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Abstract: In this paper, an extension of the output-only subspace identification, to the class of linear periodically time-varying (LPTV) systems, is proposed. The goal is to identify a useful information about the system's stability using the Floquet theory which gives a necessary and sufficient condition for stability analysis. This information is retrieved from a matrix called the monodromy matrix, which is extracted by some simultaneous singular value decompositions (SVD) and from a resolution of a least squares criterion. The method is, finally, illustrated by a simulation of a model of a helicopter with a hinged-blades rotor.

Keywords: Helicopter dynamics, Time-varying systems, Periodic motions, Subspace methods, Stability analysis

1. INTRODUCTION

Over the last forty decades, subspace identification methods have enjoyed some popularity and numerous applications of these methods have emerged in civil engineering, aeronautics and many other fields. The stochastic subspace identification (SSI) consists in obtaining the modal parameters (natural frequencies, modal damping ratios and mode shapes) of a system subject to ambient excitations, by some geometric manipulations and projections of data given by sensors measurements or input measurements. There exist, then, two types of identification algorithms: output-only and input-output algorithms. A comprehensive overview of different approaches of SSI can be found in Van Overschee and De Moor [1996].

Unfortunately, most of research interest on subspace methods has been given to linear time-invariant (LTI) systems. In contrast, the literature on linear time-varying (LTV) case is not abundant. However, most physical phenomena exhibit time varying behaviors, mainly due to internal (fatigue...) or external (disturbances...) operating conditions. One particular subclass of LTV systems is the linear periodically time varying (LPTV) systems which are widely common in communications, circuit modeling and rotating machines such as wind turbines and helicopters' rotors.

In time-domain, the few works that have been carried out, in order to extend the subspace-based methods to these LPTV cases, can be categorized into two main approaches. The first approach consists in identifying the considered system recursively using adaptive algorithms which is appropriate only for slowly-varying dynamics and when a priori information about the variation behavior is available, whereas, the second approach suggests to find a set of output subsequences that have time-invariant behaviors. This makes it possible to derive an identification algorithm for these subsequences, which is close to the classical time-invariant algorithm. Among the few attempts on the subject, one can cite Liu [1997] where a discrete time domain state space model, transition matrices and pseudo-modal parameters are used to describe and identify recursively periodic systems, and Verhaegen and Yu [1995] in which subspace model identification algorithms that allow the identification of an LPTV state space model from a set of input-output measurements are presented.

In this paper, the focus is to develop a new identification algorithm for LPTV systems, with two main features: first, the algorithm can be applied when the system is subject to unknown excitations and only output data are available. Second, it aims at identifying useful parameters for stability analysis, by the Floquet theory, with no need to identify the system over a whole period as it is usually done in periodic cases.

The paper is organized as follows: in Section 2, a typical SSI algorithm is presented. Section 3 gives the essential elements of the Floquet theory. Then, Section 4 is devoted to the design of the new LPTV extended method. Finally, in Section 5 the efficiency of the method is tested on a numerical simulation of a model of a helicopter with a hinged-blades rotor.

2. TYPICAL OUTPUT-ONLY SUBSPACE IDENTIFICATION

In this section, a typical output-only SSI algorithm, based on covariance-driven data, from Benveniste and Fuchs [1985] is recalled. Let consider the LTI discrete-time state space model of a given system:

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$$\begin{cases} z_{k+1} = F\ z_k + v_k \\ y_k = H\ x_k + w_k \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^r$ the output vector or the observation, $F \in \mathbb{R}^{n\times n}$ the state transition matrix and $H \in \mathbb{R}^{r\times n}$ the observation matrix. The vectors v and w are noises assumed to be white Gaussian.
For chosen parameters p and q such that $min\{pr, qr\} \geq n$, the covariance-driven Hankel matrix below is built:

$$\mathcal{H}_{p,q} = \begin{bmatrix} R_1 & R_2 & \cdots & R_q \\ R_2 & R_3 & \cdots & R_{q+1} \\ \vdots & \vdots & \vdots & \vdots \\ R_{p+1} & R_{p+2} & \cdots & R_{p+q} \end{bmatrix} \tag{2}$$

The covariances of the output data write $R_i = \mathbf{E}(y_k y_{k-i}^T)$, where $\mathbf{E}$ is the expectation operator. The R_i's can be estimated by $\hat{R}_i = \frac{1}{N}\sum_{k=i+1}^{N} y_k y_{k-i}^T$, where N is the number of output measurements that are available ($N \gg 1$). Then, let the estimated Hankel matrix be:

$$\hat{\mathcal{H}}_{p,q} = \frac{1}{N}\sum_{k=q}^{N-p} \mathcal{Y}_k{}^{+}\mathcal{Y}_k{}^{-T} \tag{3}$$

where:

$$\mathcal{Y}_k{}^{+} = \begin{bmatrix} y_k^T & \cdots & y_{k+p}^T \end{bmatrix}^T, \quad \mathcal{Y}_k{}^{-} = \begin{bmatrix} y_{k-1}^T & \cdots & y_{k-q}^T \end{bmatrix}^T \tag{4}$$

Let $G = \mathbf{E}(z_k y_k^T)$ be the correlation between the state and the observation, $\mathcal{O}_p \hat{=} \left[\, H^T,\ (HF)^T,\ \cdots,\ (HF^p)^T \,\right]^T$ and $\mathcal{C}_q \hat{=} \left[\, FG,\ F^2G,\ \cdots,\ F^qG \,\right]$ the p-th order observability matrix and the q-th order (shifted) controllability matrix. The computation of the R_i's leads to the decomposition (see Benveniste and Fuchs [1985]):

$$\mathcal{H}_{p,q} = \mathcal{O}_p\mathcal{C}_q \tag{5}$$

Therefore, the observability matrix $\mathcal{O}_p$ can be obtained with a thin singular value decomposition (SVD) of the Hankel matrix $\mathcal{H}_{p,q}$ and its truncation at the desired model order n (Basseville et al. [2000]):

$$\mathcal{H}_{p,q} = U\Delta V^T = [\,U_1\ \ U_2\,]\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}\begin{bmatrix} V_1{}^T \\ V_2{}^T \end{bmatrix} \tag{6}$$

$$\mathcal{O}_p = U_1\Delta_1{}^{\frac{1}{2}} \tag{7}$$

where Δ_1 contains the first n singular values and U_1 the n first columns of U. The observation matrix H is extracted from the first r rows of the observability matrix $\mathcal{O}_p$. The state transition matrix F is obtained from a least squares resolution of (Benveniste and Fuchs [1985]):

$$\mathcal{O}_p^{\uparrow}F = \mathcal{O}_p^{\downarrow} \tag{8}$$

where:

$$\mathcal{O}_p^{\uparrow} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{p-1} \end{bmatrix},\ \mathcal{O}_p^{\downarrow} = \begin{bmatrix} HF \\ HF^2 \\ \vdots \\ HF^p \end{bmatrix} \tag{9}$$

Finally, the eigenstructure of the system (1) is retrieved from:

$$det(F - I\lambda) = 0, \quad F\phi_\lambda = \lambda\phi_\lambda \tag{10}$$

$(\lambda,\ \phi_\lambda)$ denote the eigenvalues and the eigenvectors (also, called mode shapes) of the system.

3. FLOQUET THEORY

The Floquet theory is a mathematical theory of ordinary differential equations (ODEs). Introduced by Floquet [1883], it is the first complete theory for the class of periodically time-varying systems. In this section, some of its essential elements, that are related to the study hereafter, are briefly reviewed. More details can be found in Dacunha and Davis [2011].

Let consider the periodic differential system:

$$\dot{x}(t) = A(t)x(t) \tag{11}$$

where $x \in \mathbb{R}^n$ is the state vector. The state transition matrix $A \in \mathbb{R}^{n\times n}$ is continuous in time (or at least, piecewise continuous) and periodic, of period $T > 0$. If an initial condition $x(t_0) = x_0$ is fixed, a solution of (11) is guaranteed to exist.
Let $\Phi(t)$ be the matrix whose n columns are n linearly independent solutions of (11), $\Phi(t)$ is known as the *Fundamental Transition Matrix* (FTM). It has the properties: $\dot{\Phi}(t) = A(t)\Phi(t),\ \Phi(t+T) = \Phi(t)\Phi(T), \forall t$

3.1 Stability Analysis

Let Q be the value of the fundamental matrix at $t = T$:

$$Q = \Phi(T), \quad R = \frac{1}{T}log(Q) \tag{12}$$

Q is called the *Monodromy Matrix*. According to Floquet's theory, the dynamical system (11) is stable if and only if the eigenvalues of R are negative or, similarly, if the norms of the eigenvalues of Q are less than one.

3.2 Floquet Transformation

The Floquet transformation, also called the Lyapunov-Floquet transformation, gives an underlying *autonomous system* (a system with a constant state transition matrix with respect to time) that is equivalent to the initial periodic system *i.e*: the transformation is invertible.
If the change of variable $x(t) = \Phi(t)e^{-Rt}z(t)$ is made, the theory insures that:

$$\dot{z}(t) = Rz(t) \tag{13}$$

Assume that the equation of observation of the considered system is: $y(t) = Cx(t)$, where $C \in \mathbb{R}^{r\times n}$. The equivalent equation for the new variable z is:

$$\dot{y}(t) = Cx(t) = C\Phi(t)e^{-Rt}z(t) \tag{14}$$

It is easy to demonstrate that $\Phi(t+T)e^{-R(t+T)} = \Phi(t)e^{-Rt}$ using the fact that $R = \frac{1}{T}log(\Phi(T))$. Therefore, any periodic system can be transformed into an equivalent autonomous system with an equivalent periodic observation matrix.

4. SUBSPACE IDENTIFICATION FOR LPTV

In this section, an extension of SSI to the linear periodically time-varying case is suggested. Let consider the periodic state-space system, of a period T:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + v(t), \quad A(t+T) = A(t), \forall t \in \mathbb{R} \\ y(t) = Cx(t) + w(t) \end{cases} \tag{15}$$

where v and w are two white noises. As outlined in Section 3, an equivalent representation can be given by the Floquet's transformation:

$$\begin{cases} \dot{z}(t) = Rz(t) + \left(\Phi(t)e^{-Rt}\right)^{-1} v(t) \\ y(t) = C^{'}(t)z(t) + w(t), \quad C^{'}(t+T) = C^{'}(t), \forall t \end{cases} \tag{16}$$

where $C^{'}(t) = C\Phi(t)e^{-Rt}$ is periodic of period T. Sampling at rate $\frac{1}{\tau}$ ($\tau > 0$ and T assumed to be a multiple of τ), yields the discrete time model below (Kim et al. [2006]):

$$\begin{cases} z_{k+1} = Fz_k + \Gamma_k w_k \\ y_k = H_k z_k + v_k \end{cases} \tag{17}$$

The discretized monodromy matrix $F = e^{R\tau}$, the discretized matrix of observation $H_k = C^{'}(k\tau)$ and $\Gamma_k = \int_{k\tau}^{(k+1)\tau} e^{R\gamma} d\gamma$ are periodic. The discrete period is $T_d = \frac{T}{\tau}$. If μ is an eigenvalue of the continuous system R, the corresponding eigenvalue λ of F is such that: $\lambda = e^{\tau\mu}$. According to the theory of Floquet, the continuous system is stable when all the eigenvalues μ of R are negative. The discrete system is, then, stable when the norms of the eigenvalues λ are inferior to one. The goal herein is to identify these eigenvalues and, thus, to draw conclusions about the stability.

Following the lines of Meyer and Burrus [1975], a set of time-invariant data subsequences exists in the case of LPTV systems. It means that if a system is periodic of period T_d, the ensembles $(z_{k_0+iT_d})_{i\in\mathbb{N}}$ and $(y_{k_0+iT_d})_{i\in\mathbb{N}}$ are time-invariant series for any $k_0 \in \mathbb{N}$. A total of T_d different time-invariant subsequences exists. One of these subsequences (denoted the j-th subsequence) is illustrated in Fig. 1. The Hankel matrix can not be computed as in the

y_j	y_{j+Td}	$y_{j+2\,Td}$	$y_{j+3\,Td}$	...
z_j	z_{j+Td}	$z_{j+2\,Td}$	$z_{j+3\,Td}$	...

Fig. 1. The j-th time-invariant subsequence

time-invariant case, because each subsequence describes a different dynamical behavior from the others. The Hankel matrix should be redefined using each one of these time-invariant subsequences, separately. For any $j \geq q$ and if $(NT_d + p)$ data samples are available, the j-th Hankel matrix $\mathcal{H}_{p,q}^{(j)}$ is defined as:

$$\mathcal{H}_{p,q}^{(j)} = \frac{1}{N}\sum_{i=0}^{N-1} \mathcal{Y}_{j+iT_d}^{+} \mathcal{Y}_{j+iT_d}^{-T} \tag{18}$$

Fact 1. Notice that T is assumed to be a multiple of τ. Otherwise, the time-invariant subsequences are the ensembles $(y_{\lfloor \frac{j\tau+iT}{\tau} \rfloor})_i$ and the data $\mathcal{Y}_{j+iT_d}^{+}$ and $\mathcal{Y}_{j+iT_d}^{-}$ are replaced by $\mathcal{Y}_{\lfloor \frac{j\tau+iT}{\tau} \rfloor}^{+}$ and $\mathcal{Y}_{\lfloor \frac{j\tau+iT}{\tau} \rfloor}^{-}$ (where $\lfloor\rfloor$ is the floor operator), in order to get the sum for approximately the same subsequence.

Proposition 2. From the periodicity of the observation matrix H, i.e. $H_{j+iT_d} = H_j$ for all non negative integer i, one deduce the following property for the Hankel matrix:

$$\begin{aligned} \mathcal{H}_{p,q}^{(j)} = & \begin{bmatrix} H_j \\ H_{j+1}F \\ \vdots \\ H_{j+p}F^p \end{bmatrix} \Bigg[\frac{1}{N} F \sum_{i=0}^{N-1} z_{j-1+iT_d} y_{j-1+iT_d}^T \\ & \cdots \qquad \frac{1}{N} F^q \sum_{i=0}^{N-1} z_{j-q+iT_d} y_{j-q+iT_d}^T \Bigg] \\ & + \epsilon_N \end{aligned} \tag{19}$$

where ϵ_N is a function converging to zero when N goes to infinity.

Proof. See the appendix.

Corollary 3. As in the time-invariant case, the Hankel matrix $\mathcal{H}_{p,q}^{(j)}$ converges to the decomposition below:

$$\mathcal{H}_{p,q}^{(j)} = \mathcal{O}_p^{(j)} \mathcal{C}_q^{(j)} \tag{20}$$

The j-th observability and (shifted) controllability matrices are defined as:

$$\mathcal{O}_p^{(j)} = \left[\, H_j^T, \, (H_{j+1}F)^T, \, \cdots, \, (H_{j+p}F^p)^T \,\right]^T \tag{21}$$

$$\mathcal{C}_q^{(j)} = \left[\, FG^{(j-1)}, \, F^2G^{(j-2)}, \, \cdots, \, F^qG^{(j-q)} \,\right] \tag{22}$$

where $G^{(j)}$ is the state-output cross correlation of the j-th invariant subsequence. It can be estimated by $G^{(j)} = \frac{1}{N}\sum_{i=0}^{N-1} z_{j+iT_d} y_{j+iT_d}^T$. The observability matrix $\mathcal{O}_p^{(j)}$ can be obtained as in Section 2 via an SVD of $\mathcal{H}_{p,q}^{(j)}$ and its truncation at the desired model order n:

$$\begin{aligned} \mathcal{H}_{p,q}^{(j)} &= U^{(j)}\Delta^{(j)}(V^{(j)})^T \\ &= \left[\, U_1^{(j)} \;\; U_2^{(j)} \,\right] \begin{bmatrix} \Delta_1^{(j)} & 0 \\ 0 & \Delta_2^{(j)} \end{bmatrix} (V^{(j)})^T \end{aligned} \tag{23}$$

where $U_1^{(j)}$ is the n first columns of $U^{(j)}$, and $\Delta_1^{(j)}$ the n upper singular values of $\Delta^{(j)}$. The size of the Hankel matrix is $(p+1)r \times qr$. The complexity of the SVD does not depend on N and therefore, the computation is not cumbersome. The observation matrix H_j is obtained from the first r rows of $\mathcal{O}_p^{(j)}$. Notice that the extraction of the transition matrix from one observability matrix is no longer possible. Indeed, in order to get F, two successive matrices of Hankel should be computed. For the $(j+1)$-th time-invariant subsequence, $\mathcal{H}_{p,q}^{(j+1)}$ writes:

$$\begin{aligned} \mathcal{H}_{p,q}^{(j+1)} = & \begin{bmatrix} H_{j+1} \\ H_{j+2}F \\ \vdots \\ H_{j+p+1}F^p \end{bmatrix} \Big[FG^{(j)} \cdots \\ & \qquad \cdots \qquad F^qG^{(j+1-q)} \Big] \end{aligned}$$

Differently from Section 2, the transition matrix F is a least squares solution of an equation involving two different observability matrices:

$$\mathcal{O}_p^{\uparrow(j+1)} F = \mathcal{O}_p^{\downarrow(j)} \tag{24}$$

where:

$$\mathcal{O}_p^{\downarrow(j)} = \begin{bmatrix} H_{j+1}F \\ H_{j+2}F^2 \\ \vdots \\ H_{j+p}F^p \end{bmatrix}, \mathcal{O}_p^{\uparrow(k+1)} = \begin{bmatrix} H_{j+1} \\ H_{j+2}F \\ \vdots \\ H_{j+p+1}F^{p-1} \end{bmatrix}$$

Both $\mathcal{O}_p^{(j)}$ and $\mathcal{O}_p^{(j+1)}$ should be expressed in the same basis of the state space system. In fact, the left part of the singular value decomposition gives the observability matrix up to some invertible (change of basis) matrix. For instance, in the time-invariant case, the estimated observability matrix $\tilde{\mathcal{O}}_p = U_1 {\Delta_1}^{\frac{1}{2}} = \mathcal{O}_p T$, where T is a non-singular transformation. In the case of time-varying systems, two successive SVD are needed. The corresponding invertible matrices $T^{(j)}$ and $T^{(j+1)}$ will be different:

$$\tilde{\mathcal{O}}_p^{(j)} = U_1^{(j)} (\Delta_1^{(j)})^{1/2} = \mathcal{O}_p^{(j)} T^{(j)} \tag{25}$$

$$\tilde{\mathcal{O}}_p^{(j+1)} = U_1^{(j+1)} (\Delta_1^{(j+1)})^{1/2} = \mathcal{O}_p^{(j+1)} T^{(j+1)} \tag{26}$$

where none of the matrices $T^{(j)}$ or $T^{(j+1)}$ is known. The least squares resolution will results in:

$$\begin{aligned}\tilde{F} &= (\tilde{\mathcal{O}}_p^{\uparrow(j+1)})^{\dagger} \tilde{\mathcal{O}}_p^{\downarrow(j)} \\ &= (T^{(j+1)})^{-1} (\mathcal{O}_p^{\uparrow(j+1)})^{\dagger} \mathcal{O}_p^{\downarrow(j)} T^{(j)} \end{aligned} \tag{27}$$

where the symbol † denotes the Moore-Penrose pseudo-inverse. The left and the right bases of the identified transition matrix are not the same. A solution for this problem is given in Liu [1997]:

Let $L^{(j)}$ and $L^{(j+1)}$ the first l ($l \geq \lfloor \frac{n}{r} \rfloor + 1$) block rows of $\tilde{\mathcal{O}}_p^{(j)}$ and $\tilde{\mathcal{O}}_p^{(j+1)}$. For $l = 2$, and without loss of generality:

$$L^{(j)} = \begin{bmatrix} H_j \\ H_{j+1}F \end{bmatrix} T^{(j)} \tag{28}$$

$$L^{(j+1)} = \begin{bmatrix} H_{j+1} \\ H_{j+2}F \end{bmatrix} T^{(j+1)} \tag{29}$$

The 3 successive observations matrices can be considered as close: $H_{j+2} \simeq H_{j+1} \simeq H_j$. Therefore:

$$\begin{aligned}(L^{(j)})^{\dagger} L^{(j+1)} &= (T^{(j)})^{-1} \left(\begin{bmatrix} H_j \\ H_{j+1}F \end{bmatrix} \right)^{\dagger} \begin{bmatrix} H_{j+1} \\ H_{j+2}F \end{bmatrix} T^{(j+1)} \\ &\simeq (T^{(j)})^{-1} T^{(j+1)} \end{aligned} \tag{30}$$

The number of block rows l is chosen such that $L^{(j)}$ is full rank. Finally, an approximate of the state transition matrix, with the same left and right transformation, writes:

$$\begin{aligned} F &= (L^{(j)})^{\dagger} L^{(j+1)} \tilde{F} \\ &\simeq (T^{(j)})^{-1} (\mathcal{O}_p^{\uparrow(j+1)})^{\dagger} \mathcal{O}_p^{\downarrow(j)} T^{(j)} \end{aligned} \tag{31}$$

Once F and H_j are computed, the modal structure of the system (17) are obtained from the resolution of $det(F - I\lambda) = 0$, $F\phi_\lambda = \lambda\phi_\lambda$ and $\varphi_\lambda = H_k \phi_\lambda$.

Algorithm: To sum up, here are the steps of the suggested LPTV identification method:

- $(NT_d + p + 1)$ output-measurements are available. The data are set in ensembles of time-invariant series (y_{j+iT_d})
- compute, for the j-th and the $(j+1)$-th subsequences, the two successive Hankel matrices $\mathcal{H}_{p,q}^{(j)}$ and $\mathcal{H}_{p,q}^{(j+1)}$ using the formula (18)
- compute the SVD's of $\mathcal{H}_{p,q}^{(j)}$ and $\mathcal{H}_{p,q}^{(j+1)}$. Then, retrieve $\tilde{\mathcal{O}}_p^{(j)}$ and $\tilde{\mathcal{O}}_p^{(j+1)}$ as in (25) and (26)
- compute the discrete transition (monodromy) matrix F, using (31) and (30)
- given F and H_j, compute the eigenstructure of the system (eigenvalues, observed eigenvectors)

5. ILLUSTRATIVE EXAMPLE

In this section, two illustrative examples are studied in order to test the suggested identification algorithm.

5.1 Simulation Model

The studied system herein is a model of a helicopter with a hinged-blades rotor. The system is modeled as in Coleman and Feingold [1958] and Sanches et al. [2011]. The helicopter's fuselage is considered to be a rigid body with mass M, attached to a flexible LG (landing gear) which is modeled by two springs K_{bx} and K_{by}, and two viscous dampers C_x and C_y as illustrated in Fig. 2. The

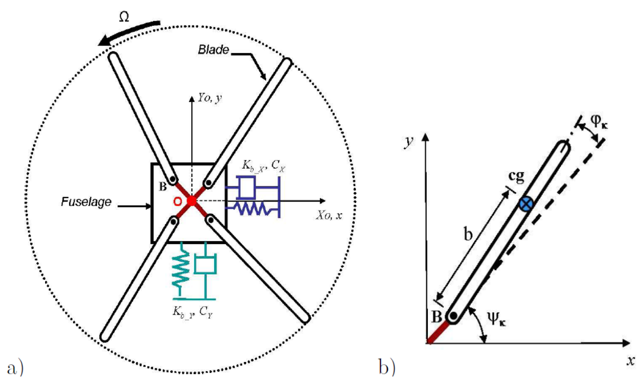

Fig. 2. Mechanical model of a helicopter with 4 blades

rotor spinning with a velocity Ω, is articulated and the offset between the MR (main rotor) and each articulation is noted a. The blades are modeled by a concentrated mass m at a distance b of the articulation point. Torque stiffness and a viscous damping K_ϕ and C_ϕ are present into each articulation. The moment of inertia around the articulation point is I_z. The degrees of freedom are the lateral displacements of the fuselage x and y, and the out-of-phase angles $(\phi_k)_{k=1\cdots N_b}$, with N_b the number of blades. Let $\mathcal{Z} = \left[x \; y \; \phi_1 \; \cdots \; \phi_{N_b} \right]^T$. A linear model for such rotating systems under free vibrations is defined by (see Christensen and Santos [2005]):

$$\mathcal{M}(t)\ddot{\mathcal{Z}}(t) + \mathcal{C}(t)\dot{\mathcal{Z}}(t) + \mathcal{K}(t)\mathcal{Z}(t) = 0 \tag{32}$$

where $\mathcal{M}(t)$, $\mathcal{C}(t)$ and $\mathcal{K}(t)$ are the system's mass, damping and stiffness matrices. The system can be written as in (15), with $x = \left[\mathcal{Z}^T \; \dot{\mathcal{Z}}^T \right]^T$ and

$$A(t) = \begin{bmatrix} 0 & \mathcal{I} \\ -\mathcal{M}(t)^{-1}\mathcal{K}(t) & -\mathcal{M}(t)^{-1}\mathcal{D}(t) \end{bmatrix}$$

$A(t) = A(t + \frac{2\pi}{\Omega})$. It is periodic of period $T = \frac{2\pi}{\Omega}$. For the study herein, the number of blades is fixed to $N_b = 4$ and the numerical values used for the application are reported in Table 1.

Table 1. Structural properties for hinged-blades helicopter with 4 blades

Structural variable name	Numerical value (SI)
m	$31.9Kg$
M	$2902.9Kg$
K_ϕ	$200N/m$
K_{bx}	$3200N/m$
K_{by}	$3200N/m$
C_ϕ	$15Ns/m$
C_x	$300Ns/m$
C_y	$300Ns/m$
a	$0.2m$
b	$2.5m$
I_z	$259Kg/m^2$

The helicopter model is simulated for $\Omega = 10 \;\; rad.s^{-1}$. To get precise results, the subspace identification should be processed on very large dataset, in order to overcome the problem of bias due to the noise. Data points are generated over a samples number of $2000.T_d$, with the discrete period $T_d = 100$ (the sampling period is $\tau = T/T_d$). The order of the system is known and is equal to $n = 12$. The suggested identification algorithm is applied to the data as explained in Section 4 (with chosen $q = p+1 = 11$). The summary of the identified eigenvalues, and the true values (the true values are computed from a numerical resolution of the state equation of the system and the corresponding monodromy matrix, using the Matlab function for ordinary differential equation's resolution ode45), are given in Table 2.

Table 2. Identified eigenvalues vs. true ones

Identified eigenvalues	True eigenvalues
$0.9985 \pm 0.0225i$	$0.9987 \pm 0.0218i$
$0.9988 \pm 0.0196i$	$0.9988 \pm 0.0198i$
$0.9988 \pm 0.0138i$	$0.9995 \pm 0.0132i$
$0.9998 \pm 0.0151i$	$0.9996 \pm 0.0150i$
$0.9992 \pm 0.0149i$	$0.9996 \pm 0.0150i$
$0.9995 \pm 0.0142i$	$0.9995 \pm 0.0141i$

We get 12 eigenvalues that are conjugate. The differences between the obtained real parts and the true ones are less than 0.1 %. For the imaginary parts, the errors are greater. The identification goes more accurate when the number of data samples is larger.

The SSI algorithm presented in Section 2 (from equation (3) to (10)), is also applied in order to compare it to the new suggested method. A total of 12 non conjugate modes are identified (see Table 3). Some of the identified eigenvalues have negative real parts. For a discrete system these real parts must be positive, because the discrete state transition matrix is an exponential of the continuous one. Therefore, applying the time-invariant SSI algorithm gives irrelevant results. This was predictable because a LTI algorithm is applied to a set of subsequences that have different dynamical behaviors.

Table 3. Real parts of the identified eigenvalues using typical SSI

0.8644	0.8644	-0.1108	-0.1108	-0.1948	0.0383
0.0383	0.2527	0.2527	-0.0698	0.1752	0.0981

Finally, the suggested algorithm in Section (4) is applied to the case where $\Omega = 0.1 \; rad.s^{-1}$ in order to compare the identified eigenvalues to the true eigenvalues of the system when the rotor is turned off ($\Omega = 0$). In fact, when $\Omega = 0$, the state matrix is no longer time-dependent $A(t) = A$. Therefore, the *Fundamental Transition Matrix* is equal to $\Phi(t) = e^{At}$ and $R = A$. If the rotation velocity $\Omega = 0.1 \; rad.s^{-1}$, the identified eigenvalues should then be close to the ones of $e^{A\tau}$ corresponding to the discrete state transition matrix for $\Omega = 0$.
The helicopter model is simulated for $\Omega = 0.1 \; rad.s^{-1}$. Data points are generated over a samples number of $2000.T_d$, with the discrete period $T_d = 200$. The results of the identification are reported in Table 4.

Table 4. Identified eigenvalues at $\Omega = 0.1 \; rad.s^{-1}$ vs. True eigenvalues at $\Omega = 0$

Identified eigenvalues for $\Omega = 0.1$	True eigenvalues for $\Omega = 0$
$0.5374 \pm i0.7769$	$0.552 \pm i0.7663$
$0.5428 \pm i0.7654$	$0.552 \pm i0.7663$
$0.8798 \pm i0.4441$	$0.879 \pm i0.4468$
$0.8766 \pm i0.4469$	$0.879 \pm i0.4468$
$0.8770 \pm i0.4507$	$0.8786 \pm i0.4474$
$0.8759 \pm i0.4489$	$0.8786 \pm i0.4474$

The identified eigenvalues for $\Omega = 0.1 \; rad.s^{-1}$ are, indeed, close to the true ones when the rotor is turned off. When Ω converges to 0, the results of the identification in the periodic case converges to the true time-invariant system.

Another feature of interest is to show that the identified state transition matrix is, indeed, invariant with respect to time. To verify it, the system is identified at different instants of time. The variations of the real parts and the imaginary parts of the identified eigenvalues with respect to time are reported in Fig. 3 and Fig. 4. These values are, as shown in the figure, constant over time. Therefore, the identified state transition matrix is constant as stated in (17).

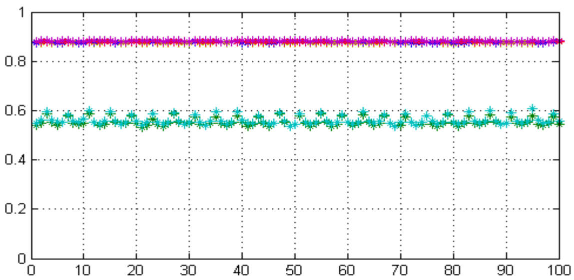

Fig. 3. Eigenvalues variations: real parts of the eigenvalues vs. sample number

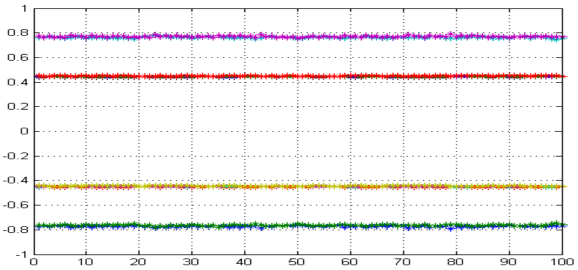

Fig. 4. Eigenvalues variations: imaginary parts of the eigenvalues vs. sample number

6. CONCLUSION

The problem of identification for linear periodically time-varying is addressed. It has been shown that, in order to get an information about the stability of such systems, a so-called monodromy matrix should be identified. This matrix is derived from the Floquet transformation which gives an equivalent description of periodic systems. Based on this description, an extension of the covariance-driven SSI algorithm is suggested. The algorithm was successfully tested for simulation data.

Future works encompasses the estimation of the uncertainties about the identified parameters for the suggested identification algorithm, given herein.

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Appendix A. PROOF OF PROPOSITION 2

The elements of the Hankel matrix in (18) are products between lagged outputs y_{k+j} and y_{k-l}. These products write:

$$\begin{aligned} y_{k+j}y_{k-l}^T = & H_{k+j}F^{j+l}z_{k-l}y_{k-l}^T \\ & + H_{k+j}F^{j+l-1}\Gamma_{k-l+1}w_{k-l+1}y_{k-l}^T \\ & + H_{k+j}F^{j+l-2}\Gamma_{k-l+2}w_{k-l+2}y_{k-l}^T \\ & + \cdots \\ & + H_{k+j}F\Gamma_{k+j-1}w_{k+j-1}y_{k-l}^T \\ & + H_{k+j}\Gamma_{k+j}w_{k+j}y_{k-l}^T \\ & + v_{k+j}y_{k-l}^T \end{aligned} \tag{A.1}$$

The observation and the control matrices are periodic. Then $H_{k+iT_d} = H_k$ and $\Gamma_{k+iT_d} = \Gamma_k$ $\forall i > 0$. Now, let compute the sum:

$$\begin{aligned} \sum_{i=0}^{N-1} y_{k+j+iT_d}y_{k-l+iT_d}^T = & H_{k+j}F^{j+l}\sum_{i=0}^{N-1} z_{k-l+iT_d}y_{k-l+iT_d}^T \\ & + H_{k+j}F^{j+l-1}\Gamma_{k-l+1}\sum_{i=0}^{N-1} w_{k-l+1+iT_d}y_{k-l+iT_d}^T \\ & + \cdots \\ & + H_{k+j}F\Gamma_{k+j-1}\sum_{i=0}^{N-1} w_{k+j-1+iT_d}y_{k-l+iT_d}^T \\ & + H_{k+j}\Gamma_{k+j}\sum_{i=0}^{N-1} w_{k+j+iT_d}y_{k-l+iT_d}^T \\ & + \sum_{i=0}^{N-1} v_{k+j+iT_d}y_{k-l+iT_d}^T \end{aligned} \tag{A.2}$$

Dividing by N, leads to:

- $H_{k+j}F^{j+l}\frac{1}{N}\sum_{i=0}^{N-1} z_{k-l+iT_d}y_{k-l+iT_d}^T$ converges to $H_{k+j}F^{j+l}G^{(k-l)}$ when N goes large, where $G^{(k-l)}$ is the correlation between the $(k-l)$-th time-invariant subsequences of the state vector and the output vector
- $H_{k+j}F^{j+l-1}\Gamma_{k-l+1}\frac{1}{N}\sum_{i=0}^{N-1} w_{k-l+1+iT_d}y_{k-l+iT_d}^T$ converges to 0 when N goes large, because w is a white noise which is uncorrelated of any time-invariant subsequence of the output, *idem* for $H_{k+j}F^{j+l-2}\Gamma_{k-l+2}\frac{1}{N}\sum_{i=0}^{N-1} w_{k-l+2+iT_d}y_{k-l+iT_d}^T, \cdots,$ $H_{k+j}F\Gamma_{k+j-1}\frac{1}{N}\sum_{i=0}^{N-1} w_{k+j-1+iT_d}y_{k-l+iT_d}^T$ and $H_{k+j}\Gamma_{k+j}\frac{1}{N}\sum_{i=0}^{N-1} w_{k+j+iT_d}y_{k-l+iT_d}^T$
- $\frac{1}{N}\sum_{i=0}^{N-1} v_{k+j+iT_d}y_{k-l+iT_d}^T$ converges to 0 too (unless when $j = l = 0$, which is not the case herein), for the same reason of uncorrelation