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Abstract: Most subspace-based methods enabling instability monitoring are restricted to the linear time-invariant (LTI) systems. In this paper, a new subspace method of instability monitoring is proposed for the linear periodically time-varying (LPTV) case. For some LPTV systems, the system transition matrices may depend on some parameter and are also periodic in time. A certain range of values for the parameter leads to an unstable transition matrix. Early warning should be given when the system gets close to that region, taking into account the time variation of the system. Using the theory of Floquet, some symptom parameter of stability- or residual- is defined. Then, the parameter variation is tracked by performing a set of parallel cumulative sum (CUSUM) tests. Finally, the method is tested on a simulated model of a helicopter with hinged blades, for monitoring the ground resonance phenomenon.

Keywords: Helicopter dynamics, Time-varying systems, Periodic motion, Resonance, Subspace methods, Statistical inference, Fault detection, Stability limits

1. INTRODUCTION

Instability monitoring is currently the subject of extensive research activities motivated by an increasing requirement on the design of highly reliable control systems, for numerous applications such as civil engineering and aeronautics. Under some assumptions, such structures can be modeled by a linear time-invariant (LTI) model. The goal is to detect any anomalous behavior on these structures, and to alarm the user before any dysfunction or destruction occurs. There is a large amount of literature on the subject of instability monitoring, which has been handled with different approaches (see Angeli and Chatzinikolaou [2004] for details). One particular method, namely the subspace-based method, consists in comparing the characteristics of a system at a reference state with a subspace matrix given by new data corresponding to an unknown, possibly different state, as explained in Basseville et al. [2000]. To check whether a change has occurred or not, a quasi-distance between the two states - called residual- is defined. Then, a statistical test decides if this residual is significantly different from zero. If some threshold is exceeded, the system has changed with respect to the reference. An overview of the different approaches of this general approach can be found in Basseville and Nikiforov [1993]. This method has been largely investigated and successfully tested on LTI systems. In some complex systems, the characterizing model may change due to some internal commands or parameters (Mach number for aircrafts, blade rotational velocity for helicopters...). In Basseville et al. [2007], the system is labeled as LTI, even if it depends on those internal commands (this class of systems is called linear parameter-varying (LPV) in control theory, but it is still considered time-invariant for each fixed parameter). This assumption holds since for each command's value, data has a time-invariant behavior, and since the command varies slowly as a step function, generally. The problem will be to detect a significant change of the structural characteristics leading to instability as those commands fluctuate and some system eigenvalue starts to cross the Nyquist circle. This problem has been solved for aircraft flutter monitoring in Mevel et al. [2005].

The subject of this paper is the extension of this approach to periodically time-varying (varying, even for fixed internal command's values) systems such as wind turbines and rotor systems, replacing the residual in Mevel et al. [2005] by a robust one considered in Döhler and Mevel [2011]. To the best of the knowledge of the authors, the bibliography on subspace methods for LPTV systems is not abundant. For instance, in Verhaegen and Yu [1995], some attempts have been carried out in subspace-based identification. Some other works have dealt with recursive tracking of periodic subspaces, as in Buzzi et al. [2003]. Otherwise, Bayesian filtering such as Kalman and particle filters, or robust and adaptive observers are also used. Two key features motivate the study reported herein:

- A simple, necessary and sufficient criterion of stability can be extracted from periodic data, using the theory of Floquet (Dacunha and Davis [2009]). Therefore, a simple residual based on this criterion can be built
- A set of time-invariant data subsequences that have the same time-varying behavior can be found for such

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periodic systems, as outlined in Meyer and Burrus [1975], which makes it possible to apply a detection algorithm similar to the classical time-invariant one (Mevel et al. [2005]), for each of these sequences

Once the criterion of stability is found, a set of residuals, corresponding to each time-invariant subsequence, is defined. Then, parallel statistical tests are performed to decide if an eventual change in the system characteristics has occurred. The statistical test used herein is the cumulative sum (CUSUM) because it is adequate for online implementation and for the detection of the instant of change (see Basseville and Nikiforov [1993]). The paper is organized as follows: in Section 2, the essential elements of the subspace methods for identification and instability monitoring are recalled. Then, Section 3 is devoted to the design of the new LPTV-extended method of instability monitoring. Finally, in Section 4 an illustrative simulation case is studied in order to test the efficiency of the suggested method.

2. SUBSPACE-BASED METHODS FOR LTI SYSTEMS

2.1 Subspace Identification

Consider the discrete time model, describing a system in the state space form:

$$\begin{cases} z_{k+1} = Fz_k + w_k \\ y_k = Hz_k + v_k \end{cases} \tag{1}$$

with the state vector $z \in \mathbb{R}^n$, the output vector $y \in \mathbb{R}^r$, the state transition matrix $F \in \mathbb{R}^{n\times n}$ and the observation matrix $H \in \mathbb{R}^{r\times n}$. The vectors w and v are unmeasured noises, assumed to be white Gaussian. The eigenstructure of (1) is given by the roots (λ, ϕ_λ) of the equations, below:

$$det(F - \lambda I) = 0, F\phi_\lambda = \lambda\phi_\lambda \tag{2}$$

Let p and q be chosen parameters such that $min\{pr, qr\} \geq n$. From the output data (y_k), a Hankel matrix $\mathcal{H}_{p,q} \in \mathbb{R}^{(p+1)r\times qr}$ is built:

$$\mathcal{H}_{p,q} \hat{=} \begin{bmatrix} R_1 & R_2 & \cdots & R_q \\ R_2 & R_3 & \cdots & R_{q+1} \\ \vdots & \vdots & \vdots & \vdots \\ R_{p+1} & R_{p+2} & \cdots & R_{p+q} \end{bmatrix} \tag{3}$$

where the covariances of the output data write $R_i = \mathbf{E}(y_k y_{k-i}^T)$ and $\mathbf{E}$ is the expectation operator. When the number of data N goes to infinity, the approximations [1] below hold: $\hat{R}_i = \frac{1}{N}\sum_{k=i+1}^{N} y_k y_{k-i}^T$ and $\hat{\mathcal{H}}_{p,q} = \frac{1}{N}\sum_{k=q}^{N-p} \mathcal{Y}_k{}^+ \mathcal{Y}_k{}^{-T}$, where $\mathcal{Y}_k{}^+ = (y_k^T \cdots y_{k+p}^T)^T$ and $\mathcal{Y}_k{}^- = (y_{k-1}^T \cdots y_{k-q}^T)^T$. The Hankel matrix can be factorized as below (see Benveniste and Fuchs [1985]):

$$\mathcal{H}_{p,q} = \mathcal{O}_p \mathcal{C}_q \tag{4}$$

where $\mathcal{O}_p \hat{=} \begin{bmatrix} H \\ HF \\ \vdots \\ HF^p \end{bmatrix}$, $\mathcal{C}_q \hat{=} \left[FG,\ F^2G,\ \cdots,\ F^qG \right]$ and $G = \mathbf{E}(z_k y_k^T)$. Therefore, the observability matrix $\mathcal{O}_p$ is obtained from a thin singular value decomposition (SVD) of the matrix $\mathcal{H}_{p,q}$ and its truncation at the desired order n (Basseville et al. [2000]):

$$\mathcal{H}_{p,q} = U\Delta V^T = [\,U_1\ U_2\,] \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \begin{bmatrix} V_1{}^T \\ V_2{}^T \end{bmatrix} \tag{5}$$

$$\mathcal{O}_p = U_1 \Delta_1^{1/2} \tag{6}$$

with U_1 the left subspace of $\mathcal{H}_{p,q}$ found in the n first columns of U, and Δ_1 the n upper singular values of Δ. The observation matrix H is found in the first block-row of O_p. To get the transition matrix F, the shift invariance property of $\mathcal{O}_p$ is used in resolving the least square solution of (Basseville et al. [2007]):

$$\mathcal{O}_p^{\uparrow} F = \mathcal{O}_p^{\downarrow} \tag{7}$$

where:

$$\mathcal{O}_p^{\uparrow} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{p-1} \end{bmatrix},\ \mathcal{O}_p^{\downarrow} = \begin{bmatrix} HF \\ HF^2 \\ \vdots \\ HF^p \end{bmatrix} \tag{8}$$

Once F is computed, the eigenvalues λ and the eigenvectors ϕ_λ can easily be derived from the resolution of (2). Then, the mode shapes (the observed eigenvectors), defined by $\Phi_\lambda = H\phi_\lambda$, are deduced. Let the eigenstructure be the vector $\theta = \begin{pmatrix} \Lambda \\ vec(\Phi) \end{pmatrix}$ considered as a canonical parametrization of (1), where Λ is the vector whose elements are the eigenvalues and Φ is the matrix of the observed eigenvectors $\Phi_\lambda = H\phi_\lambda$.

2.2 Subspace Instability Monitoring

In this section, the statistical subspace-based instability monitoring from Basseville et al. [2007] and Mevel et al. [2005] is recalled, where a residual function for change detection is associated with the subspace identification described in the previous section. This function, or residual, compares the system parameter θ_0 of a reference with a block Hankel matrix computed on new data $(y_k)_{k=1\cdots N}$ corresponding to an unknown, possibly different parameter θ. To decide whether a significant change has occurred or not, a local approach is used to decide between two hypotheses:

$$\mathbf{H}_0 : \quad \theta = \theta_0 \quad and \quad \mathbf{H}_1 : \quad \theta = \theta_0 + \delta\theta/\sqrt{N} \tag{9}$$

where $\delta\theta$ is unknown but fixed. Let K be a left kernel of the Hankel matrix at the reference $\mathcal{H}_{p,q}^0$, such that:

$$K^T(\theta_0)\mathcal{H}_{p,q}^0 = 0,\ K^T(\theta_0)K(\theta_0) = I \tag{10}$$

To check whether new data agree with the reference state corresponding to θ_0, the residual function below is introduced in Basseville et al. [2000]: $\zeta_N(\theta_0) = \sqrt{N}\ \ vec(K^T(\theta_0)\mathcal{H}_{p,q})$, where $\mathcal{H}_{p,q}$ is the Hankel matrix built with new data from the current parameter θ, possibly different from θ_0. From now on, we will use another residual that is robust to noise changes, recently suggested in Döhler and Mevel [2011]:

[1] In the following, we drop the ˆ notation for sake of simplicity

$$\zeta_N(\theta_0) = \sqrt{N} \quad vec(K^T(\theta_0)U_1) \tag{11}$$

where K is the left kernel of the left subspace U_1^0 of $\mathcal{H}_{p,q}^0$ at the reference, and U_1 the left subspace of Hankel matrix $\mathcal{H}_{p,q}$ built on the current data. Let $\mathcal{J}(\theta_0)$ be the matrix of partial derivatives of the vectorized residual with respect to the vector of parameters (Mevel et al. [2005]) and $\Sigma(\theta_0)$ is the residual covariance matrix, which can be estimated as an empirical sum (Basseville et al. [2000]). The matrices $\Sigma(\theta_0)$ and $\mathcal{J}(\theta_0)$ associated to the residual in (11) are independent of the possibly time varying noise properties (Döhler and Mevel [2011]). If $\Sigma(\theta_0)$ is positive definite and assuming θ as in (9), the central limit theorem (CLT) insures that the residual $\zeta_N(\theta_0)$ in (11) is asymptotically Gaussian distributed (Döhler and Mevel [2011]):

$$\zeta_N(\theta_0) \underset{N\to\infty}{\longrightarrow} \mathcal{N}(\mathcal{J}(\theta_0)\delta\theta, \Sigma(\theta_0)) \tag{12}$$

which implies that the mean of $\zeta_N(\theta_0)$ converges to 0 only under $\mathbf{H}_0$ (i.e. $\delta\theta = 0$). In other words, detecting if the eigenstructure has changed from the reference is equivalent to detecting a change in the mean of $\zeta_N(\theta_0)$. In order to detect the direction of the change, a normalized residual $\overline{\zeta}_N(\theta_0)$ is introduced, such that: $\overline{\zeta}_N(\theta_0) = \mathcal{J}^T(\theta_0)\Sigma^{-1}(\theta_0)\zeta_N(\theta_0)$, which leads to the normal distribution:

$$\overline{\zeta}_N(\theta_0) \underset{N\to\infty}{\longrightarrow} \mathcal{N}(\overline{\Sigma}(\theta_0)\delta\theta, \overline{\Sigma}(\theta_0)) \tag{13}$$

where: $\overline{\Sigma}(\theta_0) = \mathcal{J}^T(\theta_0)\Sigma^{-1}(\theta_0)\mathcal{J}(\theta_0)$. This covariance matrix is positive definite. Therefore, a positive (resp. negative) change in the mean of the normalized residual $\overline{\zeta}_N(\theta_0)$ reflects a positive (resp. negative) change in $\delta\theta$. A recursive decision function, or residual, is introduced in order to design an on-line monitoring algorithm:

$$\mathcal{Z}_k = \mathcal{J}(\theta_0)^T\Sigma(\theta_0)^{-1} vec\left(K^T(\theta_0)U_{1,k} \right) \tag{14}$$

where $U_{1,k}$ is the current left subspace of the Hankel matrix at the k-th sample data y_k, obtained by an update of the SVD. Based on the recursive residual in (14) and considering that the Z_k's are independent Gaussian which describe themselves a change in θ by a change in their mean, a commonly used CUSUM test (Basseville and Nikiforov [1993]) is applied. In general, only the part Λ of θ changes with respect to time. The eigenvectors are assumed to be constant, considering the small displacement hypothesis (SDH). Now, denote ρ as the norm of one of the eigenvalues λ's of F and notice that when one of the ρ's increases to 1, the system goes toward instability (Basseville and Nikiforov [1993]). The CUSUM test described below allows to detect an increase in any component of the parameter by using as Jacobian matrix the corresponding submatrix of $J(\theta_0)$: the sensitivity $J(\theta_0)$ is replaced in (14) by the column $J(\rho_0)$ corresponding to the sensitivity with respect to ρ at ρ equal to ρ_0: $S_k(\rho_0) = \overline{\Sigma}(\rho_0)^{-1/2} \sum_{j=q}^{k} \mathcal{Z}_j(\rho_0)$, $T_k(\rho_0) = \min_{q\le j\le k} S_j(\rho_0)$ and $g_k(\rho_0) = S_k(\rho_0) - T_k(\rho_0)$. The two hypotheses $\mathbf{H}_{0,k}$ and $\mathbf{H}_{1,k}$ to decide between, at each instant k, are now:

$$\begin{cases} \mathbf{H}_{0,k} : \mathbf{E}g_k(\rho_0) \approx 0 \\ \mathbf{H}_{1,k} : \mathbf{E}g_k(\rho_0) > \epsilon \end{cases} \tag{15}$$

where ϵ is some empirically tuned threshold (Mevel et al. [2005]) that allows to detect only the significant changes. If this threshold is exceeded, the test reacts and the current data are describing a new value for the parameter $\rho > \rho_0$. The test described above is applied to all the norms of the eigenvalues $(\lambda_i)_{i=1\cdots n}$, and when a change is detected in one of the ρ_i, the system is considered to be unstable.

3. SUBSPACE INSTABILITY MONITORING IN LPTV CASE

Linear periodically time-varying systems are encountered in many different fields. Some of the most relevant applications that involve periodic behaviors are rotor systems such as helicopters and wind turbines, satellite control of attitude or also in communications. This class of systems is considered as an intermediate class bridging the time-invariant case to the time-varying one. For the latter class, the stability analysis is generally based on complex algebraic-analytic tools (see Bourles and Bogdan [2011] for details). Finding a criterion of stability is a crucial step in subspace instability monitoring, because a residual based on this criterion has to be built and then tracked by the statistical test presented in the last section. It seems difficult to get a simple criterion for time-varying systems in general. Hopefully, there exist specific methods for periodic systems that give relatively simple criteria about stability. One of these methods is the modal analysis using the transformation of Floquet, described hereafter.

3.1 Transformation of Floquet

Let consider the continuous LPTV system below, with a period T:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + w(t), \quad A(t+T) = A(t), \forall t \in \mathbb{R} \\ y(t) = Cx(t) + v(t) \end{cases} \tag{16}$$

with $x \in \mathbb{R}^n$ the state vector, $A \in \mathbb{R}^{n\times n}$ the state matrix, $y \in \mathbb{R}^r$ the output vector and $C \in \mathbb{R}^{r\times n}$ the observation matrix. The vectors w and v are two additive noises assumed to be white Gaussian. The idea of the Floquet transformation is to replace this system by an equivalent autonomous (i.e. the matrix of transition is no more function of time) one.

Theorem 1. If A is continuous in time- or at least piecewise continuous- and an initial condition $x(t_0) = x_0$ is fixed, then a solution of $\dot{x}(t) = A(t)x(t)$ is guaranteed to exist. Let $\Phi(t)$ be the matrix whose n columns are n linearly independent solutions, $\Phi(t)$ is known as the Fundamental Transition Matrix (FTM) and $\dot{\Phi}(t) = A(t)\Phi(t)$, $\Phi(t + T) = \Phi(t)\Phi(T)$.

Proof. see Dacunha and Davis [2009]

Corollary 2. (Floquet Transformation) The value of the fundamental matrix at $t = T$ is called the Monodromy Matrix.

$$Q = \Phi(T) \tag{17}$$

Let $R = \frac{1}{T}log(Q)$ and $x(t) = \Phi(t)e^{-Rt}z(t)$. Then, the first-order ordinary differential equation above can be transformed into the equivalent equation:

$$\dot{z}(t) = Rz(t) \tag{18}$$

The equation of observation for the new variable z is $y(t) = C^{'}(t)z(t)$ where $C^{'}(t) = C\Phi(t)e^{-Rt}$ and $C^{'}(t+T) = C\Phi(t+T)e^{-R(t+T)} = C\Phi(t)\Phi(T)e^{-RT}e^{-Rt} = C\Phi(t)e^{-Rt} = C^{'}(t)$. Therefore, any system that writes as in (16) can be transformed into an equivalent autonomous system with a periodic matrix of observation:

$$\begin{cases} \dot{z}(t) = Rz(t) + \left(\Phi(t)e^{-Rt}\right)^{-1} w(t) \\ y(t) = C^{'}(t)z(t) + v(t), \quad C^{'}(t+T) = C^{'}(t), \forall t \end{cases} \tag{19}$$

According to Floquet theory, the continuous system (19) is stable if and only if the real parts of eigenvalues of R are negative or, similarly, if the norms of the eigenvalues of Q in (17) are inferior to one. Let consider the discretized form of (19), at a sampling rate $\frac{1}{\tau}$ (where T is assumed to be a multiple of τ) (Ma and Iglesias [2002]):

$$\begin{cases} z_{k+1} = Fz_k + \Gamma_k w_k \\ y_k = H_k z_k + v_k \end{cases} \tag{20}$$

The discretized monodromy matrix $F = e^{R\tau}$, the discretized matrix of observation $H_k = C^{'}(k\tau)$ and $\Gamma_k = \int_{k\tau}^{(k+1)\tau} e^{R\gamma} d\gamma$ are periodic. The discrete period is $T_d = \frac{T}{\tau}$.

3.2 Residuals Generation

As outlined in Meyer and Burrus [1975], the data (y_k) have different varying dynamics in the periodic case. However, the subsequences $(y_{k_0+iT_d})_{i\in\mathbb{N}}$ and $(z_{k_0+iT_d})_{i\in\mathbb{N}}$ are shown to have time-invariant behaviors for any $k_0 \in \mathbb{N}$. A total of T_d different time-invariant subsequences exists. Let the eigenstructure below be the parameter function of one of these subsequences (denoted the j-th subsequence, with $j \in [\![1, T_d]\!]$):

$$\theta^{(j)} = \begin{pmatrix} \Lambda \\ vec(\Phi_j) \\ vec(\Phi_{j+1}) \\ \vdots \\ vec(\Phi_{j+p}) \end{pmatrix} \tag{21}$$

where Λ is the vector of the eigenvalues λ_i of F and the mode shapes Φ are the observed eigenvectors of F, $\Phi_j = [H_j\phi_1, \cdots, H_j\phi_n]$ with $F\phi_i = \lambda_i\phi_i$. The condition in the discrete form is that the eigenvalues of F are in the Nyquist circle. The Hankel matrices are built using the time-invariant subsequences as described hereafter.

Lemma 3. For any $j \geq q$ and if $(NT_d + p)$ data samples are available, the j-th Hankel matrix $\mathcal{H}_{p,q}^{(j)}$ is defined as:

$$\mathcal{H}_{p,q}^{(j)} = \frac{1}{N}\sum_{i=0}^{N-1} \mathcal{Y}_{j+iT_d}^{+} \mathcal{Y}_{j+iT_d}^{-T} \tag{22}$$

This matrix can be factorized as in (4) to the product:

$$\mathcal{H}_{p,q}^{(j)} = \mathcal{O}_p^{(j)} \mathcal{C}_q^{(j)} \tag{23}$$

where $\mathcal{O}_p^{(j)}$ is the observability matrix for the j-th subsequence, such that:

$$\mathcal{O}_p^{(j)} = \left[H_j^T \ (H_{j+1}F)^T \ \cdots \ (H_{j+p}F^p)^T \right]^T \tag{24}$$

Notice that unlike the LTI case, the observation matrix H_{j+i} is varying with $i = 1, \cdots, p$.

Proof. see Jhinaoui et al. [2012]

Fact 4. For notation simplicity, T is assumed to be a multiple of τ. In general, this is not the case and the time-invariant subsequences are the ensembles $(y_{\lfloor \frac{j\tau+iT}{\tau} \rfloor})_i$ and the data $\mathcal{Y}_{j+iT_d}^{+}$ and $\mathcal{Y}_{j+iT_d}^{-}$ are replaced by $\mathcal{Y}_{\lfloor \frac{j\tau+iT}{\tau} \rfloor}^{+}$ and $\mathcal{Y}_{\lfloor \frac{j\tau+iT}{\tau} \rfloor}^{-}$ (where $\lfloor\rfloor$ is the floor operator), in order to get the sum for approximately the same subsequence.

The observability matrix can be obtained, similarly to Section 2, from a thin singular value decomposition (SVD) of the Hankel matrix $\mathcal{H}_{p,q}^{(j)}$:

$$\mathcal{H}_{p,q}^{(j)} = \left[U_1^{(j)} \ U_2^{(j)} \right] \begin{bmatrix} \Delta_1^{(j)} & 0 \\ 0 & \Delta_2^{(j)} \end{bmatrix} \begin{bmatrix} (V_1^{(j)})^T \\ (V_2^{(j)})^T \end{bmatrix} \tag{25}$$

$$\mathcal{O}_p^{(j)} = U_1^{(j)} (\Delta_1^{(j)})^{1/2} \tag{26}$$

Writing $\mathcal{O}_p^{(j)}$ in the modal basis gives:

$$\begin{aligned} \overline{\mathcal{O}}_p^{(j)} &= \left[\Phi_j^T \ (\Phi_{j+1}D)^T \ \cdots \ (\Phi_{j+p}D^p)^T \right]^T . \ T \\ &= \mathcal{O}_p^{(j)}. \ T \end{aligned} \tag{27}$$

where diagonal matrix D is defined as $D = diag(\Lambda)$, and T is an invertible transformation. When p is large enough (if the rank of the observability matrix is n), the j-th subsequence of the periodic system (20) is described by its observability matrix $\mathcal{O}_p^{(j)}$. For a full characterization at a reference $\theta_0 = (\theta_0^{(j)})_{j=1,\cdots,T_d}$, the observability matrices should be given for all the T_d subsequences $(y_{1+iT_d})_i$, $(y_{2+iT_d})_i$, $(y_{3+iT_d})_i$, $\cdots$, $(y_{T_d+iT_d})_i$. For each subsequence $(j)_{j=1\cdots T_d}$, the Hankel matrix is computed as in (22). Then, a left kernel $K^{(j)}$ of the left subspace $U_1^{(j)}$ of $\mathcal{H}_{p,q}^{(j)}$ is deduced:

$$\left(K^{(j)}\right)^T U_1^{(j)} = 0, \qquad \forall j \in [\![1, T_d]\!] \tag{28}$$

The T_d equations in (28) are satisfied if current output data describe well the reference parameters $\theta_0^{(j)}$. These functions are the candidates to be the residuals that compare current data to reference parameters. In the periodic case, there are T_d residuals, one for each of the T_d time-invariant subsequences. At an instant k, these residuals write in recursive forms $\forall j \in [\![1, T_d]\!]$, as in (14):

$$Z_k^{(j)} = (\mathcal{J}^{(j)}(\theta_0^{(j)}))^T (\Sigma^{(j)}(\theta_0^{(j)}))^{-1} vec({K^{(j)}}^T U_{1,k}^{(j)}) \tag{29}$$

where $K^{(j)}$ is the left kernel at the reference $\theta_0^{(j)}$, $U_{1,k}^{(j)}$ the left subspace update of the Hankel matrix $\mathcal{H}_{p,q}^{(j)}$, when the k-th data sample of the (j)-th subsequence is available, and $\mathcal{J}^{(j)}(\theta_0^{(j)})$ and $\Sigma^{(j)}(\theta_0^{(j)})$ are the residual's sensitivity and covariance matrices as in Döhler and Mevel [2011].

3.3 Parallel CUSUM Tests

Let ρ_0 be the norm of one of the eigenvalues contained in Λ. The goal herein is to decide whether the value of ρ_0 has increased. For each time-invariant subsequence, the hypotheses to decide between at the time sample k are:

$$\mathbf{H}_{0,k}^{(j)}: \quad \mathbf{E}g_k^{(j)}(\rho_0) \approx 0$$
$$\mathbf{H}_{1,k}^{(j)}: \quad \mathbf{E}g_k^{(j)}(\rho_0) > \epsilon \tag{30}$$

where ϵ is some predefined threshold that allows to detect only the significant changes. The T_d CUSUM tests to perform, corresponding to each time-invariant subsequence, are:

$$S_k^{(j)}(\rho_0) = \overline{\Sigma}^{(j)}(\rho_0)^{-1/2} \sum_{i=q}^{k} \mathcal{Z}_i^{(j)}(\rho_0)$$
$$T_k^{(j)}(\rho_0) = \min_{q \le i \le k} S_i^{(j)}(\rho_0)$$
$$g_k^{(j)}(\rho_0) = S_k^{(j)}(\rho_0) - T_k^{(j)}(\rho_0) \tag{31}$$

$\mathcal{Z}_i^{(j)}(\rho_0)$ is the recursive residual with respect to ρ_0 and replacing, in (29), the sensitivity $\mathcal{J}^{(j)}(\theta_0^{(j)})$ by the sensitivity $\mathcal{J}^{(j)}(\rho_0)$. Notice that when the monitoring starts, no information is available about the relation in time between the identified reference and the current data. In other words, if the monitoring starts at $t = t_0$ (with $t_0 \ge q$) and $(NT_d + p)$ data are available, it is not known to which kernel $K^{(j)}$ the Hankel matrix $\mathcal{H}_{p,q}^{(t_0)} = \frac{1}{N} \sum_{i=0}^{N-1} \mathcal{Y}_{t_0+iT_d}^{+} \mathcal{Y}_{t_0+iT_d}^{-T}$ corresponds, because the phase of identification and the phase of monitoring may have not the same time origin. The current data is synchronized with the reference parameter kernel (denoted $K^{(j_*)}$) as follows:

$$K^{(j_*)} := \underset{j=1\cdots T_d}{\operatorname{argmin}}\left(\left\| (K^{(j)})^T \mathcal{H}_{p,q}^{(t_0)} \right\|\right) \tag{32}$$

To summarize the algorithm:

- using the identification algorithm described in Jhinaoui et al. [2012], a set of T_d kernels $K^{(j)}$, corresponding to the T_d time-invariant subsequences is computed at a fixed reference $\theta_0 = (\theta_0^{(j)})_{j=1,\cdots,T_d}$
- the Jacobians and the covariances matrices are computed (see Döhler and Mevel [2011])
- start the monitoring. If it starts at $t = t_0$, the current data are synchronized with the identified kernels using (32)
- for $t \ge t_0$, the CUSUM tests are performed for each subsequence, in parallel, as in (31)
- the instant time of change is defined as the instant when one (or more) of the parallel CUSUM tests exceeds some fixed threshold (Mevel et al. [2005])

Notice also, that for some complex systems, the mechanical model is function of some internal parameters, such as rotational velocity Ω for helicopters' blades. In this case, the system is periodic of period $T = \frac{2\pi}{\Omega}$. And then, the period of the discrete system is $T_d = \frac{2\pi}{\Omega\tau}$. When $\Omega = \Omega_1$, the system is LPTV and there exist $T_{d1} = \frac{2\pi}{\Omega_1\tau}$ time-invariant subsequences, and when $\Omega = \Omega_2 > \Omega_1$, the system is LPTV and there exist $T_{d2} = \frac{2\pi}{\Omega_2\tau}$ time-invariant subsequences, with $T_{d2} < T_{d1}$. As in the case reported in Basseville et al. [2007] and assuming that the velocity is slowly varying as step function, for two different velocities Ω_1 and Ω_2, the T_{d2} subsequences $(y_{1+iT_{d2}})$, $(y_{2+iT_{d2}})$, $\cdots$, $(y_{T_{d2}+iT_{d2}})$ from $\Omega = \Omega_2$ and $(y_{1+iT_{d1}})$, $(y_{2+iT_{d1}})$, $\cdots$, $(y_{T_{d2}+iT_{d1}})$ from $\Omega = \Omega_1$ are considered as coming from the same subsystem, whereas, the $T_{d1} - T_{d2}$ other subsequences vanish when the velocity changes to $\Omega = \Omega_2$.

4. ILLUSTRATIVE EXAMPLE

4.1 Helicopter Model

The proposed method is applied to simulation data of a helicopter on the ground. The goal is to detect the ground resonance before it occurs. Similar to that proposed in Byers and Gandhi [2009], the present mechanical model is developed to characterize the dynamic behavior of a helicopter with a hinged rotor (see Fig. 1). The fuselage is considered to be a rigid body with mass M. The body is connected to springs (K_{bX}, K_Y) and viscous dampers (C_X, C_Y) which represent the flexibility and the damping of the landing skid. The rotor head system, rotating at a speed Ω, consists of an assembly of one rigid rotor hub with N_b blades. Each blade is represented by a concentrated mass m located at a distance b from the lag articulation (point B) and, on each k-th articulation, a torsional spring K_{ϕ_k} and viscous damper C_{ϕ_k} are present. The degrees of freedom of the system are the two lateral displacements of the fuselage x and y, and the lag angular motions ϕ_k for each of the N_b blades. The equations of motion of this

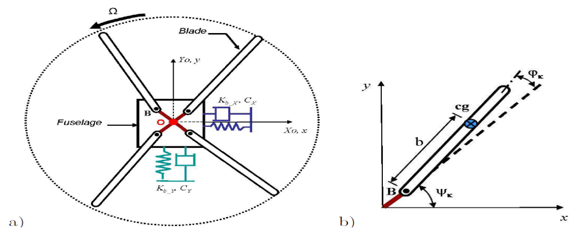

Fig. 1. Helicopter's mechanical model

system writes:

$$\mathcal{M}(t)\ddot{X}(t) + \mathcal{C}(t)\dot{X}(t) + \mathcal{K}(t)X(t) = 0 \tag{33}$$

The mass, the damping and the stiffness matrices $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K}$ are periodic of period $T = \frac{2\pi}{\Omega}$ and $X = \begin{bmatrix} x & y & \phi_1 & \phi_2 & \cdots & \phi_{N_b} \end{bmatrix}^T$. Let $\mathcal{X} = \begin{bmatrix} X^T & \dot{X}^T \end{bmatrix}^T$, then (33) can be written in a state space form:

$$\dot{\mathcal{X}}(t) = A(t)\mathcal{X}(t) \tag{34}$$

Where the periodic state transition matrix writes:

$$A(t) = \begin{bmatrix} 0 & I \\ -\mathcal{M}^{-1}(t)\mathcal{K}(t) & -\mathcal{M}^{-1}(t)\mathcal{C}(t) \end{bmatrix} \tag{35}$$

The equation of observation is $y(t) = C\mathcal{X}(t)$ where $C = [\,0 \ \mathcal{I}\,]$. The numerical values used for simulation are those reported in Byers and Gandhi [2009] and $N_b = 4$.

4.2 Simulation

Using Matlab, time series data are simulated from the mechanical model above with a sampling rate $\frac{1}{\tau} = 50Hz$. The scenario consists in simulating an angular velocity's acceleration from $\Omega_0 = 3.1rad/s$ to $\Omega_{final} = 4.5rad/s$ (close to resonance) with a step of $0.1rad/s$. For each value of Ω, 1000 samples are simulated and the eigenvalues of F

in (20) are computed. The imaginary parts of these eigenvalues are negligible. Therefore, the real parts $\rho_1 \cdots \rho_n$, which are positive, are close approximations of the norms of these eigenvalues. The evolution of the ρ_i's with respect to airspeeds Ω is plotted in Fig. 2. The resonance onset can be observed when one of the ρ_i's exceeds 1. Notice that the ρ_i's vary very close to instability.

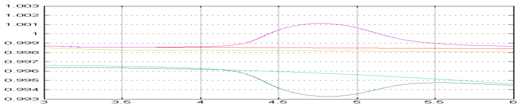

Fig. 2. Real parts ρ_1 to ρ_6 vs. angular velocity (N.B: $\rho_1 \simeq \rho_2$)

4.3 Results

At the chosen reference corresponding to $\Omega_0 = 3.1 rad/s$, the system is identified on a large data set from the subspace identification algorithm described in Jhinaoui et al. [2012]. The kernels, Jacobians and covariance matrices of the $T_{d0} = 102$ time-invariant data subsequences are computed at this reference (Döhler and Mevel [2011]). At the beginning of the monitoring phase, the Hankel matrix computed on current data is synchronized with respect to its proper reference kernel (as given in (32)). This synchronization is performed using (32) whose minimum is plotted in Fig. 3. The x-axis represents the index of the reference kernels, and the y-axis the value of the cost in equation (32). It corresponds to the correct phase (as defined by the simulation) at an index equal to 50. Once

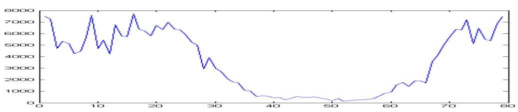

Fig. 3. Cost function vs. indexes of reference kernels

the synchronization is done, the monitoring phase begins. For each of the T_d time-invariant subsequences, a CUSUM test is started. The number of tests T_d is decreasing with respect to the angular velocity Ω which means that some time-invariant subsequences have vanished. At Ω_{final}, only 70 subsequences (then, 70 parallel tests) will remain. A choice of $\epsilon = 1000$ was taken based on the fluctuation of the tests in the reference zone for Ω (Mevel et al. [2005]). If one of the tests reacts (if some ρ_i's have significantly increased. Here it is ρ_5 that increases), monitoring is stopped. Two of these test evolutions are reported in Fig. 4. All tests have the same behavior and their reactions to the increase in parameter value are clearly observed. They all react and the earliest response corresponds to $\Omega = 4.1 rad/s$, which is close to the zone of resonance.

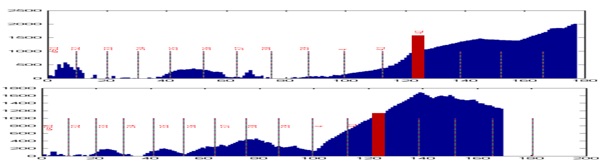

Fig. 4. CUSUM test evolution over time

5. CONCLUSION

The problem of instability monitoring for linear periodically time-varying systems is addressed. The most challenging point is to define a simple criterion of stability for these LPTV systems. This problem is solved using the Floquet theory. Then, new instability monitoring tests are developed, similarly to the time-invariant case. The suggested method is successfully tested on simulation data of a helicopter model. Current works are carried out to apply the same method to real industrial applications.

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