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Black holes and higher depth mock modular forms

Sergei Alexandrov^{1,2} and Boris Pioline³

¹ *Laboratoire Charles Coulomb (L2C), Université de Montpellier, CNRS, F-34095, Montpellier, France*

² *Department of High Energy and Elementary Particle Physics, Saint Petersburg State University, 7/9 Universitetskaya nab., St. Petersburg 199034, Russia*

³ *Laboratoire de Physique Théorique et Hautes Energies (LPTHE), UMR 7589 CNRS-Sorbonne Université, Campus Pierre et Marie Curie, 4 place Jussieu, F-75005 Paris, France*

e-mail: sergey.alexandrov@umontpellier.fr, pioline@lpthe.jussieu.fr

ABSTRACT: By enforcing invariance under S-duality in type IIB string theory compactified on a Calabi-Yau threefold, we derive modular properties of the generating function of BPS degeneracies of D4-D2-D0 black holes in type IIA string theory compactified on the same space. Mathematically, these BPS degeneracies are the generalized Donaldson-Thomas invariants with support on a fixed divisor \mathcal{D} , at the large volume attractor point. For \mathcal{D} irreducible, this function is closely related to the elliptic genus of the superconformal field theory obtained by wrapping M5-brane on \mathcal{D} and is therefore known to be modular. Instead, when \mathcal{D} is the sum of n irreducible divisors \mathcal{D}_i , we show that the generating function acquires a modular anomaly. We characterize it explicitly for arbitrary n by providing an explicit expression for a non-holomorphic modular completion in terms of generalized error functions. As a result, the generating function turns out to be a (mixed) mock modular form of depth $n - 1$.

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1. Introduction and summary

The degeneracies of BPS black holes in string vacua with extended supersymmetry possess remarkable modular properties, which have been instrumental in recent progress on explaining the statistical origin of the Bekenstein-Hawking entropy in [1] and many subsequent works. Namely, the indices $\Omega(\gamma)$ counting — with sign — microstates of BPS black holes with electromagnetic charge γ may often be collected into a suitable generating function which exhibits modular invariance, providing powerful constraints on its Fourier coefficients and enabling direct access to their growth. When the black holes can be realized as black strings wrapped on a circle, a natural candidate for such a generating function is the elliptic genus of the superconformal field theory supported by the black string, which is modular invariant by construction [2, 3, 4]. Equivalently, one may consider the partition function of the effective three-dimensional gravity living on the near-horizon geometry of the black string [5].

In most cases however, the BPS indices depend not only on the charge γ but also on the moduli z^a at spatial infinity, due to the wall-crossing phenomenon: some of the BPS bound states with total charge γ may only exist in a certain chamber in moduli space, and decay as the moduli are varied across ‘walls of marginal stability’ which delimit this chamber. At strong coupling where the black hole description is accurate, this phenomenon has a transparent explanation in terms of the (dis)appearance of multi-centered black hole configurations, which can be used to derive a universal wall-crossing formula [6, 7, 8, 9].

In the case of four-dimensional string vacua with $\mathcal{N} = 4$ supersymmetry, where the BPS index is sensitive only to single-centered 1/4-BPS black holes and to bound states of two 1/2-BPS black holes, the resulting moduli dependence is reflected in poles in the generating function, requiring a suitable choice of contour for extracting the Fourier coefficients in a given chamber [10, 11, 12]. Upon subtracting contributions of two-centered bound states, the generating function of single-centered indices is no longer modular in the usual sense but it transforms as a mock Jacobi form with specific ‘shadow’ — a property which is almost as constraining as standard modular invariance [13].

In four-dimensional string vacua with $\mathcal{N} = 2$ supersymmetry, the situation is much more complicated, firstly due to the fact that the moduli space of scalars receives quantum corrections, and secondly due to BPS bound states potentially involving an arbitrary number of constituents, resulting in an extremely intricate pattern of walls of marginal stability. Thus, it does not seem plausible that a single generating function may capture the BPS indices $\Omega(\gamma, z^a)$ in all chambers. Nevertheless, modular invariance is still expected to constrain the BPS indices — which are known in this context as generalized Donaldson-Thomas (DT) invariants. In particular, D4-D2-D0 black holes in type IIA string theory compactified on a generic compact Calabi-Yau (CY) threefold \mathfrak{Y} can be lifted to an M5-brane wrapped on a divisor $\mathcal{D} \subset \mathfrak{Y}$ [2]. If the divisor \mathcal{D} labelled by the D4-brane charge p^a is *irreducible*, the indices $\Omega(\gamma, z^a)$ are independent of the moduli of \mathfrak{Y} and their generating function is known to be a holomorphic (vector valued) modular form of weight $-\frac{1}{2}b_2(\mathfrak{Y}) - 1$ [14, 3, 4]. But if the divisor \mathcal{D} is a sum of n effective divisors \mathcal{D}_i , the indices $\Omega(\gamma, z^a)$ do depend on the Kähler moduli z^a , even in the large volume limit. In general however, the black string SCFT is supposed to capture the states associated to a single AdS_3 throat, while for generic values

of the moduli multiple AdS_3 throats can usually contribute [15, 16]. It is thus natural to consider the modular properties of the generalized DT invariants $\Omega(\gamma, z^a)$ at the large volume attractor point¹

$$z_\infty^a(\gamma) = \lim_{\lambda \rightarrow +\infty} (-q^a + i\lambda p^a), \quad (1.1)$$

where only a single AdS_3 throat is allowed [17]. Following [18] we denote these invariants by $\Omega^{\text{MSW}}(\gamma) = \Omega(\gamma, z_\infty^a(\gamma))$ and call them Maldacena-Strominger-Witten (MSW) invariants. As we discuss in Section 2, the DT invariants $\Omega(\gamma, z^a)$ can be recovered from the MSW invariants $\Omega^{\text{MSW}}(\gamma)$ by using a suitable version of the split attractor flow conjecture [19, 6], revisited recently in [20].

The case where \mathcal{D} is the sum of two irreducible divisors was first studied in [17], and more recently in [21, 22], where it was shown that the generating function of MSW invariants is a mock modular form, with a specific non-holomorphic completion obtained by smoothing out the sign function entering in the bound state contributions, recovering the prescription of [23, 17]. The goal of this paper is to extend this result to the general case where \mathcal{D} is the sum of n irreducible divisors \mathcal{D}_i , where n can be arbitrarily large. In such generic situation, we characterize the modular properties of the generating function $h_{p,\mu}$ of MSW invariants and find an explicit expression for its non-holomorphic completion $\widehat{h}_{p,\mu}$ in terms of the generating functions h_{p_i,μ_i} associated to the n constituents, multiplied by certain iterated integrals introduced in [24, 25], which generalize the usual error function appearing with $n = 2$. This result implies that in this case $h_{p,\mu}$ is a (mixed, vector valued) mock modular form of depth $n - 1$, in the sense that its shadow is itself a mock modular form of lower depth (with the depth 1 case reducing to the standard mock modular forms introduced in [23, 26]).

In order to establish this result, we follow the same strategy as in our earlier works [21, 22] and analyse D3-D1-D(-1) instanton corrections to the metric on the hypermultiplet moduli space \mathcal{M}_H in type IIB string theory compactified on \mathfrak{Y} , at arbitrary order in the instanton expansion. After reducing on a circle and T-dualizing, this moduli space is identical to the vector multiplet moduli space in type IIA string theory compactified on $\mathfrak{Y} \times S_1$, where it receives instanton corrections from D4-D2-D0 black holes winding around the circle. In either case, each instanton contribution is weighted by the same generalized DT invariant $\Omega(\gamma)$ which counts the number of BPS black hole microstates with electromagnetic charge γ . The modular properties of the generalized DT invariants are fixed by requiring that the quaternion-Kähler (QK) metric on \mathcal{M}_H admits an isometric action of $SL(2, \mathbb{Z})$, which comes from S-duality in type IIB, or equivalently from large diffeomorphisms of the torus appearing when viewing type IIA/ $\mathfrak{Y} \times S_1$ as M-theory on $\mathfrak{Y} \times T^2$ [27]. This QK metric is most efficiently described using a contact structure on the associated twistor space, a $\mathbb{C}P^1$ -bundle over \mathcal{M}_H [28, 29], the latter being specified by a set of gluing conditions determined by the DT invariants [27, 30], which can themselves be expressed in terms of the MSW invariants.

An important quantity appearing in this twistorial formulation² is the so called *contact*

¹Here p^a and q_a are D4 and D2-brane charges, respectively, and the index of q is raised with help of the inverse of the metric $\kappa_{ab} = \kappa_{abc}P^a$ with κ_{abc} being the triple intersection numbers of \mathfrak{Y} .

²A familiarity with the twistorial formulation is not required for the present work. Here we use only two equations relevant for the twistorial description of D-instantons: the integral equation (3.2) for certain Darboux coordinates, which appears also in the study of four-dimensional $N = 2$ gauge theory on a circle

potential e^ϕ , a real function on \mathcal{M}_H related to the Kähler potential on \mathcal{Z} , and afforded by the existence of a continuous isometry unbroken by D-instantons. On the type IIB side, e^ϕ can be identified with the four-dimensional dilaton $1/g_4^2$. When expressed in terms of the ten-dimensional type IIB axio-dilaton $\tau = c_0 + i/g_s$, it becomes a complicated function having a classical contribution, a one-loop correction, and a series of instanton corrections expressed as contour integrals on the CP^1 fiber. The importance of the contact potential stems from the fact that the existence of an isometric action of $SL(2, \mathbb{Z})$ on \mathcal{M}_H restricts it to be a non-holomorphic modular form of weight $(-\frac{1}{2}, -\frac{1}{2})$ in the variable τ [27]. As a result, this imposes a very non-trivial constraint on the instanton contributions to e^ϕ which can be used to deduce the modular properties of generating functions of DT invariants, order by order in the instanton expansion. This strategy was used in [21] at two-instanton order to characterize the modular behavior of the generating function of MSW invariants in the case of a divisor equal to the sum of $n = 2$ irreducible divisors. In this paper we generalize this result to all n , by analyzing the instanton expansion to all orders. Below we summarize the main steps and the main results which we find on this way.

1. First, we show that the contact potential e^ϕ in the large volume limit, where the Kähler parameters of the CY are sent to infinity, can be expressed (see (3.24)) through another (complex valued) function \mathcal{G} (3.22) on the moduli space, which we call the instanton generating function. In particular, its instanton expansion, i.e. the expansion in powers of DT invariants, is captured by a simple sum over unrooted trees (see (3.25) and (3.26)). The modularity of the contact potential requires \mathcal{G} to be a modular function of weight $(-\frac{3}{2}, \frac{1}{2})$.
2. Expressing the DT invariants $\Omega(\gamma, z^a)$ through the moduli independent MSW invariants $\Omega^{\text{MSW}}(\gamma)$ using the tree flow formula of [20], and rearranging the expansion of \mathcal{G} as an expansion in powers of $\Omega^{\text{MSW}}(\gamma)$, each order of this expansion, similarly to the standard Jacobi forms, can be represented as a sum of products of certain indefinite theta series and holomorphic generating functions h_{p_i, μ_i} of $\Omega^{\text{MSW}}(\gamma)$ (see (4.2)). Thus, modular properties of the theta series fix the modular properties of these generating functions.
3. In order for a theta series to be modular, its kernel must satisfy a certain differential equation [33], which we call Vignéras' equation. By construction, the kernels appearing in our problem are given by iterated contour integrals on the CP^1 fiber of the twistor space, multiplied by the so-called 'tree indices' $g_{\text{tr}, n}$ coming from the expression of $\Omega(\gamma, z^a)$ in terms of $\Omega^{\text{MSW}}(\gamma)$. We evaluate the twistorial integrals in terms of the generalized error functions of [24, 25], and show that the resulting kernels satisfy Vignéras' equation away from certain loci where they have discontinuities. Furthermore, we prove that the discontinuities corresponding to walls of marginal stability cancel between the integrals and the tree indices. But there are additional discontinuities coming from certain moduli independent contributions to the tree index. They spoil Vignéras' equation

[31]; and the expression for the contact potential (3.20) in terms of these Darboux coordinates. Note that the contact potential was recently extended to the context of gauge theories and interpreted there as a supersymmetric index [32]. These two equations lead to (3.15) and (3.24), respectively, which can be taken as the starting point of our analysis.

at the multi-instanton level so that the theta series is *not* modular. In turn, this implies that h_{p_i, μ_i} are also not modular.

4. However, it turns out to be possible to construct a *non-holomorphic completion* of $h_{p, \mu}$ by adding to it a series of non-linear terms (see (5.1)). The non-holomorphic functions R_n , fixing the completion, are determined by the condition that the expansion of \mathcal{G} rewritten in terms of $\widehat{h}_{p, \mu}$ gives rise to non-anomalous theta series. Equivalently, one can work with functions \widehat{g}_n appearing in the expansion (5.2) of the generating function of DT invariants $h_{p, q}^{\text{DT}}$ in terms of $\widehat{h}_{p, \mu}$. Then the modularity imposes certain recursive equations on them.
5. Using these equations, we find that \widehat{g}_n can be represented either in an iterative form (5.7), or more explicitly as a sum (5.17) over rooted trees with valency ≥ 3 at each vertex (known as Schröder trees), where $g_n^{(0)}$ are certain combinations of sign functions (5.12), whereas \mathcal{E}_n are smooth solutions of the Vignéras' equation constructed out of the generalized error functions. The functions R_n are then also given by a sum over Schröder trees (5.18) in terms of exponentially decreasing and non-decreasing parts of \mathcal{E}_n . These equations represent the main result of this work.

Thus, we have found a modular completion of the generating function of the MSW invariants for an arbitrary divisor, i.e. decomposable into a sum of *any* number of irreducible divisors. The result is expressed through sums of products of generalized error functions labelled by trees of various types. For the reader's convenience, in Fig. 1 we list the various trees which appear in our construction, up to $n = 4$.

The organization of the paper is as follows. In section 2 we review the known results about DT invariants, their expression in terms of MSW invariants, and specialize them to the case of D4-D2-D0 black holes in type IIA string theory on a Calabi-Yau threefold. In section 3 we present the twistorial description of the D-instanton corrected hypermultiplet moduli space in the dual type IIB string theory, evaluate the contact potential in the large volume approximation expressing it through a function \mathcal{G} , and provide an instanton expansion for this function via unrooted labelled trees. Then in section 4 we obtain a theta series decomposition for each order of the expansion of \mathcal{G} in MSW invariants and analyze the modular properties of the resulting theta series, which leads us to conclude that they possess a modular anomaly thereby implying a modular anomaly for the generating functions $h_{p, \mu}$. In section 5 we construct the non-holomorphic completion $\widehat{h}_{p, \mu}$, for which the anomaly is cancelled, and determine its explicit form. Section 6 is devoted to discussion of the obtained results. Finally, several appendices contain details of our calculations and proofs of various propositions. Besides, in the last appendix we provide explicit results for various functions appearing in our construction up to order $n = 4$.

2. BPS indices and wall-crossing

In this section, we review some aspects of BPS indices in theories with $\mathcal{N} = 2$ supersymmetry, including the tree flow formula relating the moduli-dependent index $\Omega(\gamma, z^a)$ to the attractor






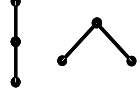
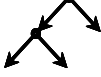
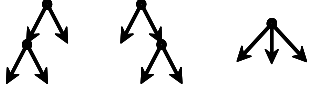
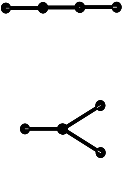
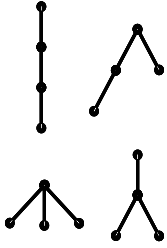
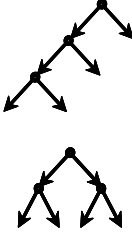
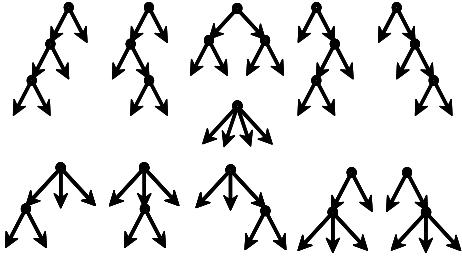
	\mathbb{T}_n	\mathbb{T}_n^r	\mathbb{T}_n^{af}	\mathbb{T}_n^{S}
$n = 2$				
$n = 3$				
$n = 4$				

Figure 1: Various types of trees arising in the multi-instanton expansion of various quantities. \mathbb{T}_n denotes the set of (unrooted) trees with n vertices. \mathbb{T}_n^r is the set of rooted trees with n vertices. \mathbb{T}_n^{af} is the set of attractor flow trees with n leaves (of which we only draw the different topologies). And \mathbb{T}_n^{S} denotes the set of Schröder trees with n leaves. In addition to these sets, an important rôle is played by the set \mathbb{T}_n^ℓ of labelled unrooted trees which is obtained from \mathbb{T}_n by assigning different labels to the vertices.

index $\Omega_\star(\gamma)$. We then apply this formalism in the context of Calabi-Yau string vacua, and express the generalized DT invariants $\Omega(\gamma, z^a)$ in terms of their counterparts evaluated at the large volume attractor point (1.1), known as MSW invariants.

2.1 Wall-crossing and attractor flows

The BPS index $\Omega(\gamma, z^a)$ counts (with sign) micro-states of BPS black holes with total electromagnetic charge $\gamma = (p^\Lambda, q_\Lambda)$, for a given value z^a of the moduli at spatial infinity. While $\Omega(\gamma, z^a)$ is a locally constant function over the moduli space, it can jump across certain real codimension-one loci where certain black hole bound states, represented by multi-centered black hole solutions of $N = 2$ supergravity, become unstable. The positions of these loci, known as *walls of marginal stability*, are determined by the central charge $Z_\gamma(z^a)$, a complex-valued linear function of γ whose modulus gives the mass of a BPS state of charge γ , while the phase determines the supersymmetry subalgebra preserved by the state. Since a bound state can only decay when its mass becomes equal to the sum of masses of its constituents, it is apparent that the walls correspond to hypersurfaces where the phases of two central charges, say $Z_{\gamma_L}(z^a)$ and $Z_{\gamma_R}(z^a)$, with non-vanishing Dirac-Schwinger-Zwanziger (DSZ) pairing $\langle \gamma_L, \gamma_R \rangle \neq 0$, align. The bound states which may become unstable are then those whose constituents have charges in the positive cone spanned by γ_L and γ_R .

The general relation between the values of $\Omega(\gamma, z^a)$ on the two sides of a wall has been found in the mathematics literature by Kontsevich–Soibelman [34] and Joyce–Song [35, 36], and justified physically in a series of works [6, 7, 8, 9]. However, in this work we require a

somewhat different result: an expression of $\Omega(\gamma, z^a)$ in terms of moduli-independent indices. One such representation, known as the Coulomb branch formula, was developed in a series of papers [37, 38, 39] (see [40] for a review) where the moduli-independent index is the so-called ‘single-centered invariant’, counting single-centered, spherically symmetric BPS black holes. Unfortunately, this representation (and its inverse) is quite involved, as it requires disentangling genuine single-centered solutions from so-called scaling solutions, i.e. multi-centered solutions with $n \geq 3$ constituents which can become arbitrarily close to each other [6, 41]. A simpler alternative is to consider the *attractor index*, i.e. the value of the BPS index in the attractor chamber $\Omega_*(\gamma) \equiv \Omega(\gamma, z_\gamma^a)$, where z_γ^a is fixed in terms of the charge γ via the attractor mechanism [42] (recall that for a spherically symmetric BPS black hole with charge γ , the scalars in the vector multiplets have fixed value z_γ^a at the horizon independently of their value z^a at spatial infinity). By definition, the attractor indices are of course moduli independent. The problem of expressing $\Omega(\gamma, z^a)$ in terms of the attractor indices was addressed recently in [20]. Relying on the split attractor flow conjecture [19, 6], it was argued that the rational BPS index

$$\bar{\Omega}(\gamma, z^a) = \sum_{d|\gamma} \frac{1}{d^2} \Omega(\gamma/d, z^a) \quad (2.1)$$

can be expanded in powers of $\Omega_*(\gamma_i)$

$$\bar{\Omega}(\gamma, z^a) = \sum_{\sum_{i=1}^n \gamma_i = \gamma} g_{\text{tr},n}(\{\gamma_i\}, z^a) \prod_{i=1}^n \bar{\Omega}_*(\gamma_i), \quad (2.2)$$

where the sum over $\{\gamma_i\}$ runs over ordered³ decompositions of γ into sums of vectors $\gamma_i \in \Gamma_+$, with Γ_+ being the set of all vectors γ whose central charge $Z_\gamma(z_\infty^a)$ lies in a fixed half-space defining the splitting between BPS particles ($\gamma \in \Gamma_+$) and anti-BPS particles ($\gamma \in -\Gamma_+$). Such decompositions correspond to contributions of multi-centered black hole solutions with constituents carrying charges γ_i .

The coefficient $g_{\text{tr},n}$, called the *tree index*, is defined as

$$g_{\text{tr},n}(\{\gamma_i\}, z^a) = \frac{1}{n!} \sum_{T \in \mathbb{T}_n^{\text{af}}} \Delta(T) \kappa(T), \quad (2.3)$$

where the sum goes over *attractor flow trees*. These are (unordered, full) rooted binary trees⁴ T with vertices decorated by electromagnetic charges γ_v , from the total charge γ at the root v_0 to the constituent charges γ_i at the leaves of the tree, with condition that $\gamma_v = \gamma_{L(v)} + \gamma_{R(v)}$ where $L(v)$, $R(v)$ are the two children of vertex v (see Fig. 2). Note that when all the charges γ_i are distinct, flow trees are in one-to-one correspondence with 2-bracketings of the unordered set $\{1, \dots, n\}$, e.g. the tree displayed in Fig. 2 corresponds to $((13)(2(45)))$. The idea of the split attractor flow conjecture is that each tree represents a nested sequence of two-centered bound states describing a multi-centered solution built out of constituents with

³In [20] this formula was written as a sum over *unordered* decompositions. Thus, the definition of $g_{\text{tr},n}$ here differs from the one in [20] by the factor of $1/n!$ which appears explicitly in (2.3).

⁴The number of such trees with n leaves is $b_n = (2n - 3)!! = (2n - 3)!/[2^{n-2}(n - 2)!] = \{1, 1, 3, 15, 105, 945, \dots\}$ for $n \geq 1$.

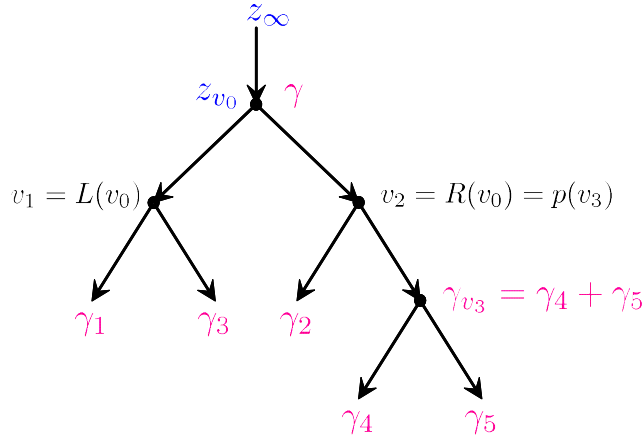


Figure 2: An example of attractor flow tree corresponding to the bracketing $((13)(2(45)))$.

charges γ_i . With this interpretation, the edges of the graph represent the evolution of the moduli under attractor flow, so that one starts from the moduli at spatial infinity $z_\infty^a \equiv z^a$ and assigns to the root v_0 the point in the moduli space $z_{v_0}^a$ where the attractor flow with charge γ crosses the wall of marginal stability where $\text{Im} [Z_{\gamma_{L(v_0)}} \bar{Z}_{\gamma_{R(v_0)}}(z_{v_0}^a)] = 0$. Then one repeats this procedure for every edge, obtaining a set of charges and moduli (γ_v, z_v^a) assigned to each vertex, with the bound state constituents and their attractor moduli $(\gamma_i, z_{\gamma_i}^a)$ assigned to the leaves. Given these data, the factor $\Delta(T)$ is given by

$$\Delta(T) = \prod_{v \in V_T} \Delta_{\gamma_{L(v)} \gamma_{R(v)}}^{z_{p(v)}}, \quad \Delta_{\gamma_L \gamma_R}^z = \frac{1}{2} \left[\text{sgn} \text{Im} [Z_{\gamma_L} \bar{Z}_{\gamma_R}(z^a)] + \text{sgn}(\gamma_{LR}) \right], \quad (2.4)$$

where V_T denotes the set of vertices of T excluding the leaves, $p(v)$ is the parent of vertex v , and $\gamma_{LR} = \langle \gamma_L, \gamma_R \rangle$. This factor vanishes unless the stability condition⁵

$$\gamma_{L(v)R(v)} \text{Im} [Z_{\gamma_{L(v)}} \bar{Z}_{\gamma_{R(v)}}(z_{p(v)}^a)] > 0 \quad (2.5)$$

is satisfied for all $v \in V_T$, which ensures admissibility of the flow tree T , i.e. the existence of the corresponding nested bound state. The second factor

$$\kappa(T) \equiv (-1)^{n-1} \prod_{v \in V_T} \kappa(\gamma_{L(v)R(v)}), \quad \kappa(x) = (-1)^x x. \quad (2.6)$$

is simply the product of the BPS indices of the nested two-centered solutions associated with the tree T . Note that the signs of $\Delta(T)$ and $\kappa(T)$ are separately ambiguous, due to the choice of ordering $(\gamma_{L(v)}, \gamma_{R(v)})$ at each vertex, but the sign of their product is unambiguous.

Sometimes, it is useful to consider the refined BPS index $\Omega(\gamma, z^a, y)$ which carries additional dependence on the fugacity y conjugate to the spin J_3 . All the above equations remain valid in this case as well (except the definition of the rational invariant (2.1) which must be slightly modified), but now the function $\kappa(x)$ appearing in (2.6) becomes a Laurent polynomial in y

$$\kappa(x) = (-1)^x \frac{y^x - y^{-x}}{y - y^{-1}}, \quad (2.7)$$

⁵In fact, the admissibility also requires $\text{Re} [Z_{\gamma_{L(v)}} \bar{Z}_{\gamma_{R(v)}}(z_v^a)] > 0$ at each vertex. This condition will hold automatically for the case of our interest, namely, D4-D2-D0 black holes in the large volume limit, so we do not impose it explicitly.

reducing to the unrefined case in the limit $y \rightarrow 1$.

It is easy to see that the ‘flow tree formula’ (2.2) is consistent with the primitive wall-crossing formula [6, 7]

$$\Delta\bar{\Omega}(\gamma_L + \gamma_R) = -\text{sgn}(\gamma_{LR}) \kappa(\gamma_{LR}) \bar{\Omega}(\gamma_L, z^a) \bar{\Omega}(\gamma_R, z^a), \quad (2.8)$$

which gives the jump of the BPS index due to the decay of bound states after crossing the wall defined by a pair of primitive⁶ charges γ_L and γ_R . To this end, it is enough to consider all flow trees which start with the splitting $\gamma \rightarrow \gamma_L + \gamma_R$ at the root of the tree. It is also useful to note that, assuming that all charges γ_i are distinct, the sum over splittings and flow trees in (2.2) can be generated by iterating the quadratic equation [20]

$$\bar{\Omega}(\gamma, z^a) = \bar{\Omega}_*(\gamma) - \frac{1}{2} \sum_{\substack{\gamma=\gamma_L+\gamma_R \\ (\gamma_L, \gamma_R) \neq 0}} \Delta_{\gamma_L \gamma_R}^z \kappa(\gamma_{LR}) \bar{\Omega}(\gamma_L, z_{LR}^a) \bar{\Omega}(\gamma_R, z_{LR}^a), \quad (2.9)$$

where z_{LR}^a is the point where the attractor flow of charge γ crosses the wall of marginal stability $\text{Im} [Z_{\gamma_L} \bar{Z}_{\gamma_R}(z_{LR}^a)] = 0$.

2.2 Partial tree index

While the representation of the BPS index based on attractor flows is useful for many purposes, it seems to require evaluating the central charges along the attractor flow (see (2.4)), which is in general difficult. In fact, in order to determine if a given tree is stable, only the signs of the relative phases between the two central charges at the vertices of the tree are required, and it turns out to be possible to compute all of them recursively [20]. Nevertheless, this still leaves us with sign functions depending on non-linear combinations of central charges and DSZ products, which are very difficult to work with.

A solution to overcome this problem was found in [20]. The key idea is to introduce a refined index with a fugacity y conjugate to angular momentum, and define

$$g_{\text{tr},n}(\{\gamma_i\}, z^a, y) = \frac{(-1)^{n-1+\sum_{i<j} \gamma_{ij}}}{(y-y^{-1})^{n-1}} \text{Sym} \{F_{\text{tr},n}(\{\gamma_i\}, z^a) y^{\sum_{i<j} \gamma_{ij}}\}, \quad (2.10)$$

where Sym denotes symmetrization (with weight $1/n!$) in all charges and the ‘partial tree index’ is defined by

$$F_{\text{tr},n}(\{\gamma_i\}, z^a) \equiv \sum_{T \in \mathbb{T}_n^{\text{af-pl}}} \Delta(T), \quad (2.11)$$

where the sum runs over the set $\mathbb{T}_n^{\text{af-pl}}$ of *planar* flow trees with n leaves⁷ carrying ordered charges $\gamma_1, \dots, \gamma_n$. Although this is not manifest, the refined tree index (2.10) is regular at $y = 1$, and its value (computed e.g. using l’Hôpital rule) reduces to the tree index (2.3). The advantage of the representation (2.10) is that the partial tree index $F_{\text{tr},n}$ does not involve the κ -factors (2.6) and is independent of the refinement parameter.

⁶Here, by primitive we mean that all charges with non-zero index in the two-dimensional lattice spanned by γ_L and γ_R are linear combinations $N_L \gamma_L + N_R \gamma_R$ with coefficients N_L, N_R of the same sign.

⁷The number of such trees is the $n-1$ -th Catalan number $|\mathbb{T}_n^{\text{af-pl}}| = \frac{(2n-2)!}{n[(n-1)!^2]} = \{1, 1, 2, 5, 14, 42, 132, \dots\}$ for $n \geq 1$.

There are two important results about the partial index $F_{\text{tr},n}$. To formulate them, let us introduce a few convenient notations:

$$c_i(z^a) = \text{Im} [Z_{\gamma_i} \bar{Z}_{\gamma}(z^a)], \quad S_k = \sum_{i=1}^k c_i(z^a), \quad \beta_{k\ell} = \sum_{i=1}^k \gamma_{i\ell}, \quad \Gamma_{k\ell} = \sum_{i=1}^k \sum_{j=1}^{\ell} \gamma_{ij}. \quad (2.12)$$

Note that due to $\gamma = \sum_{i=1}^n \gamma_i$, the parameters c_i satisfy $\sum_{i=1}^n c_i = 0$. In terms of these notations, the partial index satisfies the iterative equation [20, (2.59)],

$$F_{\text{tr},n}(\{\gamma_i\}, z^a) = \frac{1}{2} \sum_{\ell=1}^{n-1} (\text{sgn}(S_\ell) - \text{sgn}(\Gamma_{n\ell})) F_{\text{tr},\ell}(\{\gamma_i\}_{i=1}^{\ell}, z_\ell^a) F_{\text{tr},n-\ell}(\{\gamma_i\}_{i=\ell+1}^n, z_\ell^a), \quad (2.13)$$

where z_ℓ^a is the value of the moduli where the attractor flow crosses the wall for the decay $\gamma \rightarrow (\gamma_1 + \dots + \gamma_\ell, \gamma_{\ell+1} + \dots + \gamma_n)$ and it can be shown that for this value

$$c_i(z_\ell^a) = c_i^{(\ell)} \equiv c_i(z^a) - \frac{\beta_{ni}}{\Gamma_{n\ell}} S_\ell. \quad (2.14)$$

It is clear that this iterative equation is in a sense a realization of the quadratic equation (2.9). A similar equation can be written for the tree index $g_{\text{tr},n}$ itself, but it will involve the κ -factors and symmetrization.

According to the result obtained in [20, Prop. 2], the partial tree index satisfies another recursion relation

$$F_{\text{tr},n}(\{\gamma_i\}, z^a) = F_n^{(0)}(\{\gamma_i, c_i\}) - \sum_{\substack{n_1 + \dots + n_m = n \\ n_k \geq 1, m < n}} F_{\text{tr},m}(\{\gamma'_k\}, z^a) \prod_{k=1}^m F_{n_k}^{(\star)}(\gamma_{j_{k-1}+1}, \dots, \gamma_{j_k}), \quad (2.15)$$

where the sum runs over ordered partitions of n , m is the number of parts, and for $k = 1, \dots, m$ we defined

$$j_0 = 0, \quad j_k = n_1 + \dots + n_k, \quad \gamma'_k = \gamma_{j_{k-1}+1} + \dots + \gamma_{j_k}. \quad (2.16)$$

The two functions appearing in (2.15) are simply products of signs

$$F_n^{(0)}(\{\gamma_i, c_i\}) = \frac{1}{2^{n-1}} \prod_{i=1}^{n-1} \text{sgn}(S_i), \quad F_n^{(\star)}(\{\gamma_i\}) = \frac{1}{2^{n-1}} \prod_{i=1}^{n-1} \text{sgn}(\Gamma_{ni}). \quad (2.17)$$

This new recursive relation allows to express the partial index in the way which does not involve sign functions depending on non-linear combinations of parameters, in contrast to the previous relation (2.13) where such sign functions arise due to the attractor flow given by (2.14).

2.3 D4-D2-D0 black holes and BPS indices

The results presented above are applicable in any theory with $N = 2$ supersymmetry. Let us now specialize to the BPS black holes obtained as bound states of D4-D2-D0-branes in type IIA string theory compactified on a CY threefold \mathfrak{Y} . In this case the moduli $z^a = b^a + it^a$ (with $a = 1, \dots, b_2(\mathfrak{Y})$) running over a basis of $H_2(\mathfrak{Y}, \mathbb{Z})$, where t^a are Kähler moduli and b^a are components of the Kalb-Ramond field, parametrize the complexified Kähler moduli space $\mathcal{M}_\mathcal{K}$

of \mathfrak{Y} . The charge vectors in a fixed basis of even cohomology have the form $\gamma = (0, p^a, q_a, q_0)$ where zero corresponds to the vanishing D6-brane charge, whereas other components satisfy the following quantization conditions [43]

$$p^a \in \mathbb{Z}, \quad q_a \in \mathbb{Z} + \frac{1}{2}(p^2)_a, \quad q_0 \in \mathbb{Z} - \frac{1}{24}p^a c_{2,a}. \quad (2.18)$$

Here $c_{2,a}$ are components of the second Chern class of \mathfrak{Y} , and to write the second relation we introduced notations $(kp)_a = \kappa_{abc}k^b p^c$ and $(lkp) = \kappa_{abc}l^a k^b p^c$ (recall that κ_{abc} are intersection numbers of \mathfrak{Y}) which will be extensively used below. The lattice of charges satisfying (2.18) will be denoted by Γ . A smaller lattice Γ_+ is obtained by further restriction that the D4-brane charge p^a corresponds to an effective divisor in \mathfrak{Y} and belongs to the Kähler cone, i.e.

$$p^3 > 0, \quad (rp^2) > 0, \quad k_a p^a > 0, \quad (2.19)$$

for all effective divisors $r^a \gamma_a \in H_4^+(\mathfrak{Y}, \mathbb{Z})$ and effective curves $k_a \gamma^a \in H_2^+(\mathfrak{Y}, \mathbb{Z})$, where γ_a denotes irreducible divisors giving an integer basis of $\Lambda = H_4(\mathfrak{Y}, \mathbb{Z})$, whereas γ^a an integer basis of $\Lambda^* = H_2(\mathfrak{Y}, \mathbb{Z})$. Such charge p^a gives rise to the quadratic form $\kappa_{ab} = \kappa_{abc} p^c$ on $\Lambda \otimes \mathbb{R} \simeq \mathbb{R}^{b_2}$ of signature $(1, b_2 - 1)$. It allows to embed Λ into Λ^* , but since in general the map $\epsilon^a \mapsto \kappa_{ab} \epsilon^b$ is not surjective, the quotient Λ^*/Λ is a finite group of order $|\det \kappa_{ab}|$.

The holomorphic central charge for this system is given by

$$Z_\gamma(z^a) = q_\Lambda X^\Lambda(z^a) - p^\Lambda F_\Lambda(z^a), \quad (2.20)$$

where we used the special coordinates $X^\Lambda(z^a) = (1, z^a)$ and $F_\Lambda = \partial_{X^\Lambda} F(X)$ is the derivative of the holomorphic prepotential $F(X)$ on \mathcal{M}_K . In the large volume limit $t^a \rightarrow \infty$, the prepotential reduces to the classical cubic contribution

$$F(X) \approx F^{\text{cl}}(X) = -\kappa_{abc} \frac{X^a X^b X^c}{6X^0}, \quad (2.21)$$

and the central charge can be approximated as

$$Z_\gamma \approx -\frac{1}{2}(pt^2) + i(q_a t^a + (pb)t) + q_0 + q_a b^a + \frac{1}{2}(pb^2). \quad (2.22)$$

Note, in particular, that it always has a large negative real part. Another useful observation is that both quantities appearing in the definition of $\Delta_{\gamma_L \gamma_R}^z$ (2.4) are independent of the q_0 component of the charge. Indeed,

$$\begin{aligned} \langle \gamma, \gamma' \rangle &= q_a p'^a - q'_a p^a, \\ \text{Im} [Z_\gamma \bar{Z}_{\gamma'}] &= -\frac{1}{2} \left((p't^2)(q_a + (pb)_a)t^a - (pt^2)(q'_a + (p'b)_a)t^a \right). \end{aligned} \quad (2.23)$$

The BPS spectrum $\bar{\Omega}(\gamma, z^a)$ of D4-D2-D0 black holes is captured by Donaldson-Thomas invariants of \mathfrak{Y} . An important property of the DT invariants considered for charges of the above form is that they are invariant under the combination of the shift of the Kalb-Ramond field, $b^a \mapsto b^a + \epsilon^a$, and the spectral flow transformation acting on the D2 and D0 charges

$$q_a \mapsto q_a - \kappa_{abc} p^b \epsilon^c, \quad q_0 \mapsto q_0 - \epsilon^a q_a + \frac{1}{2}(p\epsilon\epsilon). \quad (2.24)$$

The shift of b^a cannot be ignored since the DT invariants are only piecewise constant as functions of the complexified Kähler moduli $z^a = b^a + it^a$ due to wall-crossing.

On the other hand, the MSW invariants $\bar{\Omega}^{\text{MSW}}(\gamma)$, defined as the generalized DT invariants $\bar{\Omega}(\gamma, z^a)$ evaluated at their respective large volume attractor point (1.1), are by construction independent of the moduli, and therefore invariant under (2.24). As a result, they only depend on p^a, μ_a and \hat{q}_0 , where we traded the electric charges (q_a, q_0) for $(\epsilon^a, \mu_a, \hat{q}_0)$. The latter comprise the spectral flow parameter ϵ^a , the residue class $\mu_a \in \Lambda^*/\Lambda$ defined by the decomposition

$$q_a = \mu_a + \frac{1}{2} \kappa_{abc} p^b p^c + \kappa_{abc} p^b \epsilon^c, \quad \epsilon^a \in \Lambda, \quad (2.25)$$

and the invariant charge (κ^{ab} is the inverse of κ_{ab})

$$\hat{q}_0 \equiv q_0 - \frac{1}{2} \kappa^{ab} q_a q_b, \quad (2.26)$$

which is invariant under (2.24). This allows to write $\bar{\Omega}^{\text{MSW}}(\gamma) = \bar{\Omega}_{p,\mu}(\hat{q}_0)$.

An important fact is that the invariant charge \hat{q}_0 is bounded from above by $\hat{q}_0^{\text{max}} = \frac{1}{24}((p^3) + c_{2,a} p^a)$. This allows to define two generating functions

$$h_{p,q}^{\text{DT}}(\tau, z^a) = \sum_{\hat{q}_0 \leq \hat{q}_0^{\text{max}}} \bar{\Omega}(\gamma, z^a) \mathbf{e}(-\hat{q}_0 \tau), \quad (2.27)$$

$$h_{p,\mu}(\tau) = \sum_{\hat{q}_0 \leq \hat{q}_0^{\text{max}}} \bar{\Omega}_{p,\mu}(\hat{q}_0) \mathbf{e}(-\hat{q}_0 \tau), \quad (2.28)$$

where we used notation $\mathbf{e}(x) = e^{2\pi i x}$. Whereas the generating function of DT invariants $h_{p,q}^{\text{DT}}$ depends on the full electric charge q_a and also carries a piecewise constant dependence on the moduli z^a , the generating function of MSW invariants $h_{p,\mu}(\tau)$, due to the spectral flow symmetry, depends only on the residue class μ_a . This generating function will be the central object of interest in this paper. In particular, our main goal is to understand its behavior under modular transformations of τ , without any assumptions on the degree of reducibility of the divisor $p^a \gamma_a$.

In general, the MSW invariants $\bar{\Omega}^{\text{MSW}}(\gamma)$ are distinct from the attractor moduli $\bar{\Omega}_*(\gamma)$, since the latter coincide with the generalized DT invariants $\bar{\Omega}(\gamma, z^a)$ evaluated at the true attractor point $z_*^a(\gamma)$ for the charge γ , while the former are the generalized DT invariants evaluated at the large volume attractor point $z_\infty^a(\gamma)$ defined in (1.1). Nevertheless, we claim that in the large volume limit $t^a \rightarrow \infty$, the tree flow formula reviewed in the previous subsections still allows to express $\bar{\Omega}(\gamma, z^a)$ in terms of the MSW invariants, namely

$$\bar{\Omega}(\gamma, z^a) = \sum_{\sum_{i=1}^n \gamma_i = \gamma} g_{\text{tr},n}(\{\gamma_i\}, z^a) \prod_{i=1}^n \bar{\Omega}^{\text{MSW}}(\gamma_i) \quad (\text{Im } z^a \rightarrow \infty). \quad (2.29)$$

The point is that the only walls of marginal stability which extend to infinite volume are those where the constituents carry only D4-brane charge, and that non-trivial bound states involving constituents with D4-brane charge are ruled out at the large volume attractor point, similarly to the usual attractor chamber. Since the r.h.s. of (2.29) is consistent with wall-crossing in the infinite volume limit and agrees with the left-hand side at $z^a = z_\infty^a(\gamma)$, it

must therefore hold everywhere at large volume. Of course, some of the states contributing to $\bar{\Omega}^{\text{MSW}}(\gamma_i)$ may have some substructure, e.g. be realized as D6- $\bar{\text{D}}6$ bound states, but this structure cannot be probed in the large volume limit. Importantly, since the quantities (2.23) entering in the definition of the tree index are independent of the D0-brane charge q_0 , the formula flow tree formula (2.29) may be applied to relate the generating functions of DT and MSW invariants,

$$h_{p,q}^{\text{DT}}(\tau, z^a) = \sum_{\sum_{i=1}^n \check{\gamma}_i = \check{\gamma}} g_{\text{tr},n}(\{\check{\gamma}_i\}, z^a) e^{\pi i \tau Q_n(\{\check{\gamma}_i\})} \prod_{i=1}^n h_{p_i, \mu_i}(\tau), \quad (2.30)$$

where $\check{\gamma} = (p^a, q_a)$ denotes a part of the charge vector and

$$Q_n(\{\check{\gamma}_i\}) = \kappa^{ab} q_a q_b - \sum_{i=1}^n \kappa_i^{ab} q_{i,a} q_{i,b} \quad (2.31)$$

appears due to the quadratic term in the definition of \hat{q}_0 .

3. D3-instantons and contact potential

In this section, we switch to the dual setup⁸ of type IIB string theory compactified on the same CY manifold \mathfrak{Y} . The DT invariants, describing the BPS degeneracies of D4-D2-D0 black holes in type IIA, now appear as coefficients in front of the D3-D1-D(-1) instanton effects affecting the metric on the hypermultiplet moduli space \mathcal{M}_H . The main idea of our approach is that these instanton effects are strongly constrained by demanding that \mathcal{M}_H admits an isometric action of the type IIB S-duality $SL(2, \mathbb{Z})$. This constraint uniquely fixes the modular behavior of the generating functions introduced in the previous section. Here we recall the twistorial construction of D-instanton corrections to the hypermultiplet metric, describe the action of S-duality, and analyze the instanton expansion of a particular function on \mathcal{M}_H known as contact potential.

3.1 \mathcal{M}_H and twistorial description of instantons

The moduli space of four-dimensional $N = 2$ supergravity is a direct product of vector and hypermultiplet moduli spaces, $\mathcal{M}_V \times \mathcal{M}_H$. The former is a (projective) special Kähler manifold, whereas the latter is a quaternion-Kähler (QK) manifold. In type IIB string theory compactified on a CY threefold \mathfrak{Y} , \mathcal{M}_H is a space of real dimension $4b_2(\mathfrak{Y}) + 4$, which is fibered over the complexified Kähler moduli space $\mathcal{M}_{\mathcal{K}}(\mathfrak{Y})$ of dimension $2b_2(\mathfrak{Y})$. In addition to the Kähler moduli $z^a = b^a + it^a$, it describes the dynamics of the ten-dimensional axio-dilaton $\tau = c^0 + i/g_s$, the Ramond-Ramond (RR) scalars $c^a, \tilde{c}_a, \tilde{c}_0$, corresponding to periods of the RR 2-form, 4-form and 6-form on a basis of $H^{\text{even}}(\mathfrak{Y}, \mathbb{Z})$, and finally, the NS-axion ψ , dual to the Kalb-Ramond two-form B in four dimensions.

⁸Our preference for the type IIB set-up is merely for consistency with our earlier works on hypermultiplet moduli spaces in $d = 4$ Calabi-Yau vacua. The same considerations apply verbatim, with minor changes of wording, to the vector multiplet in type IIA string theory compactified on $\mathfrak{Y} \times S^1$, which is more directly related to the counting of D4-D2-D0 black holes in four dimensions.

At tree-level, the QK metric on \mathcal{M}_H is obtained from the moduli space $\mathcal{M}_{\mathcal{K}}$ via the c -map construction [44, 45] and thus is completely determined by the holomorphic prepotential $F(X)$. But this metric receives g_s -corrections, both perturbative and non-perturbative. The latter can be of two types: either from Euclidean D-branes wrapping even dimensional cycles on \mathfrak{Y} , or from NS5-branes wrapped around the whole \mathfrak{Y} . In this paper we shall be interested only in the effects of D3-D1-D(-1) instantons, and ignore the effects of NS5 and D5-instantons, which are subleading in the large volume limit. Since NS5-instantons only mix with D5-instantons under S-duality, this truncation does not spoil modular invariance.

The most concise way to describe the D-instanton corrections is to consider type IIA string theory compactified on the mirror CY and to the twistor formalism for quaternionic geometries [28, 29]. In this approach the metric is encoded in the contact structure on a $\mathbb{C}P^1$ -bundle over \mathcal{M}_H , known as twistor space. The D-instanton corrected contact structure has been constructed to all orders in the instanton expansion in [27, 30], and an explicit expression for the metric has been derived recently in [46, 47]. Here we will present only those elements of the construction which are relevant for the subsequent analysis, and refer to reviews [48, 49] for more details.

The crucial point is that, locally, the contact structure is determined by a set of holomorphic Darboux coordinates $(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$ on the twistor space, considered as functions of coordinates on \mathcal{M}_H and a coordinate t on the $\mathbb{C}P^1$ fiber. Although all Darboux coordinates are important for recovering the metric from the holomorphic one-form $d\alpha + \tilde{\xi}_\Lambda d\xi^\Lambda$, for the purposes of this paper the coordinate α is irrelevant. Therefore, we consider only ξ^Λ and $\tilde{\xi}_\Lambda$ which can be conveniently packed into symplectic invariant exponential functions $\mathcal{X}_\gamma = \mathbf{e}\left(p^\Lambda \tilde{\xi}_\Lambda - q_\Lambda \xi^\Lambda\right)$ labelled by a charge vector $\gamma = (p^\Lambda, q_\Lambda)$.

At tree level, the Darboux coordinates (multiplied by t) are known to be simple quadratic polynomials in t so that \mathcal{X}_γ take the form⁹

$$\mathcal{X}_\gamma^{\text{sf}}(t) = \mathbf{e}\left(\frac{\tau_2}{2} \left(\bar{Z}_\gamma(\bar{u}^a) t - \frac{Z_\gamma(u^a)}{t}\right) + p^\Lambda \tilde{\xi}_\Lambda - q_\Lambda \xi^\Lambda\right), \quad (3.1)$$

where $Z_\gamma(u^a)$ is the central charge (2.20) but written in terms of the complex structure moduli u^a of the mirror CY, ζ^Λ and $\tilde{\zeta}_\Lambda$ are periods of the RR 3-form of the type IIA formulation, and $\tau_2 = g_s^{-1}$ is the ten-dimensional string coupling. At non-perturbative level, this expression gets modified and the Darboux coordinates are determined by the integral equation

$$\mathcal{X}_\gamma(t) = \mathcal{X}_\gamma^{\text{sf}}(t) \mathbf{e}\left(\frac{1}{8\pi^2} \sum_{\gamma'} \sigma_{\gamma'} \bar{\Omega}(\gamma') \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{dt'}{t'} \frac{t+t'}{t-t'} \mathcal{X}_{\gamma'}(t')\right), \quad (3.2)$$

where the sum goes over all charges labeling cycles wrapped by D-branes,

$$\ell_\gamma = \{t \in \mathbb{C}P^1 : Z_\gamma/t \in i\mathbb{R}^-\} \quad (3.3)$$

is the so called BPS ray, a contour on $\mathbb{C}P^1$ extending from $t = 0$ to $t = \infty$ along the direction fixed by the central charge, and σ_γ is a quadratic refinement of the DSZ product, i.e. a sign factor satisfying the defining relation

$$\sigma_{\gamma_1} \sigma_{\gamma_2} = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \sigma_{\gamma_1 + \gamma_2}. \quad (3.4)$$

⁹The superscript ‘sf’ means ‘semi-flat’, a type of geometry of the c -map.

The system of integral equations (3.2) can be solved iteratively by first substituting $\mathcal{X}_{\gamma'}(t')$ on the r.h.s. with its zero-th order value $\mathcal{X}_{\gamma'}^{\text{sf}}(t')$ in the weak coupling limit, computing the leading correction from the integral and iterating this process. This produces an asymptotic series at weak coupling, in powers of the DT invariants $\bar{\Omega}(\gamma)$. Using the saddle point method, it is easy to check that the coefficient of each monomial $\prod_i \bar{\Omega}(\gamma_i)$ is suppressed by a factor $e^{-\pi\tau_2 \sum_i |Z_{\gamma_i}|}$, corresponding to an n -instanton effect [31, 50]. Note that multi-instanton effects become of the same order as one-instanton effects on walls of marginal stability where the phases of Z_{γ_i} become aligned.

3.2 D3-instantons in the large volume limit

The above construction of D-instantons is adapted to the type IIA formulation because the equation (3.2) defines the Darboux coordinates in terms of the type IIA fields appearing explicitly in the tree level expression (3.1). To pass to the mirror dual type IIB formulation, one should apply the mirror map, a coordinate transformation from the type IIA to the type IIB physical fields. The problem is that this transformation itself gets instanton corrections. A way to fix it is to require that the metric on \mathcal{M}_H carries an isometric action of S-duality group $SL(2, \mathbb{Z})$ of type IIB string theory, which acts on the type IIB fields by an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the following way

$$\begin{aligned} \tau &\mapsto \frac{a\tau + b}{c\tau + d}, & t^a &\mapsto |c\tau + d| t^a, & \begin{pmatrix} c^a \\ b^a \end{pmatrix} &\mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} c^a \\ b^a \end{pmatrix}, \\ \tilde{c}_a &\mapsto \tilde{c}_a - c_{2,a} \varepsilon(g), & \begin{pmatrix} \tilde{c}_0 \\ \psi \end{pmatrix} &\mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \tilde{c}_0 \\ \psi \end{pmatrix}, \end{aligned} \tag{3.5}$$

where $\varepsilon(g)$ is the logarithm of the multiplier system of the Dedekind eta function [43].

It is important to note that any isometric action on a quaternion-Kähler manifold can be lifted to a *holomorphic* isometric action on its twistor space. In the present case, $SL(2, \mathbb{Z})$ acts on the fiber coordinate t by fractional-linear transformations with τ -dependent coefficients. This transformation takes a much simpler form when formulated in terms of another coordinate on $\mathbb{C}P^1$ related to t by a Cayley transformation

$$z = \frac{t + i}{t - i}. \tag{3.6}$$

Then the action of $SL(2, \mathbb{Z})$ on the fiber is given by a simple phase rotation¹⁰

$$z \mapsto \frac{c\bar{\tau} + d}{|c\tau + d|} z. \tag{3.7}$$

Using the non-trivial constraint of holomorphicity for the $SL(2, \mathbb{Z})$ isometry on the twistor space, quantum corrections to the classical mirror map, originally found in [52], were identified in [53, 54, 18, 22]. In fact, D3-instanton corrections to the mirror map are known only in the large volume limit where the Kähler moduli are taken to be large, $t^a \rightarrow \infty$. In the following,

¹⁰Actually, this is true only when five-brane instanton corrections are ignored. Otherwise, the lift also gets a non-trivial deformation [51].

we restrict ourselves to this approximation so that we can use the results in *loc. cit.*, which can be written as

$$\begin{aligned}
u^a &= b^a + it^a - \frac{i}{2\tau_2} \sum_{\gamma \in \Gamma_+} p^a \left[\int_{\ell_\gamma} dz (1-z) H_\gamma + \int_{\ell_{-\gamma}} \frac{dz}{z^3} (1-z) H_{-\gamma} \right] \\
\zeta^0 &= \tau_1, \quad \zeta^a = -(c^a - \tau_1 b^a) - 3 \sum_{\gamma \in \Gamma_+} p^a \operatorname{Re} \int_{\ell_\gamma} dz z H_\gamma, \\
\tilde{\zeta}_a &= \tilde{c}_a + \frac{1}{2} \kappa_{abc} b^b (c^c - \tau_1 b^c) + \kappa_{abc} t^b \sum_{\gamma \in \Gamma_+} p^c \operatorname{Im} \int_{\ell_\gamma} dz H_\gamma,
\end{aligned} \tag{3.8}$$

where we introduced a convenient notation¹¹

$$H_\gamma(t) = \frac{\bar{\Omega}(\gamma)}{(2\pi)^2} \sigma_\gamma \mathcal{X}_\gamma(t). \tag{3.9}$$

Similar results are known for $\tilde{\zeta}_0$ and the NS-axion dual to the B -field, but we will not need them in this paper.

Note that the integral contributions to the mirror map are written in terms of the coordinate z (3.6). Besides simplifying the study of modular properties, there is another reason for this: in the large volume limit, the integrals along BPS rays ℓ_γ are dominated by the saddle point [18]

$$z'_\gamma \approx -i \frac{(q+b)_+}{\sqrt{(pt^2)}}, \quad \text{where } q_+ = \frac{q_a t^a}{\sqrt{(pt^2)}}, \tag{3.10}$$

for $(pt^2) > 0$ and $z'_{-\gamma} = 1/z'_\gamma$ in the opposite case. This shows that all integrands can be expanded in Fourier series either around $z = 0$ or $z = \infty$, keeping constant $t^a z$ or t^a/z , respectively. This allows to extract the leading order in the large volume limit in a simple way.

Let us evaluate the combined limit $t^a \rightarrow \infty$, $z \rightarrow 0$ of the part of the integral equation (3.2) relevant for D3-instantons. As a first step, we rewrite the tree level expression (3.1) in terms of the type IIB fields. To this end, we restrict the charge γ to the lattice Γ_+ , take the central charge as in (2.20) with the cubic¹² prepotential (2.21), and substitute the mirror map (3.8). Furthermore, we change the coordinate t to z and take the combined limit. In this way one finds

$$\mathcal{X}_\gamma^{\text{sf}}(z) = \mathcal{X}_\gamma^{\text{cl}}(z) \exp \left[2\pi \sum_{\gamma' \in \Gamma_+} (t p p') \int_{\ell_{\gamma'}} dz' H_{\gamma'} \right], \tag{3.11}$$

where

$$\mathcal{X}_\gamma^{\text{cl}}(z) = \mathbf{e}(-\hat{q}_0 \mathcal{T}) \mathcal{X}_{p,q}^{(\theta)}(z) \tag{3.12}$$

¹¹The functions H_γ have a simple geometric meaning [27, 55]: they generate contact transformations (i.e. preserving the contact structure) relating the Darboux coordinates living in the patches separated by the BPS rays. In fact, these functions together with the contours ℓ_γ are the fundamental data fixing the contact structure on the twistor space.

¹²The other contributions to the prepotential, representing perturbative and non-perturbative α' -corrections, combine with D(-1) and D1-instantons, but are irrelevant for our discussion of instantons with non-vanishing D3-brane charge.

is the classical part of the Darboux coordinates which we represented as a product of two factors: exponential of the invariant charge (2.26) and the remaining q_0 -independent exponential

$$\mathcal{X}_{p,q}^{(\theta)}(z) = e^{-S_p^{\text{cl}}} \mathbf{e} \left(-\frac{\tau}{2} (q+b)^2 + c^a (q_a + \frac{1}{2}(pb)_a) + i\tau_2 (pt^2) (z^2 - 2zz_\gamma) \right), \quad (3.13)$$

with S_p^{cl} being the leading part of the Euclidean D3-brane action in the large volume limit given by $S_p^{\text{cl}} = \pi\tau_2(pt^2) - 2\pi ip^a \tilde{c}_a$. Next, we can approximate

$$\frac{dt'}{t'} \frac{t+t'}{t-t'} = \frac{2dz'}{1-z'^2} \frac{1-zz'}{z-z'} \approx \begin{cases} \frac{2dz'}{z-z'}, & \gamma' \in \Gamma_+, \\ \frac{2(1-zz')dz'}{z'^3}, & \gamma' \in -\Gamma_+. \end{cases} \quad (3.14)$$

This shows that the contribution of $\gamma' \in -\Gamma_+$ is suppressed comparing to $\gamma' \in \Gamma_+$ and therefore can be neglected. As a result, we arrive at the integral equation which is conveniently written in terms of H_γ

$$H_\gamma(z) = H_\gamma^{\text{cl}} \exp \left[\sum_{\gamma' \in \Gamma_+} \int_{\ell_{\gamma'}} dz' K_{\gamma\gamma'}(z, z') H_{\gamma'}(z') \right], \quad (3.15)$$

where the effective kernel is

$$K_{\gamma_1\gamma_2}(z_1, z_2) = 2\pi \left((tp_1p_2) + \frac{i\langle \gamma_1, \gamma_2 \rangle}{z_1 - z_2} \right). \quad (3.16)$$

Below we will need a perturbative solution of the integral equation (3.15). Applying the iterative procedure outlined below (3.4), or equivalently using the Lagrange inversion theorem, such solution can be written as a sum over *rooted trees* [31, §C],

$$H_{\gamma_1}(z_1) = H_{\gamma_1}^{\text{cl}}(z_1) \sum_{n=1}^{\infty} \left(\prod_{i=2}^n \sum_{\gamma_i \in \Gamma_+} \int_{\ell_{\gamma_i}} dz_i H_{\gamma_i}^{\text{cl}}(z_i) \right) \sum_{\mathcal{T} \in \mathbb{T}_n^r} \frac{\mathcal{A}(\mathcal{T})}{|\text{Aut}(\mathcal{T})|}, \quad (3.17)$$

where \mathbb{T}_n^r is the set of rooted trees with n vertices and $\mathcal{A}(\mathcal{T})$ is the ‘amplitude’

$$\mathcal{A}(\mathcal{T}) = \prod_{e \in E_{\mathcal{T}}} K_{\gamma_{s(e)}, \gamma_{t(e)}}(z_{s(e)}, z_{t(e)}). \quad (3.18)$$

A rooted tree¹³ \mathcal{T} consists of n vertices joined by directed edges so that the root vertex, corresponding to $\mathbf{v} = 1$, has only outgoing edges, whereas all other vertices have one incoming edge and an arbitrary number of outgoing ones. We label the vertices of the tree by $\mathbf{v} = 1, \dots, n$ in an arbitrary fashion, except for the root which is labelled by $\mathbf{v} = 1$. The symmetry factor $|\text{Aut}(\mathcal{T})|$ is the order of the symmetry group which permutes the labels $2, \dots, n$ without changing the topology of the tree. Each vertex is decorated by a charge vector $\gamma_{\mathbf{v}}$ and a complex variable $z_{\mathbf{v}}$. We denote by $E_{\mathcal{T}}$ the set of edges, $V_{\mathcal{T}}$ the set of vertices, $s(e)$ and $t(e)$

¹³We will use calligraphic letters \mathcal{T} for trees where charges γ_i are assigned to vertices to distinguish them from rooted trees T where the charges are assigned to leaves (hence T has always more than n vertices). Similarly, we will use notations \mathbf{v} and v for vertices of these two types of trees, respectively. Note also that whereas $V_{\mathcal{T}}$ denotes the set of all vertices, V_T does not include the leaves. An example of trees of the latter type are attractor flow trees.

the source and target vertex of edge e , respectively, and by $n_{\mathbf{v}}$ the number of outgoing edges at vertex \mathbf{v} . Unpacking these notations, we get

$$\begin{aligned}
H_{\gamma_1} &= H_{\gamma_1}^{\text{cl}} + \sum_{\gamma_2} K_{12} H_{\gamma_1}^{\text{cl}} H_{\gamma_2}^{\text{cl}} + \sum_{\gamma_2, \gamma_3} \left(\frac{1}{2} K_{12} K_{13} + K_{12} K_{23} \right) H_{\gamma_1}^{\text{cl}} H_{\gamma_2}^{\text{cl}} H_{\gamma_3}^{\text{cl}} \\
&+ \sum_{\gamma_2, \gamma_3, \gamma_4} \left(\frac{1}{6} K_{12} K_{13} K_{14} + \frac{1}{2} K_{12} K_{23} K_{24} + K_{12} K_{13} K_{24} + K_{12} K_{23} K_{34} \right) H_{\gamma_1}^{\text{cl}} H_{\gamma_2}^{\text{cl}} H_{\gamma_3}^{\text{cl}} H_{\gamma_4}^{\text{cl}} + \dots
\end{aligned} \tag{3.19}$$

where we omitted the integrals and denoted $K_{ij} = K_{\gamma_i, \gamma_j}(z_i, z_j)$. The expansion (3.17) is effectively a multi-instanton expansion in powers of the DT invariants $\bar{\Omega}(\gamma_i)$, which is asymptotic to the exact result at weak coupling.

3.3 From the contact potential to the instanton generating function

Let us recall that our goal is to derive constraints imposed by S-duality on the DT invariants $\bar{\Omega}(\gamma)$ appearing as coefficients in the multi-instanton expansion. To achieve this goal, it is not necessary to study the full metric on \mathcal{M}_H . It is enough to consider a suitable function on this moduli space which has a non-trivial dependence on $\bar{\Omega}(\gamma)$ and specified transformations under S-duality.

Remarkably, on \mathcal{M}_H there is a natural candidate with the above properties — the so-called contact potential e^ϕ . This is a real function which can be defined on any quaternion-Kähler manifold with a continuous isometry in terms of the contact structure on its twistor space [29]. Furthermore, there is a general expression for the contact potential in terms of Penrose-type integrals on the $\mathbb{C}P^1$ fiber. In the case of \mathcal{M}_H , the required isometry is the shift of the NS-axion, which survives all quantum corrections as long as NS5-instantons are kept switched off. In the case at hand, the contact potential is given exactly by [27]

$$e^\phi = \frac{i\tau_2^2}{16} (\bar{u}^\Lambda F_\Lambda - u^\Lambda \bar{F}_\Lambda) - \frac{\chi_{\mathfrak{Y}}}{192\pi} + \frac{i\tau_2}{16} \sum_{\gamma} \int_{\ell_\gamma} \frac{dt}{t} (t^{-1} Z_\gamma(u^a) - t \bar{Z}_\gamma(\bar{u}^a)) H_\gamma, \tag{3.20}$$

where $\chi_{\mathfrak{Y}}$ is the Euler characteristic of \mathfrak{Y} . This formula has a non-trivial contribution due to D-instantons given by the last term.

On the other hand, in the classical, large volume limit one finds $e^\phi = \frac{\tau_2^2}{12}(t^3)$, which shows that the contact potential can be identified with the four-dimensional dilaton and in this approximation behaves as a modular form of weight $(-\frac{1}{2}, -\frac{1}{2})$ under S-duality transformations (3.5). In fact, one can show [27] that $SL(2, \mathbb{Z})$ preserves the contact structure, i.e. it is an isometry of \mathcal{M}_H , only if the *full non-perturbative* contact potential transforms in this way,

$$e^\phi \mapsto \frac{e^\phi}{|c\tau + d|}. \tag{3.21}$$

Furthermore, since S-duality acts by rescaling on both Kähler moduli t^a and fiber coordinate z (see (3.7)), it preserves each order in the expansion around the large volume limit. This implies that the large volume limit of the D3-instanton contribution to e^ϕ , which we denote by $(e^\phi)_{\text{D3}}$, must itself transform as (3.21). It is this condition that we shall exploit to derive modularity constraints on the DT invariants.

To make this condition more explicit, let us extract the D3-instanton contribution to (3.20). The procedure is the same as was used to get (3.15), and we relegate the details of the calculation to appendix A. The result can be written in a concise way using the complex function defined by

$$\mathcal{G} = \sum_{\gamma \in \Gamma_+} \int_{\ell_\gamma} dz H_\gamma(z) - \frac{1}{2} \sum_{\gamma_1, \gamma_2 \in \Gamma_+} \int_{\ell_{\gamma_1}} dz_1 \int_{\ell_{\gamma_2}} dz_2 K_{\gamma_1 \gamma_2}(z_1, z_2) H_{\gamma_1}(z_1) H_{\gamma_2}(z_2) \quad (3.22)$$

and the modular covariant derivative operator

$$\mathcal{D}_{\mathfrak{h}} = \frac{1}{2\pi i} \left(\partial_\tau + \frac{\mathfrak{h}}{2i\tau_2} + \frac{it^a}{4\tau_2} \partial_{t^a} \right), \quad (3.23)$$

which maps modular functions of weight $(\mathfrak{h}, \bar{\mathfrak{h}})$ to modular functions of weight $(\mathfrak{h}+2, \bar{\mathfrak{h}})$. Then one has (generalizing [21, (4.5)] to all orders in the instanton expansion)

$$(e^\phi)_{\text{D3}} = \frac{\tau_2}{2} \text{Re} \mathcal{D}_{-\frac{3}{2}} \mathcal{G} + \frac{1}{32\pi^2} \kappa_{abc} t^c \partial_{\bar{c}_a} \mathcal{G} \partial_{\bar{c}_b} \bar{\mathcal{G}}. \quad (3.24)$$

It is immediate to see that $(e^\phi)_{\text{D3}}$ transforms under S-duality as (3.21) provided the function \mathcal{G} transforms as a modular form of weight $(-\frac{3}{2}, \frac{1}{2})$. In order to derive the implications of this fact, we need to express \mathcal{G} in terms of the generalized DT invariants.

For this purpose, we substitute the multi-instanton expansion (3.17) into (3.22). We claim that the result takes the simple form

$$\mathcal{G} = \sum_{n=1}^{\infty} \left[\prod_{i=1}^n \sum_{\gamma_i \in \Gamma_+} \int_{\ell_{\gamma_i}} dz_i H_{\gamma_i}^{\text{cl}}(z_i) \right] \mathcal{G}_n(\{\gamma_i, z_i\}), \quad (3.25)$$

where $\mathcal{G}_n(\{\gamma_i, z_i\})$ is now a sum over *unrooted* trees with n vertices,

$$\mathcal{G}_n(\{\gamma_i, z_i\}) = \sum_{\mathcal{T} \in \mathbb{T}_n} \frac{\mathcal{A}(\mathcal{T})}{|\text{Aut}(\mathcal{T})|} = \frac{1}{n!} \sum_{\mathcal{T} \in \mathbb{T}_n^\ell} \mathcal{A}(\mathcal{T}), \quad (3.26)$$

where in the second equality we rewrote the result as a sum over *unrooted labelled* trees.¹⁴ To see why this is the case, observe that under the action of the operator $\hat{D} = H^{\text{cl}} \partial_{H^{\text{cl}}}$ rescaling all functions H_γ^{cl} , the function \mathcal{G} maps to the first term in (3.22), which we denote by \mathcal{F} ,

$$\hat{D} \cdot \mathcal{G} = \mathcal{F}, \quad \mathcal{F} \equiv \sum_{\gamma \in \Gamma_+} \int_{\ell_\gamma} dz H_\gamma, \quad (3.27)$$

as can be easily verified with the help of the integral equation (3.15). The multi-instanton expansion of \mathcal{F} follows immediately from (3.17),

$$\mathcal{F} = \sum_{n=1}^{\infty} \left(\prod_{i=1}^n \sum_{\gamma_i \in \Gamma_+} \int_{\ell_{\gamma_i}} dz_i H_{\gamma_i}^{\text{cl}}(z_i) \right) \sum_{\mathcal{T} \in \mathbb{T}_n} \frac{\mathcal{A}(\mathcal{T})}{|\text{Aut}(\mathcal{T})|}. \quad (3.28)$$

¹⁴The number of such trees is $|\mathbb{T}_n^\ell| = n^{n-2} = \{1, 1, 3, 16, 125, 1296, \dots\}$. Such trees also appear in the Joyce-Song wall-crossing formula [35, 36] and are conveniently labelled by their Prüfer code.

Integrating back the action of the derivative operator \hat{D} , we see that the sum over rooted trees in (3.28) turns into the sum over unrooted trees in (3.26). At the first few orders we get, using the same shorthand notation as in (3.19),

$$\begin{aligned} \mathcal{G} = & \sum_{\gamma} H_{\gamma}^{\text{cl}} + \frac{1}{2} \sum_{\gamma_1, \gamma_2} K_{12} H_{\gamma_1}^{\text{cl}} H_{\gamma_2}^{\text{cl}} + \frac{1}{2} \sum_{\gamma_1, \gamma_2, \gamma_3} K_{12} K_{23} H_{\gamma_1}^{\text{cl}} H_{\gamma_2}^{\text{cl}} H_{\gamma_3}^{\text{cl}} \\ & + \sum_{\gamma_1, \gamma_2, \gamma_3, \gamma_4} \left(\frac{1}{6} K_{12} K_{13} K_{14} + \frac{1}{2} K_{12} K_{23} K_{34} \right) H_{\gamma_1}^{\text{cl}} H_{\gamma_2}^{\text{cl}} H_{\gamma_3}^{\text{cl}} H_{\gamma_4}^{\text{cl}} + \dots \end{aligned} \quad (3.29)$$

The simplicity of the expansion (3.26), and the relation (3.24) to the contact potential, show that the function \mathcal{G} is very natural and, in some sense, more fundamental¹⁵ than the naive instanton sum \mathcal{F} . We shall henceforth refer to \mathcal{G} as the ‘instanton generating function’. In the following we shall postulate that \mathcal{G} transforms as a modular form of weight $(-\frac{3}{2}, \frac{1}{2})$, and analyze the consequences of this assumption for the DT invariants.

4. Theta series decomposition and modularity

4.1 Factorisation

To derive modularity constraints on the DT invariants, we need to perform a theta series decomposition of the generating function \mathcal{G} defined in (3.22). To this end, let us make use of the fact noticed in (2.23) that the DSZ products $\langle \gamma, \gamma' \rangle$ and hence the kernels (3.16) do not depend on the q_0 charge. Choosing then the quadratic refinement σ_{γ} as in (C.5), which is also q_0 -independent, and using the factorization (3.12) of $\mathcal{X}_{\gamma}^{\text{cl}}$, one can rewrite the expansion (3.25) as follows

$$\mathcal{G} = \sum_{n=1}^{\infty} \frac{1}{(2\pi)^{2n}} \left[\prod_{i=1}^n \sum_{p_i, q_i} \sigma_{p_i, q_i} h_{p_i, q_i}^{\text{DT}} \int_{\ell_{\gamma_i}} dz_i \mathcal{X}_{p_i, q_i}^{(\theta)}(z_i) \right] \mathcal{G}_n(\{\gamma_i, z_i\}), \quad (4.1)$$

where the sum over the invariant charges $\hat{q}_{i,0}$ gave rise to the generating functions of DT invariants defined in (2.27). This is not yet the desired form because these generating functions depend non-trivially on the remaining electric charges $q_{i,a}$. If it were not for this dependence, the sum over $q_{i,a}$ would produce certain non-Gaussian theta series, and at each order we would have a product of this theta series and n generating functions. Then the modular properties of the theta series would dictate the modular properties of the generating function.

Such a theta series decomposition can be achieved by expressing the DT invariants in terms of the MSW invariants, for which the dependence on electric charges $q_{i,a}$ reduces to the dependence on the residue classes $\mu_{i,a}$ due to the spectral flow symmetry. Substituting the expansion (2.30) of $h_{p,q}^{\text{DT}}$ in terms of $h_{p,\mu}$, the expansion (4.1) of the function \mathcal{G} can be brought to the following factorized form

$$\mathcal{G} = \sum_{n=1}^{\infty} \frac{2^{-\frac{n}{2}}}{\pi \sqrt{2T_2}} \left[\prod_{i=1}^n \sum_{p_i, \mu_i} \sigma_{p_i} h_{p_i, \mu_i} \right] e^{-S_p^{\text{cl}}} \vartheta_{\mathbf{p}, \boldsymbol{\mu}}(\Phi_n^{\text{tot}}, n-2), \quad (4.2)$$

¹⁵In [21], it was noticed that the function \mathcal{G} , denoted by $\tilde{\mathcal{F}}$ in that reference and computed at second order in the multi-instanton expansion, could be obtained from the seemingly simpler function \mathcal{F} by *halving* the coefficient of its second order contribution. Now we see that this *ad hoc* prescription is the consequence of going from rooted to unrooted trees, as a result of adding the second term in (3.22).

where $\vartheta_{\mathbf{p},\boldsymbol{\mu}}$ is the theta series (C.1) with parameter $\lambda = n - 2$ and the kernel which has the following structure

$$\Phi_n^{\text{tot}}(\mathbf{x}) = \text{Sym} \left\{ \sum_{\substack{n_1+\dots+n_m=n \\ n_k \geq 1}} \Phi_m(\mathbf{x}') \prod_{k=1}^m \Phi_{n_k}^g(x_{j_{k-1}+1}, \dots, x_{j_k}) \right\}, \quad (4.3)$$

where the sum runs over ordered partitions of n , whose number of parts is denoted by m , and we adopted notations from (2.16) for indices j_k . The argument \mathbf{x} of the kernel encodes the electric components of the charges (shifted by the B-field and rescaled by $\sqrt{2\tau_2}$) and lives in a vector space $(\oplus_{i=1}^n \Lambda_i) \otimes \mathbb{R}$ of dimension $d = nb_2(\mathfrak{Y})$, given by $b_2(\mathfrak{Y})$ copies of the lattice Λ , where the i -th copy Λ_i carries the bilinear form $\kappa_{i,ab} = \kappa_{abc} p_i^c$ of signature $(1, b_2 - 1)$. Therefore, $\vartheta_{\mathbf{p},\boldsymbol{\mu}}$ is an indefinite theta series of signature $(n, n(b_2 - 1))$ with the bilinear form given explicitly in (C.4).

Finally, the kernel (4.3) is constructed from two other functions. The first, Φ_n , is the multiple integral of the coefficient \mathcal{G}_n in the expansion (3.25),

$$\Phi_n(\mathbf{x}) = \left(\frac{\sqrt{2\tau_2}}{2\pi} \right)^{n-1} \left[\prod_{i=1}^n \int_{\ell_{\gamma_i}} \frac{dz_i}{2\pi} W_{p_i}(x_i, z_i) \right] \mathcal{G}_n(\{\gamma_i, z_i\}), \quad (4.4)$$

with measure factor

$$W_p(x, z) = e^{-2\pi\tau_2 z^2 (pt^2) - 2\pi i \sqrt{2\tau_2} z (pxt)} \quad (4.5)$$

coming from the z -dependent part of (3.13). Although this function is written in terms of \mathcal{G}_n depending on full electromagnetic charge vectors γ_i , it is actually independent of their q_0 components. Indeed, using the result (3.26), it can be rewritten as

$$\Phi_n(\mathbf{x}) = \frac{1}{n!} \left[\prod_{i=1}^n \int_{\ell_{\gamma_i}} \frac{dz_i}{2\pi} W_{p_i}(x_i, z_i) \right] \sum_{\mathcal{T} \in \mathbb{T}_n^\ell} \prod_{e \in E_{\mathcal{T}}} \hat{K}_{s(e)t(e)}, \quad (4.6)$$

where we introduced a rescaled version of the kernel (3.16)

$$\hat{K}_{ij}(z_i, z_j) = \left(\left(\sqrt{2\tau_2} t + i \frac{x_i - x_j}{z_i - z_j} \right) p_i p_j \right). \quad (4.7)$$

Note that τ_2 and t^a appear only in the modular invariant combination $\sqrt{2\tau_2} t^a$. In (4.3) this function appears with the argument \mathbf{x}' and carries a dependence on \mathbf{p}' (not indicated explicitly) which are both mb_2 -dimensional vectors with components (cf. (2.16))

$$p_k'^a = \sum_{i=j_{k-1}+1}^{j_k} p_i^a, \quad x_k'^a = \kappa_k'^{ab} \sum_{i=j_{k-1}+1}^{j_k} \kappa_{i,bc} x_i^c, \quad (4.8)$$

where $\kappa_k'^{ab} = \kappa_{abc} p_k'^c$.

The second function, Φ_n^g , appears due to the expansion of DT invariants in terms of the MSW invariants and is given by a suitably rescaled tree index

$$\Phi_n^g(\mathbf{x}) = \frac{\sigma_\gamma (\sqrt{2\tau_2})^{n-1}}{\prod_{i=1}^n \sigma_{\gamma_i}} g_{\text{tr},n}(\{\gamma_i\}, z^a). \quad (4.9)$$

It is also written in terms of functions depending on the full electromagnetic charge vectors γ_i (with $\gamma = \gamma_1 + \dots + \gamma_n$). However, using (3.4), all quadratic refinements can be expressed through $(-1)^{\langle \gamma_i, \gamma_j \rangle}$ which cancel the corresponding factors in the tree index (see (2.6)). Furthermore, as was noticed in the end of section 2.3, the tree index is independent of the q_0 components of the charge vectors. Therefore, it can be written as a function of p_i^a , $\mu_{i,a}$ and $x_i^a = \sqrt{2\tau_2}(\kappa_i^{ab}q_{i,b} + b^a)$. Then, since after canceling the sign factors $g_{\text{tr},n}$ is homogeneous of degree $n - 1$ in $q_{i,a}$, all factors of $\sqrt{2\tau_2}$ in (4.9) cancel as well.

4.2 Modularity and Vignéras' equation

As explained in appendix C.1, the theta series $\vartheta_{\mathbf{p},\boldsymbol{\mu}}(\Phi, \lambda)$ is a vector-valued modular form of weight $(\lambda + d/2, 0)$ provided the kernel Φ satisfies Vignéras' equation (C.3). In our case $\lambda = n - 2$ and the dimension of the lattice $d = nb_2(\mathfrak{Y})$ so that the expected weight of the theta series is $(2(n - 1) + nb_2/2, 0)$. It is consistent with weight $(-\frac{3}{2}, \frac{1}{2})$ of \mathcal{G} given in (4.2) only if $h_{\mathbf{p},\boldsymbol{\mu}}^{\text{DT}}$ is a vector-valued holomorphic modular form of weight $(-b_2/2 - 1, 0)$. However for this to be true, the kernel Φ_n^{tot} ought to satisfy Vignéras' equation. Let us check whether or not this is the case.

To this end, we first consider the kernel Φ_n . In appendix D we evaluate the multiple integrals defining this kernel explicitly. To present the final result, let us introduce the following d -dimensional vectors $\mathbf{v}_{ij}, \mathbf{u}_{i,j}$:

$$\begin{aligned} (\mathbf{v}_{ij})_k^a &= \delta_{ki}p_j^a - \delta_{kj}p_i^a & \text{such that } \mathbf{v}_{ij} \cdot \mathbf{x} &= (p_i p_j (x_i - x_j)), \\ (\mathbf{u}_{ij})_k^a &= \delta_{ki}(p_j t^2) t^a - \delta_{kj}(p_i t^2) t^a & \text{such that } \mathbf{u}_{ij} \cdot \mathbf{x} &= (p_j t^2)(p_i x_i t) - (p_i t^2)(p_j x_j t), \end{aligned} \quad (4.10)$$

where k labels the copy in $\bigoplus_{k=1}^n \Lambda_k$, $a = 1, \dots, b_2(\mathfrak{Y})$, and the bilinear form is given in (C.4). This first scalar product $\mathbf{v}_{ij} \cdot \mathbf{x}$ corresponds to the DSZ product $\langle \gamma_i, \gamma_j \rangle$, whereas the second $\mathbf{u}_{ij} \cdot \mathbf{x}$ to $-2 \text{Im}[Z_{\gamma_i} \bar{Z}_{\gamma_j}]$ (2.23), both rescaled by $\sqrt{2\tau_2}$ and expressed in terms of x_i^a . From these vectors we also construct two sets of vectors which are assigned to the edges of an unrooted labelled tree \mathcal{T} , such as the trees appearing in (3.26) and (4.6). Namely,

$$\mathbf{v}_e = \sum_{i \in V_{\mathcal{T}_e^s}} \sum_{j \in V_{\mathcal{T}_e^t}} \mathbf{v}_{ij}, \quad \mathbf{u}_e = \sum_{i \in V_{\mathcal{T}_e^s}} \sum_{j \in V_{\mathcal{T}_e^t}} \mathbf{u}_{ij}, \quad (4.11)$$

where $\mathcal{T}_e^s, \mathcal{T}_e^t$ are the two disconnected trees obtained from the tree \mathcal{T} by removing the edge e . Then the kernel Φ_n can be expressed as follows,

$$\Phi_n(\mathbf{x}) = \frac{\Phi_1(x)}{2^{n-1} n!} \sum_{\mathcal{T} \in \mathbb{T}_n^e} \tilde{\Phi}_{n-1}^M(\{\mathbf{u}_e\}, \{\mathbf{v}_{s(e)t(e)}\}; \mathbf{x}). \quad (4.12)$$

Here the first factor is simply a Gaussian

$$\Phi_1(x) = \frac{e^{-\frac{\pi(pxt)^2}{(pt^2)}}}{2\pi\sqrt{2\tau_2}(pt^2)} \quad (4.13)$$

which ensures the suppression along the direction of the total charge in the charge lattice. In the second factor one sums over unrooted labelled trees \mathcal{T} with n vertices, with summand

given by a function $\tilde{\Phi}_n^M$ defined as in (C.15), upon replacing Φ_n^E by Φ_n^M in that expression. Both Φ_n^E and Φ_n^M are the so-called generalized (complementary) error functions introduced in [24] and further studied in [25], whose definitions are recalled in (C.10) and (C.14). The functions $\tilde{\Phi}_{n-1}^M$ in (4.12) depend on two sets of $n-1$ d -dimensional vectors: the vectors from the first set are given by \mathbf{u}_e defined above, whereas $\mathbf{v}_{s(e)t(e)}$ coincide with \mathbf{v}_{ij} for i and j corresponding to the source and target vertices of edge e of the labelled tree.

The remarkable property of the generalized error functions $\tilde{\Phi}_{n-1}^M$ is that, away from certain loci where these functions are discontinuous, they satisfy Vignéras' equation for $\lambda = 0$ and $n-1$, respectively. Given that Φ_1 is also a solution for $\lambda = -1$, and the vector $\mathbf{t} = (t^a, \dots, t^a)$ (such that $\mathbf{t} \cdot \mathbf{x} = (pxt)$) is orthogonal to all vectors \mathbf{u}_e and $\mathbf{v}_{s(e)t(e)}$, the kernel (4.12) solves this equation for $\lambda = n-2$.

However, as mentioned above, it fails to do so at the discontinuity loci. These discontinuities arise due to dependence of the integration contours ℓ_γ on moduli and electric charges. Of course, since the integrands are meromorphic functions, the integrals do not depend on deformations of the contours provided they do not cross the poles. But this is exactly what happens when two BPS rays, say ℓ_γ and $\ell_{\gamma'}$, exchange their positions, which in turn takes place when the phases of the corresponding central charges Z_γ and $Z_{\gamma'}$ align, as follows from (3.3). The loci where such alignment takes place are nothing else but the walls of marginal stability. This point will play an important role in the next subsection since it makes possible to recombine the discontinuities of the generalized error functions with discontinuities of the tree indices.

Next, we evaluate the action of Vignéras' operator on Φ_n^g . To this end, it is convenient to use the representation of the tree index as a sum over attractor flow trees (2.3). Let us assign a nb_2 -dimensional vector $\tilde{\mathbf{v}}_v$ to each vertex v of a flow tree. Denoting by \mathcal{I}_v the set of leaves which are descendants of vertex v , we set

$$\tilde{\mathbf{v}}_v = \sum_{i \in \mathcal{I}_L(v)} \sum_{j \in \mathcal{I}_R(v)} \mathbf{v}_{ij}. \quad (4.14)$$

With these definitions the kernel (4.9) can be written as

$$\Phi_n^g(\mathbf{x}) = (-1)^{n-1} \sum_{T \in \mathbb{T}_n^{\text{af}}} \prod_{v \in V_T} (\tilde{\mathbf{v}}_v, \mathbf{x}) \Delta_{\gamma_{L(v)} \gamma_{R(v)}}^{z_{p(v)}}. \quad (4.15)$$

The factors $\Delta_{\gamma_L \gamma_R}^z$ are locally constant and therefore, if we stay away from the loci of their discontinuities, the action of Vignéras operator reduces to its action on the scalar products $(\tilde{\mathbf{v}}_v, \mathbf{x})$. For a single such factor one finds

$$V_\lambda(\tilde{\mathbf{v}}_v, \mathbf{x}) = (\tilde{\mathbf{v}}_v, \mathbf{x}) V_{\lambda-1} + 2\tilde{\mathbf{v}}_v \cdot \partial_{\mathbf{x}}. \quad (4.16)$$

The crucial observation is that all vectors $\tilde{\mathbf{v}}_v$ appearing in the product (4.15) for a single tree are mutually orthogonal $(\tilde{\mathbf{v}}_v, \tilde{\mathbf{v}}_{v'}) = 0$, which is clear because $\langle \gamma_{L(v)}, \gamma_{R(v)} \rangle$ is antisymmetric in charges $\gamma_{L(v)}, \gamma_{R(v)}$, whereas the factors associated with vertices which are not descendants of v either depend on their sum or do not depend on them at all. Therefore, one obtains

$$V_\lambda \Phi_{n_k}^g(\mathbf{x}) = \Phi_{n_k}^g(\mathbf{x}) V_{\lambda-n_k+1} + 2(-1)^{n_k-1} \sum_{T \in \mathbb{T}_{n_k}^{\text{af}}} \Delta(T) \sum_{v \in V_T} \left[\prod_{v' \in V_T \setminus \{v\}} (\tilde{\mathbf{v}}_{v'}, \mathbf{x}) \right] \tilde{\mathbf{v}}_v \cdot \partial_{\mathbf{x}}. \quad (4.17)$$

Let us now evaluate the action of V_{n-2} on the full kernel Φ_n^{tot} . Applying the result (4.17), we observe that the second term vanishes on the other factors in (4.3) due to the same reason that they either do not depend ($\Phi_{n_{k'}^g}$, $k' \neq k$) or depend (Φ_m) only on the sum of charges entering $\Phi_{n_k^g}$. Therefore, one finds that, *away from discontinuities of generalized error functions and $\Delta(T)$* , one has $V_{n-2} \cdot \Phi_n^{\text{tot}} = 0$.

4.3 Discontinuities and the anomaly

Let us now turn to the discontinuities of Φ_n^{tot} which we ignored so far and which spoil Vignéras' equation and hence modularity of the theta series. There are three potential sources of these discontinuities:

1. walls of marginal stability — at these loci Φ_m are discontinuous due to exchange of integration contours and $\Phi_{n_k^g}$ jump due to factors $\Delta_{\gamma_{L(v)}\gamma_{R(v)}}^{z_{p(v)}}$ assigned to the root vertices of attractor flow trees;
2. ‘fake walls’ — these are loci in the moduli space where $\text{Im} [Z_{\gamma_{L(v)}} \bar{Z}_{\gamma_{R(v)}} (z_{p(v)}^a)] = 0$, and hence the corresponding Δ -factor jumps, where v is not a root vertex — they have the meaning of walls of marginal stability for the intermediate bound states appearing in the attractor flow;
3. moduli independent loci where $(\tilde{\mathbf{v}}_v, \mathbf{x}) = 0$ — at these loci the factors $\Delta_{\gamma_{L(v)}\gamma_{R(v)}}^{z_{p(v)}}$ and hence $\Phi_{n_k^g}$ are discontinuous due to the second term in (2.4).

Remarkably, the two effects due to the non-trivial charge and moduli dependence of the DT invariants and the contours cancel each other and the function \mathcal{G} turns out to be smooth at loci of the first type. This is expected because the whole construction of D-instantons has been designed to make the resulting metric on the moduli space smooth across these loci, which required the cancelation of these two types of discontinuities [31, 30]. Besides, in [32] it was proven that the contact potential is also smooth, which suggests that the function \mathcal{G} must be smooth as well. In appendix B we present an explicit proof of this fact based on the representation in terms of trees.

Furthermore, in [20] it was shown that the discontinuities across ‘fake walls’ cancel in the sum over flow trees as well. In fact, this cancelation is explicit in the representation of the partial tree index given by the recursive formula (2.15) where the signs generating such ‘fake discontinuities’ do not arise at all. As a result, it remains to consider only the discontinuities of the third type corresponding to the moduli independent loci.

It is straightforward to check that already for $n = 2$ these discontinuities are present and do spoil modularity of the theta series. For small n one can explicitly evaluate the anomaly in Vignéras' equation. It is given by a series of terms proportional to $\delta(\tilde{\mathbf{v}}_v, \mathbf{x})$. Note that no derivatives of delta functions or products of two delta functions arise despite the presence of the second derivative in Vignéras' operator. This is because each $\text{sgn}(\tilde{\mathbf{v}}_v, \mathbf{x})$ from $\Delta(T)$ is multiplied by $(\tilde{\mathbf{v}}_v, \mathbf{x})$ from $\kappa(T)$ in (2.3) and one gets a non-vanishing result only if the second order derivative operator acts on both factors. In particular, this implies that the anomaly is completely characterized by the action of V_λ on Φ_n^g .

5. Modular completion

Since the theta series $\vartheta_{\mathbf{p},\mu}(\Phi_n^{\text{tot}})$ are not modular for $n \geq 2$, the analysis of the previous section implies that the generating function $h_{p,\mu}$ of the MSW invariants is not modular either as soon as the divisor $\mathcal{D} = p^a \gamma_a$ is reducible. However, its modular anomaly has a definite structure. In particular, in [21] it was shown that for $n = 2$, $h_{p,\mu}$ must be a vector-valued *mixed mock* modular form, i.e. it has a non-holomorphic completion $\widehat{h}_{p,\mu}$ constructed in a specific way from a set of holomorphic modular forms and their integrals of Eichler type [26, 13]. In this section we generalize this result for arbitrary n , i.e. for any degree of reducibility of the divisor.

5.1 Completion of the generating function

Let us recall the notations $\check{\gamma} = (p^a, q_a)$ and Q_n from (2.31), and decompose the electric component q_a as in (2.25). Then we define

$$\widehat{h}_{p,\mu}(\tau) = h_{p,\mu}(\tau) - \sum_{n=2}^{\infty} \sum_{\sum_{i=1}^n \check{\gamma}_i = \check{\gamma}} R_n(\{\check{\gamma}_i\}, \tau) e^{\pi i \tau Q_n(\{\check{\gamma}_i\})} \prod_{i=1}^n h_{p_i, \mu_i}(\tau). \quad (5.1)$$

We are looking for *non-holomorphic* functions R_n , exponentially suppressed for large charges, such that $\widehat{h}_{p,\mu}$ transforms as a modular form. The condition for this to be true can be found along the same lines as before: one needs to rewrite the expansion of the function \mathcal{G} as a series in $\widehat{h}_{p,\mu}$ and require that at each order the coefficient is given by a modular covariant theta series. For such a theta series decomposition to be possible however, it is important that $\widehat{h}_{p,\mu}$ be invariant under the spectral flow, which implies that the functions R_n be *independent* of the spectral flow parameter ϵ^a of the total charge $\check{\gamma}$. This condition will be an important consistency requirement on our construction.

Rather than inverting (5.1) and substituting the result into (4.2), we can consider the generating function of DT invariants $h_{p,\mu}^{\text{DT}}$ and, as a first step, rewrite it as a series in $\widehat{h}_{p,\mu}$. Let us denote the coefficient of such expansion by $\widehat{g}_n(\{\check{\gamma}_i\}, z^a)$ (with $\widehat{g}_1 \equiv 1$), i.e. we take (cf. (2.30))

$$h_{p,q}^{\text{DT}}(\tau, z^a) = \sum_{\sum_{i=1}^n \check{\gamma}_i = \check{\gamma}} \widehat{g}_n(\{\check{\gamma}_i\}, z^a) e^{\pi i \tau Q_n(\{\check{\gamma}_i\})} \prod_{i=1}^n \widehat{h}_{p_i, \mu_i}(\tau). \quad (5.2)$$

It is clear that \widehat{g}_n is a direct analogue of the tree index $g_{\text{tr},n}$, and to get the expansion of \mathcal{G} in terms of $\widehat{h}_{p,\mu}$, it suffices to replace $g_{\text{tr},n}$ by \widehat{g}_n in (4.9). Our first problem is to express these coefficients in terms of the functions R_n .

The result can be nicely formulated using the so-called Schröder trees, which are rooted ordered trees such that all vertices (except the leaves) having at least 2 children. Their vertices are decorated by charges in such way that the leaves have charges γ_i , whereas the charges at other vertices are defined by the inductive rule $\gamma_v = \sum_{v' \in \text{child}(v)} \gamma_{v'}$. Note that these trees should not be confused with flow trees, since they are not restricted to be binary nor do they carry moduli at the vertices. We denote the set of Schröder trees¹⁶ with n leaves by \mathbb{T}_n^{S} .

¹⁶The number of Schröder trees with n leaves is the $n - 1$ -th super-Catalan number, $\mathbb{T}_n^{\text{S}} = \{1, 1, 3, 11, 45, 197, 903, \dots\}$ for $n \geq 1$ (sequence A001003 on the Online Encyclopedia of Integer Sequences).

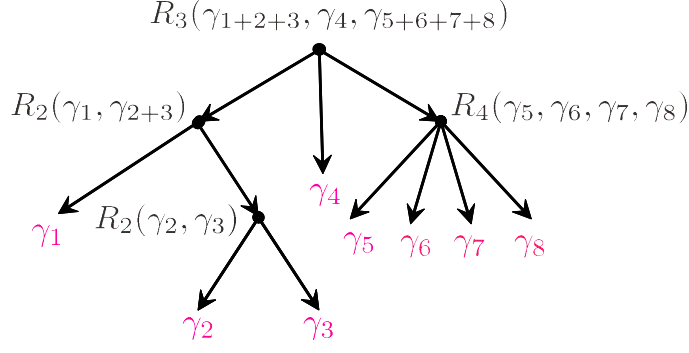


Figure 3: An example of Schröder tree contributing to W_8 . Near each vertex we showed the corresponding factor using the shorthand notation $\gamma_{i+j} = \gamma_i + \gamma_j$.

Let us also introduce a convenient notation: for any set of functions $f_n(\{\tilde{\gamma}_i\})$ depending on n charges, we set $f_v \equiv f_{n_v}(\{\tilde{\gamma}_{v'}\})$ where $v' \in \text{child}(v)$ and n_v is their number. Using these notations, the expression of \hat{g}_n in terms of R_n can be encoded into a recursive equation provided by the following proposition, whose proof we relegate to appendix E:

Proposition 1. *The coefficients \hat{g}_n are determined recursively by the following equation*

$$\hat{g}_n(\{\tilde{\gamma}_i\}, z^a) = -\frac{1}{2} \text{Sym} \left\{ \sum_{\ell=1}^{n-1} \Delta_{\gamma_L^\ell \gamma_R^\ell}^z \kappa(\gamma_{LR}^\ell) \hat{g}_\ell(\{\tilde{\gamma}_i\}_{i=1}^\ell, z_\ell^a) \hat{g}_{n-\ell}(\{\tilde{\gamma}_i\}_{i=\ell+1}^n, z_\ell^a) \right\} + W_n(\{\tilde{\gamma}_i\}), \quad (5.3)$$

where $\gamma_L^\ell = \sum_{i=1}^\ell \gamma_i$, $\gamma_R^\ell = \sum_{i=\ell+1}^n \gamma_i$, $\gamma_{LR}^\ell = \langle \gamma_L^\ell, \gamma_R^\ell \rangle$, z_ℓ^a is the value of the moduli where the attractor flow crosses the wall for the decay $\gamma \rightarrow \gamma_L^\ell + \gamma_R^\ell$ (cf. (2.13)), and W_n is a function given by the sum over Schröder trees

$$W_n(\{\tilde{\gamma}_i\}) = \text{Sym} \left\{ \sum_{T \in \mathbb{T}_n^S} \prod_{v \in V_T} R_v \right\}. \quad (5.4)$$

What are conditions on \hat{g}_n for the corresponding theta series to be modular? Let us denote by $\Phi_n^{\hat{g}}$ the kernel defined by \hat{g}_n analogously to (4.9). Then the above analysis implies that the modularity requires from $\Phi_n^{\hat{g}}$ to satisfy Vignéras equation away from walls of marginal stability, whereas at these walls its discontinuities should follow the same pattern as before to cancel the discontinuities from the contour exchange in Φ_m . Thus, the completion should smoothen out the discontinuities from the moduli independent signs $\text{sgn}(\tilde{\mathbf{v}}_v, \mathbf{x})$, but otherwise leave the action of Vignéras' operator intact. Formally, this means that $\Phi_n^{\hat{g}}$ must satisfy the following equation

$$V_{n-1} \cdot \Phi_n^{\hat{g}} = \text{Sym} \sum_{\ell=1}^{n-1} \left[\tilde{\mathbf{u}}_\ell^2 \Delta_{n,\ell}^{\hat{g}} \delta'(\tilde{\mathbf{u}}_\ell \cdot \mathbf{x}) + 2\tilde{\mathbf{u}}_\ell \cdot \partial_{\mathbf{x}} \Delta_{n,\ell}^{\hat{g}} \delta(\tilde{\mathbf{u}}_\ell \cdot \mathbf{x}) \right], \quad \Delta_{n,\ell}^{\hat{g}} = \frac{1}{2} (\tilde{\mathbf{v}}_\ell, \mathbf{x}) \Phi_\ell^{\hat{g}} \Phi_{n-\ell}^{\hat{g}}, \quad (5.5)$$

where we introduced two vectors constructed from the vectors (4.10),

$$\tilde{\mathbf{v}}_\ell = \sum_{i=1}^\ell \sum_{j=\ell+1}^n \mathbf{v}_{ij}, \quad \tilde{\mathbf{u}}_\ell = \sum_{i=1}^\ell \sum_{j=\ell+1}^n \mathbf{u}_{ij}. \quad (5.6)$$

Note that $(\tilde{\mathbf{u}}_\ell, \mathbf{x})$ corresponds to the quantity S_ℓ (2.12).

To solve the above constraints, let us consider the following iterative ansatz

$$\widehat{g}_n(\{\check{\gamma}_i\}, z^a) = g_n^{(0)}(\{\check{\gamma}_i, c_i\}) - \text{Sym} \left\{ \sum_{\substack{n_1+\dots+n_m=n \\ n_k \geq 1, m < n}} \widehat{g}_m(\{\check{\gamma}'_k\}, z^a) \prod_{k=1}^m \mathcal{E}_{n_k}(\check{\gamma}_{j_{k-1}+1}, \dots, \check{\gamma}_{j_k}) \right\}, \quad (5.7)$$

where the notations for indices and primed charges are the same as in (2.16). This ansatz is motivated by analogy with the iterative equation for the (partial) tree index (2.15). It involves two yet unknown sets of functions, $g_n^{(0)}$ and \mathcal{E}_n . The former carry the moduli dependence through the variables c_i (2.12), whereas the latter are moduli independent. We set $g_1^{(0)} = \mathcal{E}_1 = 1$ and also assume that $g_n^{(0)}$ have discontinuities only at walls of marginal stability, i.e. at $\sum_{i \in \mathcal{I}} c_i = 0$ for various subsets \mathcal{I} of indices.

The unknown functions together with the functions R_n defining the completion, or their combinations (5.4), are supposed to be fixed by the conditions (5.3) and (5.5). In appendix E we prove the following result:

Proposition 2. *Let us split $\mathcal{E}_n = \mathcal{E}_n^{(0)} + \mathcal{E}_n^{(+)}$ into $\mathcal{E}_n^{(+)}$ which is exponentially decreasing for large charges part and the non-decreasing remainder $\mathcal{E}_n^{(0)}$. Then the ansatz (5.7) satisfies the recursive equation (5.5) provided*

1. the functions $g_n^{(0)}$ are subject to a similar recursive relation

$$\begin{aligned} \frac{1}{4} \text{Sym} \left\{ \sum_{\ell=1}^{n-1} (\text{sgn}(S_\ell) - \text{sgn}(\Gamma_{n\ell})) \kappa(\Gamma_{n\ell}) g_\ell^{(0)}(\{\check{\gamma}_i, c_i^{(\ell)}\}_{i=1}^\ell) g_{n-\ell}^{(0)}(\{\check{\gamma}_i, c_i^{(\ell)}\}_{i=\ell+1}^n) \right\} \\ = g_n^{(0)}(\{\check{\gamma}_i, c_i\}) - g_n^{(0)}(\{\check{\gamma}_i, \beta_{ni}\}), \end{aligned} \quad (5.8)$$

where $S_k, \beta_{k\ell}, \Gamma_{k\ell}$ and $c_i^{(\ell)}$ were defined in (2.12), (2.14);

2. the non-decreasing part of \mathcal{E}_n is fixed in terms of $g_n^{(0)}(\{\check{\gamma}_i, c_i\})$ as

$$\mathcal{E}_n^{(0)}(\{\check{\gamma}_i\}) = g_n^{(0)}(\{\check{\gamma}_i, \beta_{ni}\}); \quad (5.9)$$

3. its decreasing part is given by

$$\mathcal{E}_n^{(+)}(\{\check{\gamma}_i\}) = -\text{Sym} \left\{ \sum_{\substack{n_1+\dots+n_m=n \\ n_k \geq 1, m > 1}} W_m(\{\check{\gamma}'_k\}) \prod_{k=1}^m \mathcal{E}_{n_k}(\check{\gamma}_{j_{k-1}+1}, \dots, \check{\gamma}_{j_k}) \right\}. \quad (5.10)$$

Furthermore, if the functions \mathcal{E}_n depend on electric charges $q_{i,a}$ only through the DSZ products γ_{ij} and their kernels $\Phi_n^\mathcal{E}(\mathbf{x})$ defined as in (4.9) are smooth solutions of Vignéras' equation,

$$V_{n-1} \cdot \Phi_n^\mathcal{E} = 0, \quad (5.11)$$

then the ansatz (5.7) also satisfies the modularity constraint (5.5).

This proposition allows in principle to fix all unknown functions. Indeed, the recursive relation (5.8) determines all $g_n^{(0)}$. Then equations (5.9) and (5.10) give the two parts of $\mathcal{E}_n = \mathcal{E}_n^{(0)} + \mathcal{E}_n^{(+)}$ in terms of $g_n^{(0)}$ and W_n . At this point the latter are still undetermined and are defined in terms of the unknown functions R_n . But there is an additional constraint that \mathcal{E}_n should solve Vignéras' equation. This constraint has a unique solution for $\mathcal{E}_n^{(+)}$ for a given $\mathcal{E}_n^{(0)}$ and thus establishes a relation between W_n and $g_n^{(0)}$. Finally, inverting (5.4) generates a solution for R_n .

5.2 Generalized error functions and the completion

We see that the first step to be done is to provide an explicit expression for $g_n^{(0)}$. Once such an expression is known, all other functions can be determined algebraically.

A solution for $g_n^{(0)}$ can be guessed on the basis of (4.12). The point is that these two sets of functions both have discontinuities at walls of marginal stability which, as we know, must cancel each other. Furthermore, they should recombine into a smooth solution of Vignéras' equation. Thus, Φ_n is expected to encode at least a part of the completion of $g_n^{(0)}$. On the other hand, as explained in appendix C.2, the function $\tilde{\Phi}_{n-1}^M$, from which Φ_n is constructed, appears as one of the terms (the one with the fastest decay) in the expansion of the function $\tilde{\Phi}_{n-1}^E$ (C.15) with the arguments $\mathcal{V} = \{\mathbf{u}_e\}$, $\tilde{\mathcal{V}} = \{\mathbf{v}_{s(e)t(e)}\}$. The latter is a smooth solution of Vignéras' equation having the asymptotics (C.16). This means that $\tilde{\Phi}_{n-1}^M$ is a part of the completion for the kernel given by this asymptotics. This naturally leads us to the following ansatz which is proven in appendix E:

Proposition 3. *The function $g_n^{(0)}(\{\tilde{\gamma}_i, c_i\})$ is given by the sum over unrooted labelled trees with n leaves,*

$$g_n^{(0)}(\{\tilde{\gamma}_i, c_i\}) = \frac{(-1)^{n-1+\sum_{i<j} \gamma_{ij}}}{2^{n-1}n!} \sum_{\mathcal{T} \in \mathbb{T}_n^\ell} \prod_{e \in E_{\mathcal{T}}} \gamma_{s(e)t(e)} \operatorname{sgn}(S_e), \quad (5.12)$$

where¹⁷

$$S_e = \sum_{i \in V_{\mathcal{T}_e^s}} c_i = \sum_{i \in V_{\mathcal{T}_e^s}} \sum_{j \in V_{\mathcal{T}_e^t}} \operatorname{Im} [Z_{\gamma_i} \bar{Z}_{\gamma_j}]. \quad (5.13)$$

Given the result (5.12) and the relation (5.9), one obtains an explicit expression for the non-decreasing part of \mathcal{E}_n ,

$$\mathcal{E}_n^{(0)}(\{\tilde{\gamma}_i\}) = \frac{(-1)^{\sum_{i<j} \gamma_{ij}}}{2^{n-1}n!} \sum_{\mathcal{T} \in \mathbb{T}_n^\ell} \prod_{e \in E_{\mathcal{T}}} \gamma_{s(e)t(e)} \operatorname{sgn}(\Gamma_e), \quad \Gamma_e = \sum_{i \in V_{\mathcal{T}_e^s}} \sum_{j \in V_{\mathcal{T}_e^t}} \gamma_{ij}. \quad (5.14)$$

From Proposition 2 we know that the kernel $\Phi_n^\mathcal{E}$ corresponding to the full function \mathcal{E}_n must solve Vignéras' equation. Thus, we come to the problem of finding a smooth function $\Phi_n^\mathcal{E}$ annihilated by V_{n-1} , which asymptotes to a locally polynomial function at infinity,

$$\lim_{\mathbf{x} \rightarrow \infty} \Phi_n^\mathcal{E}(\mathbf{x}) = \frac{1}{2^{n-1}n!} \sum_{\mathcal{T} \in \mathbb{T}_n^\ell} \prod_{e \in E_{\mathcal{T}}} (\mathbf{v}_{s(e)t(e)}, \mathbf{x}) \operatorname{sgn}(\mathbf{v}_e, \mathbf{x}), \quad (5.15)$$

¹⁷Note that the sign factor $(-1)^{\sum_{i<j} \gamma_{ij}}$ is equal to the ratio of quadratic refinements appearing in (4.9).

where the vectors $\mathbf{v}_{ij}, \mathbf{v}_e$ are defined in (4.10) and (4.11). From the above discussion, it is clear that such function can be obtained from (4.12) by replacing $\tilde{\Phi}_{n-1}^M$ by $\tilde{\Phi}_{n-1}^E$ and \mathbf{u}_e by \mathbf{v}_e . Thus, we arrive at the following result

$$\Phi_n^{\mathcal{E}}(\mathbf{x}) = \frac{1}{2^{n-1}n!} \sum_{T \in \mathbb{T}_n^{\ell}} \tilde{\Phi}_{n-1}^E(\{\mathbf{v}_e\}, \{\mathbf{v}_{s(e)t(e)}\}; \mathbf{x}). \quad (5.16)$$

In words, \mathcal{E}_n is a sum over unrooted labelled trees of solutions $\tilde{\Phi}_{n-1}^E$ of Vignéras equation with $\lambda = n - 1$, obtained from standard generalized error functions Φ_{n-1}^E by acting with $n - 1$ raising operators.

After substituting into the iterative ansatz (5.7), the two results (5.12) and (5.16) completely specify the coefficients of the expansion (5.2) of the generating function of DT invariants in terms of $\hat{h}_{p,\mu}$. The result of the iteration can in fact be written explicitly as a sum over Schröder trees. Adopting the same notations as in Proposition 1, one has

Proposition 4. *The function $\hat{g}_n(\{\tilde{\gamma}_i, c_i\})$ is given by the sum over Schröder trees with n leaves,*

$$\hat{g}_n = \text{Sym} \left\{ \sum_{T \in \mathbb{T}_n^{\mathfrak{S}}} (-1)^{n_T-1} (g_{v_0}^{(0)} - \mathcal{E}_{v_0}) \prod_{v \in V_T \setminus \{v_0\}} \mathcal{E}_v \right\}, \quad (5.17)$$

where v_0 is the root of the tree and n_T is the number of its vertices (excluding the leaves).

The properties of $g_n^{(0)}$ and \mathcal{E}_n ensure that the theta series appearing in the corresponding decomposition of \mathcal{G} is modular so that the modularity of \mathcal{G} requires the function $\hat{h}_{p,\mu}$ to be a vector valued (non-holomorphic) modular form of weight $(-b_2/2 - 1, 0)$. The functions R_n entering in the non-holomorphic completion $\hat{h}_{p,\mu}$, are then given by the following proposition, whose proof can again be found in appendix E:

Proposition 5. *Inverting the relations (5.4) and (5.10), one obtains*

$$R_n = \text{Sym} \left\{ \sum_{T \in \mathbb{T}_n^{\mathfrak{S}}} (-1)^{n_T} \mathcal{E}_{v_0}^{(+)} \prod_{v \in V_T \setminus \{v_0\}} \mathcal{E}_v^{(0)} \right\}. \quad (5.18)$$

It is important to check that the resulting functions R_n are invariant under the spectral flow of the total charge. This is in fact a simple consequence of the fact that the functions \mathcal{E}_n entering their definition depend on the electric charges only through the DSZ products γ_{ij} . As follows from (2.23), all such products are invariant under the spectral flow of the total charge, hence R_n are invariant as well.

In appendix F we present explicit expressions for various elements of our construction up to order $n = 4$. Based on these results, we conjecture a general formula for the kernel $\hat{\Phi}_n^{\text{tot}}$ of the theta series appearing in the expansion of \mathcal{G} in terms of $\hat{h}_{p,\mu}$:

Conjecture 1. *The generating function of multi-instantons is given by*

$$\mathcal{G} = \sum_{n=1}^{\infty} \frac{2^{-\frac{n}{2}}}{\pi \sqrt{2\tau_2}} \left[\prod_{i=1}^n \sum_{p_i, \mu_i} \sigma_{p_i} \hat{h}_{p_i, \mu_i} \right] e^{-S_p^{\text{cl}}} \vartheta_{\mathbf{p}, \boldsymbol{\mu}}(\hat{\Phi}_n^{\text{tot}}, n-2), \quad (5.19)$$

where the kernels of the theta series are given by

$$\widehat{\Phi}_n^{\text{tot}} = \Phi_1 \text{Sym} \left\{ \sum_{T \in \mathbb{T}_n^S} (-1)^{n_T-1} \left(\widehat{\Phi}_{v_0}^\mathcal{E} - \Phi_{v_0}^\mathcal{E} \right) \prod_{v \in V_T \setminus \{v_0\}} \Phi_v^\mathcal{E} \right\}. \quad (5.20)$$

Here $\Phi_1(x)$ is the Gaussian factor (4.13), and $\widehat{\Phi}_n^\mathcal{E}$ is obtained from $\Phi_n^\mathcal{E}$ by replacing the vectors \mathbf{v}_e in the arguments of generalized error functions by the vectors \mathbf{u}_e , i.e.

$$\widehat{\Phi}_n^\mathcal{E}(\mathbf{x}) = \frac{1}{2^{n-1}n!} \sum_{\mathcal{T} \in \mathbb{T}_n^\mathcal{E}} \widetilde{\Phi}_{n-1}^E(\{\mathbf{u}_e\}, \{\mathbf{v}_{s(e)t(e)}\}; \mathbf{x}). \quad (5.21)$$

Note that this result automatically implies that the theta series $\vartheta_{\mathbf{p},\boldsymbol{\mu}}(\widehat{\Phi}_n^{\text{tot}}, n-2)$ is modular, since its kernel is a solution of Vignéras' equation.

6. Discussion

In this paper we studied the modular properties of the generating function $h_{\mathbf{p},\boldsymbol{\mu}}(\tau)$ of MSW invariants encoding BPS degeneracies of D4-D2-D0 black holes in Type IIA string theory on a Calabi-Yau threefold, with fixed D4-charge p^a , D2-brane charge (up to spectral flow) μ_a and invariant D0-brane charge \hat{q}_0 (defined in (2.26)) conjugate to the modular parameter τ . Our main result is an explicit formula for the non-holomorphic modular completion $\widehat{h}_{\mathbf{p},\boldsymbol{\mu}}$ (5.1), where the functions $R_n(\{\gamma_i\}, \tau)$ are given by Eq. (5.18), with $\mathcal{E}_v = \mathcal{E}_v^{(0)} + \mathcal{E}_v^{(+)}$ specified by (5.14) and (5.16). This result applies for D4-branes wrapping a general effective divisor which may be the sum of an arbitrary number n of irreducible divisors, and it follows from demanding that the vector multiplet moduli space in $D = 3$ (or the hypermultiplet moduli space in the dual type IIB picture) admits an isometric action of $SL(2, \mathbb{Z})$.

The existence of such completion is far from obvious. We arrived at this result by canceling the modular anomaly of an indefinite theta series of signature $(n, nb_2(\mathfrak{Q})) - n$ with a complicated kernel (4.3). In particular, the Vignéras equation encoding this anomaly involves Kähler moduli in a non-trivial way through the walls of marginal stability, whereas the functions R_n in our ansatz (5.1) can only depend (non-holomorphically) on the modular parameter τ and on the charges γ_i . We regard this cancellation as one of many string theory ‘miracles’.

On the other hand, the arguments in the present work were somewhat formal, since we did not control the convergence of the various generating functions and theta series. Technically, we constructed a theta series $\vartheta_{\mathbf{p},\boldsymbol{\mu}}(\widehat{\Phi}_n^{\text{tot}}, n-2)$ whose kernel satisfies Vignéras' equation (C.3), but we did not demonstrate that it decays as required by Vignéras' criterium. (For $n = 2$ this is easy to show [21] since the kernel is a product of an exponentially suppressed factor and a difference of two error functions defined by two positive vectors.) In fact, convergence issues already arise when expressing the generating function of DT invariants in terms of MSW invariants in (2.30), and are related to the problem of convergence of the BPS black hole partition function. The latter was proven to converge in the large volume limit for the case involving up to $n \leq 3$ centers in [56]. We believe that the results for the tree index obtained in [20] will allow to extend this result to arbitrary n .

The structure of the completion suggests that, for a divisor decomposable into a sum of n effective divisors, the holomorphic generating function $h_{p,\mu}$ is a mixed vector valued mock modular form of higher depth, equal to $n - 1$. Such objects have recently appeared in the mathematical literature [57, 58, 59, 60] and correspond to holomorphic functions whose completion is constructed from period (or Eichler) integrals of mock modular forms but smaller depth (with depth 0 mock modular forms being synonymous with ordinary modular forms). For instance, in the case of standard mock modular forms of depth one, the completion is given by an Eichler integral of a modular function [23]. In our case this structure follows from the fact that the completion $\widehat{h}_{p,\mu}$ is built from the generalized error functions which have a representation in terms of iterated period integrals [24, 57]. Note however that in order to make manifest the mixed mock modular nature of $h_{p,\mu}$, one should represent its completion as a sum of products of holomorphic modular forms times iterated period integral of anti-holomorphic modular forms. This is a non-trivial task which involves the technique of lattice factorization for indefinite theta series, which we leave for future research.

An interesting side remark is that the modular completion depends on a choice of function $g_n^{(0)}$ which satisfies the recursive relation (5.8). The solution to this equation is not unique, since at each step one may add a moduli-independent contribution to $g_n^{(0)}$ without affecting the r.h.s. of (5.8). We suspect that this ambiguity amounts to shifting $\widehat{h}_{p_i,\mu}$ by a mock modular form of lower depth, but it would be interesting to flesh this out.

As for the physics of BPS black holes, an outstanding question is to understand the physical significance of the instanton generating function \mathcal{G} defined in (3.22). Its modular properties are exactly those expected from the elliptic genus of a superconformal field theory, except for the fact that it is not holomorphic in τ . It is natural to expect that this non-holomorphy can be traced to the existence of a continuum of states with a non-trivial spectral asymmetry between bosons and fermions. It would be very interesting to understand the origin of this continuum in terms of the worldvolume theory of an M5-brane wrapped on a reducible divisor. Moreover, \mathcal{G} is closely related to the index considered in [32], for special $\mathcal{N} = 2$ theories in $D = 4$ obtained by circle compactification of an $\mathcal{N} = 1$ theory in $D = 5$. It would be interesting to understand the physical origin of its modular invariance in this context.

As a technical byproduct of this work, it is worth pointing out a new representation of the tree index. Indeed, the index \widehat{g}_n plays the same role in (5.2) as the tree index $g_{\text{tr},n}$ in (2.30), upon replacing the holomorphic partition function $h_{p,\mu}$ by its completion $\widehat{h}_{p,\mu}$. Therefore it must reduce to $g_{\text{tr},n}$ after dropping all terms exponentially suppressed for large charges. To this end, it is enough to replace \mathcal{E}_v by its asymptotic expression $\mathcal{E}_v^{(0)}$ in (5.17). Thus, one arrives at the following representation

$$g_{\text{tr},n} = \text{Sym} \left\{ \sum_{T \in \mathbb{T}_n^S} (-1)^{n_T-1} (g_{v_0}^{(0)} - \mathcal{E}_{v_0}^{(0)}) \prod_{v \in V_T \setminus \{v_0\}} \mathcal{E}_v^{(0)} \right\}, \quad (6.1)$$

where $g_v^{(0)}$ and $\mathcal{E}_v^{(0)}$ are constructed from (5.12) and (5.14), respectively. Note that this representation is more explicit than the ones given in section 2.2 since it does not require neither iterations, nor taking the limit $y \rightarrow 1$.

Finally, we expect that the structure found in the context of generalized DT-invariants on Calabi-Yau threefolds will also arise in the study of other types of BPS invariants where higher depth mock modular forms are expected to occur, such as Vafa-Witten invariants and Donaldson invariants of four-folds with $b_2^+ = 1$ [24, 58]. In particular, it would be interesting to determine the modular completion of the generating function of Vafa-Witten invariants on the complex projective plane computed for all ranks in [61], and compare with the completion found in the present paper.

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A. D3-instanton contribution to the contact potential

To evaluate $(e^\phi)_{\text{D3}}$, we first replace in (3.20) the full prepotential F by its classical, cubic part F^{cl} (2.21) and take into account that the sum over $\gamma \in -\Gamma_+$ is complex conjugate to the sum over $\gamma \in \Gamma_+$. This gives

$$e^\phi \approx \frac{\tau_2^2}{12} ((\text{Im } u)^3) + \frac{i\tau_2}{8} \sum_{\gamma \in \Gamma_+} \text{Re} \int_{\ell_\gamma} \frac{dt}{t} (t^{-1} Z_\gamma(u^a) - t \bar{Z}_\gamma(\bar{u}^a)) H_\gamma. \quad (\text{A.1})$$

Next, we substitute the quantum corrected mirror map (3.8), change the integration variable t to z , and take the combined limit $t^a \rightarrow \infty$, $z \rightarrow 0$. Keeping only the leading contributions, one obtains

$$\begin{aligned} (e^\phi)_{\text{D3}} &= -\frac{\tau_2}{2} \sum_{\gamma \in \Gamma_+} \text{Re} \int_{\ell_\gamma} dz \left[\hat{q}_0 + \frac{1}{2} (q + b)^2 + 2(pt^2)zz_\gamma - \frac{3}{2} z^2(pt^2) \right] H_\gamma \\ &\quad - \frac{1}{4} \sum_{\gamma_1, \gamma_2 \in \Gamma_+} (tp_1 p_2) \left(\text{Re} \int_{\ell_{\gamma_1}} dz_1 H_{\gamma_1} \right) \left(\text{Re} \int_{\ell_{\gamma_2}} dz_2 H_{\gamma_2} \right). \end{aligned} \quad (\text{A.2})$$

To further simplify this expression, we note that

$$\begin{aligned} 0 &= \frac{1}{4\pi} \sum_{\gamma \in \Gamma_+} \int_{\ell_\gamma} dz \partial_z (z H_\gamma) \\ &= \sum_{\gamma \in \Gamma_+} \int_{\ell_\gamma} dz \left[\frac{1}{4\pi} + \tau_2(pt^2)(zz_\gamma - z^2) - \frac{i}{4} \sum_{\gamma' \in \Gamma_+} \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{dz'}{z - z'} H_{\gamma'} \right] H_\gamma, \end{aligned} \quad (\text{A.3})$$

where we used the integral equation (3.15). Multiplying this identity by 3/4 and adding its real part to (A.2), one finds

$$\begin{aligned}
(e^\phi)_{\text{D3}} &= \frac{\tau_2}{2} \sum_{\gamma \in \Gamma_+} \text{Re} \int_{\ell_\gamma} dz a_{\gamma, -\frac{3}{2}}(z) H_\gamma \\
&\quad - \frac{1}{4} \sum_{\gamma_1, \gamma_2 \in \Gamma_+} \left[(tp_1 p_2) \left(\text{Re} \int_{\ell_{\gamma_1}} dz_1 H_{\gamma_1} \right) \left(\text{Re} \int_{\ell_{\gamma_2}} dz_2 H_{\gamma_2} \right) \right. \\
&\quad \left. + \frac{3}{4} \text{Re} \left(\int_{\ell_{\gamma_1}} dz_1 H_{\gamma_1} \int_{\ell_{\gamma_2}} dz_2 H_{\gamma_2} \frac{i\langle \gamma, \gamma' \rangle}{z - z'} \right) \right], \tag{A.4}
\end{aligned}$$

where we introduced

$$a_{\gamma, \mathfrak{h}}(z) = - \left(\hat{q}_0 + \frac{1}{2} (q + b)^2 + \frac{1}{2} (pt^2) z z_\gamma + \frac{\mathfrak{h}}{4\pi\tau_2} \right). \tag{A.5}$$

The meaning of this function is actually very simple: it gives the action of the modular covariant derivative operator $\mathcal{D}_\mathfrak{h}$ (3.23) on the classical part of the Darboux coordinate (3.12),

$$\mathcal{D}_\mathfrak{h} \mathcal{X}_\gamma^{\text{cl}} = a_{\gamma, \mathfrak{h}}(z) \mathcal{X}_\gamma^{\text{cl}}. \tag{A.6}$$

Combining this fact with equation (3.15), it is easy to check that the function (3.22) satisfies

$$\begin{aligned}
\mathcal{D}_\mathfrak{h} \mathcal{G} &= \sum_{\gamma \in \Gamma_+} \int_{\ell_\gamma} dz a_{\gamma, \mathfrak{h}}(z) H_\gamma \\
&\quad + \frac{1}{8\tau_2} \sum_{\gamma_1, \gamma_2 \in \Gamma_+} \int_{\ell_{\gamma_1}} dz_1 \int_{\ell_{\gamma_2}} dz_2 \left[(tp_1 p_2) + 2\mathfrak{h} \left((tp_1 p_2) + \frac{i\langle \gamma_1, \gamma_2 \rangle}{z_1 - z_2} \right) \right] H_{\gamma_1}(z_1) H_{\gamma_2}(z_2), \tag{A.7}
\end{aligned}$$

$$\partial_{\tilde{z}_a} \mathcal{G} = 2\pi i \sum_{\gamma \in \Gamma_+} p^a \int_{\ell_\gamma} dz H_\gamma, \tag{A.8}$$

which allows to rewrite (A.4) exactly as in (3.24).

B. Smoothness of the instanton generating function

In this appendix we prove the smoothness of the function \mathcal{G} across walls of marginal stability. The starting point is the representation (4.2) where the potential discontinuities are hidden in the kernel of the theta series (4.3). Given that Φ_m can be represented as a sum over unrooted labelled trees (see (4.6)), whereas $\Phi_{n_k}^g$ appear as sums over attractor flow trees (see (4.15)), we can view the kernel Φ_n^{tot} as a sum over ‘blooming trees’ which are unrooted trees with a flow tree (the ‘flower’) growing from each vertex. Then the idea is that the discontinuities due to flow trees (i.e. due to DT invariants) of a blooming tree with m vertices are cancelled by the discontinuities due to exchange of integration contours in Φ_{m+1} corresponding to blooming trees with $m + 1$ vertices.

Following this idea, let us consider a tree which has an attractor flow tree $T_\mathfrak{v}$ growing from vertex \mathfrak{v} (see Fig. 4). We denote by \mathcal{T}_i , $i = 1, \dots, n_\mathfrak{v}$, the blooming subtrees connected

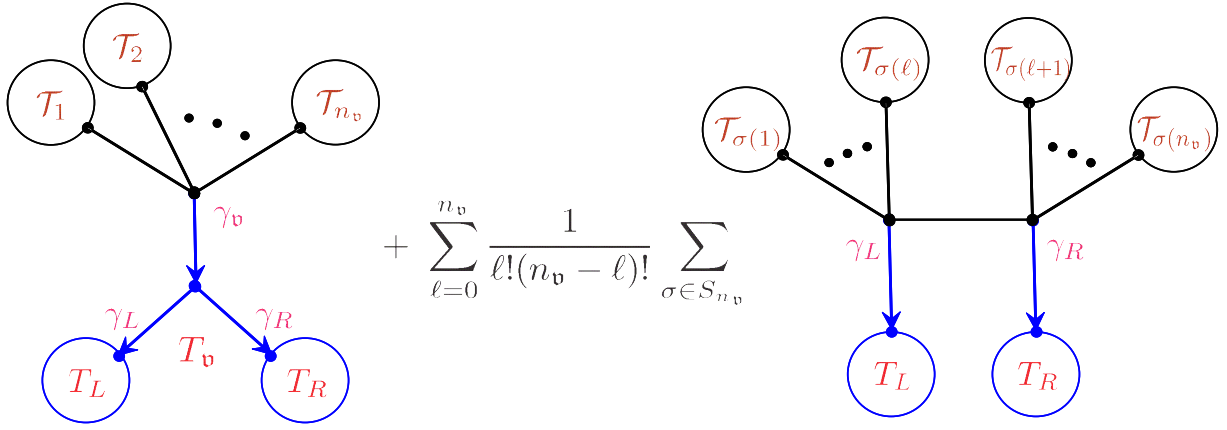


Figure 4: Combination of trees showing cancellation of discontinuities across the wall of marginal stability corresponding to the decay $\gamma_{\mathbf{v}} \rightarrow \gamma_L + \gamma_R$. The parts corresponding to attractor flow trees are drawn in blue.

to this vertex and by T_L and T_R the two parts (which may be trivial) of $T_{\mathbf{v}}$ with the total charges γ_L and γ_R so that $\gamma_{\mathbf{v}} = \gamma_L + \gamma_R$. Together with the contribution of this tree, we consider the contributions of the trees obtained by splitting the vertex \mathbf{v} into two vertices \mathbf{v}_L and \mathbf{v}_R connected by an edge, carrying charges γ_L and γ_R and all possible allocations of the subtrees \mathcal{T}_i to these two vertices. Different allocations are accounted for by the sum over permutations, where the weight $\frac{1}{\ell!(n_{\mathbf{v}}-\ell)!}$ takes into account the fact that permutations between subtrees connected to one vertex are redundant. The attractor flow trees T_L and T_R are then connected to \mathbf{v}_L and \mathbf{v}_R , respectively, as shown in the picture.

The contribution corresponding to the first blooming tree has a discontinuity at the wall of marginal stability for the bound state $\gamma_L + \gamma_R$ and originating from the factor $\Delta_{\gamma_L \gamma_R}^z$ assigned to the root vertex of $T_{\mathbf{v}}$. The other contributions have discontinuities at the same wall due to the exchange of the contours ℓ_{γ_L} and ℓ_{γ_R} for the integrals assigned to \mathbf{v}_L and \mathbf{v}_R , respectively. They are given by the residues at the pole of the integration kernel $K_{\gamma_L \gamma_R}$. It is clear that the structure of all jumps is very similar since different subtrees produce essentially the same weights. Let us analyze which differences they may have.

- First, the contributions of the flow trees T_L and T_R could differ in the two cases because they have different starting points for the attractor flows: for the trees on the right side of Fig. 4 this is $z^a \in \mathcal{M}_K$, whereas for the tree on the left this is the point on the wall for $\gamma_L + \gamma_R$ reached by the flow from z^a . But we are evaluating the discontinuity exactly on the wall where the two points coincide. Thus, the contributions are the same.
- Although the subtrees \mathcal{T}_i give rise to the same contributions for all blooming trees shown in Fig. 4, the contributions of the edges connecting them to either \mathbf{v} or \mathbf{v}_L , \mathbf{v}_R are not exactly the same. Each of them contributes the factor given by the kernel (3.16). After taking the residue, the z -dependence of these kernels for the trees shown on the left and the right sides of the picture coincide. However, their charge dependence is different: for the tree on the left they depend on $\gamma_{\mathbf{v}}$, whereas for the trees on the right they depend on γ_L or γ_R , depending on which vertex they are connected to. But it is easy to see that the sums over ℓ and permutations produce the standard binomial expansion of a

single product of kernels which all depend on $\gamma_L + \gamma_R = \gamma_{\mathbf{v}}$ and thus coinciding with the contribution of the tree on the left.

- Finally, one should take into account that the discontinuity of $\Delta_{\gamma_L \gamma_R}^z$ in the first contribution gives the factor $\langle \gamma_L, \gamma_R \rangle$. But exactly the same factor arises as the residue of $K_{\gamma_L \gamma_R}$ corresponding to the additional edge.

Thus, it remains only to check that all numerical factors work out correctly. Except the factors which are common to both contributions, we have

$$-\frac{(-1)^{\langle \gamma_L, \gamma_R \rangle}}{2} \frac{\sigma_{\gamma_{\mathbf{v}}}}{(2\pi)^2} \frac{2m}{m!} + (2\pi i)(-2\pi i) \frac{\sigma_{\gamma_L} \sigma_{\gamma_R}}{(2\pi)^4} \frac{m(m+1)}{(m+1)!}. \quad (\text{B.1})$$

Here $-\frac{(-1)^{\langle \gamma_L, \gamma_R \rangle}}{2}$ comes from the factor $-\Delta_{\gamma_L \gamma_R}^z$ in $T_{\mathbf{v}}$, the factors with quadratic refinement are due to functions H_{γ} (3.9) assigned to \mathbf{v} or $\mathbf{v}_L, \mathbf{v}_R$, $(2\pi i)$ is the standard weight of the residue, $(-2\pi i)$ is the residue of $K_{\gamma_L \gamma_R}$ (3.16), and factorials are the weights of the trees in the expansion (3.26). Finally, the factors $2m$ and $m(m+1)$ arise due to the freedom to relabel charges assigned to the marked vertices: on the left these are vertex \mathbf{v} and the two children of the root in $T_{\mathbf{v}}$, whereas on the right these are \mathbf{v}_L and \mathbf{v}_R . It is immediate to check that all these numerical weights cancel, which ensures that the function \mathcal{G} is continuous across walls of marginal stability. Moreover, in this cancelation the condition that we sit on the wall was used only in locally constant factors. Therefore, it proves not only that \mathcal{G} is continuous, but that it is actually smooth around these loci.

C. Indefinite theta series and generalized error functions

C.1 Vignéras' theorem

Let Λ be a d -dimensional lattice equipped with a bilinear form $(\mathbf{x}, \mathbf{y}) \equiv \mathbf{x} \cdot \mathbf{y}$, where $\mathbf{x}, \mathbf{y} \in \Lambda \otimes \mathbb{R}$, such that its associated quadratic form has signature $(n, d-n)$ and is integer valued, i.e. $\mathbf{k}^2 \equiv \mathbf{k} \cdot \mathbf{k} \in \mathbb{Z}$ for $\mathbf{k} \in \Lambda$. Furthermore, let $\mathbf{p} \in \Lambda$ be a characteristic vector (such that $\mathbf{k} \cdot (\mathbf{k} + \mathbf{p}) \in 2\mathbb{Z}, \forall \mathbf{k} \in \Lambda$), $\boldsymbol{\mu} \in \Lambda^*/\Lambda$ a glue vector, and λ an arbitrary integer. We consider the following family of theta series

$$\vartheta_{\mathbf{p}, \boldsymbol{\mu}}(\Phi, \lambda; \tau, \mathbf{b}, \mathbf{c}) = \tau_2^{-\lambda/2} \sum_{\mathbf{k} \in \Lambda + \boldsymbol{\mu} + \frac{1}{2}\mathbf{p}} (-1)^{\mathbf{k} \cdot \mathbf{p}} \Phi(\sqrt{2\tau_2}(\mathbf{k} + \mathbf{b})) \mathbf{e}\left(-\frac{\tau}{2}(\mathbf{k} + \mathbf{b})^2 + \mathbf{c} \cdot (\mathbf{k} + \frac{1}{2}\mathbf{b})\right) \quad (\text{C.1})$$

defined by a kernel $\Phi(\mathbf{x})$ such that the function $f(\mathbf{x}) \equiv \Phi(\mathbf{x}) e^{\frac{\pi}{2}\mathbf{x}^2} \in L^1(\Lambda \otimes \mathbb{R})$ so that the sum is absolutely convergent. Irrespective of the choice of this kernel and of the parameter λ , any such theta series satisfies the following elliptic properties

$$\begin{aligned} \vartheta_{\mathbf{p}, \boldsymbol{\mu}}(\Phi, \lambda; \tau, \mathbf{b} + \mathbf{k}, \mathbf{c}) &= (-1)^{\mathbf{k} \cdot \mathbf{p}} \mathbf{e}\left(-\frac{1}{2}\mathbf{c} \cdot \mathbf{k}\right) \vartheta_{\mathbf{p}, \boldsymbol{\mu}}(\Phi, \lambda; \tau, \mathbf{b}, \mathbf{c}), \\ \vartheta_{\mathbf{p}, \boldsymbol{\mu}}(\Phi, \lambda; \tau, \mathbf{b}, \mathbf{c} + \mathbf{k}) &= (-1)^{\mathbf{k} \cdot \mathbf{p}} \mathbf{e}\left(\frac{1}{2}\mathbf{b} \cdot \mathbf{k}\right) \vartheta_{\mathbf{p}, \boldsymbol{\mu}}(\Phi, \lambda; \tau, \mathbf{b}, \mathbf{c}). \end{aligned} \quad (\text{C.2})$$

Now let us require that in addition the kernel satisfies the following two conditions:

1. Let $D(\mathbf{x})$ be any differential operator of order ≤ 2 , and $R(\mathbf{x})$ any polynomial of degree ≤ 2 . Then $f(\mathbf{x})$ defined above must be such that $f(\mathbf{x})$, $D(\mathbf{x})f(\mathbf{x})$ and $R(\mathbf{x})f(\mathbf{x}) \in L^2(\Lambda \otimes \mathbb{R}) \cap L^1(\Lambda \otimes \mathbb{R})$.
2. $\Phi(\mathbf{x})$ must satisfy

$$V_\lambda \cdot \Phi(\mathbf{x}) = 0, \quad V_\lambda = \partial_{\mathbf{x}}^2 + 2\pi(\mathbf{x} \cdot \partial_{\mathbf{x}} - \lambda). \quad (\text{C.3})$$

Then in [33] it was proven that the theta series (C.1) transforms as a vector-valued modular form of weight $(\lambda + d/2, 0)$ (see Theorem 2.1 in [24] for the detailed transformation under $\tau \rightarrow -1/\tau$). We refer to V_λ as Vignéras' operator. The simplest example is the Siegel theta series for which the kernel is $\Phi(\mathbf{x}) = e^{-\pi \mathbf{x}_+^2}$ where \mathbf{x}_+ is the projection of \mathbf{x} on a fixed positive plane of dimension n . This kernel is annihilated by V_{-n} .

In this paper we apply the Vignéras' theorem to the case of $\Lambda = \oplus_{i=1}^n \Lambda_i$. Thus, the charges appearing in the description of the theta series (C.1) are of the type $\mathbf{k} = (k_1^a, \dots, k_n^a)$, whereas the vectors \mathbf{b} and \mathbf{c} are taken with i -independent components, namely, $\mathbf{b}_i^a = b^a$, $\mathbf{c}_i^a = c^a$ for $i = 1, \dots, n$. The lattices Λ_i carry the bilinear forms $\kappa_{i,ab} = \kappa_{abc} p_i^c$ which are all of signature $(1, b_2 - 1)$. This induces a natural bilinear form on Λ :

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n (p_i x_i y_i). \quad (\text{C.4})$$

Note also that the sign factor $(-1)^{\mathbf{k} \cdot \mathbf{p}}$ in (C.1) can be traced back to the quadratic refinement provided we choose it as

$$\sigma_\gamma = \sigma_{p,q} \equiv \mathbf{e} \left(\frac{1}{2} p^a q_a \right) \sigma_p, \quad \sigma_p = \mathbf{e} \left(\frac{1}{2} A_{ab} p^a p^b \right). \quad (\text{C.5})$$

It involves the matrix A_{ab} satisfying

$$A_{ab} p^p - \frac{1}{2} \kappa_{abc} p^b p^c \in \mathbb{Z} \quad \text{for } \forall p^a \in \mathbb{Z}. \quad (\text{C.6})$$

This matrix appears due to the non-trivial quantization of charges on the type IIB side (2.18) and can be used to perform a symplectic rotation to identify them with mirror dual integer charges on the type IIA side [43]. It is easy to check that the quadratic refinement (C.5) satisfies (3.4).

C.2 Generalized error functions

An important class of solutions of Vignéras' equation is given by the error function and its generalizations constructed in [24] and further elaborated in [25]. Let us take

$$M_1(u) = -\text{sgn}(u) \text{Erfc}(|u|\sqrt{\pi}) = \frac{i}{\pi} \int_{\ell} \frac{dz}{z} e^{-\pi z^2 - 2\pi i z u}, \quad (\text{C.7})$$

$$\begin{aligned} E_1(u) &= \text{sgn}(u) + M_1(u) \\ &= \text{Erf}(u\sqrt{\pi}) = \int_{\mathbb{R}} du' e^{-\pi(u-u')^2} \text{sgn}(u'), \end{aligned} \quad (\text{C.8})$$

where the contour $\ell = \mathbb{R} - iu$ runs parallel to the real axis through the saddle point at $z = -iu$. Then, given a vector with a positive norm $\mathbf{v}^2 > 0$ so that $|\mathbf{v}| = \sqrt{\mathbf{v}^2}$, we define

$$\Phi_1^E(\mathbf{v}; \mathbf{x}) = E_1\left(\frac{\mathbf{v} \cdot \mathbf{x}}{|\mathbf{v}|}\right), \quad \Phi_1^M(\mathbf{v}; \mathbf{x}) = M_1\left(\frac{\mathbf{v} \cdot \mathbf{x}}{|\mathbf{v}|}\right). \quad (\text{C.9})$$

It is easy to check that the first function is a smooth solution of (C.3) with $\lambda = 0$, whereas the second is exponentially suppressed at large \mathbf{x} and also solves the same equation, but only away from the locus $\mathbf{v} \cdot \mathbf{x} = 0$ where it has a discontinuity.

Generalizing the integral representations (C.7) and (C.8), we define

$$M_n(\mathcal{M}; \mathbf{u}) = \left(\frac{i}{\pi}\right)^n |\det \mathcal{M}|^{-1} \int_{\mathbb{R}^{n-iu}} d^n z \frac{e^{-\pi \mathbf{z}^{\text{tr}} \mathbf{z} - 2\pi i \mathbf{z}^{\text{tr}} \mathbf{u}}}{\prod(\mathcal{M}^{-1} \mathbf{z})}, \quad (\text{C.10})$$

$$E_n(\mathcal{M}; \mathbf{u}) = \int_{\mathbb{R}^n} d\mathbf{u}' e^{-\pi(\mathbf{u}-\mathbf{u}')^{\text{tr}}(\mathbf{u}-\mathbf{u}')} \text{sgn}(\mathcal{M}^{\text{tr}} \mathbf{u}'), \quad (\text{C.11})$$

where $\mathbf{z} = (z_1, \dots, z_n)$ and $\mathbf{u} = (u_1, \dots, u_n)$ are n -dimensional vectors, \mathcal{M} is $n \times n$ matrix of parameters, and we used the shorthand notations $\prod \mathbf{z} = \prod_{i=1}^n z_i$ and $\text{sgn}(\mathbf{u}) = \prod_{i=1}^n \text{sgn}(u_i)$. The detailed properties of these functions can be found in [25]. Here we mention only a few:

- M_n are exponentially suppressed for large \mathbf{u} as $M_n \sim \frac{(-1)^n}{\pi^n} |\det \mathcal{M}|^{-1} \frac{e^{-\pi \mathbf{u}^{\text{tr}} \mathbf{u}}}{\prod(\mathcal{M}^{-1} \mathbf{u})}$, whereas E_n are locally constant for large \mathbf{u} as $E_n \sim \text{sgn}(\mathcal{M}^{\text{tr}} \mathbf{u})$.
- More generally, E_n can be expressed as a linear combination of M_k , $k = 0, \dots, n$, multiplied by $n - k$ sign functions, generalizing the first relation in (C.8).
- From (C.11) it follows that every identity between products of sign functions implies an identity between generalized error functions E_n . Moreover, expanding the E_n functions in terms of M_k 's and sign functions, one obtains similar identities for functions (C.10). For instance, the identity

$$(\text{sgn}(x_1) + \text{sgn}(x_2)) \text{sgn}(x_1 + x_2) = 1 + \text{sgn}(x_1) \text{sgn}(x_2) \quad (\text{C.12})$$

implies

$$\begin{aligned} E_2((\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2); \mathbf{u}) + E_2((\mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2); \mathbf{u}) &= 1 + E_2((\mathbf{v}_1, \mathbf{v}_2); \mathbf{u}), \\ M_2((\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2); \mathbf{u}) + M_2((\mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2); \mathbf{u}) &= M_2((\mathbf{v}_1, \mathbf{v}_2); \mathbf{u}), \end{aligned} \quad (\text{C.13})$$

where $\mathbf{v}_1, \mathbf{v}_2$ are two dimensional vectors.

The main reason to introduce these functions is that, similarly to the usual error and complementary error functions, they can be used to produce solutions of Vignéras' equation on $\mathbb{R}^{n,d-n}$. To write them down, let us consider $d \times n$ matrix \mathcal{V} which can be viewed as a collection of n vectors, $\mathcal{V} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$. We assume that these vectors span a positive definite subspace, i.e. $\mathcal{V}^{\text{tr}} \cdot \mathcal{V}$ is a positive definite matrix. Let \mathcal{B} be $n \times d$ matrix whose rows define an orthonormal basis for this subspace. Then we take

$$\Phi_n^M(\mathcal{V}; \mathbf{x}) = M_n(\mathcal{B} \cdot \mathcal{V}, \mathcal{B} \cdot \mathbf{x}), \quad \Phi_n^E(\mathcal{V}; \mathbf{x}) = E_n(\mathcal{B} \cdot \mathcal{V}, \mathcal{B} \cdot \mathbf{x}). \quad (\text{C.14})$$

It can be shown that both these functions satisfy Vignéras' equation (for Φ_n^M one should stay away from its discontinuities, i.e. loci where $\text{sgn}((\mathcal{B} \cdot \mathcal{V})^{-1} \mathcal{B} \cdot \mathbf{x}) = 0$). Since at large \mathbf{x} one has $\Phi_n^E \sim \text{sgn}(\mathcal{V}^{\text{tr}} \cdot \mathbf{x}) = \prod_{i=1}^n \text{sgn}(\mathbf{v}_i \cdot \mathbf{x})$, one can think about this function as providing the modular completion for (indefinite) theta series with kernel given by a product of signs.

Finally, note that a solution of Vignéras' equation with a given λ can be upgraded to a solution with $\lambda + 1$ by the action of the differential operator $\tilde{\mathbf{v}} \cdot (\mathbf{x} + \frac{1}{2\pi} \partial_{\mathbf{x}})$, which realizes the action of the covariant derivative raising the holomorphic weight by 1. In particular, we can construct solutions with $\lambda = n$ which behave for large \mathbf{x} as products of n sign functions. To this end, it is enough to act on Φ_n^E by this operator n times. Thus, we define

$$\tilde{\Phi}_n^E(\mathcal{V}, \tilde{\mathcal{V}}; \mathbf{x}) = \left[\prod_{i=1}^n \tilde{\mathbf{v}}_i \cdot \left(\mathbf{x} + \frac{1}{2\pi} \partial_{\mathbf{x}} \right) \right] \Phi_n^E(\mathcal{V}; \mathbf{x}), \quad (\text{C.15})$$

where $\tilde{\mathcal{V}} = (\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_n)$ encodes the vectors contracted with the covariant derivatives. In the case where all $\tilde{\mathbf{v}}_i$ are mutually orthogonal, one finds the following asymptotics at large \mathbf{x}

$$\tilde{\Phi}_n^E(\mathcal{V}, \tilde{\mathcal{V}}; \mathbf{x}) \sim \prod_{i=1}^n (\tilde{\mathbf{v}}_i, \mathbf{x}) \text{sgn}(\mathbf{v}_i, \mathbf{x}). \quad (\text{C.16})$$

Note that the derivative $\partial_{\mathbf{x}}$ in (C.15) does not act on sign functions since Φ_n^E is smooth and all discontinuities due to signs are guaranteed to cancel. Similarly to (C.15), we can also define $\tilde{\Phi}_n^M(\mathcal{V}, \tilde{\mathcal{V}}; \mathbf{x})$ where the action of derivatives on the discontinuities of Φ_n^M is ignored as well.

D. Twistorial integrals and generalized error functions

In this appendix we evaluate the kernels Φ_n (4.6) and show that they can be expressed through the generalized error functions introduced in appendix C.2. To this end, let us note the following identity

$$i \mathbf{v}_{ij} \cdot \left(\mathbf{x} + \frac{1}{2\pi} \partial_{\mathbf{x}} \right) \frac{W_{p_i}(x_i, z_i) W_{p_j}(x_j, z_j)}{z_i - z_j} = \hat{K}_{ij} W_{p_i}(x_i, z_i) W_{p_j}(x_j, z_j). \quad (\text{D.1})$$

Due to it, the kernel can be represented as

$$\Phi_n(\mathbf{x}) = \frac{1}{n!} \sum_{\mathcal{T} \in \mathbb{T}_n^\ell} \left[\prod_{e \in E_{\mathcal{T}}} \mathbf{v}_{s(e)t(e)} \cdot \left(\mathbf{x} + \frac{1}{2\pi} \partial_{\mathbf{x}} \right) \right] \Phi_{\mathcal{T}}(\mathbf{x}), \quad (\text{D.2})$$

where

$$\Phi_{\mathcal{T}}(\mathbf{x}) = \frac{i^{n-1}}{(2\pi)^n} \left[\prod_{i=1}^n \int_{\ell_{\gamma_i}} dz_i W_{p_i}(x_i, z_i) \right] \frac{1}{\prod_{e \in E_{\mathcal{T}}} (z_{s(e)} - z_{t(e)})}. \quad (\text{D.3})$$

One could may that the representation (D.2) is not complete because it misses contributions from the covariant derivatives acting on each other. Such contributions are proportional to the scalar products of two vectors $\mathbf{v}_{s(e)t(e)}$ and are non-vanishing provided the two edges have a common vertex. As a result, the two edges generate the following factor

$$\frac{(p_{\mathbf{v}_1} p_{\mathbf{v}_2} p_{\mathbf{v}_3})}{(z_{\mathbf{v}_3} - z_{\mathbf{v}_1})(z_{\mathbf{v}_2} - z_{\mathbf{v}_3})}, \quad (\text{D.4})$$

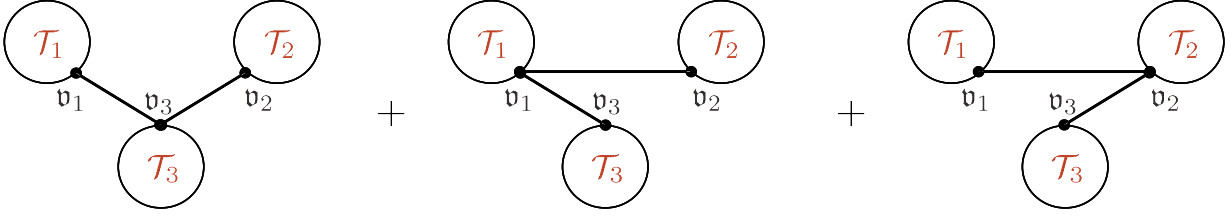


Figure 5: Three trees generating contributions to the action of covariant derivative operators which cancel each other.

where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the tree vertices joint by the edges and $\mathbf{v}_3 = e_1 \cap e_2$. The crucial observation is that if we pick up 3 subtrees, each with a marked vertex, there are exactly 3 ways to form a labelled tree out of them by joining the marked vertices (see Fig. 5). Each subtree contributes the same factor in all 3 cases, whereas the joining edges and the sum over trees give rise to the vanishing factor

$$\frac{(p_1 p_2 p_3)}{(z_3 - z_1)(z_2 - z_3)} + \frac{(p_1 p_2 p_3)}{(z_1 - z_2)(z_3 - z_1)} + \frac{(p_1 p_2 p_3)}{(z_2 - z_3)(z_1 - z_2)} = 0. \quad (\text{D.5})$$

This ensures that no additional contributions arise and thereby proves (D.2). Note that for this proof it was crucial that inequivalent labelled trees enter the sum with the same weight.

Next, note that the contours ℓ_{γ_i} in the z -plane go along the arcs running from -1 to 1 and passing through the saddle points z_{γ_i} (3.10). However, in the large volume limit one can deform the contours into the straight lines $z_{\gamma_i} + \mathbb{R}$ since this changes the integrals by exponentially small contributions.

Then let us do the change of variables

$$z_i = z' - \frac{i(pxt)}{\sqrt{2\tau_2}(pt^2)} + \sum_{\alpha=1}^{n-1} e_i^\alpha z'_\alpha, \quad (\text{D.6})$$

where $x^a = \kappa^{ab} \sum \kappa_{i,bc} x_i^c$ (cf. (4.8)), and e_i^α are such that

$$\sum_{i=1}^n \mathbf{p}_i e_i^\alpha = 0, \quad \sum_{i=1}^n \mathbf{p}_i e_i^\alpha e_i^\beta = 0, \quad \alpha \neq \beta \quad (\text{D.7})$$

and we introduced the convenient notation $\mathbf{p}_i = (p_i t^2)$. Labeling the $n - 1$ edges of the tree by the same index α , one can rewrite the function (D.3) in the new variables as

$$\Phi_{\mathcal{T}}(\mathbf{x}) = \frac{i^{n-1} \sqrt{\Delta}}{(2\pi)^n} e^{-\frac{\pi(pxt)^2}{(pt^2)^2}} \int dz' e^{-2\pi\tau_2(pt^2)z'^2} \prod_{\alpha=1}^{n-1} \int \frac{dz'_\alpha e^{-2\pi\tau_2 \Delta_\alpha (z'_\alpha)^2 - 2\pi i \sqrt{2\tau_2} w^\alpha z'_\alpha}}{\sum_{\beta=1}^{n-1} (e_{s(\alpha)}^\beta - e_{t(\alpha)}^\beta) z'_\beta}, \quad (\text{D.8})$$

where

$$w^\alpha = \sum_{i=1}^n (p_i x_i t) e_i^\alpha, \quad \Delta_\alpha = \sum_{i=1}^n \mathbf{p}_i (e_i^\alpha)^2, \quad \Delta = \frac{\mathbf{p} \prod_{\alpha=1}^{n-1} \Delta_\alpha}{\prod_{i=1}^n \mathbf{p}_i}. \quad (\text{D.9})$$

The integral over z' is Gaussian and is easily evaluated. In the remaining integrals, rescaling the integration variables by $\sqrt{2\tau_2 \Delta_\alpha}$, one recognizes the generalized error functions (C.10).

Thus, one obtains

$$\Phi_{\mathcal{T}}(\mathbf{x}) = \frac{1}{2^{n-1}} \frac{\sqrt{\Delta} |\det \mathcal{M}|}{\prod_{\alpha=1}^{n-1} \Delta_{\alpha}} \Phi_1(x) M_{n-1} \left(\mathcal{M}; \left\{ \frac{w^{\alpha}}{\sqrt{\Delta_{\alpha}}} \right\} \right). \quad (\text{D.10})$$

where we used the function Φ_1 (4.13) and introduced the matrix \mathcal{M} such that

$$\mathcal{M}_{\alpha\beta}^{-1} = (\Delta_{\alpha}\Delta_{\beta})^{-1/2} \left(e_{s(\alpha)}^{\beta} - e_{t(\alpha)}^{\beta} \right). \quad (\text{D.11})$$

There is a simple solution to the conditions (D.7). Let T be a rooted ordered binary tree with n leaves labelled by γ_i , $i = 1 \dots n$. As usual for such trees, other vertices v carry charges given by the sum of charges of their children, i.e. $\gamma_v = \sum_{i \in \mathcal{I}_v} \gamma_i$ where \mathcal{I}_v is the set of leaves which are descendants of v . There are $n - 1$ such vertices which we label by index α . Then we can choose

$$e_i^{\alpha} = \sum_{j \in \mathcal{I}_L(v_{\alpha})} \sum_{k \in \mathcal{I}_R(v_{\alpha})} (\delta_{ij} \mathbf{p}_k - \delta_{ik} \mathbf{p}_j), \quad (\text{D.12})$$

which satisfy (D.7) as can be easily checked. For this choice (cf. (4.14))

$$w^{\alpha} = (\tilde{\mathbf{u}}_{\alpha}, \mathbf{x}), \quad \text{where} \quad \tilde{\mathbf{u}}_{\alpha} = \sum_{i \in \mathcal{I}_L(v_{\alpha})} \sum_{j \in \mathcal{I}_R(v_{\alpha})} \mathbf{u}_{ij}, \quad (\text{D.13})$$

$$\Delta_{\alpha} = \tilde{\mathbf{u}}_{\alpha}^2 = \mathbf{p}_{v_{\alpha}} \mathbf{p}_{L(v_{\alpha})} \mathbf{p}_{R(v_{\alpha})}.$$

Note that the vectors $\tilde{\mathbf{u}}_{\alpha}$ are mutually orthogonal.

In principle, any rooted binary tree T is suitable for the above construction. However, given the unrooted tree \mathcal{T} , there is a simple (but non-unique) choice of T which simplifies the resulting matrix \mathcal{M} . For this purpose, let us construct a partially increasing family of subtrees of \mathcal{T} , such that two members in this family are either disjoint, or contained in one another. Moreover, we require that the largest subtree is \mathcal{T} itself, while each subtree containing more than one vertex is obtained by joining two smaller subtrees along an edge of \mathcal{T} . Any such family contains $2n - 1$ subtrees $\mathcal{T}_{\hat{\alpha}}$ labelled by $\hat{\alpha} = 1, \dots, 2n - 1$. Among them, $n - 1$ subtrees, which we label by $\alpha = \hat{\alpha} = 1, \dots, n - 1$, contain several vertices, while the remaining n subtrees with label $\hat{\alpha} = n, \dots, 2n - 1$ have only one vertex. For each subtree T_{α} , we denote by e_{α} the edge of \mathcal{T} which is used to reconstruct T_{α} from two smaller subtrees $\mathcal{T}_{\alpha_L}, \mathcal{T}_{\alpha_R}$. From this data, we construct a rooted binary tree T with $n - 1$ vertices in one-to-one correspondence with the subtrees \mathcal{T}_{α} and n leaves in one-to-one correspondence with the one-vertex subtrees $\mathcal{T}_{\hat{\alpha}}$ with $\hat{\alpha} \geq n$. In this correspondence, the two children of a vertex associated to \mathcal{T}_{α} are the vertices associated to the two subtrees $\mathcal{T}_{\alpha_L}, \mathcal{T}_{\alpha_R}$. The ordering at each vertex is defined to be such that the subtree containing the source/target vertex of the corresponding edge e_{α} is on the left/right.¹⁸ Of course, this construction is not unique since there are many ways to decompose \mathcal{T} into such a set of subtrees (see Fig. 6).

Applying the above construction to this particular choice of rooted tree, one finds

$$\sqrt{\Delta_{\alpha}\Delta_{\beta}} \mathcal{M}_{\alpha\beta}^{-1} = \begin{cases} \mathbf{p}_{v_{\alpha}}, & \alpha = \beta, \\ \epsilon_{\alpha\beta} \mathbf{p}_{L(v_{\beta})}, & e_{\alpha} \cap \mathcal{T}_{\beta_R} \neq \emptyset, e_{\alpha} \not\subseteq \mathcal{T}_{\beta}, \\ \epsilon_{\alpha\beta} \mathbf{p}_{R(v_{\beta})}, & e_{\alpha} \cap \mathcal{T}_{\beta_L} \neq \emptyset, e_{\alpha} \not\subseteq \mathcal{T}_{\beta}, \\ 0, & e_{\alpha} \cap \mathcal{T}_{\beta} = \emptyset \text{ or } e_{\alpha} \subset \mathcal{T}_{\beta}, \end{cases} \quad (\text{D.14})$$

¹⁸The orientation of the edges of \mathcal{T} is fixed already in (D.2), but the full kernel Φ_n does not depend on its choice.

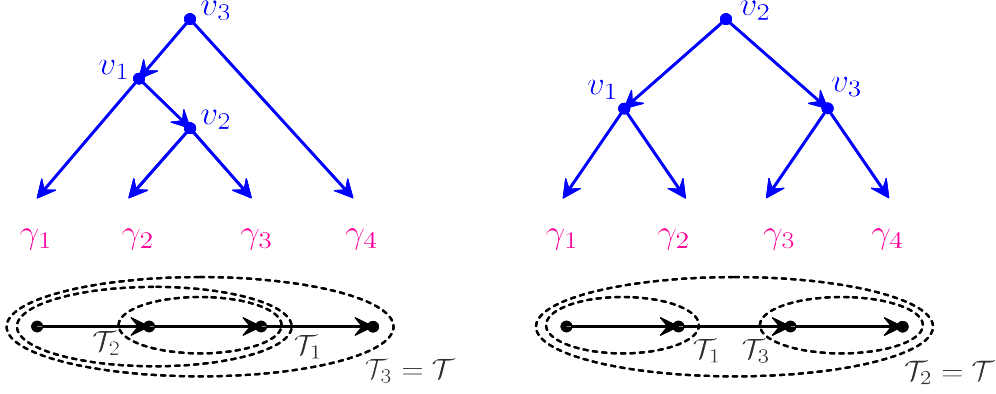


Figure 6: An example of an unrooted labelled tree with 4 vertices and two choices of decompositions into subtrees with the corresponding rooted binary trees. The edges are labelled e_1, e_2, e_3 from left to right along \mathcal{T} .

where $\epsilon_{\alpha\beta} = -1$ if the orientations of e_α and e_β on the path joining them are the same and $+1$ otherwise. This result shows that the matrix \mathcal{M}^{-1} turns out to be triangular which makes it much simpler to find its inverse. On the basis of (D.14), below we will prove the following

Lemma 1. *One has $\mathcal{B} \cdot \mathcal{V} = \mathcal{M}$ and $\mathcal{B} \cdot \mathbf{x} = \left\{ \frac{w^\alpha}{\sqrt{\Delta_\alpha}} \right\}$ provided*

$$\begin{aligned} \mathcal{B} &= \left(\frac{\tilde{\mathbf{u}}_1}{\sqrt{\Delta_1}}, \dots, \frac{\tilde{\mathbf{u}}_{n-1}}{\sqrt{\Delta_{n-1}}} \right)^{\text{tr}}, \\ \mathcal{V} &= \mathbf{p}^{-1} \left(\sqrt{\Delta_1} \mathbf{u}_1, \dots, \sqrt{\Delta_{n-1}} \mathbf{u}_{n-1} \right), \end{aligned} \quad (\text{D.15})$$

where the vectors \mathbf{u}_α are defined in (4.11). Moreover, the vectors $\tilde{\mathbf{u}}_\alpha$ form an orthogonal basis in the subspace spanned by \mathbf{u}_α .

This lemma allows to reexpress the kernel $\Phi_{\mathcal{T}}$ (D.10) in terms of the generalized error function Φ_{n-1}^M (C.14). It is important that its argument \mathcal{V} does not depend on the choice of the binary tree T , but only on \mathcal{T} . In addition, the function actually does not depend on the normalization of the vectors composing \mathcal{V} . Given also that the determinant of \mathcal{M} is found to be

$$|\det \mathcal{M}| = \prod_{\alpha=1}^{n-1} \frac{\Delta_\alpha}{\mathbf{p}^{v_\alpha}} = \prod_{i=1}^n \mathbf{p}_i \prod_{v \in V_T \setminus \{v_0\}} \mathbf{p}_v = \sqrt{\mathbf{p}^{-1} \prod_{i=1}^n \mathbf{p}_i \prod_{\alpha=1}^{n-1} \Delta_\alpha}, \quad (\text{D.16})$$

so that the prefactor in (D.10) cancels, one arrives at

$$\Phi_T(\mathbf{x}) = \frac{1}{2^{n-1}} \Phi_1(x) \Phi_{n-1}^M(\{\mathbf{u}_e\}; \mathbf{x}). \quad (\text{D.17})$$

Finally, since the differential operator in (D.2) commutes with functions of x due to orthogonality of \mathbf{v}_{ij} and \mathbf{t} , one can write the kernel Φ_n as

$$\Phi_n(\mathbf{x}) = \frac{\Phi_1(x)}{2^{n-1} n!} \sum_{\mathcal{T} \in \mathbb{T}_n^\ell} \left[\prod_{e \in E_{\mathcal{T}}} \mathbf{v}_{s(e)t(e)} \cdot \left(\mathbf{x} + \frac{1}{2\pi} \partial_x \right) \right] \Phi_{n-1}^M(\{\mathbf{u}_\alpha\}; \mathbf{x}). \quad (\text{D.18})$$

which is the same as (4.12).

D.1 Proof of the Lemma

We start by proving that the vectors $\tilde{\mathbf{u}}_\alpha$ form an orthogonal basis in the subspace spanned by \mathbf{u}_α . Since the orthogonality is ensured by construction based on a rooted binary tree, it remains to show that any vector \mathbf{u}_α can be decomposed as a linear combination of $\tilde{\mathbf{u}}_\alpha$. To this end, we show that the determinant of the Gram matrix constructed from the set of vectors $\{\tilde{\mathbf{u}}_\alpha\}_{\alpha=1}^{n-1} \cup \{\mathbf{u}_\beta\}$ vanishes. This requires to calculate the scalar product $(\tilde{\mathbf{u}}_\alpha, \mathbf{u}_\beta)$ which can be done using

$$(\mathbf{u}_{ij}, \mathbf{u}_{kl}) = \begin{cases} \mathfrak{p}_i \mathfrak{p}_j \mathfrak{p}_{i+j}, & i = k, j = l, \\ \mathfrak{p}_i \mathfrak{p}_j \mathfrak{p}_l, & i = k, j \neq l, \\ 0, & i, j \neq k, l. \end{cases} \quad (\text{D.19})$$

Summing i, j, k, l over appropriate subsets, it is immediate to see that $(\tilde{\mathbf{u}}_\alpha, \mathbf{u}_\beta) = 0$ if $\mathcal{T}_\alpha \subset \mathcal{T}_\beta^s$ or \mathcal{T}_β^t , which are the two trees obtained by dividing the tree \mathcal{T} into two parts by cutting the edge e_α . In other words, it is non-vanishing only if $e_\beta \subseteq \mathcal{T}_\alpha$. Then there are two cases which give

$$(\tilde{\mathbf{u}}_\alpha, \mathbf{u}_\beta) = \begin{cases} \mathfrak{p}_\beta^L \mathfrak{p}_\beta^R \mathfrak{p}, & \alpha = \beta, \\ \mathfrak{p}_{\alpha\beta}^{Ls} \mathfrak{p}_\alpha^R \mathfrak{p}_\beta^t - \mathfrak{p}_{\alpha\beta}^{Lt} \mathfrak{p}_\alpha^R \mathfrak{p}_\beta^s + \mathfrak{p}_\alpha^L \mathfrak{p}_\alpha^R \mathfrak{p}_\beta^s = \mathfrak{p}_{\alpha\beta}^{Ls} \mathfrak{p}_\alpha^R \mathfrak{p}, & e_\beta \subset \mathcal{T}_\alpha, \end{cases} \quad (\text{D.20})$$

where we introduced

$$\mathfrak{p}_\alpha^L = \sum_{i \in \mathcal{I}_L(v_\alpha)} \mathfrak{p}_i, \quad \mathfrak{p}_\alpha^s = \sum_{i \in V_{\mathcal{T}_\alpha^s}} \mathfrak{p}_i, \quad \mathfrak{p}_{\alpha\beta}^{Ls} = \sum_{i \in \mathcal{I}_L(v_\alpha) \cap V_{\mathcal{T}_\beta^s}} \mathfrak{p}_i, \quad \mathfrak{p}_{\alpha\beta}^{st} = \sum_{i \in V_{\mathcal{T}_\alpha^s} \cap V_{\mathcal{T}_\beta^t}} \mathfrak{p}_i, \quad (\text{D.21})$$

and similarly for variables with labels R and t . In (D.20) in the second case we assumed that the orientation of edges is such that $e_\beta \subset \mathcal{T}_\alpha^s$ and $e_\alpha \subset \mathcal{T}_\beta^t$. If this is not the case, one should replace s by t , L by R and flip the sign for each change of orientation. Below we use the same assumption, but the computation can easily be generalized to a more general situation.

Given the result (D.20), $(\tilde{\mathbf{u}}_\alpha, \tilde{\mathbf{u}}_\beta) = \Delta_\alpha \delta_{\alpha\beta}$ and $(\mathbf{u}_\beta, \mathbf{u}_\beta) = \mathfrak{p}_\beta^s \mathfrak{p}_\beta^t \mathfrak{p}$, the determinant of the Gram matrix is easily found to be

$$\det \text{Gram}(\tilde{\mathbf{u}}_1, \dots, \tilde{\mathbf{u}}_{m-1}, \mathbf{u}_\beta) = \mathfrak{p} \prod_{\alpha=1}^{n-1} \Delta_\alpha \left[\mathfrak{p}_\beta^s \mathfrak{p}_\beta^t - \mathfrak{p} \sum_{\mathcal{T}_\alpha \supseteq e_\beta} \frac{(\mathfrak{p}_{\alpha\beta}^{Ls})^2 \mathfrak{p}_\alpha^R}{\mathfrak{p}_{v_\alpha} \mathfrak{p}_\alpha^L} \right]. \quad (\text{D.22})$$

Note that the subtrees \mathcal{T}_α containing the edge e_β form an ordered set so that the sum in the square brackets goes over α_ℓ , $\ell = 1, \dots, m$, such that $\mathcal{T}_{\alpha_\ell} \subset \mathcal{T}_{\alpha_{\ell+1}}$. The first element of this set $\alpha_1 = \beta$, whereas the last corresponds to the total tree, $\mathcal{T}_{\alpha_m} = \mathcal{T}$. Due to $\mathfrak{p}_{\alpha_m}^L = \mathfrak{p}_{\alpha_m}^s$, $\mathfrak{p}_{\alpha_m}^R = \mathfrak{p}_{\alpha_m}^t$ and $\mathfrak{p}_{\alpha_m\beta}^{Ls} = \mathfrak{p}_\beta^s$, the first term in the square brackets together with the term in the sum corresponding to α_m gives

$$\mathfrak{p}_\beta^s \left(\mathfrak{p}_\beta^t - \frac{\mathfrak{p}_\beta^s \mathfrak{p}_{\alpha_m}^t}{\mathfrak{p}_{\alpha_m}^s} \right) = \frac{\mathfrak{p} \mathfrak{p}_\beta^s \mathfrak{p}_{\alpha_m\beta}^{st}}{\mathfrak{p}_{\alpha_m}^s}, \quad (\text{D.23})$$

where we have used that $\mathfrak{p}_\beta^t = \mathfrak{p}_{\alpha_m\beta}^{st} + \mathfrak{p}_{\alpha_m}^t$ and $\mathfrak{p}_{\alpha_m}^s = \mathfrak{p}_{\alpha_m\beta}^{st} + \mathfrak{p}_\beta^s$. Thus, the expression in the square brackets in (D.22) becomes

$$\frac{\mathfrak{p}}{\mathfrak{p}_{\alpha_m}^s} \left[\mathfrak{p}_\beta^s \mathfrak{p}_{\alpha_m\beta}^{st} - \mathfrak{p}_{\alpha_m}^s \sum_{\ell=1}^{m-1} \frac{(\mathfrak{p}_{\alpha_\ell\beta}^{Ls})^2 \mathfrak{p}_{\alpha_\ell}^R}{\mathfrak{p}_{v_{\alpha_\ell}} \mathfrak{p}_{\alpha_\ell}^L} \right]. \quad (\text{D.24})$$

The new expression in the square brackets is exactly the same as in (D.22) where tree \mathcal{T} was replaced by subtree $\mathcal{T}_{\alpha_m}^s = \mathcal{T}_{\alpha_{m-1}}$. Thus, one can repeat the above manipulation until one exhausts all terms in the sum. As a result, the determinant of the Gram matrix turns out to be proportional to $\mathbf{p}_{\alpha_1\beta}^{st}$. But since $\alpha_1 = \beta$, this quantity, and hence the whole determinant, trivially vanish.

Next, we prove that (D.15) is consistent with $\mathcal{B} \cdot \mathbf{x} = \left\{ \frac{w^\alpha}{\sqrt{\Delta_\alpha}} \right\}$ and $\mathcal{B} \cdot \mathcal{V} = \mathcal{M}$. The first relation is a direct consequence of (D.13). The second relation requires to show that $(\tilde{\mathbf{u}}_\alpha, \mathbf{u}_\beta) = \mathbf{p} \Delta_\alpha^{1/2} \Delta_\beta^{-1/2} \mathcal{M}_{\alpha\beta}$ or equivalently $\sum_\gamma (\tilde{\mathbf{u}}_\alpha, \mathbf{u}_\gamma) \sqrt{\Delta_\gamma \Delta_\beta} \mathcal{M}_{\gamma\beta}^{-1} = \mathbf{p} \Delta_\alpha \delta_{\alpha\beta}$. Using the result (D.14) for the matrix $\mathcal{M}_{\gamma\beta}^{-1}$, this relation can be written explicitly as

$$\mathbf{p}_{v_\beta}(\tilde{\mathbf{u}}_\alpha, \mathbf{u}_\beta) + \left(\sum_{e_\gamma \in E_\beta^R} \epsilon_{\gamma\beta} \mathbf{p}_{L(v_\beta)} + \sum_{e_\gamma \in E_\beta^L} \epsilon_{\gamma\beta} \mathbf{p}_{R(v_\beta)} \right) (\tilde{\mathbf{u}}_\alpha, \mathbf{u}_\gamma) = \mathbf{p} \Delta_\alpha \delta_{\alpha\beta}, \quad (\text{D.25})$$

where $E_\beta^L = \{e \in E_\mathcal{T} : e \cap \mathcal{T}_{\beta_L} \neq \emptyset, e \not\subseteq \mathcal{T}_\beta\}$ and similarly for E_β^R .

Consider first the case $\alpha = \beta$. From (D.20), it immediately follows that the first term gives $\mathbf{p} \Delta_\alpha$. On the other hand, the second contribution sums over edges for which $\mathcal{T}_\alpha \subseteq \mathcal{T}_\gamma^s$ or \mathcal{T}_γ^t , and as noted above (D.20) this leads to vanishing of the scalar product. Thus, in this case the relation (D.25) indeed holds.

Let us now show that it holds as well for $\alpha \neq \beta$. To this end, one should consider several options. If $\mathcal{T}_\alpha \cap \mathcal{T}_\beta = \emptyset$, then $\mathcal{T}_\alpha \subset \mathcal{T}_\beta^s$ or \mathcal{T}_β^t which implies vanishing of the first term. But the second term vanishes as well because the conditions $e_\gamma \subseteq \mathcal{T}_\alpha$ and $e_\gamma \cap \mathcal{T}_\beta \neq \emptyset$ are inconsistent with $\mathcal{T}_\alpha \cap \mathcal{T}_\beta = \emptyset$.

Similarly, if $\mathcal{T}_\alpha \subset \mathcal{T}_\beta$, one has $\mathcal{T}_\alpha \subset \mathcal{T}_\beta^s$ or \mathcal{T}_β^t which again leads to the vanishing of the first term, whereas the vanishing of the second is a consequence of that $e_\gamma \subseteq \mathcal{T}_\alpha$ implies $e_\gamma \subset \mathcal{T}_\beta$ so that the sum over e_γ is empty.

It remains to consider the case $\mathcal{T}_\beta \subset \mathcal{T}_\alpha$. It is clear that $\mathcal{T}_\beta \subset \mathcal{T}_\alpha^s$ or \mathcal{T}_α^t . Without loss of generality, let us assume that $\mathcal{T}_\beta \subset \mathcal{T}_\alpha^s$ and $e_\alpha \subset \mathcal{T}_\beta^t$. Then according to (D.20), the first term gives $\mathbf{p}_{v_\beta} \mathbf{p}_{\alpha\beta}^{Ls} \mathbf{p}_\alpha^R \mathbf{p}$. If instead we have chosen the orientation such that $e_\alpha \subset \mathcal{T}_\beta^s$, then we would find $-\mathbf{p}_{v_\beta} \mathbf{p}_{\alpha\beta}^{Lt} \mathbf{p}_\alpha^R \mathbf{p}$. Similar results are obtained for each term in the sum of the second contribution. Again without loss of generality we assume that for all relevant edges e_γ one has $e_\alpha \subset \mathcal{T}_\gamma^t$, otherwise one flips their orientation. Then the l.h.s. of (D.25) is proportional to

$$\mathbf{p}_{v_\beta} \mathbf{p}_{\alpha\beta}^{Ls} + \sum_{e_\gamma \in E_\beta^R} \epsilon_{\gamma\beta} \mathbf{p}_\beta^L \mathbf{p}_{\alpha\gamma}^{Ls} + \sum_{e_\gamma \in E_\beta^L} \epsilon_{\gamma\beta} \mathbf{p}_\beta^R \mathbf{p}_{\alpha\gamma}^{Ls}. \quad (\text{D.26})$$

Let $e_\alpha^* \in E_\beta^R$ is the edge belonging to the path from \mathcal{T}_β to e_α (which may coincide with e_α). Then our choice of orientation implies that $\epsilon_{\gamma\beta} = -1$ for $e_\gamma \in E_\beta^L \cup \{e_\alpha^*\}$ and $\epsilon_{\gamma\beta} = 1$ for $e_\gamma \in E_\beta^R \setminus \{e_\alpha^*\}$. Furthermore, one has

$$\mathbf{p}_{\alpha\gamma_\alpha^*}^{Ls} = \mathbf{p}_{v_\beta} + \sum_{e_\gamma \in E_\beta^L \cup E_\beta^R \setminus \{e_\alpha^*\}} \mathbf{p}_{\alpha\gamma}^{Ls}, \quad \mathbf{p}_{\alpha\beta}^{Ls} = \mathbf{p}_\beta^L + \sum_{e_\gamma \in E_\beta^L} \mathbf{p}_{\alpha\gamma}^{Ls}. \quad (\text{D.27})$$

As a result, one finds

$$(\mathfrak{p}_\beta^L + \mathfrak{p}_\beta^R) \left(\mathfrak{p}_\beta^L + \sum_{e_\gamma \in E_\beta^L} \mathfrak{p}_{\alpha\gamma}^{Ls} \right) - \mathfrak{p}_\beta^L \mathfrak{p}_{\alpha\gamma^*}^{Ls} + \sum_{e_\gamma \in E_\beta^R \setminus \{e_\alpha^*\}} \mathfrak{p}_\beta^L \mathfrak{p}_{\alpha\gamma}^{Ls} - \sum_{e_\gamma \in E_\beta^L} \mathfrak{p}_\beta^R \mathfrak{p}_{\alpha\gamma}^{Ls} = 0. \quad (\text{D.28})$$

This completes the proof of the required statement.

E. Proofs of propositions

In this appendix we fill the gap and prove several propositions which we stated in the main text.

Proposition 1

To prove the recursive equation (5.3), let us note that the tree index $g_{\text{tr},n}$ satisfies a very similar equation (cf. (2.9) or (2.15)) which can be seen as the origin of its representation (2.3) through the attractor flow trees. The only difference is the absence of the last term in (5.3). Therefore, it is easy to see that this equation implies a simple relation between \widehat{g}_n and $g_{\text{tr},n}$

$$\widehat{g}_n(\{\check{\gamma}_i\}, z^a) = \sum_{\substack{n_1 + \dots + n_m = n \\ n_k \geq 1}} g_{\text{tr},m}(\{\check{\gamma}'_k\}, z^a) \prod_{k=1}^m W_{n_k}(\check{\gamma}_{j_{k-1}+1}, \dots, \check{\gamma}_{j_k}), \quad (\text{E.1})$$

where as usual we use notations from (2.16).

Next, we substitute this relation into the expansion (5.2). The result can be represented in the following form

$$h_{p,q}^{\text{DT}} = \sum_{\sum_{i=1}^n \check{\gamma}_i = \check{\gamma}} g_{\text{tr},n}(\{\check{\gamma}_i\}, z^a) e^{\pi i \tau Q_n(\{\check{\gamma}_i\})} \prod_{i=1}^n h_{p_i, \mu_i}^R(\tau), \quad (\text{E.2})$$

where we introduced

$$\begin{aligned} h_{p,\mu}^R &= \sum_{\sum_{i=1}^n \check{\gamma}_i = \check{\gamma}} W_n(\{\check{\gamma}_i\}) e^{\pi i \tau Q_n(\{\check{\gamma}_i\})} \prod_{i=1}^n \widehat{h}_{p_i, \mu_i}(\tau) \\ &= \sum_{\sum_{i=1}^n \check{\gamma}_i = \check{\gamma}} \left[\sum_{T \in \mathbb{T}_n^S} \prod_{v \in V_T} R_v \right] e^{\pi i \tau Q_n(\{\check{\gamma}_i\})} \\ &\quad \times \prod_{i=1}^n \left(h_{p_i, \mu_i} - \sum_{n_i=2}^{\infty} \sum_{\sum_{j=1}^{n_i} \check{\gamma}'_j = \check{\gamma}} R_{n_i}(\{\check{\gamma}'_j\}, \tau, \bar{\tau}) e^{\pi i \tau Q_{n_i}(\{\check{\gamma}'_j\})} \prod_{j=1}^{n_i} h_{p'_{j_i}, \mu'_{j_i}} \right) \end{aligned} \quad (\text{E.3})$$

and in the last relation we used the definition of W_n (5.4) and the expansion of $\widehat{h}_{p,\mu}$ (5.1). The crucial observation is that if one picks up a factor R_{n_i} from the second line of (E.3), appearing due to the expansion of \widehat{h}_{p_i, μ_i} , and combines it with the contribution of a tree T from the first line, it is equal to *minus* the contribution of another tree obtained from T by

adding n_i children to its i th leaf. As a result, all such contributions cancel and the function (E.3) reduces to the trivial term

$$h_{p,\mu}^R = h_{p,\mu}. \quad (\text{E.4})$$

Substituting this into (E.2), it gives back the original expansion (2.30) of the generating function of DT invariants, which proves the recursive equation (5.3).

Proposition 2

We will prove proposition 2 by induction. For $n = 2$ the recursive relation (5.8) reads

$$g_2^{(0)}(\check{\gamma}_1, \check{\gamma}_2; c_1) - g_2^{(0)}(\check{\gamma}_1, \check{\gamma}_2; \beta_{21}) = -\frac{1}{4} (\text{sgn}(c_1) - \text{sgn}(\beta_{21})) \kappa(\gamma_{12}), \quad (\text{E.5})$$

where we took into account that $S_1 = c_1$ and $\Gamma_{21} = \beta_{21} = -\gamma_{12}$. Since $g_2^{(0)}$ is supposed to have discontinuities only at walls of marginal stability, it must not involve signs of DSZ products. Therefore, we are led to take

$$g_2^{(0)}(\check{\gamma}_1, \check{\gamma}_2; c_1) = -\frac{1}{4} \text{sgn}(c_1) \kappa(\gamma_{12}). \quad (\text{E.6})$$

Then (5.9) and (5.10) imply

$$\mathcal{E}_2 = \frac{1}{4} \text{sgn}(\gamma_{12}) \kappa(\gamma_{12}) - R_2, \quad (\text{E.7})$$

so that the ansatz (5.7) reads

$$\widehat{g}_2(\check{\gamma}_1, \check{\gamma}_2; z) = g_2^{(0)}(\check{\gamma}_1, \check{\gamma}_2; c_1) - \mathcal{E}_2(\check{\gamma}_1, \check{\gamma}_2) = -\frac{1}{4} \left[\text{sgn}(c_1) + \text{sgn}(\gamma_{12}) \right] \kappa(\gamma_{12}) + R_2(\check{\gamma}_1, \check{\gamma}_2), \quad (\text{E.8})$$

which reproduces the recursive equation (5.3). Furthermore, in appendix C.2 it is shown that there is a smooth solution of Vignéras' equation which asymptotes the function $(\mathbf{v}, \mathbf{x}) \text{sgn}(\mathbf{v}, \mathbf{x})$, coinciding with the (rescaled) first term in \mathcal{E}_2 for $\mathbf{v} = \mathbf{v}_{12}$ (4.10). It is given by

$$\tilde{\Phi}_1^E(\mathbf{v}, \mathbf{v}; \mathbf{x}) = \mathbf{v} \cdot \left(\mathbf{x} + \frac{1}{2\pi} \partial_{\mathbf{x}} \right) \text{Erf} \left(\frac{\sqrt{\pi} \mathbf{v} \cdot \mathbf{x}}{|\mathbf{v}|} \right), \quad (\text{E.9})$$

which corresponds to the following choice of R_2 [21]

$$R_2 = \frac{(-1)^{1+\gamma_{12}}}{8\pi} |\gamma_{12}| \beta_{\frac{3}{2}} \left(\frac{2\tau_2 \gamma_{12}^2}{(pp_1 p_2)} \right), \quad (\text{E.10})$$

where $\beta_{\frac{3}{2}}(x^2) = 2|x|^{-1} e^{-\pi x^2} - 2\pi \text{Erfc}(\sqrt{\pi}|x|)$. Note that the resulting \mathcal{E}_2 depends on the electric charges only through the DSZ product γ_{12} . Finally, it is immediate to see that the kernel $\Phi_2^{\widehat{g}}$ corresponding to \widehat{g}_2 satisfies (5.5).

Now we assume that (5.3) is consistent with the ansatz (5.7) for all orders up to $n-1$ and check it at order n . Denoting the second term in (5.7) by $\widehat{g}_n^{(+)}$ and substituting this ansatz into the r.h.s. of (5.3), one finds

$$\text{Sym} \left\{ \sum_{\ell=1}^{n-1} g_2^\ell \left[g_\ell^{(0)} g_{n-\ell}^{(0)} - \widehat{g}_\ell^{(+)} \widehat{g}_{n-\ell} - \widehat{g}_\ell \widehat{g}_{n-\ell}^{(+)} - \widehat{g}_\ell^{(+)} \widehat{g}_{n-\ell}^{(+)} \right]_{z^a \rightarrow z_\ell^a} \right\} + W_n, \quad (\text{E.11})$$

where

$$g_2^\ell = -\frac{1}{2} \Delta_{\gamma_L^\ell \gamma_R^\ell}^z \kappa(\gamma_{LR}^\ell) = \frac{1}{4} (\text{sgn}(S_\ell) - \text{sgn}(\Gamma_{n\ell})) \kappa(\Gamma_{n\ell}). \quad (\text{E.12})$$

In the first term proportional to $g_\ell^{(0)} g_{n-\ell}^{(0)}$, one can apply the relation (5.8), which together with (5.9) gives $g_n^{(0)} - \mathcal{E}_n^{(0)}$. The other terms in the sum over ℓ can be reorganized as follows

$$\begin{aligned} & - \text{Sym} \left\{ \sum_{\ell=1}^{n-1} g_2^\ell \sum_{\substack{n_1+\dots+n_m=n \\ n_k \geq 1, m < n, \ell \in \{j_k\}}} \widehat{g}_{k_0}(z_\ell^a) \widehat{g}_{m-k_0}(z_\ell^a) \prod_{k=1}^m \mathcal{E}_{n_k} \right\} \\ & = - \text{Sym} \left\{ \sum_{\substack{n_1+\dots+n_m=n \\ n_k \geq 1, 1 < m < n}} \left[\sum_{k_0=1}^{m-1} g_2^{j_{k_0}} \widehat{g}_{k_0}(z_\ell^a) \widehat{g}_{m-k_0}(z_\ell^a) \right] \prod_{k=1}^m \mathcal{E}_{n_k} \right\}. \end{aligned} \quad (\text{E.13})$$

Here we first combined three contributions into one sum over splittings by adding the condition $\ell \in \{j_k\}$, with k_0 being the index for which $j_{k_0} = \ell$, and then interchanged the two sums which allows to drop the condition $\ell \in \{j_k\}$, but adds the requirement $m > 1$ (following from $\ell \in \{j_k\}$ in the previous representation). In square brackets one recognizes the first contribution from the r.h.s. of (5.3) with n replaced by $m < n$. Hence, it is subject to the induction hypothesis which allows to replace this expression by $W_m(\{\gamma'_k\}) - \widehat{g}_m(\{\gamma'_k\}, z^a)$. Combining all contributions together, one concludes that (E.11) is equal to

$$g_n^{(0)} - \mathcal{E}_n^{(0)} - \text{Sym} \left\{ \sum_{\substack{n_1+\dots+n_m=n \\ n_k \geq 1, 1 < m < n}} (\widehat{g}_m - W_m) \prod_{k=1}^m \mathcal{E}_{n_k} \right\} + W_n. \quad (\text{E.14})$$

The contributions with W can be combined into one sum by dropping the condition $m < n$. The resulting sum coincides with the r.h.s. of (5.10) so that these contributions can be replaced by $-\mathcal{E}_n^{(+)}$. Combined with $-\mathcal{E}_n^{(0)}$, this gives $-\mathcal{E}_n$ and allows to drop the condition $m > 1$ in the remaining sum with \widehat{g}_m . As a result, (E.14) becomes equivalent to the r.h.s. of (5.7), which proves the consistency of this ansatz with the recursive equation.

Finally, let us show that the ansatz satisfies the modularity constraint (5.5). The crucial observation is that the vectors \mathbf{v}_{ij} and \mathbf{u}_{ij} (4.10) satisfy

$$(\mathbf{v}_{i+j,k}, \mathbf{v}_{ij}) = (\mathbf{u}_{i+j,k}, \mathbf{v}_{ij}) = 0, \quad (\text{E.15})$$

where we abused notation with $\mathbf{v}_{i+j,k} = \mathbf{v}_{ik} + \mathbf{v}_{jk}$, etc. These orthogonality relations together with assumption that \mathcal{E}_n depend on electric charges only through the DSZ products $\gamma_{ij} \sim (\mathbf{v}_{ij}, \mathbf{x})$ imply factorization of the action of Vignéras' operator on the kernel corresponding to the second term in (5.7). Indeed, all contributions of $\partial_{\mathbf{x}}^2$ where two derivatives act on different factors vanish and the action reduces to the sum of terms where Vignéras' operator acts on one of the factors. But since it is supposed to vanish on $\Phi_n^\mathcal{E}$, one obtains the simple result

$$V_{n-1} \cdot \Phi_n^{\widehat{g}} = V_{n-1} \cdot \Phi_n^{g^{(0)}} - \text{Sym} \left\{ \sum_{\substack{n_1+\dots+n_m=n \\ n_k \geq 1, m < n}} (V_{m-1} \cdot \Phi_m^{\widehat{g}}) \prod_{k=1}^m \Phi_{n_k}^\mathcal{E} \right\}. \quad (\text{E.16})$$

Since for $n = 2$ the constraint was already shown to hold, one can proceed by induction. Then in the second term one can substitute the r.h.s. of (5.5), whereas the first term can be evaluated using the recursive relation (5.8). First of all, by the same reasoning as above, away from discontinuities, the action of Vignéras' operator is factorized and actually vanishes. Furthermore, since $g_n^{(0)}$ have discontinuities only at walls of marginal stability, to obtain the complete action, it is enough to consider it only on $\text{sgn}(S_\ell)$. Since at $S_\ell = 0$ one has $c_i^{(\ell)} = c_i$ (see (2.14)), one finds that $\Phi_n^{g^{(0)}}$ satisfy exactly the same constraint as (5.5). Thus, one can rewrite (E.16) as

$$V_{n-1} \cdot \Phi_n^{\hat{g}} = \text{Sym} \sum_{\ell=1}^{n-1} \left(\mathbf{u}_\ell^2 \Delta_{n,\ell}^{g^{(0)}} \delta'(\mathbf{u}_\ell \cdot \mathbf{x}) + 2\mathbf{u}_\ell \cdot \partial_{\mathbf{x}} \Delta_{n,\ell}^{g^{(0)}} \delta(\mathbf{u}_\ell \cdot \mathbf{x}) \right) \quad (\text{E.17})$$

$$- \text{Sym} \left\{ \sum_{\substack{n_1+\dots+n_m=n \\ n_k \geq 1, 1 < m < n}} \left[\sum_{\ell=1}^{n-1} \left(\mathbf{u}_\ell^2 \Delta_{m,\ell}^{\hat{g}} \delta'(\mathbf{u}_\ell \cdot \mathbf{x}) + 2\mathbf{u}_\ell \cdot \partial_{\mathbf{x}} \Delta_{m,\ell}^{\hat{g}} \delta(\mathbf{u}_\ell \cdot \mathbf{x}) \right) \right] \prod_{k=1}^m \Phi_{n_k}^{\mathcal{E}} \right\}.$$

Note that the orthogonality relation allows to include $\Phi_{n_k}^{\mathcal{E}}$ under the derivative in the last term. Then one can perform the same manipulations with the sum over splittings as in (E.13) but in the inverse direction, which directly leads to the constraint (5.5).

Proposition 3

To prove the proposition, one needs to verify that the ansatz (5.12) satisfies the recursive equation (5.8). To this end, we substitute the ansatz into the l.h.s. of this equation and decompose $\Gamma_{n\ell} = -\sum_{i=1}^{\ell} \sum_{j=\ell+1}^n \gamma_{ij}$. Then the crucial observation is that this double sum, the sum over ℓ and the two sums over trees (over \mathbb{T}_ℓ^ℓ and $\mathbb{T}_{n-\ell}^\ell$) are equivalent to a single sum over trees with n vertices, i.e. over \mathbb{T}_n^ℓ , supplemented by the sum over edges. Namely, one can do the following replacement

$$\frac{1}{2} \sum_{\ell=1}^{n-1} \frac{1}{\ell!(n-\ell)!} \sum_{\mathcal{T}_L \in \mathbb{T}_\ell^\ell} \sum_{\mathcal{T}_R \in \mathbb{T}_{n-\ell}^\ell} \sum_{i=1}^{\ell} \sum_{j=\ell+1}^n = \frac{1}{n!} \sum_{\mathcal{T} \in \mathbb{T}_n^\ell} \sum_{e \in E_{\mathcal{T}}} \quad (\text{E.18})$$

The idea is that on the l.h.s. one sums over all possible splittings of unrooted labelled trees with n vertices into two trees with ℓ and $n - \ell$ vertices. Such splitting can be done by cutting an edge and then i, j correspond to the labels of the vertices joined by the cutting edge. The relative coefficient $\frac{n!}{\ell!(n-\ell)!}$ takes into account that after splitting the vertices of the first tree can have arbitrary labels from the set $\{1, \dots, n\}$ and not necessarily $\{1, \dots, \ell\}$, whereas $\frac{1}{2}$ avoids doubling due to the symmetry between \mathcal{T}_L and \mathcal{T}_R .

It is easy to check that all factors in (5.8) fit this interpretation and the l.h.s. takes the following form

$$\frac{(-1)^{n-1+\sum_{i<j} \gamma_{ij}}}{2^{n-1} n!} \sum_{\mathcal{T} \in \mathbb{T}_n^\ell} \prod_{e \in E_{\mathcal{T}}} \gamma_{s(e)t(e)} \sum_{e \in E_{\mathcal{T}}} (\text{sgn}(S_e) - \text{sgn}(\Gamma_e)) \prod_{e' \in E_{\mathcal{T}} \setminus \{e\}} \text{sgn} \left(S_{e'} - \frac{\Gamma_{e'}}{\Gamma_e} S_e \right), \quad (\text{E.19})$$

where S_e and Γ_e were defined in (5.13) and (5.14). This expression immediately reproduces the r.h.s. of (5.8) upon applying the following sign identity, established in [20, Eq.(A.7)],

$$\sum_{\beta=1}^m (\text{sgn}(x_\beta) - 1) \prod_{\substack{\alpha=1 \\ \alpha \neq \beta}}^m \text{sgn}(x_\alpha - x_\beta) = \prod_{\alpha=1}^m \text{sgn}(x_\alpha) - 1, \quad (\text{E.20})$$

where one should take the label α to run over $m = n-1$ edges of a tree \mathcal{T} , identify $x_\alpha = S_\alpha/\Gamma_\alpha$, and multiply it by $\prod_{\alpha=1}^{n-1} \text{sgn}(\Gamma_\alpha)$. This completes the proof of the proposition.

Proposition 4

This Proposition trivially follows from (5.7).

Proposition 5

The easiest way to prove (5.18) is to substitute it into (5.4) and then to check that the result is consistent with the constraint (5.10).

The substitution generates a sum over trees which resemble the blooming trees of appendix B: these are trees with vertices from which other trees grows. But this time the two types of trees, representing the ‘base’ and the ‘flowers’, are actually the same – both of them are Schröder trees. The only difference between them is that to vertices of the ‘base’ one assigns the factors $\mathcal{E}_v^{(+)}$, whereas the vertices of ‘flowers’ have weights $\mathcal{E}_v^{(0)}$. Equivalently, this can be written as

$$W_n = \text{Sym} \left\{ \sum_{T \in \mathbb{T}_n^S} (-1)^{n_T} \sum_{T' \subseteq T} \prod_{v' \in V_{T'}} \mathcal{E}_{v'}^{(+)} \prod_{v \in V_T \setminus V_{T'}} \mathcal{E}_v^{(0)} \right\}, \quad (\text{E.21})$$

where the second sum goes over all subtrees of T containing its root. Its clear that the two contributions differing only by whether a vertex v is included into T' or not, can be combined producing the factor \mathcal{E}_v assigned to this vertex. Thus, the sum over subtrees can be evaluated explicitly and gives

$$W_n = \text{Sym} \left\{ \sum_{T \in \mathbb{T}_n^S} (-1)^{n_T} \mathcal{E}_{v_0}^{(+)} \prod_{v \in V_T \setminus \{v_0\}} \mathcal{E}_v \right\}. \quad (\text{E.22})$$

Given this result, the check of the constraint (5.10) is analogous to the proof of the relation (E.4): the contribution of each tree T (from the sum in (E.22)) and a splitting with $n_k > 1$ (from the sum in (5.10)) is cancelled by the contribution of another tree obtained from T by adding n_k children to its k th leaf and the same splitting but with $n_k = 1$. The only contribution which survives is the one generated by the tree with one vertex and n leaves and the splitting with all $n_k = 1$. It is given by $\mathcal{E}_n^{(+)}$, which verifies the constraint and proves the proposition.

F. Explicit results up to 4th order

In this appendix we provide explicit expressions for various functions appearing in our construction up to the forth order. To write them down, we will use the shorthand notation

$\gamma_{i+j} = \gamma_i + \gamma_j$, $c_{i+j} = c_i + c_j$, etc. as well as indicate the arguments of functions through their indices, for instance, $\mathcal{E}_{i_1 \dots i_n} = \mathcal{E}_n(\tilde{\gamma}_{i_1}, \dots, \tilde{\gamma}_{i_n})$. These expressions are obtained by simple substitutions using the results found in the main text and the sets of trees shown in Fig. 1.

The results (5.17) and (5.18) generate the following expansions

$$\begin{aligned}
h_{p,q}^{\text{DT}} &= \widehat{h}_{p,\mu} + \sum_{\tilde{\gamma}_1 + \tilde{\gamma}_2 = \tilde{\gamma}} \left[g_{12}^{(0)} - \mathcal{E}_{12} \right] e^{\pi i \tau Q_2(\{\tilde{\gamma}_i\})} \widehat{h}_{p_1, \mu_1} \widehat{h}_{p_2, \mu_2} \\
&+ \sum_{\sum_{i=1}^3 \tilde{\gamma}_i = \tilde{\gamma}} \left[g_{123}^{(0)} - \mathcal{E}_{123} - 2 \left(g_{1+2,3}^{(0)} - \mathcal{E}_{1+2,3} \right) \mathcal{E}_{12} \right] e^{\pi i \tau Q_3(\{\tilde{\gamma}_i\})} \prod_{i=1}^3 \widehat{h}_{p_i, \mu_i} \\
&+ \sum_{\sum_{i=1}^4 \tilde{\gamma}_i = \tilde{\gamma}} \left[g_{1234}^{(0)} - \mathcal{E}_{1234} - 2 \left(g_{1+2+3,4}^{(0)} - \mathcal{E}_{1+2+3,4} \right) \left(\mathcal{E}_{123} - 2\mathcal{E}_{1+2,3} \mathcal{E}_{12} \right) \right. \\
&\quad \left. - 3 \left(g_{1+2,34}^{(0)} - \mathcal{E}_{1+2,34} \right) \mathcal{E}_{12} + \left(g_{1+2,3+4}^{(0)} - \mathcal{E}_{1+2,3+4} \right) \mathcal{E}_{12} \mathcal{E}_{34} \right] e^{\pi i \tau Q_4(\{\tilde{\gamma}_i\})} \prod_{i=1}^4 \widehat{h}_{p_i, \mu_i} + \dots,
\end{aligned} \tag{F.1}$$

$$\begin{aligned}
\widehat{h}_{p,q} &= h_{p,\mu} + \sum_{\tilde{\gamma}_1 + \tilde{\gamma}_2 = \tilde{\gamma}} \mathcal{E}_{12}^{(+)} e^{\pi i \tau Q_2(\{\tilde{\gamma}_i\})} h_{p_1, \mu_1} h_{p_2, \mu_2} \\
&+ \sum_{\sum_{i=1}^3 \tilde{\gamma}_i = \tilde{\gamma}} \left[\mathcal{E}_{123}^{(+)} - 2\mathcal{E}_{1+2,3}^{(+)} \mathcal{E}_{12}^{(0)} \right] e^{\pi i \tau Q_3(\{\tilde{\gamma}_i\})} \prod_{i=1}^3 h_{p_i, \mu_i} \\
&+ \sum_{\sum_{i=1}^4 \tilde{\gamma}_i = \tilde{\gamma}} \left[\mathcal{E}_{1234}^{(+)} - 2\mathcal{E}_{1+2+3,4}^{(+)} \left(\mathcal{E}_{123}^{(0)} - 2\mathcal{E}_{1+2,3}^{(0)} \mathcal{E}_{12}^{(0)} \right) - 3\mathcal{E}_{1+2,34}^{(+)} \mathcal{E}_{12}^{(0)} \right. \\
&\quad \left. + \mathcal{E}_{1+2,3+4}^{(+)} \mathcal{E}_{12}^{(0)} \mathcal{E}_{34}^{(0)} \right] e^{\pi i \tau Q_4(\{\tilde{\gamma}_i\})} \prod_{i=1}^4 h_{p_i, \mu_i} + \dots,
\end{aligned} \tag{F.2}$$

where the functions $g_n^{(0)}$ and \mathcal{E}_n can be read off from (5.12) and (5.16)

$$\begin{aligned}
g_2^{(0)} &= \frac{(-1)^{1+\gamma_{12}}}{4} \gamma_{12} \text{sgn}(c_1), \\
g_3^{(0)} &= \frac{(-1)^{1+\gamma_{12}+\gamma_{1+2,3}}}{8} \text{Sym} \left\{ \gamma_{12} \gamma_{23} \text{sgn}(c_1) \text{sgn}(c_3) \right\}, \\
g_4^{(0)} &= \frac{(-1)^{\gamma_{12}+\gamma_{1+2,3}+\gamma_{1+2+3,4}}}{16} \text{Sym} \left\{ \gamma_{12} \gamma_{23} \gamma_{34} \text{sgn}(c_1) \text{sgn}(c_{1+2}) \text{sgn}(c_4) \right. \\
&\quad \left. - \frac{1}{3} \gamma_{12} \gamma_{23} \gamma_{24} \text{sgn}(c_1) \text{sgn}(c_3) \text{sgn}(c_4) \right\},
\end{aligned} \tag{F.3}$$

$$\begin{aligned}
\mathcal{E}_{12} &= \frac{(-1)^{\gamma_{12}}}{4} \left[\gamma_{12} E_1 \left(\frac{\sqrt{2\tau_2} |\gamma_{12}|}{\sqrt{(pp_1 p_2)}} \right) + \frac{\sqrt{(pp_1 p_2)}}{\pi \sqrt{2\tau}} e^{-\frac{2\pi \tau_2 \gamma_{12}^2}{(pp_1 p_2)}} \right], \\
\mathcal{E}_{123} &= \frac{(-1)^{\gamma_{12}+\gamma_{1+2,3}}}{8(2\tau_2)} \text{Sym} \left\{ \tilde{\Phi}_2^E \left((\mathbf{v}_{1,2+3}, \mathbf{v}_{1+2,3}), (\mathbf{v}_{12}, \mathbf{v}_{23}); \sqrt{2\tau_2}(\mathbf{q} + \mathbf{b}) \right) \right\}, \\
\mathcal{E}_{1234} &= \frac{(-1)^{\gamma_{12}+\gamma_{1+2,3}+\gamma_{1+2+3,4}}}{16(2\tau_2)^{3/2}} \text{Sym} \left\{ \tilde{\Phi}_3^E \left((\mathbf{v}_{1,2+3+4}, \mathbf{v}_{1+2+3,4}, \mathbf{v}_{1+2+3,4}), (\mathbf{v}_{12}, \mathbf{v}_{23}, \mathbf{v}_{34}); \sqrt{2\tau_2}(\mathbf{q} + \mathbf{b}) \right) \right. \\
&\quad \left. + \frac{1}{3} \tilde{\Phi}_3^E \left((\mathbf{v}_{1,2+3+4}, \mathbf{v}_{1+2+4,3}, \mathbf{v}_{1+2+3,4}), (\mathbf{v}_{12}, \mathbf{v}_{23}, \mathbf{v}_{24}); \sqrt{2\tau_2}(\mathbf{q} + \mathbf{b}) \right) \right\}.
\end{aligned} \tag{F.4}$$

Finally, we provide expressions for the kernels $\widehat{\Phi}_n^{\text{tot}}$ of the theta series appearing in the expansion (5.19) of \mathcal{G} in powers of $\widehat{h}_{p,\mu}$:

$$\begin{aligned}
\widehat{\Phi}_1^{\text{tot}} &= \Phi_1, \\
\widehat{\Phi}_2^{\text{tot}} &= \Phi_2 + \Phi_1 \Phi_2^{\widehat{g}} = \frac{1}{4} \Phi_1 \left(\widetilde{\Phi}_1^E(\mathbf{u}_{12}, \mathbf{v}_{12}) - \widetilde{\Phi}_1^E(\mathbf{v}_{12}, \mathbf{v}_{12}) \right), \\
\widehat{\Phi}_3^{\text{tot}} &= \Phi_3 + 2 \text{Sym} \left\{ \Phi_2(x_{1+2}, x_3) \Phi_2^{\widehat{g}}(x_1, x_2) \right\} + \Phi_1 \Phi_3^{\widehat{g}} \\
&= \frac{1}{8} \Phi_1 \text{Sym} \left\{ \widetilde{\Phi}_2^E((\mathbf{u}_{1,2+3}, \mathbf{u}_{1+2,3}), (\mathbf{v}_{12}, \mathbf{v}_{23})) - \widetilde{\Phi}_2^E((\mathbf{v}_{1,2+3}, \mathbf{v}_{1+2,3}), (\mathbf{v}_{12}, \mathbf{v}_{23})) \right. \\
&\quad \left. - \left(\widetilde{\Phi}_1^E(\mathbf{u}_{1+2,3}, \mathbf{v}_{1+2,3}) - \widetilde{\Phi}_1^E(\mathbf{v}_{1+2,3}, \mathbf{v}_{1+2,3}) \right) \widetilde{\Phi}_1^E(\mathbf{u}_{12}, \mathbf{v}_{12}) \right\}, \tag{F.5} \\
\widehat{\Phi}_4^{\text{tot}} &= \Phi_4 + \text{Sym} \left\{ 3 \Phi_3(x_{1+2}, x_3, x_4) \Phi_2^{\widehat{g}}(x_1, x_2) + \Phi_2(x_{1+2}, x_{3+4}) \Phi_2^{\widehat{g}}(x_1, x_2) \Phi_2^{\widehat{g}}(x_3, x_4) \right. \\
&\quad \left. + 2 \Phi_2(x_{1+2+3}, x_3) \Phi_3^{\widehat{g}}(x_1, x_2, x_3) \right\} + \Phi_1 \Phi_4^{\widehat{g}} \\
&= \frac{1}{16} \Phi_1 \text{Sym} \left\{ \widetilde{\Phi}_3^E((\mathbf{u}_{1,2+3+4}, \mathbf{u}_{1+2,3+4}, \mathbf{u}_{1+2+3,4}), (\mathbf{v}_{12}, \mathbf{v}_{23}, \mathbf{v}_{34})) \right. \\
&\quad \left. - \widetilde{\Phi}_3^E((\mathbf{v}_{1,2+3+4}, \mathbf{v}_{1+2,3+4}, \mathbf{v}_{1+2+3,4}), (\mathbf{v}_{12}, \mathbf{v}_{23}, \mathbf{v}_{34})) \right. \\
&\quad \left. + \frac{1}{3} \left(\widetilde{\Phi}_3^E((\mathbf{u}_{1,2+3+4}, \mathbf{u}_{1+2+4,3}, \mathbf{u}_{1+2+3,4}), (\mathbf{v}_{12}, \mathbf{v}_{24}, \mathbf{v}_{34})) \right. \right. \\
&\quad \left. \left. - \widetilde{\Phi}_3^E((\mathbf{v}_{1,2+3+4}, \mathbf{v}_{1+2+4,3}, \mathbf{v}_{1+2+3,4}), (\mathbf{v}_{12}, \mathbf{v}_{24}, \mathbf{v}_{34})) \right) \right. \\
&\quad \left. - \left(\widetilde{\Phi}_2^E((\mathbf{u}_{1+2,3+4}, \mathbf{u}_{1+2+3,4}), (\mathbf{v}_{1+2,3}, \mathbf{v}_{34})) - \widetilde{\Phi}_2^E((\mathbf{v}_{1+2,3+4}, \mathbf{v}_{1+2+3,4}), (\mathbf{v}_{1+2,3}, \mathbf{v}_{34})) \right) \right. \\
&\quad \left. + \frac{1}{2} \left(\widetilde{\Phi}_2^E((\mathbf{u}_{1+2+4,3}, \mathbf{u}_{1+2+3,4}), (\mathbf{v}_{1+2,3}, \mathbf{v}_{1+2,4})) \right. \right. \\
&\quad \left. \left. - \widetilde{\Phi}_2^E((\mathbf{v}_{1+2+4,3}, \mathbf{v}_{1+2+3,4}), (\mathbf{v}_{1+2,3}, \mathbf{v}_{1+2,4})) \right) \right) \widetilde{\Phi}_1^E(\mathbf{u}_{12}, \mathbf{v}_{12}) \\
&\quad - \left(\widetilde{\Phi}_1^E(\mathbf{u}_{1+2+3,4}, \mathbf{v}_{1+2+3,4}) - \widetilde{\Phi}_1^E(\mathbf{v}_{1+2+3,4}, \mathbf{v}_{1+2+3,4}) \right) \widetilde{\Phi}_2^E((\mathbf{v}_{1,2+3}, \mathbf{v}_{1+2,3}), (\mathbf{v}_{12}, \mathbf{v}_{23})) \\
&\quad + \left(\widetilde{\Phi}_1^E(\mathbf{u}_{1+2+3,4}, \mathbf{v}_{1+2+3,4}) - \widetilde{\Phi}_1^E(\mathbf{v}_{1+2+3,4}, \mathbf{v}_{1+2+3,4}) \right) \widetilde{\Phi}_1^E(\mathbf{u}_{12}, \mathbf{v}_{12}) \widetilde{\Phi}_1^E(\mathbf{u}_{1+2,3}, \mathbf{v}_{1+2,3}) \\
&\quad \left. + \frac{1}{4} \left(\widetilde{\Phi}_1^E(\mathbf{u}_{1+2,3+4}, \mathbf{v}_{1+2,3+4}) - \widetilde{\Phi}_1^E(\mathbf{v}_{1+2,3+4}, \mathbf{v}_{1+2,3+4}) \right) \widetilde{\Phi}_1^E(\mathbf{u}_{12}, \mathbf{v}_{12}) \widetilde{\Phi}_1^E(\mathbf{u}_{34}, \mathbf{v}_{34}) \right\},
\end{aligned}$$

where $\Phi_1(x)$ is defined in (4.13). These explicit results for $\widehat{\Phi}_n^{\text{tot}}$ are the basis for the conjectural formula (5.20).

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