,  $f$  induces an  $F$ -algebra homomorphism  $f_*$  defined on the Clifford algebra  $C(V^s, \varphi)$ . Dimension count shows that  $f_*$  is an isomorphism

$$(4.1) \quad f_*: C(V^s, \varphi) \xrightarrow{\sim} M_2(\text{Cor}_{K/F} Q).$$

The restriction to the even Clifford algebra is an isomorphism

$$C_0(V^s, \varphi) \simeq (\text{Cor}_{K/F} Q) \times (\text{Cor}_{K/F} Q),$$

hence the discriminant (or Arf invariant) of  $\varphi$  is trivial since the center of its even Clifford algebra is split. This means  $\varphi \in I_q^2(F)$ , and (4.1) shows that  $e_2(\varphi) = \text{Cor}_{K/F} Q$  in  $\text{Br}_2(F)$ , so  $\varphi$  is an Albert form of  $\text{Cor}_{K/F} Q$ .

We use the Albert form  $\varphi$  to sketch an alternative proof of Theorem 1.1. If the base field  $F$  is finite, then  $Q$  is split and all the conditions in Theorem 1.1 trivially hold. Therefore, we may assume  $F$  is infinite.

Suppose condition (i) of Theorem 1.1 holds. If  $x \in Q$  generates a quadratic  $F$ -algebra disjoint from  $K$ , then  $\text{Trd}(x) \in F$  and  $\text{Nrd}(x) \in F$  (and  $x \notin K$ ), hence  $\gamma(\kappa x) \otimes 1 + 1 \otimes (\kappa x) \in V^s$  is an isotropic vector of  $\varphi$ . Since  $\varphi$  is an Albert form of  $\text{Cor}_{K/F} Q$ , it follows that  $\text{Cor}_{K/F} Q$  is not a division algebra. Therefore, (i) implies (iii).

For the converse, suppose (iii) holds. Then there exists  $y \in Q$  such that  $\xi = \gamma y \otimes 1 + 1 \otimes y \in V^s$  is an isotropic vector for  $\varphi$ . A density argument shows that we may find such an element  $y$  with  $\text{Trd}(y) \neq 0$ . We obtain that  $\gamma(\text{Nrd}(y)) - \text{Nrd}(y) = 0$ , hence  $\text{Nrd}(y) \in F$ , and  $T_{K/F}(\text{Trd}(y)) = 0$ .

We claim that  $y \notin K$ . Suppose on the contrary that  $y \in K$ . Then we have  $\xi = 1 \otimes (y + \gamma(y)) \neq 0$  and therefore  $T_{K/F}(y) \neq 0$ . Since  $2T_{K/F}(y) = T_{K/F}(\text{Trd}(y)) = 0 \neq T_{K/F}(y)$  we conclude that  $\text{char}(F) = 2$  and  $y \notin F$ . But  $y^2 = \text{Nrd}(y)$  since  $y \in K$  and  $\text{Nrd}(y) \in F$  because  $\varphi(\xi) = 0$ , hence  $y^2 \in F$ . This is a contradiction since  $K$  is an étale  $F$ -algebra.

Thus  $y \notin K$ . Note that  $\kappa y \in Q$  satisfies

$$\text{Trd}(\kappa y) \in F^\times \quad \text{and} \quad \text{Nrd}(\kappa y) = \kappa^2 \text{Nrd}(y) \in F.$$

Therefore,  $\kappa y$  generates a quadratic étale  $F$ -subalgebra of  $Q$  linearly disjoint from  $K$ , proving that (ii) (hence also (i)) holds.

## REFERENCES

- [1] K.-J. Becher, N. Grenier-Boley, J.-P. Tignol, Involutions and stable subalgebras, in preparation.
- [2] P. Draxl, Über gemeinsame separabel-quadratische Zerfällungskörper von Quaternionenalgebren, *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II* **1975**, no. 16, 251–259.
- [3] R. Elman, N. Karpenko, A. Merkurjev. *The algebraic and geometric theory of quadratic forms*. American Mathematical Society Colloquium Publications, **56**, Amer. Math. Soc., Providence, RI, 2008.
- [4] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*. Cambridge Studies in Advanced Mathematics, 101, Cambridge Univ. Press, Cambridge, 2006.

- [5] M.-A. Knus, Sur la forme d'Albert et le produit tensoriel de deux algèbres de quaternions, *Bull. Soc. Math. Belg. Sér. B* **45** (1993), no. 3, 333–337.
- [6] M.-A. Knus, A. S. Merkurjev, M. Rost, and J.-P. Tignol. *The book of involutions. American Mathematical Society Colloquium Publications* **44**. American Mathematical Society, Providence, RI, 1998.
- [7] J. Tits, Sur les produits tensoriels de deux algèbres de quaternions, *Bull. Soc. Math. Belg. Sér. B* **45** (1993), no. 3, 329–331.

UNIVERSITEIT ANTWERPEN, DEPARTEMENT WISKUNDE EN INFORMATICA, MIDDELHEIMLAAN 1, B-2020 ANTWERPEN, BELGIUM

*E-mail address:* `KarimJohannes.Becher@uantwerpen.be`

LDAR (EA4434), UA, UCP, UPD, UPEC, URN, UNIVERSITÉ DE ROUEN NORMANDIE, MONT-SAINT-AIGNAN, FRANCE.

*E-mail address:* `nicolas.grenier-boley@univ-rouen.fr`

UNIVERSITÉ CATHOLIQUE DE LOUVAIN, ICTEAM INSTITUTE, AVENUE G. LEMÂÎTRE 4, BOX L4.05.01, B-1348 LOUVAIN-LA-NEUVE, BELGIUM.

*E-mail address:* `jean-pierre.tignol@uclouvain.be`